# Computer assisted proof of chaos in the Rössler equations and in the Hénon map

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# Computer assisted proof of chaos in the Rössler equations and in the Hénon map

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**Abstract.** We introduce horseshoe-type mappings which are geometrically similar to Smale's horseshoes. For such mappings we prove by means of the fixed point index the existence of chaotic dynamics—the semi-conjugacy to the shift on a finite number of symbols. Our theorem does not require any assumptions concerning derivatives, it is a purely topological result. The assumptions of our theorem are then rigorously verified by computer assisted computations for the classical Hénon map and for classical Rössler equations.

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## Introduction

The aim of the paper is to describe a new technique for obtaining rigorous results concerning the global dynamics of nonlinear systems. The technique combines abstract existence results based on topological invariants (the fixed point index) with finite, computer assisted computations necessary to verify the assumptions of the theorems in a concrete example.

We introduce a class of TS-maps (the topological shifts) which includes as particular cases the Smale's horseshoes [S]. For such maps we prove by means of the fixed point index the existence of chaotic dynamics—the semi-conjugacy to the shift on a finite number of symbols, with an infinite number of periodic points with unbounded periods. It should be insisted that our theorem does not require any assumptions concerning derivatives, only some simple inclusions of images of sets under considerations should be checked.

The assumptions of our theorem are then rigorously verified by computer assisted computations for the classical Hénon map and for classical Rössler equations. The necessary computer assisted computations are small enough to be performed on a PC.

We show that the seventh iterate of the Hénon map and a suitably chosen Poincaré map derived from Rössler equation have a horseshoe-type dynamics on the invariant set embedded in the numerically observed strange attractor, but the very existence of a strange attractor still remains unproven. At least for the Hénon map (with classical parameter values) there is a fundamental obstacle to such a proof. Numerical simulation suggests that the Hénon map has homoclinic tangencies (stable and unstable manifolds of fixed point have non-transversal intersections). Therefore, the Hénon map can probably be embedded into a one-parameter family of diffeomorphisms unfolding a homoclinic tangency. It follows from papers [BC] and [MV] that for such families with some additional generic assumptions the

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parameter set for which there exists a strange attractor (an attracting set with dense orbit, with positive Lapunov exponent) has a positive Lebesgue measure and the set of parameters for which there exists attractive periodic orbits is dense. So for any system in a one-parameter family, to which the theorems from [BC] and [MV] apply, it is imposible to decide rigoursly using just topological methods whether it has a strange attractor, as all statements obtained by topological mean hold for all nearby systems. Probably the same holds for the Rössler equations.

The idea of a computer assisted proof of chaotic dynamics based on topological invariants appeared first in the work of Mischaikow and Mrozek ([MM1], [MM2]). They used the discrete Conley index introduced in [M] to prove the existence of chaotic dynamics for the Hénon map and the Lorenz equations. We believe that our method is considerably easier to understand and apply—the topological machinery behind the fixed point index is relatively simple compared to the one involved in the discrete Conley index.

## 1. Topological theorem

By $\mathcal{R}$, $\mathcal{Z}$, $\mathcal{N}$ we will denote the sets of real numbers, integers and natural numbers (including zero) respectively. Let $(X, \rho)$ be a metric space. Let $Z \subset X$ and $x \in X$. By $\mathrm{int}(Z)$, $\mathrm{cl}(Z)$, $\mathrm{bd}(Z)$ we denote respectively the interior, the closure and the boundary of the set $Z$.

Let $f : X \to X$ be any continous map and $N \subset X$. By $f_{|N}$ we will denote the map obtained by restricting the domain of $f$ to the set $N$. The maximal invariant part of $N$ (with respect to $f$) is defined by

$$\mathrm{Inv}(N, f) = \bigcap_{i\in\mathcal{Z}} f_{|N}^{-i}(N).$$

For any set $P \subset \mathcal{R}^2$, which is union of disjoint rectangles $P_k = [a_k, b_k] \times [c_k, d_k]$ we set

$$L(P) := \bigcup \{a_k\} \times [c_k, d_k] \tag{1}$$
$$R(P) := \bigcup \{b_k\} \times [c_k, d_k] \tag{2}$$
$$V(P) := L(P) \cup R(P) \tag{3}$$
$$H(P) := \bigcup ([a_k, b_k] \times \{c_k\} \cup [a_k, b_k] \times \{d_k\}). \tag{4}$$

So $L(P)$, $R(P)$, $V(P)$, $H(P)$ are equal to the union of left vertical, right vertical, vertical and horizontal egdes in $P$ respectively. In the remaining part of this section we consider maps on the plane $\mathcal{R}^2$.

Let us fix $u, d \in \mathcal{R}$, $u > d$ and a sequence $a_{-1} = -\infty < a_0 < a_1 < \cdots a_{2K-2} < a_{2K-1} < a_{2K} = \infty$, where $a_i \in \mathcal{R}$ for $i = 0, 1, \ldots, 2K-1$. Let

$$N_i := [a_{2i}, a_{2i+1}] \times [d, u] \qquad \text{for } i = 0, \ldots, K-1 \tag{5}$$
$$E_i := (a_{2i-1}, a_{2i}) \times [d, u] \qquad \text{for } i = 0, \ldots, K \tag{6}$$
$$N := N_0 \cup N_1 \cup \cdots \cup N_{K-1} \tag{7}$$
$$E := E_0 \cup E_1 \cup \cdots \cup E_{K-1} \cup E_K. \tag{8}$$

The sets $E_i$, $N_i$ are contained in the horizontal strip $(-\infty, \infty) \times [d, u]$ in the following order (we compare $x$-coordinates)

$$E_0 < N_0 < E_1 < N_1 < \cdots < E_{K-1} < N_{K-1} < E_K. \tag{9}$$

Suppose further that for $i = 0, 1, \dots, K$ we have sets $E'_i$ such that

$$E'_i \cap (-\infty, \infty) \times [d, u] = E_i \tag{10}$$
$$\mathrm{cl}(E'_i) \cap (N \setminus V(N)) = \emptyset \tag{11}$$
$$\mathrm{cl}E'_i \cap \mathrm{cl}E'_j = \emptyset \qquad \text{for } i \neq j \tag{12}$$

and there exist continuous homotopies $h_i : [0, 1] \times E'_i \to E'_i$ such that

$$h_i(0, p) = p \qquad \text{for } p \in E'_i \tag{13}$$
$$h_i(1, p) \in E_i \qquad \text{for } p \in E'_i \tag{14}$$
$$h_i(t, p) = p \qquad \text{for } p \in E_i \text{ and } t \in [0, 1]. \tag{15}$$

This means that the set $E'_i$ can be continuously deformed to the set of $E_i$ without any intersection with the set $N$. $E_i$ is a *deformation retract* of $E'_i$. We set

$$E' := E'_0 \cup E'_1 \cup \dots E'_K.$$

Let us remark that from conditions (10)–(11) follows

$$E'_i \cap N_j = \emptyset \qquad \text{for } i, j = 0, 1, \dots, K - 1. \tag{16}$$

Figure 1 presents a schematic drawing of the sets $N_i$, $E_i$, $E'_i$ for $K = 3$.

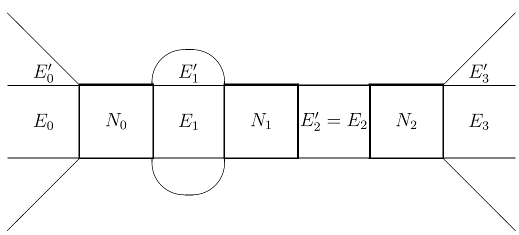


**Figure 1.** An example of sets $N_i$, $E_i$, $E'_i$ for $K = 3$.

**Definition 1.** Let the sets $E_i$, $E'_i$, $N_i$ be as above. Let $D$ be an open set such that $N \subset D$ and a map $f : D \to \mathcal{R}^2$ be continuous. We say that $f$ is TS-map (topological shift) (relatively to the sets $N$, $E$, $E'$) if there exist functions $l, r : \{0, 1, \dots, K - 1\} \to \{0, 1, \dots, K\}$ such that the following conditions hold

$$f(L(N_i)) \subset E'_{l(i)} \qquad f(R(N_i)) \subset E'_{r(i)} \tag{17}$$
$$f(N) \subset E' \cup N. \tag{18}$$

Geometrically, the above conditions mean that the image of the vertical edges does not intersect the set $N$ and the image of $N$ is contained in the set which can be continuously deformed to the horizontal strip without any intersection with horizontal egdes of $N$.

We are looking for periodic points of the TS-map $f$. We will characterize them by periodic infinite sequences $c = (c_i)_{i \in \mathcal{N}}$ of symbols $0, 1, \dots, K - 1$ with the property $f^i(x) \in N_{c_i}$ for $i \in \mathcal{N}$.

Let $\Sigma_K := \{0, 1, \dots, K - 1\}^{\mathcal{Z}}$, $\Sigma^+_K := \{0, 1, \dots, K - 1\}^{\mathcal{N}}$. $\Sigma_K$, $\Sigma^+_K$ are topological spaces with the Tichonov topology. On $\Sigma_K$, $\Sigma^+_K$ we have the shift map $\sigma$ given by

$$(\sigma(c))_i = c_{i+1}.$$

Let $A = [\alpha_{ij}]$ be a $K \times K$-matrix, $\alpha_{ij} \in \mathcal{R}_+ \cup \{0\}$, $i, j = 0, 1, \ldots, K-1$. We define $\Sigma_A \subset \Sigma_K$ and $\Sigma_A^+ \subset \Sigma_K^+$ by

$$\Sigma_A := \{c = (c_i)_{i \in \mathcal{Z}} | \alpha_{c_i c_{i+1}} > 0\} \tag{19}$$

$$\Sigma_A^+ := \{c = (c_i)_{i \in \mathcal{N}} | \alpha_{c_i c_{i+1}} > 0\}. \tag{20}$$

Obviously $\Sigma_A^+$, $\Sigma_A$ are invariant under $\sigma$.

Let $f$ be a TS-map. To relate the dynamics of $f$ on $\mathrm{Inv}(N, f)$ with shift dynamics on $\Sigma_K^+$ we introduce the *transition matrix of* $f$ denoted by $A(f)$.

We define $A(f)_{i,j}$, where $i, j = 0, 1, \ldots, K-1$ by

$$A(f)_{ij} := \begin{cases} 1 & \text{if } E_{l(i)} < N_j < E_{r(i)} \text{ or } E_{l(i)} > N_j > E_{r(i)} \\ 0 & \text{otherwise.} \end{cases}$$

It easy easy to see that $A(f)_{i,j} \neq 0$, if $N_j$ lies between the images of vertical egdes of $N_i$ (we deform the image by the homotopies $h$ if necessary).

For $i \in \mathcal{N}$ we define a map $\pi_i : \mathrm{Inv}(N, f) \to \{0, 1, \ldots, K\}$ given by $\pi_i(x) = j$ iff $f^i(x) \in N_j$. Now we define a map $\pi : \mathrm{Inv}(N, f) \to \Sigma_K^+$ by $\pi(x) := (\pi_i(x))_{i \in \mathcal{N}}$. The map $\pi$ assigns to the point $x$ the indices of rectangles $N_i$ its trajectory goes through. It is easy to see that we have

$$\pi \circ f = \sigma \circ \pi. \tag{21}$$

If $f$ is also a homeomorphism, then the definition of $\pi_i$ can be extended to all integers and the domain of $\pi$ is $\Sigma_K$.

Obviously the semiconjugacy (21) alone is not a sign of complicated dynamics. It may happen that the set $\mathrm{Inv}(N, f)$ is finite or even empty. The dynamics will be complicated if the set $\pi(\mathrm{Inv}(N, f))$ is infinite. The following theorem gives the characterization of this set for TS-maps.

**Theorem 1.** *Let $f$ be a TS-map. Then $\Sigma_{A(f)}^+ \subset \pi(Inv(N, f))$. The pre-image of any periodic sequence from $\Sigma_{A(f)}^+$ contains periodic points of $f$. If we additionally suppose that $f$ is a homeomorphism, then $\Sigma_{A(f)} \subset \pi(Inv(N, f))$.*

The proof of this theorem is postponed to the next section.

## 2. Proof of the topological theorem

Let $N \subset \mathcal{R}^d$ be a compact set and $f : N \to \mathcal{R}^d$ be a continuous map. The set $N$ is called *an isolating neighbourhood* iff $\mathrm{Inv}(N, f) \subset \mathrm{Int}(N)$.

Let $Y \subset \mathcal{R}^d$, $\mathrm{cl}(Y) \subset \mathrm{Dom}(f)$ be such that $f(x) \neq x$ for $x \in \mathrm{bd}(Y)$ then we denote by $I(f, Y)$ the fixed point index of the map $f$ relatively to the set $Y$ (see [D, ch VII.5]). In the sequel we will use the following properties of the fixed point index

$$I(f, Y) \neq 0 \Longrightarrow \exists y \in Y \qquad f(y) = y. \tag{22}$$

Let $A$ be a $n \times n$ matrix and $x_0 \in \mathcal{R}^d$. Suppose that the equation $Ax = x$ does not have any non-zero solution and $x_0 \in \mathrm{Int}(Y)$. Let us denote the identity matrix by Id. Then

$$I(A(x - x_0) + x_0, Y) = \mathrm{sgn}(\det(\mathrm{Id} - A)). \tag{23}$$

For any map $F : [0, 1] \times N \to \mathcal{R}^d$ we set $F_\lambda(x) := F(\lambda, x)$.

The following theorem was obtained by the author in [Z1]

**Theorem 2.** *Let* $N = \bigcup_{i=0}^{i=K-1} N_i$*, where* $N_i \subset \mathcal{R}^d$ *are compact and disjoint. Let* $F : [0,1] \times N \to \mathcal{R}^n$ *be a continuous map, such that* $N$ *is an isolating neighbourhood for* $F_\lambda$ *for* $\lambda \in [0,1]$*. Then for every finite sequence* $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \{0, 1, \ldots, K-1\}^{n+1}$ *the fixed point index* $I(F_\lambda^{n+1}, N_{\alpha_0} \cap F_\lambda^{-1}(N_{\alpha_1}) \cap \ldots \cap F_\lambda^{-n}(N_{\alpha_n}))$ *is defined and does not depend on* $\lambda$ *(is equal to a constant independent from* $\lambda$*).*

**Proof of theorem 1.** First we observe that the conditions (17) and (18) imply that

$$f^{-1}(N) \cap N \cap f(N) \cap \mathrm{bd}(N) = \emptyset$$

but this means that $\mathrm{Inv}(N, f) \subset \mathrm{Int}(N)$, so $N$ is an isolating neighbourhood.

We will now connect the map $f$ by the appropriate homotopy ($N$ should be an isolating neighbourhood along this homotopy) with the piecewise affine model map $m$ for which we can easily calculate various fixed point indices.

Let us fix $s$ such that $u < s < d$. Let us choose $b_i \in \mathcal{R}$, $i = 0, 1, \ldots, 2K-1$ such that

$$b_{2i} \in (a_{2l(i)-1}, a_{2l(i)}) \qquad b_{2i+1} \in (a_{2r(i)-1}, a_{2r(i)}). \tag{24}$$

The functions $l, r$ in this formula are the $l, r$ functions for $f$ from the definition of the TS-map, and the $a_i$ are as defined in the last section.

Now we define a model map $m : N \to \mathcal{R}^2$ by

$$m(x, y) := \left( \frac{b_{2i+1} - b_{2i}}{a_{2i+1} - a_{2i}} (x - a_{2i}) + b_{2i}, s \right) \qquad \text{for } (x, y) \in N_i. \tag{25}$$

With this choice of $b_i, s$ it is obvious that the map $m$ is a TS-map with the same functions $l$ and $r$, so it has the same transition matrix as map the $f$. Obviously $N$ is an isolating neighbourhood for $m$.

We will now construct the homotopy connecting $f$ and $m$. First we define a homotopy $H$ by gluing together the maps $h_i$. $H : [0,1] \times (E' \cup N) \to (E' \cup N)$ given by

$$H(\lambda, p) := \begin{cases} p & \text{if } p \in N \\ h_i(\lambda, p) & \text{if } p \in E_i'. \end{cases}$$

From (10)–(15) it immediately follows that the map $H$ is continuous.

Next we define $F(t, p) = H(t, f(p))$. For every $\lambda \in [0,1]$ $F_\lambda$ is a TS-map (relatively to the same sets $N$, $E$, $'E'$ as the map $f$), with the same functions $l$ and $r$ as for the map $f$. An important point is that $F_1(N) \subset N \cup E$. Now we connect this map with $m$ by the homotopy

$$G(\lambda, p) := (1 - \lambda) F_1(p) + \lambda m(p). \tag{26}$$

From the convexity of the sets $N_i$ and $E_i$ follows that the map $G_\lambda$ for every $\lambda \in [0,1]$ is a TS-map (relatively to the same sets $N_i$, $E_i$ as a map $f$), with the same functions $l$ and $r$ as for the map $f$.

So using $F$ and $G$ we connect $f$ by homotopy with the piecewise affine map $m$ such that $N$ is an isolating neighbourhood along this homotopy; so we can apply theorem 2.

Let $\alpha = (\alpha_0, \alpha_1, \ldots) \in \Sigma_K^+$ be periodic with period $n+1$. Due to the piecewise affine form of the map $m$ and the property (23) one can easily show that

$$I(m^{n+1}, N_{\alpha_0} \cap m^{-1}(N_{\alpha_1}) \cap \cdots \cap m^{-n}(N_{\alpha_n})) := \begin{cases} \pm 1 & \text{for } \alpha \in \Sigma_{A(m)}^+ \\ 0 & \text{otherwise.} \end{cases}$$

Obviously by construction of $m$ we have $A(f) = A(m)$, so for every $n+1$-periodic sequence $\alpha \in \Sigma_{A(f)}^+$ from theorem 2 we get

$$I(f^{n+1}, N_{\alpha_0} \cap f^{-1}(N_{\alpha_1}) \cap \cdots \cap f^{-n}(N_{\alpha_n})) \neq 0.$$

From (22) we obtain $x_\alpha \in N$ such that

$$f^i(x_\alpha) \in N_{\alpha_i} \qquad \text{for } i = 0, 1, \ldots, n, \;\; f^{n+1}(x_\alpha) = x_\alpha \in N_{\alpha_0}. \tag{27}$$

This gives us the existence of periodic points from the assertion of our theorem. Since periodic sequences are dense in $\Sigma_A^+$ for every matrix $A$ we have $\Sigma_{A(f)}^+ \subset \pi(\mathrm{Inv}(N, f))$.

In the homeomorphism case the range of the map $\pi$ is $\Sigma_K$. Let $\alpha$ be an $n+1$-periodic sequence, $\alpha \in \Sigma_{A(f)}$. $\alpha$ can be identified with an element from $\Sigma_{A(f)}^+$ by deleting elements with negative indices. From the first part of the proof it follows that there exists $x_\alpha \in N$ such that (27) holds. Obviously $\pi(x_\alpha) = \alpha$. This shows that all periodic sequences from $\Sigma_{A(f)}$ are contained in $\pi(\mathrm{Inv}(N, f))$. From compactness of $\mathrm{Inv}(N, f)$ and density of periodic sequences in $\Sigma_{A(f)}$ we get $\Sigma_{A(f)} \subset \pi(\mathrm{Inv}(N, f))$. $\square$

## 3. Chaos in the Rössler equations

The Rössler equations are given by [R]

$$\begin{aligned} \dot{x} &= -(y+z) \\ \dot{y} &= x + by \\ \dot{z} &= b + z(x-a) \end{aligned} \tag{28}$$

where $a = 5.7$, $b = 0.2$. These are the parameters values originally considered by Rössler. The flow generated by (28) exihibits a so-called strange attractor. There are no exact results concerning the structure of this attracting set. We show here an invariant set with complicated dynamics apparently contained in the numerically observed attractor.

We will apply theorem 1 to the Poincaré map $P$ generated by (28) on the section $\Theta := \{(x, y, z) | x = 0, y < 0, \dot{x} > 0\}$.

We will consider $P$ as a map $(y, z)_n \to (y, z)_{n+1}$ on $\mathcal{R}^2$. We will show that $P$ is a TS-map with a nontrivial transition matrix. First we need to define constants which describe the sets $N_i$, $E_i$. In this case we choose $K = 3$, $d = -0.02$, $u = 0.08$, $a_0 = -10.3$, $a_1 = -9.01$, $a_2 = -9.0$, $a_3 = -7$, $a_4 = -6.4$, $a_5 = -3.8$ and $E_i' = E_i$.

Figure 2 presents the image of $N$ under $P$. This figure is then decomposed into figures 3–5 showing the images of the sets $N_0$, $N_1$ and $N_2$ individually. From these figures one can

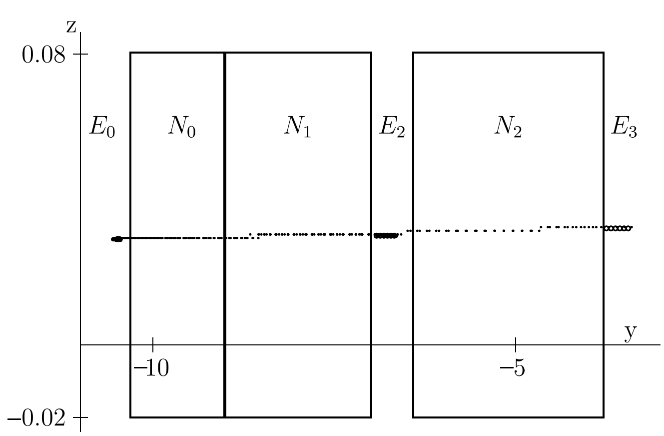


**Figure 2.** Image of $N$ for the Rössler Poincaré map (dots).

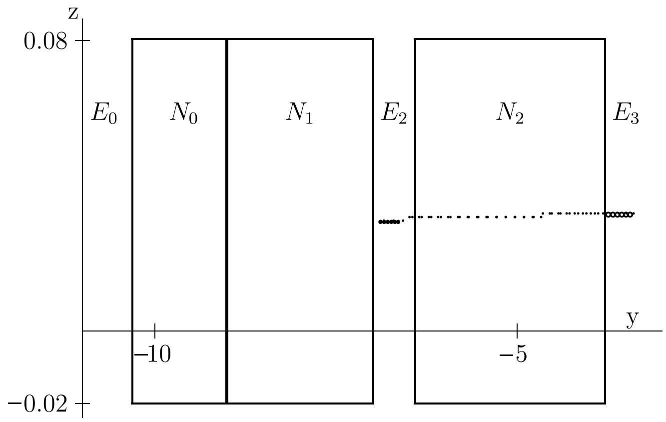


**Figure 3.** Image of $N_0$ for the Rössler Poincaré map (dots). The images of points from left and right vertical egdes are marked by circles and disks, respectively.

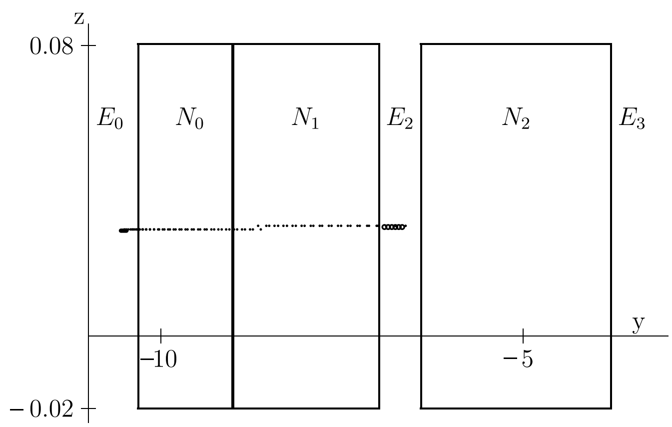


**Figure 4.** Image of $N_1$ for the Rössler Poincaré map (dots). The images of points from left and right vertical egdes are marked by circles and disks, respectively.

clearly see the difference of scales in the $y$- and $z$-direction and manifestly strong contraction in $z$ direction of map $P$.

The following lemma is proved rigorously with computer assistance (see [Z3] for details) (compare figures 3–5).

**Lemma 3.** *For all parameter values in a sufficiently small neighbourhood of* $(a, b) = (5.7, 0.2)$ *the following conditions hold*

$$N \subset \mathrm{Dom}(P) \tag{29}$$

$$P(N) \subset E \cup N \tag{30}$$

$$P(L(N_0)) \subset E_3 \qquad P(R(N_0)) \subset E_2 \tag{31}$$

$$P(L(N_1)) \subset E_2 \qquad P(R(N_1)) \subset E_0 \tag{32}$$

$$P(L(N_2)) \subset E_0 \qquad P(R(N_2)) \subset E_2. \tag{33}$$

This lemma shows that $P$ is a TS-map and enables us to calculate the transition matrix for $P$. So from theorem 1 we obtain

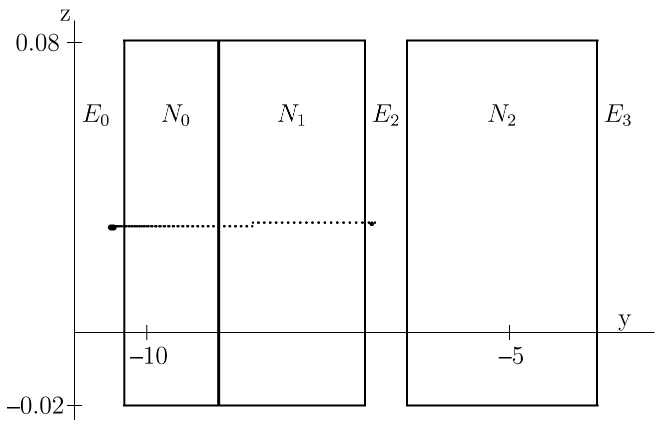


**Figure 5.** Image of $N_2$ for Rössler Poincaré map (dots). The images of points from left and right vertical egdes are marked by circles and disks, respectively.

**Theorem 4.** *For all parameter values in a sufficiently small neighbourhood of* $(a, b) = (5.7, 0.2)$ *there exists a Poincaré section* $N \subset \Theta$ *such that the Poincaré map* $P$ *induced by (28) is well defined and continuous.*

*There exists a continuous map* $\pi : Inv(N, P) \to \Sigma_3$*, such that*

$$\pi \circ P = \sigma \circ \pi.$$

$\Sigma_A \subset \pi(Inv(N, P))$*, where*

$$A := \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

*The pre-image of any periodic sequence from* $\Sigma_A$ *contains periodic points of* $P$*.*

As it was mentioned above we prove lemma 3 with computer assistance. Our program was implemented in Borland C++ 3.1. Rigorous numerical verification of conditions (30)–(33) required calculation of $P$ for around one million points. This took around 50 hours on an IBM PC 486DX 50 MHz computer.

## 4. Chaos in the Hénon map

In this section we deal with the Hènon map [H]. The Hénon map $H : \mathcal{R}^2 \to \mathcal{R}^2$ is given by

$$H_{ab}(x, y) = (1 + y - ax^2, bx).$$

We investigate this mapping for $a = 1.4$, $b = 0.3$—the parameter values considered originally by Hénon (see [H]). The long time behaviour of (4) is very complicated, one observes numerically a 'strange attractor'—an attracting set with fractal nature on which the dynamics is very complicated and chaotic. We will show here again the existence of an invariant set with complicated dynamics which is embedded the numerically observed attractor.

First we introduce new coordinates $x_1$, $y_1$ given by

$$x_1 := x - 0.5y \tag{34}$$

$$y_1 := y. \tag{35}$$

We will express the sets $E$, $E'$, $N$ in these new coordinates. Let $K = 2$, $d = 0.01$, $u = 0.28$, $a_0 = 0.455$, $a_1 = 0.551$, $a_2 = 0.583$, $a_3 = 0.615$. The sets $E'_i$ are given by

$$E'_0 = \{x_1 < a_0 \text{ and } y_1 \geqslant d\} \tag{36}$$

$$E'_1 := E_1 \tag{37}$$

$$E'_2 := \{y_1 < -0.01\} \cup \{x_1 > a_3 \text{ and } y_1 \leqslant u\}. \tag{38}$$

With computer assistance we have proved the following (see [Z2] for details)

**Lemma 5.** *For all parameter values in a sufficiently small neighbourhood of* $(a, b) = (1.4, 0.3)$ *the following conditions are satisfied*

$$H^7_{ab}(N) \subset E' \cup N \tag{39}$$

$$H^7_{ab}(L(N_0)) \subset E'_2 \qquad H^7_{ab}(R(N_0)) \subset E'_0 \tag{40}$$

$$H^7_{ab}(L(N_1)) \subset E'_0 \qquad H^7_{ab}(R(N_1)) \subset E'_2. \tag{41}$$

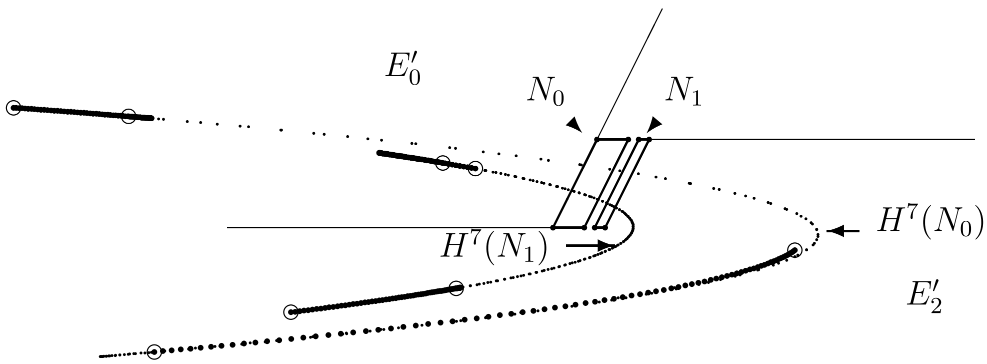


**Figure 6.** The original sets $N_0$ and $N_1$ for the Hénon map, and its seventh iterate (dots). Image of 'horizontal' edges are marked by disks. Images of cornes are marked by circles.

Figure 6 presents the image of $H^7_{ab}(N)$ in initial coordinates $x, y$. The sets $H^7(N_i)$ apparently lay along the attractor and are so thin that in this figure they appear as one-dimensional curves. The images of corners are marked by circles and the images of points belonging to $V(N)$ are enlarged, so that they form four thick arcs corresponding to the images of the vertical egdes according with lemma 5.

The above lemma shows that $H^7_{ab}$ is a TS-map for $a, b$ in a sufficiently small neighbourhood of $(a, b) = (1.4, 0.3)$ with the transition matrix

$$A := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

For such a matrix $A$ we have $\Sigma_A = \Sigma_2$. Since $H_{ab}$ is an homeomorphism lemma 5 and theorem 1 imply

**Theorem 6.** *For all parameter values in a sufficiently small neighbourhood of* $(a, b) = (1.4, 0.3)$ *there exists a set* $N$ *and a continuous surjection* $\pi : Inv(N, H^7_{ab}) \to \Sigma_2$*, such that*

$$\pi \circ P = \sigma \circ \pi.$$

*Moreover, the pre-image of any periodic sequence from* $\Sigma_2$ *contains periodic points of* $H^7_{ab}$.

Rigourous numerical verification of lemma 5 required the calculation of $H^7$ for around 60 000 points. This took about 10 seconds on an IBM PC 486 66 MHZ. Our program has been implemented in Turbo Pascal 7.0.

## 5. Conclusion

In this paper we have presented a simple topological method to establish rigoursly the existence of horseshoe dynamics in typical dynamical systems. We applied this method to two two-dimensional systems which numerically exhibit a strange attractor, to obtain a symbolic dynamics on an invariant set. But the existence of a strange attractor still remains unproven.

Our method can be easily generalized to higher dimensions. The number of 'vertical directions' (in our paper the $y$-direction) can be arbitrary (finite). Also the number of 'horizontal directions' (in the paper the $x$-direction) can be increased, but in this case some substantial changes are required in the definitions of a transition matrix and of model maps.

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