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CYCLOTRON RADIATION FROM A HOT PLASMA*†

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Abstract—In their Geneva paper, TRUBNIKOV and KUDRYAVTSEV calculated the cyclotron radiation from a hot plasma. In doing this, the approximation was made that the individual particles radiated as though they were in a vacuum. We have investigated this approximation by calculating the absorption length directly from the Boltzmann equation and we find that indeed this approximation is correct whenever $\omega_p^2/\Omega^2 \ll n^2$ where n is the harmonic number of the radiation in question, ω_p is the plasma frequency, and Ω is the cyclotron frequency. For a contained plasma, the left-hand side of this inequality is of the order of magnitude of one and thus the inequality is well satisfied for the dominant radiation from a plasma at high temperature.

The physical reason for this inequality can be investigated by solving the test-charge problem to find the transverse response current of the plasma to the motion of an electron. The result is that the transverse organizing length in the plasma is c/ω_p, and thus one may picture an electron in its Larmor orbit as being surrounded by a co-moving current cloud of radius c/ω_p. Classical electrodynamics then leads to the above inequality.

AT the *Geneva Conference on the Peaceful Uses of Atomic Energy*, a paper by TRUBNIKOV and KUDRYAVTSEV (1958) on the cyclotron radiation from a hot plasma was given. Their results were that the cyclotron radiation is unexpectedly large and that it is essentially a surface effect and can be minimized with respect to volume effects such as thermonuclear-energy production by simply increasing the size of a controlled thermonuclear reactor. Their estimates of the critical size of a controlled thermonuclear reactor, i.e. the size at which the thermonuclear energy production is just balanced by the radiation energy loss, were in the neighbourhood of 10–100 meters.

Because of the serious ramifications these results have on the future of controlled thermonuclear power considerable controversy has arisen over their method of calculation (see e.g. BEARD, 1959). In particular they assume that an individual electron in a plasma will radiate as though it is in a vacuum and this assumption has been challenged by several people. For example, it has been suggested that each electron in a plasma is surrounded by a 'Debye cloud' of charge which would inhibit the radiation from this electron. Rosenbluth and I have considered this problem and conclude that indeed an electron is surrounded by a co-moving cloud of current, but the important part of this current cloud for inhibiting radiation is the transverse part and we find that the transverse part has a radius of approximately c/ω_p. This will inhibit radiation at wavelengths long compared to c/ω_p but radiation at wavelengths short compared to c/ω_p will be essentially unaffected. This means that for frequencies large compared to the plasma frequency an electron in a plasma does indeed radiate as though it were in a vacuum and the Russian assumption is correct if the frequencies of interest, i.e. multiples of the cyclotron frequency, are large compared to the plasma frequency. To show that this is so we consider the quantity $\beta_H = 8\pi nkT/B^2$. For plasmas of interest β_H is a number between 1/10 and 1 and this means that for temperatures in the neighbourhood of 50 to 200 kilovolts the plasma frequency and the cyclotron frequency are roughly equal. It follows that multiples of the cyclotron frequency are large compared to the plasma frequency. Thus we agree with the basic assumption of TRUBNIKOV and KUDRYAVTSEV.

Before proceeding to a derivation of our results, I wish first however to give a short summary of the method used by TRUBNIKOV and KUDRYAVTSEV. Consider an electron moving in a circular orbit the radius of which is given by the Larmor formula $r = |v_\perp/\omega_c|$ where $v_\perp$ is the magnitude of the projection of the velocity vector on to the plane normal to **B**. To calculate the vector potential of such an electron we have the expression

$$A(\omega) \propto \int \exp(i\omega t) \exp[i\mathbf{k} \cdot \mathbf{r}(t)] v(t) dt.$$

In many radiation problems one makes the dipole approximation, i.e. one neglects $|kr|$ compared to 1. For this case we see that we obtain only radiation at

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the cyclotron frequency. For our case however $|kr| = |(\omega/\omega_c)(v_\perp/c)|$ which cannot be neglected compared to 1 for a hot plasma. Thus it is the failure of the dipole approximation that is responsible for the harmonics of the cyclotron frequency. What TRUBNIKOV and KUDRYAVTSEV did was to calculate from this the energy radiated from a single electron and to average this over a Maxwellian distribution of electrons to obtain the source function S which is the energy radiated per second per unit solid angle per unit frequency per unit volume. Now the cyclotron frequency of a relativistic electron depends upon the electron momentum. Thus, when we average over a Maxwellian distribution of electrons, we smear out the spectrum of radiation into a continuum. From Kirchhoff's law we obtain a relation between the source function S and the absorption coefficient α, $S = I_{RJ}\alpha$ where I_{RJ} is the Rayleigh-Jeans distribution law. Thus one obtains in a simple way the absorption coefficient of the plasma. For low frequencies where there is considerable radiation there will also be considerable absorption and a plasma of finite thickness will appear 'optically thick.' As the frequency is increased to very high harmonics of the cyclotron frequency the source function and absorption coefficient will gradually drop off until the plasma is optically thin. Thus a plasma of finite thickness appears to be a black body for all frequencies up to the limiting frequency ω^* at which the absorption length is roughly equal to the plasma thickness. This is the result of TRUBNIKOV and KUDRYAVTSEV.

We now wish to discuss the philosophy behind our derivation. If one considers the limit in which $e = m = 1/n = T = 0$ with e/m, etc. constant, the collisionless Boltzmann equation is an exact result of the Liouville theorem. This is the fluid limit and there are no individual-particle effects such as cyclotron radiation in this limit. If we consider e, m, $1/n$ and T all to be of the same order (we refer to the order of any of these quantities as g) then cyclotron radiation appears as a first order effect in g. The straightforward way to calculate this is to introduce a test charge into the plasma. Besides the current of the test charge there will also be the reaction current of plasma. This includes effects due to two-particle correlations as well as the fluid response. This total current is then the source of cyclotron radiation. The total radiation is then given by averaging over a Maxwellian distribution of test charges and the source function so derived is of order $ne^2 = g$. A simpler way of obtaining the results is to calculate the absorption coefficient to zeroth order in g, i.e. from the collisionless Boltzmann equation, and then obtain the source function from Kirchhoff's law which, because I_{RJ} is of first order in g relates quantitites of different orders. This is what we have done in the following.

Thus we wish to find the absorption coefficient for electromagnetic waves from the collisionless Boltzmann equation. Because we are dealing with very hot plasmas we use the relativistic Boltzmann equation. If we linearize, Fourier transform with respect to space, and Laplace transform with respect to time, we obtain

$$\left[s + \frac{i\mathbf{k}\cdot\mathbf{p}}{\sqrt{1+p^2}} + \frac{e}{\sqrt{1+p^2}}(\mathbf{p}\times\mathbf{B})\cdot\nabla_p\right]f = -e\mathbf{E}\cdot\nabla_p f_0 + b.$$

Here s is the Laplace transform variable and k is the Fourier transform variable. The solution to this equation can be written formally as

$$f(k,\omega,\mathbf{v}) = \mathbf{G}\cdot\mathbf{E}(k,\omega)$$

where $\mathbf{G}$ is an operator and $m = c = 1$. Similarly from Maxwell's equations we obtain

$$(s^2 + k^2)\mathbf{E} - (k\cdot E)\mathbf{k} + 4\pi s\mathbf{j} = \mathbf{b}$$

where $\mathbf{j}$ the current is defined by

$$\mathbf{j} = \int \mathbf{v}f(\mathbf{v})d^3v = \int \mathbf{v}\mathbf{G}\cdot\mathbf{E}d^3\mathbf{v} = \boldsymbol{\sigma}\cdot\mathbf{E}$$

and b, $\mathbf{b}$ are concerned with the initial data and are of no interest for the present problem. Thus we obtain an equation of the form

$$\mathbf{AE} = \mathbf{b}$$

$$\text{where } \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} s_2 + 4\pi s\sigma_{11} & 4\pi s\sigma_{12} & 0 \\ 4\pi s\sigma_{21} & s^2 + c^2k^2 + 4\pi s\sigma_{22} & 0 \\ 0 & 0 & s^2 + c^2k^2 + 4\pi s\sigma_{33} \end{pmatrix}$$

The dispersion relations are obtained by setting the determinant of $\mathbf{A}$ equals to zero. For propagation directly across the magnetic field the 13, 23, 31, 32 elements of $\mathbf{A}$ are zero because of symmetry. We are interested in propagation at very small angles to the normal of the magnetic field and thus we neglect these elements. The dispersion relations are then obtained from the equation $a_{33}(a_{11}a_{22} - a_{12}a_{21}) = 0$. For propagation across the magnetic field with polarization along the field, the dispersion relation is given by $a_{33} = 0$. For propagation across the magnetic field with polarization perpendicular to the magnetic field, the situation is complicated by the off-diagonal elements of the conductivity tensor. However it is easy to show that these elements can be neglected compared to the diagonal elements for frequencies large compared with the plasma frequency and as we have seen, these are the frequencies of interest. For such waves the dispersion relationship is then simply given by

$$a_{22} = 0.$$

Actually it is this polarization which is the most important. The imaginary part of σ_{22} leads simply to a slight frequency shift but the real part gives us damping in time and this is related to spatial absorption by the group velocity. In turn, this is related to the source function through Kirchhoff's law and thus to find the source function we must simply evaluate the real part of σ_{22}. An elementary calculation gives us

$$S(\omega,\mathbf{\Omega}) = I_{RJ}\,\alpha(\omega,\mathbf{\Omega}) = \int f_0(p)d^3p \times \left[\frac{e^2\omega^2}{2\pi} r_\perp{}^2 \sum_m J_m{}'^2(k_\perp r_\perp)\right]$$

where the quantity in brackets is simply the energy radiated by a single particle in a vacuum, and $k_\perp$ is the magnitude of the projection of the propagation vector onto the plane normal to B and r is $v_\perp \omega_c$. And thus we see, by direct calculation, that the source function is indeed just the source function for independent electrons averaged over a Maxwellian distribution. TRUBNIKOV and KUDRYAVTSEV have evaluated this integral for the case of propagation exactly across the magnetic field. We have extended this to the case of small angles from the normal magnetic field and obtain

$$\alpha_\perp = \frac{\omega_p{}^2}{4\Omega}\sqrt{\frac{\pi}{T}}(\epsilon^2 - 1)^{3/2}\left(\frac{1}{mT}\right)^2 \times \exp\left\{\frac{1}{T} - \frac{2m}{\epsilon^2 - 1} - m\cos^2\theta\right\}$$

where

$$m = \frac{\omega}{\omega_c}$$

and

$$\frac{2\epsilon}{\epsilon^2 - 1} - \ln\frac{\epsilon + 1}{\epsilon - 1} = \frac{1}{mT}$$

which agrees with results of TRUBNIKOV and KUDRYAVTSEV for $\theta = \pi/2$. It is important to recognize here that the radiation is almost entirely contained within an angular region of the order of $1/\sqrt{m}$ about the normal to the magnetic field. And this means that the black-body spectrum will not fill up at large angles to the normal. If one now calculates the radiation from a slab of plasma of thickness L a considerable reduction occurs compared to the results of TRUBNIKOV and KUDRYAVTSEV because of this angular effect and this materially changes the critical size. Equating the total radiation out to the thermonuclear energy production we then obtain the table of critical sizes shown in Table 1.

Note added in galley.* *Due to an algebraic mistake in the evaluation of* $\alpha_\perp$, *the angular distribution given above is in error. Actually* $\alpha_\perp$ *falls off less strongly with angle and the results given in Table* 1 *are wrong and it has thus been deleted. Although it has not yet been evaluated quantitatively there should nevertheless be some reduction in the critical size, but not as much as previously stated.*

It results that for the case of $\beta_H = 1$ cyclotron radiation is not too bad a problem. However, for stability reasons it may be desirable to have β_H in the neighbourhood of 1/10 and for this case the critical size again becomes large. In order to mitigate the effects of cyclotron radiation one can use reflectors. If one considers a reflector with reflectivity R which does not change the angle the radiation makes with the magnetic field, the effect of the reflector can be analysed as follows (see Fig. 1):

$$\frac{dI}{dz} = -\alpha I + S = -\alpha(I - I_{RJ}).$$

The boundary conditions are

$$I(1) = I(3) = RI(2)$$

and these then yield

$$(1 - R)\,I(L) = (I - R)\,I_{RJ}\frac{(1 - e^{-\alpha L})}{(1 - R^{-\alpha L})}.$$

Inserting this into the critical equation one then finds that the critical size is reduced by the factor 1-R. For magnetic fields in the neighbourhood of 10^4 gauss the harmonics of the cyclotron radiation are in the millimetre and centimetre range and a reflectivity of 0·99 is not unreasonable for such a case. Thus by using a reflector the critical size can be reduced by two orders of magnitude.

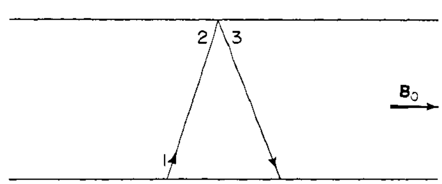

FIG. 1.—Radiation from plasma contained by reflector.

* This was pointed out to us independently by TRUBNIKOV and by HUBBARD.

To sum up I might say that we have shown three things. First, that the basic assumption of TRUBNIKOV and KUDRYAVTSEV, i.e. that electrons in a plasma radiate as though they are in a vacuum, is correct. Second, that the angular dependence of the cyclotron radiation leads to some reduction in the critical size of a controlled thermonuclear reaction compared to the results of TRUBNIKOV and KUDRYAVTSEV. Third that machines such as the mirror machine which operate at low β will necessarily use reflectors and that such reflectors can be expected to cut down the critical size by two orders of magnitude or more.

REFERENCES

BEARD D. B. (1959) *Phys. Fluids* **2**, 379; **3**, 324.
TRUBNIKOV B. A. and KUDRYAVTSEV V. S. (1958) *Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy, Geneva*, **31**, 93. United Nations N.Y.