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Analytic theory of the ground state properties of a spin glass: I. Ising spin glass

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Abstract. A general theory to count the number of 'local minimum states' in an Ising spin glass is developed. The problem is reduced to finding the partition function of an interacting non-random spin system with an imaginary spin weight function. For the infinite-ranged spin glass with Gaussian bond distribution, we find $\langle g_0 \rangle = 2^{0.28743N}$ for the average number of the local minimum states. The distribution function for the energies of these states is also studied. The upper limit of the average ground state energy per spin of the infinite-ranged spin glass is found to be $-\tilde{J}/\sqrt{2\pi}$.

1. Definition of the problem

Since the theoretical prediction of the existence of a continuous phase transition in a spin glass (Edwards and Anderson 1975), there has been considerable interest in the understanding of the low-temperature properties of a spin glass, a particular problem of which is the ground state (Edwards and Anderson 1976, Edwards 1976). The details of the low-temperature phase, however, remain largely unknown. The difficulty is due mainly to the fact that a random-ordered state has a very large number of low-lying local minimum states which are approximately degenerate in energy. One of these local minimum states is a 'global' ground state which is stable for any kind of perturbation. Since the experimental measurements are mainly concerned with metastable relaxation phenomena, a study of these local minimum states seems to be crucial for understanding the low-temperature phase of a spin glass.

The first step towards this was developed by Edwards and Anderson (1976) by counting the number of metastable states and also by finding a probability distribution of their energies in a model spin-glass system (hereafter referred to as the EA model). Later, more elaborate studies were developed in a random Ising spin system with competing exchange interactions $\pm J$ (frustration model) (Toulouse 1977, Kirkpatrick 1977, Vannimenus and Toulouse 1977). The method, however, is heavily dependent on computer calculations. The purpose of this paper is to develop a simple analytic theory to understand the above mentioned local minimum states.

We start with the EA model;

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j \tag{1.1}$$

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where σ_i is an Ising spin ($\sigma_i = \pm 1$) on a lattice point i and J_{ij} is a nearest-neighbour random exchange interaction of which the probability distribution is assumed to be Gaussian with a vanishing mean value:

$$P(J_{ij}) = \frac{1}{\sqrt{2\pi}J_0}\exp(-J_{ij}^2/2J_0^2). \tag{1.2}$$

The summation should be taken over all nearest-neighbouring pairs $\langle i,j\rangle$. For a given configuration of spins $\{\sigma_i\}$ at zero temperature, the spin on the lattice point i is affected by the internal magnetic field

$$h_i \equiv \sum_j J_{ij}\sigma_j$$

produced by surrounding spins. Our conditions of metastability then are defined by

$$h_i \equiv \sum_j J_{ij}\sigma_j = \lambda_i\sigma_i \qquad \lambda_i \geqslant 0 \tag{1.3}$$

for all spins σ_i ($i = 1, 2, \ldots, N$).

These conditions are derived by considering the stability to 'single' spin flipping. Let us consider the local exchange energy

$$\epsilon_i \equiv -\sigma_i \sum_j J_{ij}\sigma_j$$

for each spin in the starting configuration. After flipping the direction of σ_i, the local energy becomes

$$\epsilon_i' = \sigma_i \sum_j J_{ij}\sigma_j,$$

so that the energy difference is given by

$$\Delta\epsilon_i \equiv \epsilon_i' - \epsilon_i = 2\sigma_i \sum_j J_{ij}\sigma_j = 2\lambda_i \tag{1.4}$$

which must be positive by definition of the concept of the local minimum. The conditions (1.3), of course, can say nothing about stability against a simultaneous flipping of a spin cluster consisting of more than two spins. It is the first diagonal condition on the positive definiteness of the energy matrix for small deformations. Stability conditions for any type of spin flipping give a global ground state of course. The conditions (1.3) indicate that the system has a very large number of metastable configurations for a given $\{J_{ij}\}$, and the single spin-flip barriers separating one such configuration from a neighbouring one are, in general, not very high.

This argument can be applied to general n-component classical spins. In the previous paper (Edwards and Anderson 1976), a classical XY spin system ($n = 2$) was studied. For an XY model, the Hamiltonian is given by

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij}\boldsymbol{S}_i.\boldsymbol{S}_j = -\sum_{\langle i,j\rangle} J_{ij}\cos(\theta_i - \theta_j) \tag{1.5}$$

where $\boldsymbol{S}_i \equiv (\cos\theta_i, \sin\theta_i)$. The conditions of the minimum energy are then given by

$$\frac{\partial\mathcal{H}}{\partial\theta_i} = (\boldsymbol{S}_i \times \boldsymbol{h}_i)_z = 0 \tag{1.6}$$

$$\frac{\partial^2\mathcal{H}}{\partial\theta_i\partial\theta_j} \equiv \mathcal{H}_{ij} = (\boldsymbol{S}_i.\boldsymbol{h}_i)\delta_{ij} - J_{ij}\cos(\theta_i - \theta_j)$$

$$= \text{positive definite matrix} \tag{1.7}$$

where

$$\boldsymbol{h}_i \equiv \sum_j J_{ij}\boldsymbol{S}_j$$

as before and the product in (1.6) means the external product in spin space. The first condition (1.6) gives $\boldsymbol{S}_i \| \boldsymbol{h}_i$, i.e. $\boldsymbol{h}_i = \lambda_i \boldsymbol{S}_i$. The main condition for positive definiteness of the matrix $\mathscr{H}_{ij}$ is given by the positivity of the diagonal elements; $\lambda_i \geqslant 0$ for all i. Of course, these conditions are not sufficient for a true local minimum. In conclusion, the conditions

$$\boldsymbol{h}_i \equiv \sum_j J_{ij}\boldsymbol{S}_j = \lambda_i \boldsymbol{S}_i \qquad \lambda_i \geqslant 0 \tag{1.8}$$

for all spins $\boldsymbol{S}_i$ seem to be essential to characterise the local minimum states.

The following parts of this paper are devoted to counting the number of local minimum states characterised by the conditions (1.3) and to finding the energy distribution of these states.

2. The average number of the local minimum states

Let $g_0(\{J_{ij}\})$ be the number of states satisfying the condition (1.3) for a given configuration of the exchange interactions $\{J_{ij}\}$. The average number of the local minimum states is then given by the formula

$$\langle g_0 \rangle = \frac{1}{2}\int_0^\infty \prod_i \mathrm{d}\lambda_i \, \mathrm{Tr}_\sigma \left\langle \prod_i \delta\left(\sum_j J_{ij}\sigma_j - \lambda_i \sigma_i\right)\right\rangle_J . \tag{2.1}$$

In this equation, $\langle \ldots \rangle_J$ denotes the average over all configurations of $\langle J_{ij}\}$ and Tr_σ is the operation taking the trace of all configurations of the spins. The pre-factor 1/2 is included to eliminate the trivial degeneracy of a simultaneous change of the spin direction $\sigma_i \rightarrow -\sigma_i$ $(i = 1, 2, \ldots, N)$. The most important point is that the λ integrals are limited to the half space. Using the integral representation of the δ function and taking the Gaussian average for $\{J_{ij}\}$, equation (2.1) can be written as

$$2\langle g_0 \rangle = \int_0^\infty \prod_i \mathrm{d}\lambda_i \frac{1}{(2\pi)^N}\int_{-\infty}^\infty \prod_i \mathrm{d}\phi_i \, \mathrm{Tr}_\sigma \left[\exp\left(-i\sum_i \phi_i \lambda_i \sigma_i - \frac{J_0^2}{2}\sum_{\langle i,j\rangle}(\phi_i\sigma_j + \phi_j\sigma_i)^2\right)\right]. \tag{2.2}$$

The simple variable change $\phi_i \rightarrow \sigma_i\phi_i (i = 1, 2, \ldots, N)$ gives a spin-independent form of the integrand and therefore Tr_σ gives 2^N. Hence we find

$$\begin{aligned} 2\langle g_0 \rangle &= \frac{1}{\pi^N}\int_0^\infty \prod_i \mathrm{d}\lambda_i \int_{-\infty}^\infty \prod_i \mathrm{d}\phi_i \exp\left(-i\sum_i \phi_i\lambda_i - \frac{J_0^2}{2}\sum_{\langle i,j\rangle}(\phi_i+\phi_j)^2\right) \\ &= \frac{1}{\pi^N}\int_0^\infty \prod_i \mathrm{d}\lambda_i \int_{-\infty}^\infty \prod_i \mathrm{d}\phi_i \exp\left(-i\sum_i \lambda_i\phi_i - \frac{1}{2}\sum_i \phi_i^2 - \frac{1}{z}\sum_{\langle i,j\rangle}\phi_i\phi_j\right) \end{aligned} \tag{2.3}$$

where we have rescaled the variables as $\sqrt{z}\, J_0\, \phi_i \rightarrow \phi_i$ and $\lambda_i \rightarrow \sqrt{z}\, J_0\, \lambda_i$ to obtain the

final expression where z is the number of the nearest-neighbouring lattice points. In order to get a more familiar expression, we replace the half space integral as

$$\int_0^\infty \prod_i \mathrm{d}\lambda_i = \int_{-\infty}^\infty \prod_i \mathrm{d}\lambda_i \theta(\lambda_i)$$

by inserting the step function $\theta(\lambda) \equiv 1$ $(\lambda > 0) \equiv 0$ $(\lambda < 0)$. Using the integral representation of the step function

$$\theta(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\lambda\phi)}{\phi - \mathrm{i}\epsilon} \, \mathrm{d}\phi$$

we find our starting formula:

$$2\langle g_0 \rangle = \int_{-\infty}^{\infty} \prod_i \mathrm{d}\phi_i D(\phi_i) \exp\left(-\frac{1}{z} \sum_{\langle i,j \rangle} \phi_i \phi_j \right) \tag{2.4}$$

where the function $D(\phi)$ is defined by

$$D(\phi) \equiv \frac{1}{\pi \mathrm{i}} \frac{\exp(-\phi^2/2)}{\phi - \mathrm{i}\epsilon} = \delta(\phi) + \frac{\mathscr{P}}{\pi \mathrm{i}} \frac{\exp(-\phi^2/2)}{\phi} \tag{2.5}$$

where $\mathscr{P}$ denotes the principal value. Our problem, therefore, is reduced to finding a partition function of a regular system which consists of interacting 'spins' $\{\phi_i\}$ with an imaginary spin weight function $D(\phi)$. The following analysis will be based on the formula (2.4).

3. The cumulant expansion

Since our problem was converted into a familiar statistical mechanical problem of an interacting (non-random) spin system with a particular spin weight function, the methods developed in that field can be applied directly. In this section, we evaluate (2.4) in a straightforward expansion method which is quite similar to the high-temperature series expansion of the partition function of an Ising model, in order to get an overall view of the problem. Let us start by rewriting (2.4) as

$$2\langle g_0 \rangle = \left\langle \exp\left(-\frac{1}{z} \sum_{\langle i,j \rangle} \phi_i \phi_j \right) \right\rangle_0 \tag{3.1}$$

where the average $\langle \ldots \rangle_0$ is defined by

$$\langle \ldots \rangle_0 \equiv \int_{-\infty}^{\infty} \prod_i \mathrm{d}\phi_i D(\phi_i) (\ldots). \tag{3.2}$$

For example, we find

$$\begin{aligned} &\langle 1 \rangle_0 = 1 \qquad \langle \phi_i^{2m} \rangle_0 = 0 \\ &\langle \phi_i^{2m+1} \rangle_0 = -\mathrm{i} \sqrt{\frac{2}{\pi}} (2m-1)!! \qquad (m = 0, 1, 2, \ldots). \end{aligned} \tag{3.3}$$

Let X be the exponent of equation (3.1):

$$X \equiv -\frac{1}{z} \sum_{\langle i,j \rangle} \phi_i \phi_j .$$

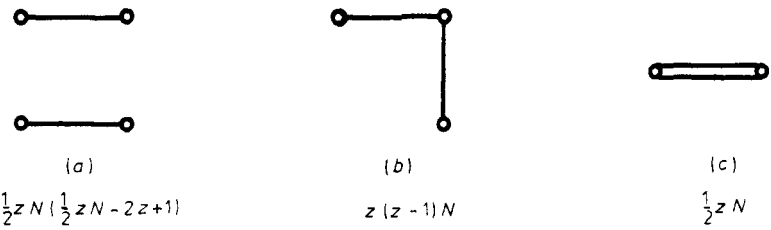

Figure 1. Three types of graphs which appear in the second-order cumulant are shown with their combinatorial factors.

We can then expand the right-hand side of equation (3.1) in powers of X by the use of the cumulant expansion formula:

$$2\langle g_0\rangle = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n!}\langle X^n\rangle_{0,c}\right) \tag{3.4}$$

where $\langle X\rangle_{0,c} = \langle X\rangle_0$, $\langle X^2\rangle_{0,c} = \langle X^2\rangle_0 - \langle X\rangle_0^2$ and so on. The first term is easily evaluated as follows:

$$\langle X\rangle_0 = -\frac{1}{z}\sum_{\langle i,j\rangle}\langle\phi_i\phi_j\rangle_0 = -\frac{1}{z}\langle\phi\rangle_0^2\frac{z}{2}N = \frac{N}{\pi} \tag{3.5}$$

where $zN/2$ is the total number of nearest-neighbouring spin pairs. In the second-order term, three distinct types of graph appear. These graphs and the associated combinatorial factors are shown in figure 1.

The cumulant can be obtained easily by extracting the $O(N)$ terms in the corresponding moments:

$$\begin{aligned}\langle X^2\rangle_{0,c} &= O(N)\text{ terms in the moment }\langle X^2\rangle_0\\ &= \frac{4}{\pi^2}\left(-1+\frac{1}{2z}\right)N.\end{aligned} \tag{3.6}$$

Combining the results of (3.5) and (3.6), we have

$$2\langle g_0\rangle = \exp\left[\frac{1}{\pi}+\frac{2}{\pi^2}\left(-1+\frac{1}{2z}\right)+\ldots\right]N. \tag{3.7}$$

Although it is straightforward to evaluate the higher order cumulants, the series (3.7) is unfortunately not a power series of $1/z$ because of the appearance of combinatorial factors which include the number z. In the next section, we collect all terms of zeroth order in $1/z$.

4. Infinite-ranged spin glass

It is possible to obtain a rigorous solution for the infinite-ranged spin-glass model proposed by Sherrington and Kirkpatrick (1975).

In this model, we have $z = N - 1 \approx N$ and

$$\begin{aligned}2\langle g_0\rangle &= \int_{-\infty}^{\infty}\prod_i d\phi_i\, D(\phi_i)\exp\left(-\frac{1}{2N}\sum_{\substack{\text{all } i,j\\ i\neq j}}\phi_i\phi_j\right)\\ &= \int_{-\infty}^{\infty}\prod_i d\phi_i\, D(\phi_i)\exp\left\{-\frac{1}{2N}\left[\left(\sum_i\phi_i\right)^2-\sum_i\phi_i^2\right]\right\}.\end{aligned} \tag{4.1}$$

Substituting the identity

$$\exp\left[-\frac{1}{2N}\left(\sum_i \phi_i\right)^2\right] = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{2\pi}} \exp\left(-\frac{N}{2}t^2 + \mathrm{i}t\sum_i \phi_i\right)$$

into (4.1) and neglecting the $O(1/N)$ term, we find

$$2\langle g_0\rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{2\pi}} \exp(-Nf(t)) \tag{4.2}$$

where the function $f(t)$ is defined as

$$f(t) \equiv \tfrac{1}{2}t^2 - \ln \int_{-\infty}^{\infty} \mathrm{d}\phi\, D(\phi)\, \mathrm{e}^{\mathrm{i}t\phi}. \tag{4.3}$$

An elementary calculation gives

$$\int_{-\infty}^{\infty} \mathrm{d}\phi\, D(\phi)\, \mathrm{e}^{\mathrm{i}t\phi} = 2\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp(-x^2/2)\,\mathrm{d}x \equiv 2\Phi(t) \tag{4.4}$$

where $\Phi(t)$ is the accumulated distribution function of the normal distribution. For $N \to \infty$, we can evaluate the integral (4.2) by the saddle-point method. The condition of the saddle point is given by

$$\mathrm{d}f/\mathrm{d}t = t - \Phi'(t)/\Phi(t) = 0 \tag{4.5}$$

which gives the unique solution:

$$t^* = 0{\cdot}5061 \qquad \Phi(t^*) = 0{\cdot}69361 \qquad f(t^*) = -0{\cdot}19923. \tag{4.6}$$

Hence we find the final solution

$$\langle g_0\rangle = \mathrm{e}^{0{\cdot}19923N} = 2^{0{\cdot}28743N}. \tag{4.7}$$

Kirkpatrick (1977) has estimated, by the Monte Carlo method, the ground-state degeneracy of an Ising spin glass, with $J_{ij} = \pm J$, to be as high as 2^{cN} where $c \approx 0{\cdot}14$ for a two-dimensional square lattice. For the model that we are considering here, one does not expect such a high degeneracy, essentially because of the fact that, for a Gaussian bond distribution, finite clusters which can be turned over with no energy cost at all form a set of measure zero. The large fraction (4.7) we obtained is therefore due to our definition of the local equilibrium states. The degeneracy will be reduced substantially if further stability conditions against the flip of clusters are taken into account. This problem remains open.

In appendix 1, correction due to the finite range of the interaction is discussed using a simple variational calculation.

5. Distribution of the local minimum energies

Our next problem is to get a distribution for the energies of the local minimum states defined by (1.3). Let $-\mathscr{E}_0$ be the energy of one of the states satisfying the condition (1.3):

$$-\mathscr{E}_0 = -\tfrac{1}{2}\sum_{i,j} J_{ij}\sigma_i\sigma_j = -\tfrac{1}{2}\sum_i \sigma_i h_i = -\tfrac{1}{2}\sum_i \lambda_i. \tag{5.1}$$

The distribution function of the local minimum energies then is given by

$$\mathcal{N}(\mathscr{E}_0) = \int_0^\infty \prod_i \mathrm{d}\lambda_i \delta\Big(\mathscr{E}_0 - \tfrac{1}{2}\sum_i \lambda_i\Big) \mathrm{Tr}_\sigma \Big\langle \prod_i \delta\Big(\sum_j J_{ij}\sigma_j - \lambda_i\sigma_i\Big)/g_0(\{J\})\Big\rangle_J . \tag{5.2}$$

To avoid any mathematical difficulty, we evaluate this by replacing $g_0(\{J\})$ by its average $\langle g_0\rangle$; $\mathcal{N}(\mathscr{E}_0) = P(\mathscr{E}_0)/\langle g_0\rangle$ where

$$P(\mathscr{E}_0) \equiv \int_0^\infty \prod_i \mathrm{d}\lambda_i \delta\Big(\mathscr{E}_0 - \tfrac{1}{2}\sum_i \lambda_i\Big) \mathrm{Tr}_\sigma \Big\langle \prod_i \delta\Big(\sum_j J_{ij}\sigma_j - \lambda_i\sigma_i\Big)\Big\rangle_J . \tag{5.3}$$

This approximation, of course, is not proper to a 'quenched' alloy, but the following picture will not be far from the real situation as far as $g_0(\{J\})$ is a mild function of $\{J\}$.

In a procedure quite similar to that used to derive (2.3), we find

$$P(\mathscr{E}_0) = \frac{1}{\pi^N \sqrt{z}\, J_0} \int_0^\infty \prod_i \mathrm{d}\lambda_i \delta\Big(\frac{\mathscr{E}_0}{\sqrt{z}\, J_0} - \tfrac{1}{2}\sum_i \lambda_i\Big) \int_{-\infty}^\infty \prod_i \mathrm{d}\phi_i$$
$$\times \exp\Big(-i\sum_i \lambda_i\phi_i - \tfrac{1}{2}\sum_i \phi_i^2 - \frac{1}{z}\sum_{\langle i,j\rangle} \phi_i\phi_j\Big). \tag{5.4}$$

By using the integral representation of the δ function again and with the scaled energy variable $E_0 \equiv \mathscr{E}_0/\sqrt{z}\, J_0$, we can rewrite equation (5.4) as

$$P(\mathscr{E}_0) = \frac{1}{\pi\sqrt{z}\, J_0} \int_{-\infty}^\infty \mathrm{d}\mu \exp(-2\mathrm{i}\mu E_0) \int_{-\infty}^\infty \prod_i \mathrm{d}\phi_i\, D(\phi_i;\mu) \exp\Big(-\frac{1}{z}\sum_{\langle i,j\rangle} \phi_i\phi_j\Big)$$
$$= \frac{1}{\pi\sqrt{z}\, J_0} \int_{-\infty}^\infty \mathrm{d}\mu \exp(-N\mu^2 - 2\mathrm{i}\mu E_0)$$
$$\times \int_{-\infty}^\infty \prod_i \mathrm{d}\phi_i\, D(\phi_i) \exp\Big(-\frac{1}{z}\sum_{\langle i,j\rangle} \phi_i\phi_j - 2\mu\sum_i \phi_i\Big) \tag{5.5}$$

where $D(\phi;\mu) \equiv \exp(-\phi^2/2)/\pi\mathrm{i}(\phi - \mu - \mathrm{i}\epsilon)$.

The problem is reduced to find a partition function of the same interacting regular system as before but under a uniform 'magnetic field' μ in this time. Equation (5.5) is our second basic formula.

It is easy to find the rigorous solution for the infinite-ranged SK model. By definition, we set

$$z = N - 1 \approx N \qquad \text{and} \qquad J_0 = \tilde{J}/\sqrt{N} \tag{5.6}$$

to find

$$P(\mathscr{E}_0) = \frac{1}{\pi\tilde{J}} \int_{-\infty}^\infty \mathrm{d}\mu \exp(-N\mu^2 - 2\mathrm{i}\mu E_0) \int_{-\infty}^\infty \frac{\mathrm{d}t}{\sqrt{2\pi}} \exp(-Nt^2/2)$$
$$\times \Big(\int_{-\infty}^\infty \mathrm{d}\phi\, D(\phi) \exp[\mathrm{i}(t + 2\mathrm{i}\mu)\phi]\Big)^N$$
$$= \frac{1}{\pi\sqrt{2\pi}\tilde{J}} \int_{-\infty}^\infty \mathrm{d}\mu\, \mathrm{d}t \exp(-Nf(t,\mu)) \tag{5.7}$$

where

$$f(t,\mu) \equiv \mu^2 + 2\mathrm{i}\mu\epsilon_0 + \frac{t^2}{2} - \ln\int_{-\infty}^\infty \mathrm{d}\phi\, D(\phi) \exp[\mathrm{i}(t + 2\mathrm{i}\mu)\phi] \tag{5.8}$$

and $\epsilon_0 \equiv E_0/N = \mathscr{E}_0/N\tilde{J}$ is the dimensionless energy per spin. Let us choose μ and $\tau \equiv t + 2i\mu$ as new independent variables. We then have

$$f(t,\mu) = f(\tau,\mu) = \tfrac{1}{2}\tau^2 - \mu^2 - 2i\mu(\tau - \epsilon_0) - \ln(2\Phi(\tau)). \tag{5.9}$$

The saddle-point conditions give

$$\partial f/\partial\tau = \tau - 2i\mu - \Phi'(\tau)/\Phi(\tau) = 0 \tag{5.10}$$

$$\partial f/\partial\mu = -2[\mu + i(\tau - \epsilon_0)] = 0. \tag{5.11}$$

Eliminating the parameter μ^* by using (5.11), we find the saddle-point equation for τ^* from (5.10):

$$\tau^* + \Phi'(\tau^*)/\Phi(\tau^*) = 2\epsilon_0. \tag{5.12}$$

Substituting the solution $\tau^* = \tau^*(\epsilon_0)$ of this equation into (5.7), the energy distribution can be expressed as

$$P(\epsilon_0) \approx \{2\Phi(\tau^*)\exp[(\tau^* - \epsilon_0)^2 - (\tau^*)^2/2]\}^N \tag{5.13}$$

for large N apart from the unimportant pre-factor. As for the normalised distribution function $\mathcal{N}(\epsilon_0)$, one can readily find $\mathcal{N}(\epsilon_0) \approx (n(\epsilon_0))^N$ where

$$n(\epsilon_0) \equiv 2\Phi(\tau^*)\exp[(\tau^* - \epsilon_0)^2 - (\tau^*)^2/2 - 0{\cdot}19923] \tag{5.14}$$

with the use of the result (4.7) for $\langle g_0 \rangle$. We have solved the equation (5.12) numerically. The result is shown in figure 2 in terms of the function $n(\epsilon_0)$.

For deep energy states $\epsilon_0 \gg 1$, Φ'/Φ is sufficiently small compared with τ^*, so that one finds $\tau^* \approx 2\epsilon_0$, which gives a Gaussian tail of $P(\epsilon_0)$; $P(\epsilon_0) \sim \exp(-N\epsilon_0^2)$, $(\epsilon_0 \gg 1)$. In the middle of the distribution, $P(\epsilon_0)$ behaves like a displaced Gaussian distribution:

$$P(\epsilon_0) \sim \exp[-N(\epsilon_0 - a)^2/2\sigma^2] \qquad (\epsilon_0 \lesssim 1). \tag{5.15}$$

We estimated the numerical parameters a and σ as $a = 0{\cdot}50$ and $\sigma = 0{\cdot}31$ from the peak position and the half-width of $n(\epsilon_0)$. The average and the variance of ϵ_0 are

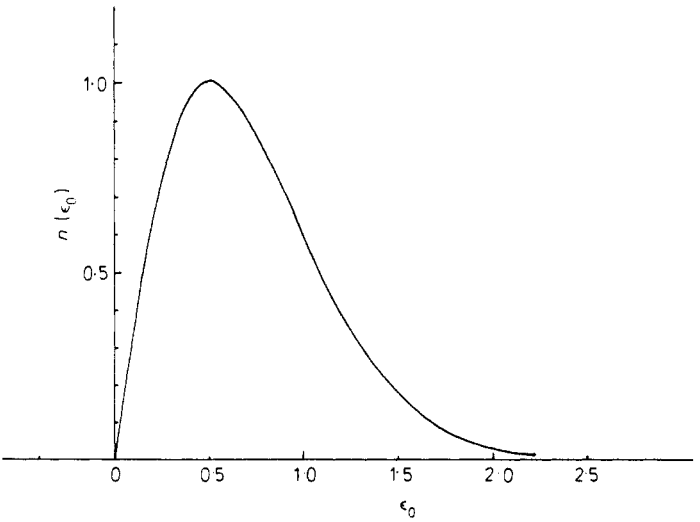

Figure 2. Energy distribution of the local minima is shown. The function $n(\epsilon_0)$ defined in (5.14) is plotted against the dimensionless energy ϵ_0 per one spin.

approximately given by

$$\langle \epsilon_0 \rangle = 0{\cdot}50 \qquad \langle (\delta\epsilon_0)^2 \rangle = \frac{\sigma^2}{N} = \frac{0{\cdot}09}{N}. \tag{5.16}$$

For comparison with other theories, we summarise the results obtained so far:

$$\langle \epsilon_0 \rangle = \begin{cases} 2/\pi = 0{\cdot}79 & \text{replica (Sherrington and Kirkpatrick 1975)} \\ 0 & \text{TAP (Thouless } et\ al \text{ 1977)} \\ 1/\sqrt{2\pi} = 0{\cdot}39 & \text{mean random field (Klein 1976)} \\ 0{\cdot}75 \sim 0{\cdot}77 & \text{Monte Carlo (Kirkpatrick and Sherrington 1978).} \end{cases}$$

Our result is lower than that of the mean random field theory (MRF) by Klein (1976) and higher than the replica calculation and the Monte Carlo result. (Our ϵ_0 is defined by changing the sign.) Since our condition for a local minimum is a necessary but not sufficient condition for a global minimum, it seems appropriate to interpret our result as giving an upper limit of the mean ground state energy.

As for the variance of the energy $\langle (\delta E_0)^2 \rangle = N\sigma^2$, most parts of the number of local minimum states are packed within a width of about $\sqrt{N}$ around the mean value. As was discussed by Edwards and Anderson (1976), this fact leads the system to the following domain structure. Let us consider a situation in which our system is divided into M subsystems, each of which is in one of the $g_0(\{J\})$ local minimum configurations defined above and has an energy lying in the middle of the distribution. The cost of energy mismatching along the boundaries is of the order of about $N^{(d-1)/d}$ in d dimensional space. The energy of the whole system therefore is still lying within the width $\sqrt{N}$ from the mean ground state energy in two dimensions. The system will have a long relaxation time to decay into the final lowest state.

6. Conclusions and discussion

On the basis of a definition of 'local minimum states' which is derived by stability conditions for flipping an individual spin, we have developed a general theory of counting the number of such states in an Ising spin glass. The energy distribution of these local minimum states is also studied. The problem has been found to be equivalent to obtaining a partition function of an interacting regular spin system with a particular spin weight function. To examine stability against any kind of flipping of a cluster of spins is a difficult problem. Our theory is the first step towards this purpose. Extension of our theory to general n component spins seems to be straightforward. A detailed study of the XY model is reported in the following paper.

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Appendix 1. Variational calculation of the O(1/z) corrections to $\langle g_0 \rangle$

In this appendix, we study the effect of short-range interaction using a simple variational method. We start from equation (2.4). Let us choose a trial single-particle

Hamiltonian

$$it \sum_i \phi_i$$

(t is a variational parameter). We then find

$$2\langle g_0 \rangle = (2\Phi(t))^N \left\langle \exp - \left(\frac{1}{z} \sum_{\langle i,j \rangle} \phi_i \phi_j + it \sum_i \phi_i \right) \right\rangle_t \tag{A.1}$$

where the average $\langle \ldots \rangle_t$ is defined by

$$\langle \ldots \rangle \equiv \int_{-\infty}^{\infty} \prod_i d\phi_i D(\phi_i)(\ldots) \exp\left(it \sum_i \phi_i \right) \left[\int_{-\infty}^{\infty} \prod_i d\phi_i D(\phi_i) \exp\left(it \sum_i \phi_i \right) \right]^{-1}. \tag{A.2}$$

For example, we have

$$\langle \phi_i \rangle_t = -iA(t) \qquad \langle \phi_i^2 \rangle_t = -B(t) \tag{A.3}$$

where $A(t) \equiv \Phi'(t)/\Phi(t)$ and $B(t) \equiv \Phi''(t)/\Phi(t)$. The first term of the cumulant expansion of the average in (A.1) gives

$$-\left\langle \frac{1}{z} \sum_{\langle i,j \rangle} \phi_i \phi_j + it \sum_i \phi_i \right\rangle_t = -NA(t)(t - A(t)/2). \tag{A.4}$$

Hence we have

$$2\langle g_0 \rangle \approx \exp - N[A(t)(t - A(t)/2) - \ln 2\Phi(t)]. \tag{A.5}$$

An extremum condition with respect to t reproduces equation (4.5), i.e. $A(t^*) = t^*$. Under this condition, the second cumulant gives

$$\frac{1}{2}\left\langle \left(\frac{1}{z} \sum_{\langle i,j \rangle} \phi_i \phi_j + it \sum_i \phi_i \right)^2 \right\rangle_{t,c} = \exp \frac{N}{4z} (t^{*4} - 2t^{*2} B(t^*) + B(t^*)^2) \tag{A.6}$$

after a similar calculation to that in §3. A numerical evaluation gives

$$\langle g_0 \rangle = e^{(0.19923 + 0.0656/z)N} = 2^{(0.28743 + 0.0946/z)N}. \tag{A.7}$$

Since each spin can move more freely for a short range interaction, $O(1/z)$ correction increases the number of local minima. The correction, however, is already small enough in any realistic lattice in space dimensions above two.

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