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ADDENDUM

Generalization of the matrix product ansatz for integrable chains

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Abstract
We present a general formulation of the matrix product ansatz for exactly integrable chains on periodic lattices. This new formulation extends the matrix product ansatz present in our previous articles (F C Alcaraz and M J Lazo 2004 *J. Phys. A: Math. Gen.* **37** L1–L7 and F C Alcaraz and M J Lazo 2004 *J. Phys. A: Math. Gen.* **37** 4149–82).

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In [1] (to which we refer hereafter as I) and [2], we formulate a matrix product ansatz (MPA) for a large family of exactly integrable spin chains such as the anisotropic Heisenberg model, Fattev–Zamolodchikov model, Izergin–Korepin model, Sutherland model, t-J model, Hubbard model, etc. In this note we present a generalization of the MPA for periodic quantum chains. The generalization is important since it allows, at least in some cases, finite-dimension representations of the matrices defining the MPA. In order to illustrate this generalization, we consider the standard XXZ quantum chain under periodic boundary condition

$$H = -\frac{1}{2}\sum_{i=1}^{L}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z\right), \tag{1}$$

where $\sigma_i^x, \sigma_i^y, \sigma_i^z$ are spin-$\frac{1}{2}$ Pauli matrices located at the L sites of the chain. An arbitrary eigenstate of (1) $|\psi_{n,p}\rangle$, in the sector with a number n of up spins ($n = 0, 1, \ldots$) and momentum $p = \frac{2\pi}{L}j$, ($j = 0, 1, \ldots, L-1$), is given by

$$|\psi_{n,p}\rangle = \sum_{1\leqslant x_1<x_2<\cdots<x_n\leqslant L} f(x_1, \ldots, x_n)|x_1, \ldots, x_n\rangle, \tag{2}$$

where $|x_1, \ldots, x_n\rangle$ denotes the coordinates of the up spins of an arbitrary configuration.

As in I we make a one-to-one correspondence between the configurations of spins and product of matrices. The matrix product associated with a given configuration is obtained by associating with the sites with down and up spins matrices E and A, respectively. The unknown amplitudes in (2) are obtained by associating them with the MPA

$$f(x_1, \ldots, x_n) \quad \Leftrightarrow \quad E^{x_1-1}AE^{x_2-x_1-1}A\cdots E^{x_n-x_{n-1}-1}AE^{L-x_n}. \tag{3}$$

Actually, E and A are abstract operators with an associative product. A well-defined eigenfunction is obtained, apart from a normalization factor, if all the amplitudes are related uniquely, due to the algebraic relations (to be fixed) among the matrices A and E. Equivalently the correspondence (3) implies that, in the subset of words (products of matrices) of the algebra containing n matrices A and $L-n$ matrices E, there exists only a single independent word ('normalization constant'). The relation of any two words is a c number that gives the ratio of the corresponding amplitudes in (3).

We could also formulate the ansatz (3) by associating a complex number with the single independent word. We can choose any operation on the matrix products that gives a non-zero scalar. In the original formulation of the MPA under periodic boundary conditions [1, 2], the trace operation was chosen to produce this scalar

$$f(x_1, \ldots, x_n) = \mathrm{Tr}[E^{x_1-1}AE^{x_2-x_1-1}A\cdots E^{x_n-x_{n-1}-1}AE^{L-x_n}\Omega_p]. \tag{4}$$

The matrix Ω_p was chosen to have a given algebraic relation with the matrices E and A. Recently, Golinelli and Mallick [4] have shown that in the particular case of the asymmetric exclusion problem in a periodic chain, it is possible to formulate the ansatz only by imposing relations between the matrices E and Ω_p, the relations of A and Ω_p being totally arbitrary. Actually, as we are going to show in this note, we do not need to impose any algebraic relation between the matrices E and A with Ω_p. The matrix Ω_p can be just any arbitrary matrix that produces a non-vanishing trace in (4). This observation is not particular for the present model. It is valid for any of the exactly integrable quantum chains solved in the original formulation of the MPA presented in I. Instead of restricting the MPA with the trace operation, as in I, we consider the more general formulation (3).

Since the eigenfunctions produced by the ansatz have a well-defined momentum, $p = \frac{2\pi}{L}j$ ($j = 0, \ldots, L-1$), the correspondence (3) implies the following constraints for the matrix products appearing in the ansatz (3):

$$E^{x_1-1}AE^{x_2-x_1-1}\cdots AE^{L-x_n} = \mathrm{e}^{-\mathrm{i}p}E^{x_1}AE^{x_2-x_1-1}\cdots AE^{L-x_n-1}, \tag{5}$$

for $x_n \leqslant L-1$ and for $x_n = L$

$$E^{x_1-1}AE^{x_2-x_1-1}\cdots A = \mathrm{e}^{-\mathrm{i}p}AE^{x_1-1}A\cdots AE^{L-x_{n-1}-1}. \tag{6}$$

The eigenvalue equation,

$$H|\psi_{n,p}\rangle = e|\psi_{n,p}\rangle, \tag{7}$$

gives us relations among the amplitudes $f(x_1, \ldots, x_n)$ defining the eigenfunctions $|\psi_{n,p}\rangle$. As a consequence of the correspondence (3), these relations give two types of constraints for the algebraic relations of the matrices A and E. The first type of relations come from the configurations where all the up spins are at distances larger than the unity. The algebraic relations coming from these relations are solved by identifying the matrix A as composed by n-spectral dependent matrices, as in (I.27):

$$A = \sum_{j=1}^{n} A_{k_j}E, \tag{8}$$

where the matrices A_{k_j} obey the commutations relations

$$EA_{k_j} = \mathrm{e}^{\mathrm{i}k_j}A_{k_j}E. \tag{9}$$

Relations (8) and (9), applied to the algebraic constraints implied by the eigenvalue equation (7) and to (5), give us the energy e and momentum p as a function of the spectral parameters:

$$e = \frac{\Delta}{2}(4n - L) - 2\sum_{j=1}^{n} \cos k_j, \qquad p = \sum_{j=1}^{n} k_j. \tag{10}$$

The second type of relations, coming from the amplitudes where the spins are at nearest-neighbour positions, imply the commuting relations among the matrices A_{k_j}:

$$A_{k_j} A_{k_l} = s(k_j, k_l) A_{k_l} A_{k_j}, \tag{11}$$

where

$$s(k_j, k_l) = -\frac{1 + \mathrm{e}^{\mathrm{i}(k_j + k_l)} - 2\Delta\, \mathrm{e}^{\mathrm{i}k_j}}{1 + \mathrm{e}^{\mathrm{i}(k_j + k_l)} - 2\Delta\, \mathrm{e}^{\mathrm{i}k_l}}. \tag{12}$$

The spectral parameters $\{k_1, \ldots, k_n\}$, free up to now, are fixed by using (8), (9) in the remaining relation (6), giving us

$$\mathrm{e}^{\mathrm{i}k_j L} = -\prod_{l=1}^{n} s(k_j, k_l). \tag{13}$$

The solutions of (13), when inserted in (10), give us the eigenenergies. The fact that the correspondence (3) is exact implies that, apart from an overall normalization constant, any amplitude $f(x_1, \ldots, x_n)$ can be calculated exactly.

In the present formulation of the MPA, it is possible to produce finite-dimensional representations for the matrices A and E.[1] For a given solution $\{k_1, \ldots, k_n\}$ of the spectral parameter (13), in the sector with n particles, the matrices E and $\{A_{k_1}, \ldots, A_{k_n}\}$ have the following finite-dimensional representation:

$$\begin{aligned} E &= \bigotimes_{l=1}^{n} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}kl} \end{pmatrix}, \\ A_{k_j} &= \left[\bigotimes_{l=1}^{j-1} \begin{pmatrix} s(k_j, k_l) & 0 \\ 0 & 1 \end{pmatrix} \right] \bigotimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \bigotimes_{l=j+1}^{n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{14}$$

where $s(k_j, k_l)$ are given by (12) and A is obtained by using (8). The dimension of the representation is 2^n and the products appearing on the ansatz have trace zero. If we want a formulation of the matrix product ansatz where the trace operation is used, as in formulation (4), it is quite simple to produce the matrix Ω_p that gives a non-zero value for the trace. We should stress that in the original formulation of the ansatz in [1, 2], it was required unnecessary algebraic relations among the matrices E and A that probably would have only infinite-dimensional representations. The existence of the finite representations, in the present formulation, simplifies the calculation of the amplitudes.

We conclude this note by mentioning that all exact solutions presented for periodic quantum chains in [1–3] can be reobtained by using the formulation of the MPA presented in this note.

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