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Periodic orbits and a correlation function for the semiclassical density of states

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Abstract. A principle of uniform density of periodic orbits in the phase space of a Hamiltonian system with bound classical motion is proposed and used to obtain information about the semiclassical quantum eigenvalue spectrum. It supplies a more refined statistic than the 'one state per Planck cell' rule for the average semiclassical density of states, namely the limiting behaviour of a certain correlation function of the density of states. Unlike the average, this correlation shows markedly different behaviour for systems with integrable and ergodic classical motion.

1. Introduction

Through the rule that there is one quantum state per Planck cell in phase space, classical mechanics supplies an 'average' semiclassical density of states $\langle n(E)\rangle$ for a system with bound motion. The question addressed here is whether it can supply a more refined 'average' like the correlation function for the density of states $\langle n(E)n(E+\Delta E)\rangle$. We shall argue that, for the two extreme cases of integrable classical motion and ergodic classical motion, information can indeed be obtained on not quite this but a closely related correlated function. The results for the two are markedly different.

The starting point is a remarkable connection established a decade or so ago by Gutzwiller (1967, 1969, 1970, 1971) (with important subsequent contributions by Balian and Bloch (1970, 1971, 1972, 1974) and Berry and Tabor (1976, 1977a, b)) between the quantum energy spectrum of bound motion and those special trajectories which close and retrace themselves—periodic orbits. For fixed energy E the number of such orbits with period less than t increases rapidly with t and their paths fill the energy surface ever more densely. The connection is a semiclassical one, applying when Planck's constant can be considered small, and supplies an infinite sum of oscillatory corrections to the average density of states $\langle n(E)\rangle$, each one associated with a particular periodic orbit (or a particular family of them). The idea is that the oscillatory corrections ultimately generate the true spiky density of states with a δ function at each energy level (figure 1).

Formally this connection makes our original question redundant—classical mechanics supplies through periodic orbits *complete* information on the semiclassical density of states. The problem is that given a Hamiltonian there is in general no way

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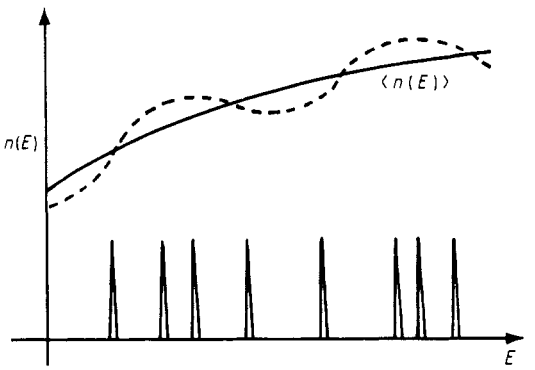

Figure 1. The average density of states (bold) with the first of an infinite set of oscillatory corrections (broken) whch yield, ultimately, the true density of states—a δ function at each energy level.

of enumerating its periodic orbits explicitly or systematically—they are deeply buried in its structure. So our question is really; can correlation information be extracted *directly* from the classical Hamiltonian *without* explicit knowledge of periodic orbits? To show that it can we will begin, in § 2, with the periodic orbit sum and show that one feature (the behaviour near the origin) of a certain correlation function does not actually depend on the details of these orbits at all—only on whether they occur in families (integrable system) or are isolated (ergodic system).

The principle that is proposed and used to derive the result on the correlation function is that for any Hamiltonian system, periodic orbits, counted with their natural weighting, are *uniformly* dense in phase space. This 'principle of uniformity' is formulated in § 3 using δ function apparatus. The interpretation of the δ function expressions for general systems is subtle. However for the two extremes of ergodic and integrable motion there is no difficulty and we restrict attention to these.

2. Correlation function

In this section we construct our correlation function and present the results on it. The principle by which they are justified is explained in the next section. The periodic orbit sum gives the semiclassical density of states

$$n(E) \equiv \sum_j \delta(E - E_j) = \langle n(E) \rangle + \tilde{n}(E), \tag{1}$$

$$\langle n(E) \rangle \equiv (2\pi\hbar)^{-N} \int \mathrm{d}^N q \, \mathrm{d}^N p \, \delta(E - H(\boldsymbol{q}, \boldsymbol{p})), \tag{2}$$

$$\tilde{n}(E) \underset{\hbar \to 0}{=} \hbar^{-\nu} \sum_{\substack{\text{periodic} \\ \text{orbits} j}} A_j(E) \exp(\mathrm{i}\alpha_j \pi/2) \exp(\mathrm{i}\hbar^{-1} S_j(E)). \tag{3}$$

The middle line defines the usual classical density of states—one state per Planck cell, volume $(2\pi\hbar)^N$, in $2N$-dimensional phase space with Hamiltonian $H(\boldsymbol{q}, \boldsymbol{p})$. The last line specifies the periodic orbit sum. The sum is over individual periodic orbits in the case of ergodic systems and periodic tori in the integrable case. In both cases *repetitions*

(multiple traversals) of primitive orbits are counted as separate orbits, and the repetition number m runs throught all positive and negative integers $m = \pm 1, \pm 2, \ldots$. The quantities A, S, α, ν are all classical ($\hbar$ independent) and have the following meanings. A_j is a real positive amplitude of the orbit j (Berry and Tabor 1977b, Gutzwiller 1971). S_j is its action $m_j \oint \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{q}$ where m_j is its repetition number. Both A_j and S_j depend on the energy E which parametrises the orbit. ν is unity for isolated periodic orbits and $\frac{1}{2}(N+1)$ for periodic tori and finally α_j is an integer connected with the Morse index of the orbit but we need not be specific because it will cancel out in our application.

It may be noted that while each individual term in $\tilde{n}(E)$ vanishes in comparison with $\langle n(E) \rangle$ as $\hbar \to 0$, periodic orbits are infinitely numerous—the sum is certainly not absolutely convergent. There are difficulties in applying the periodic orbit method to systems such as are described by KAM theory which are neither fully integrable nor fully ergodic. (It is not quite clear how to evaluate orbit amplitudes for the *stable* isolated periodic orbits which occur in such systems in contrast to the unstable ones of ergodic systems.) This is the reason for the present restriction to the two extremes.

A well defined correlation function for the density of states can now be set up. It is *not* the most obvious form $\langle \tilde{n}(E)\tilde{n}(E+\Delta E) \rangle$. Rather the correlation involves $\tilde{n}$ evaluated at two different values of the parameter $\hbar$ on which it depends implicitly. It is convenient to work with the reciprocal $\hbar^{-1}$ and write explicitly $\tilde{n}(E; \hbar^{-1})$. We define the average $\langle \cdot \rangle$ as an average over *all* values of $\hbar^{-1}$:

$$\langle \cdot \rangle \equiv \lim_{\hbar^{-1} \to \infty} \frac{1}{2\hbar^{-1}} \int_{-\hbar^{-1}}^{\hbar^{-1}} \cdot \, \mathrm{d}\hbar^{-1}. \tag{4}$$

(This procedure is reminiscent of that used by Balian and Bloch (1974) who take Fourier transforms with respect to $\hbar^{-1}$). Using the fact that $\tilde{n}$ is real we then obtain directly

$$\begin{aligned} &\langle \tilde{n}(E; \hbar^{-1})\tilde{n}(E; \hbar^{-1} + \Delta\hbar^{-1})/(\hbar^{-1})^{\nu}(\hbar^{-1} + \Delta\hbar^{-1})^{\nu} \rangle \\ &\quad = \sum_j \sum_k A_j A_k \exp(\mathrm{i}\pi(\alpha_j - \alpha_k)/2) \langle \exp(\mathrm{i}\hbar^{-1}(S_j - S_k)) \rangle \exp(\mathrm{i}\Delta\hbar^{-1} S_j) \\ &\quad = \sum_j A_j^2 \exp(\mathrm{i}\Delta\hbar^{-1} S_j). \end{aligned} \tag{5}$$

The last equality follows because the average in (4) is, in the absence of strict degeneracy among orbit actions, the Kronecker delta δ_{ij}. In the case that there is strict degeneracy among some set of orbits (or tori), as might be forced by symmetry for example, then they should be counted as a single contribution in the sum (5) with A_j^2 replaced by the square of the sum of the amplitudes of degenerate orbits.

In words then the correlation function (5) is the Fourier transform of the density of periodic orbit actions taken with weighting A_j^2 (figure 2):

$$\langle \quad \rangle = \sum_j \int_{-\infty}^{\infty} A_j^2 \delta(S - S_j) \exp(\mathrm{i}\Delta\hbar^{-1} S) \, \mathrm{d}S. \tag{6}$$

Complete knowledge of this weighted action density would, therefore, yield complete information on the correlation function. Unfortunately one has, *a priori*, no knowledge of either. We claim though that one feature of the weighted action density, its asymptotic form for large actions, is accessible directly from classical mechanics via a *uniformity* principle on the density of periodic orbits in phase space. This principle is justified in the next section. Our results are easily stated now.

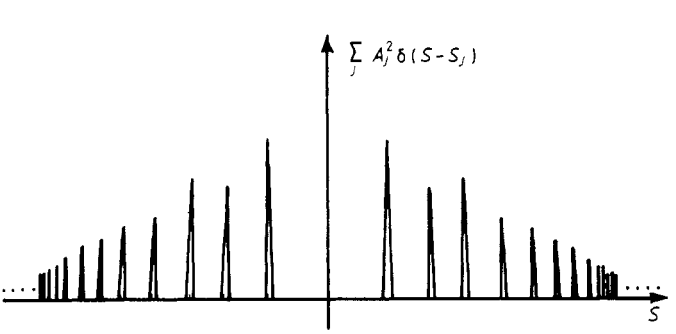

Figure 2. As they get larger, the actions S_j of periodic orbits j become more and more closely spaced, but their amplitudes A_j decrease (as indicated by the heights of the δ spikes). All repetitions of orbits (both positive and negative) count as separate orbits. The Fourier transform of the function shown is a correlation function of the semiclassical density of states.

In both integrable and ergodic systems the periodic orbits become more numerous with increasing S, but the amplitudes decrease. For integrable systems these two effects compensate so that the weighted action density tends to a constant as $S \to \infty$, while for ergodic systems it goes like $|S|$. That is, with α and β constants to be supplied in (22) and (23):

$$\sum_{\substack{j \text{ with} \\ |S_j|<|S|}} A_j^2 \underset{S\to\infty}{=} \begin{cases} \alpha|S| & \text{Integrable} \quad (7) \\ \beta S^2 & \text{Ergodic} \quad (8) \end{cases}$$

with the consequence, from the Fourier transform relationship (6),

$$\langle\ \rangle \underset{\Delta\hbar^{-1}\to 0}{=} \begin{cases} 0 & \text{Integrable} \quad (9) \\ -\beta/(\Delta\hbar^{-1})^2 & \text{Ergodic.} \quad (10) \end{cases}$$

In the integrable case this result of zero correlation is consistent with the Poisson distribution predicted by Berry and Tabor (1976). In the ergodic case the negative result is consistent with level repulsion (given one level it is less than averagely likely to find another nearby). It should be noted however that although the results apply to $\Delta\hbar^{-1} \to 0$, they are not describing structure on the scale of the level spacing, about which much is known from other arguments (Berry 1981). This is because the correlation function averages over *all* $\hbar^{-1}$ and at high $\hbar^{-1}$ the levels are closely spaced in $\hbar^{-1}$ so that there are many in the interval $\Delta\hbar^{-1}$. Indeed the ergodic result is reminiscent of that for the *tail* of the correlation function for the density of eigenvalues of a random matrix (Porter 1965, Dyson 1962) which is used with increasing confidence to model that of ergodic Hamiltonian systems (Pechukas 1984).

3. Principle of uniformity

(*Note:* a derivation of this principle in its simplest context—an area preserving map rather than the Hamiltonian flow required here—is given in appendix 2.)

Consider a Hamiltonian function $H(\boldsymbol{r})$ which defines a steady bounded flow in $2N$-dimensional phase space ($\boldsymbol{r} \equiv \boldsymbol{q}, \boldsymbol{p}$). Let $\boldsymbol{r}_t$ be the point to which $\boldsymbol{r}_0$ moves in time

t. We may then define in phase space a function of $\boldsymbol{r}$ which is a δ-function of unit total integral on the invariant manifold swept out by $\boldsymbol{r}_t$,

$$\langle \delta(\boldsymbol{r}-\boldsymbol{r}_t)\rangle_t \equiv \lim_{T\to\infty} \frac{1}{2T} \bar{\int}_{-T}^{T} \delta(\boldsymbol{r}-\boldsymbol{r}_t)\,\mathrm{d}t. \tag{11}$$

If $\boldsymbol{r}$ lies on the invariant manifold of $\boldsymbol{r}_0$ it will ultimately be hit by $\boldsymbol{r}_t$ and the integral in (11) is non-zero, otherwise it is zero. That the function has unit total integral over $\boldsymbol{r}$ easily follows by interchanging the order of integration. The bar though the integral sign indicates that, for a reason which arises later, we choose to exclude an aribitrarily small interval about $t=0$ from the range of integration.

In the two extreme cases of integrable and ergodic motion we can exhibit (11) explicitly: for almost all $\boldsymbol{r}_0$

$$\langle \delta(\boldsymbol{r}-\boldsymbol{r}_t)\rangle_t = \begin{cases} \dfrac{\delta(H(\boldsymbol{r})-H(\boldsymbol{r}_0))}{\int \delta(H(\boldsymbol{r})-H(\boldsymbol{r}_0))\,\mathrm{d}^{2N}\boldsymbol{r}_0} & \text{Ergodic} \qquad (12) \\ \dfrac{\delta(\boldsymbol{I}(\boldsymbol{r})-\boldsymbol{I}(\boldsymbol{r}_0)}{\int \delta(\boldsymbol{I}(\boldsymbol{r})-\boldsymbol{I}(\boldsymbol{r}_0))\,\mathrm{d}^{2N}\boldsymbol{r}_0} & \text{Integrable} \qquad (13) \end{cases}$$

where $\boldsymbol{I}(\boldsymbol{r})$, the N action functions, are convenient constants of the motion of the integrable system. The denominators of (12) and (13) are respectively $\mathrm{d}V/\mathrm{d}E$ at r_0, where V is the volume of phase space with energy less than E, and $(2\pi)^N$ (as is easily shown by changing to action angle variables). The content of (12) and (13) is that the time spent by the point $\boldsymbol{r}_t$ in any part of the invariant manifold is, by Liouville's theorem proportional to the 'infinitesimal separation' of neighbouring 'planes' of the confining constants.

As indicated, equations (12) and (13) apply for almost all $\boldsymbol{r}_0$—the excluded $\boldsymbol{r}_0$ being those lying on the periodic orbits of the system. We will, however, wish to use (12) and (13) specifically for such points on periodic orbits, and it is not difficult to formulate a sense in which (12) and (13) hold for these points as well. It involves slight smearing over the initial position $\boldsymbol{r}_0$ in (11). If ε measures the degree of smearing, and δ_ε denotes the consequent modified integrand, then (11) should read

$$\langle \delta(\boldsymbol{r}-\boldsymbol{r}_t)\rangle_t \equiv \lim_{\varepsilon\to 0} \lim_{T\to\infty} \frac{1}{2T} \bar{\int}_{-T}^{T} \delta_\varepsilon(\boldsymbol{r}-\boldsymbol{r}_t)\,\mathrm{d}t$$

where the limit $\varepsilon\to 0$ is taken after $T\to\infty$. The principle of uniformity will rest on the assumption that these two limits may be interchanged.

We now construct a function in phase space which picks out periodic orbits of period t irrespective of their energy. We might try

$$\delta(\boldsymbol{r}_0-\boldsymbol{r}_t) \qquad \left(\text{or strictly } \lim_{\varepsilon\to 0} \delta_\varepsilon(\boldsymbol{r}_0-\boldsymbol{r}_t)\right) \tag{14}$$

considered as a function of $\boldsymbol{r}_0$ (remembering of course that $\boldsymbol{r}_t$ depends on $\boldsymbol{r}_0$ implicitly). This function certainly takes the value *zero* for points $\boldsymbol{r}_0$ which do *not* return to themselves after time t, i.e. points not on the desired orbits, and the value infinity on them. All points on the same periodic orbit share the same period of course. But for this very reason the infinity is too strong. The integral of $\delta(\boldsymbol{r}_0-\boldsymbol{r}_t)$ over a little phase space volume containing a point for which $\boldsymbol{r}_0=\boldsymbol{r}_t$ is given by $|\det(\partial\boldsymbol{r}_t/\partial\boldsymbol{r}_0-1)|^{-1}$ which is infinite because the matrix $\partial\boldsymbol{r}_t/\partial\boldsymbol{r}_0-1$ has less than full rank—a condition associated

with the *continuum* of points satisfying $\boldsymbol{r}_0 = \boldsymbol{r}_t$ locally. The rank loss is unity for the one-dimensional continuum of points of an isolated periodic orbit and N for a periodic torus.

The difficulty is exactly rectified by replacing (14) by the following construction with *finite* integral evaluated in appendix 1

$$\delta(\boldsymbol{r}_0 - \boldsymbol{r}_t)/\langle\delta(\boldsymbol{r}_0 - \boldsymbol{r}_t)\rangle_t. \tag{15}$$

(It is in the denominator here, where $\boldsymbol{r}$ in (11) is set equal to $\boldsymbol{r}_0$ that the need to exclude $t = 0$ from the range of integration arises.) The lost rank of the matrix $(\partial \boldsymbol{r}_t/\partial \boldsymbol{r}_0 - 1)$ is balanced by the dimensionality of the δ-function in the denominator. Thus for an isolated orbit of an ergodic system the denominator contains, from (12) the one-dimensional δ-function $\delta(H(\boldsymbol{r}_0) - H(\boldsymbol{r}_t))$balancing the unit loss and leaving a $(2N-1)$-dimensional δ-function. For a periodic torus the denominator contains the N-fold δ-function $\delta(\boldsymbol{I}(\boldsymbol{r}_0) - \boldsymbol{I}(\boldsymbol{r}_t))$, which reduces (15) to an N-dimensional δ-function. The denominators are in themselves meaningless since they are infinite for *all* $\boldsymbol{r}_0$ (not just periodic orbits of period t)—every phase space point of a bound Hamiltonian system lies on its own invariant manifold. The object (15) should strictly be considered as a complete entity. Indeed in the cases (integrable and ergodic) where (15) is to be used, the denominator exactly cancels part of the numerator.

The proposed principle of uniformity can now be stated. The average density of periodic orbits in phase space as defined by the average of (15) over all possible return times t is unity:

$$\left\langle \frac{\delta(\boldsymbol{r}_0 - \boldsymbol{r}_t)}{\langle\delta(\boldsymbol{r}_0 - \boldsymbol{r}_t)\rangle_t} \right\rangle_t \equiv \frac{\lim\limits_{T\to\infty}\lim\limits_{\varepsilon\to 0}\dfrac{1}{2T}\displaystyle\int_{-T}^{T}\delta_\varepsilon(\boldsymbol{r}_0 - \boldsymbol{r}_t)\,\mathrm{d}t}{\lim\limits_{\varepsilon\to 0}\lim\limits_{T\to\infty}\dfrac{1}{2T}\displaystyle\int_{-T}^{T}\delta_\varepsilon(\boldsymbol{r}_0 - \boldsymbol{r}_t)\,\mathrm{d}t} = 1 \tag{16}$$

provided that the two limits are interchangeable. The proposal is that they almost always are so. Periodic orbits are almost always dense in phase space and (16) identifies that weighting of the orbits which renders their density uniform. There is a well known exception—the multidimensional harmonic oscillator with incommensurate frequencies which has no periodic orbits except straight lines through the origin. Evidently, there, the two limits are not interchangeable, but it is plausible that the slightest perturbation of this perfectly linear system restores the dense set of periodic orbits together with the relation (16). Obviously a proof of this conjecture would be desirable. In one special case only so far as we are aware has uniformity previously been proposed and proved—that of motion on a surface of constant negative curvature (Bowen 1971).

To obtain the results required (7), (8) some manipulation is now necessary. The first step is to examine only the periodic orbits of fixed energy that arise in (16). Fixing the energy E fixes the period of each orbit so that there is a discrete set of return times t_j. A function which is a weighted δ-function on each such time is easily constructed using (15)

$$\int \frac{\delta(\boldsymbol{r}_0 - \boldsymbol{r}_t)}{\langle\delta(\boldsymbol{r}_0 - \boldsymbol{r}_t)\rangle_t}\delta(H(\boldsymbol{r}_0) - E)\,\mathrm{d}\boldsymbol{r}_0^{2N} = \sum_j I_j\delta(t - t_j) \tag{17}$$

defining the 'intensity' weightings I_j. (Both t_j and I_j are functions of energy E.) It is

the time average of (17) which is to be used. This reads, using (16),

$$\sum_{|t_j|<T} I_j \underset{T\to\infty}{=} 2T \int \delta(H(\boldsymbol{r}_0)-E)\,\mathrm{d}r_0^{2N} \equiv 2\left|\frac{\mathrm{d}V}{\mathrm{d}E}\right| T \tag{18}$$

and constitutes a sum rule for the intensities I_j of periodic orbits. It is worth emphasising that this, like the desired sum rule (7, 8) is giving information about long orbits only—in fact any finite subset of orbits, e.g. short ones, can be ignored in the sum. To connect the two sum rules we must relate orbit intensities I_j to orbit amplitudes A_j, and time T to action S. The intensities are derived in appendix 1 and comparison with the amplitudes (Berry and Tabor 1977b, Gutzwiller 1971) yields

$$I = \begin{cases} |m/t|A^2(2\pi)^2|\mathrm{d}V/\mathrm{d}E| & \text{for an isolated periodic orbit returning in time } t \text{ after } m \text{ repetitions.} \quad (19) \\ (2\pi)^{N+1}\mathrm{A}^2 & \text{for a periodic orbit in an } N\text{-torus family.} \quad (20) \end{cases}$$

The required relation between T and S is easily derived. The ratio S/T is the average, over all periodic orbits in the energy surface, of the time averaged rate of increase of action $\boldsymbol{p}\cdot\dot{\boldsymbol{q}}$ around each (e.g. for an integrable periodic orbit this latter average is $\boldsymbol{I}\cdot\partial H/\partial \boldsymbol{I} \equiv \boldsymbol{I}\cdot\boldsymbol{\omega}$). Since periodic orbits fill the energy surface uniformly, S/T simply reduces to the average of $\boldsymbol{p}\cdot\dot{\boldsymbol{q}}$ over the energy surface

$$\frac{S}{T} = \left|\frac{\mathrm{d}V}{\mathrm{d}E}\right|^{-1} \int (\boldsymbol{p}\cdot\dot{\boldsymbol{q}})\delta(H(\boldsymbol{r})-E)\,\mathrm{d}^{2N}\boldsymbol{r} = \left|\frac{\mathrm{d}V}{\mathrm{d}E}\right|^{-1} \int \boldsymbol{p}\cdot\frac{\partial H}{\partial \boldsymbol{p}}\delta(H(\boldsymbol{r})-E)\,\mathrm{d}^{2N}\boldsymbol{r}$$

$$= \left|\frac{\mathrm{d}V}{\mathrm{d}E}\right|^{-1} \int \boldsymbol{p}\cdot\frac{\partial}{\partial \boldsymbol{p}}\theta(H(\boldsymbol{r})-E)\,\mathrm{d}^{2N}\boldsymbol{r} = \left|\frac{\mathrm{d}V}{\mathrm{d}E}\right|^{-1} NV. \tag{21}$$

With one final observation we can obtain the desired results. It concerns only the ergodic case and it is that the repetition number m in (19) can be taken as unity in the sum. This is because the periodic orbits in an ergodic system proliferate exponentially (as is indeed verifiable from the present analysis) and in consequence almost all long orbits are primitive ($|m| = 1$). Since these orbits determine the sum the replacement $|m| = 1$ is indeed justified.

Substituting (19), (20) into (18) and using (21) we have, as claimed,

$$\sum_{|Sj|<S} I_j \underset{S\to\infty}{=} \begin{cases} \dfrac{2}{(2\pi)^2}\dfrac{|\mathrm{d}V/\mathrm{d}E|^2}{N^2V^2}S^2 & \text{Ergodic.} \quad (22) \\ \dfrac{2|\mathrm{d}V/\mathrm{d}E|^2}{(2\pi)^{N+1}NV}S & \text{Integrable.} \quad (23) \end{cases}$$

It is worth noting that there is a well studied class of system, billiards (motion in a two-dimensional enclosure with hard walls), for which the ratio $S|\mathrm{d}V/\mathrm{d}E|/V$ has a simple interpretation, namely the period of the orbit. This follows from the constancy of the speed of the particle implying $\boldsymbol{p}\cdot\dot{\boldsymbol{q}} = 2E$, constant. In fact billiards admit a much simplified derivation of the results which has been presented elsewhere (Hannay 1983).

4. Summary

A uniformity principle for the density of periodic orbits in phase space has given us information on a higher statistic than the average semiclassical density of states (one

state per Planck cell). The statistic is a certain correlation function involving the density of states evaluated not at two different energies but at two different values of Planck's constant. The complete behaviour of this correlation function would require intimate knowledge of the periodic orbits, but its behaviour at the origin is governed by coarser properties supplied by the uniformity principle and is markedly different for integrable and ergodic systems.

Formally the domain of validity of the uniformity principle is the same as that of the *Poincaré recurrence theorem* (Arnold 1978), so it includes all bound Hamiltonian systems but excludes dissipative dynamical systems. We restricted attention however to the two extreme cases of integrable and ergodic systems because in order to calculate the periodic orbit intensities I one needs an interpretation of the quantity $\langle\delta(\boldsymbol{r}-\boldsymbol{r}_t)\rangle_t$ in the denominator of (15). This is not obvious in the intermediate case with its stable orbits and KAM structure. Correspondingly there is no rule for the calculation of the orbit amplitudes A in the Gutzwiller path sum in this case. It is, incidentally, the same denominator which prevents the use of the uniformity principle in dissipative systems, where its value falls to zero for typical $\boldsymbol{r}$ because $\boldsymbol{r}_t$ asymptotically approaches an attracting set. The ratio (15) is therefore indeterminate.

Acknowledgments

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Appendix 1

For the cases of (i) an isolated periodic orbit of an ergodic system and (ii) a torus of periodic orbits in an integrable system we shall evaluate the orbit (or torus) intensity I (11)

$$I=\left|\frac{\mathrm{d}t}{\mathrm{d}E}\right|\int_{\Gamma}\frac{\delta(\boldsymbol{r}-\boldsymbol{r}_t)}{\langle\delta(\boldsymbol{r}-\boldsymbol{r}_t)\rangle_t}\,\mathrm{d}\boldsymbol{r}^{2N}, \tag{A1.1}$$

where Γ is a domain of phase space containing just one periodic orbit (or one periodic torus) of period t. We shall at first consider the orbit to be primitive rather than a multiple traversal. The general case of an m-fold repetition can then be written down directly.

(i) Isolated periodic orbit of an ergodic system

In the neighbourhood of an arbitrary point of the orbit we introduce phase space coordinates $H, T, \boldsymbol{Q}, \boldsymbol{P}$ as follows. H is the Hamiltonian function. Let it have the value E on the orbit concerned. T measures time along orbits—the plane $H=E$, $T=0$ is chosen to intersect the orbit transversely (but otherwise arbitrarily). This ensures that H and T are canonically conjugate, i.e. the Poisson brackets $\{H, T\}$ equals unity. $\boldsymbol{P}$ and $\boldsymbol{Q}$ are a complementary canonical pair of vectors: $\{P_i, Q_j\}=\delta_{i,j}$ for $1\leqslant i,j\leqslant N-1$, and $\{P_i, H\}=\{P_i, T\}=\{Q_i, H\}=\{Q_i, T\}=0$. They are zero on the orbit, thus measuring deviation from it, but otherwise arbitrary.

From (12) then (A1.1) becomes, in obvious notation,

$$I = \left|\frac{\mathrm{d}V}{\mathrm{d}E}\right| \left|\frac{\mathrm{d}t}{\mathrm{d}E}\right| \int_\Gamma \frac{\delta(H-H_t)}{\delta(H-H_t)} \delta(T-T_t)\delta(\boldsymbol{Q}-\boldsymbol{Q}_t)\delta(\boldsymbol{P}-\boldsymbol{P}_t)\,\mathrm{d}H\,\mathrm{d}T\,\mathrm{d}\boldsymbol{Q}^{N-1}\,\mathrm{d}\boldsymbol{P}^{N-1}. \tag{A1.2}$$

Now the last two δ-functions isolate the local *family* of periodic orbits in phase space. They do not merely select the orbit of period exactly t because both this orbit and its neighbouring periodic orbits have *transverse* coordinates $\boldsymbol{Q}, \boldsymbol{P}$ which are, to first order, unchanged after time t. Only the one has exactly closed at time t, its neighbours have either not yet closed or are already retracing themselves. On the local family we have

$$\delta(T-T_t) = |\mathrm{d}t/\mathrm{d}E|^{-1}\delta(H-E), \tag{A1.3}$$

which removes the $\mathrm{d}H$ integral, and the $\mathrm{d}T$ integral trivially yields the period t because the argument of the integral is T independent. Thus (A1.2) reduces to

$$I = |\mathrm{d}V/\mathrm{d}E|\, t \int\int \delta(\boldsymbol{Q}-\boldsymbol{Q}_t)\delta(\boldsymbol{P}-\boldsymbol{P}_t)\,\mathrm{d}^{N-1}\boldsymbol{Q}\,\mathrm{d}^{N-1}\boldsymbol{P} \tag{A1.4}$$

$$= |\mathrm{d}V/\mathrm{d}E|t|\det(\mathbf{M}-\mathbf{1})|^{-1}, \tag{A1.5}$$

where $\mathbf{M}$ is the $2(N-1)\times 2(N-1)$ matrix of unit determinant:

$$\mathbf{M} \equiv \begin{bmatrix} \partial \boldsymbol{Q}_t/\partial \boldsymbol{Q} & \partial \boldsymbol{Q}_t/\partial \boldsymbol{P} \\ \partial \boldsymbol{P}_t/\partial \boldsymbol{Q} & \partial \boldsymbol{P}_t/\partial \boldsymbol{P} \end{bmatrix}. \tag{A1.6}$$

(This matrix is actually the same as that describing the linear part of the induced map on the Poincaré surface of section $H = E$, $T = 0$.)

Finally if the orbit is an mth repetition rather than a primitive orbit the $\mathrm{d}T$ integral yields not the full period t but the primitive period t/m so

$$I = |\mathrm{d}V/\mathrm{d}E||t/m||\det(\mathbf{M}-\mathbf{1})|^{-1}, \tag{A1.7}$$

where $\mathbf{M}$ is the mth power of the primitive mapping matrix.

(ii) Torus of periodic orbits

Here it is convenient to use action angle variables $\boldsymbol{I}, \boldsymbol{\theta} \equiv \{I_1, \ldots, I_N, \theta_1, \ldots, \theta_N\}$. From the appropriate form (13), (A1.1) becomes directly

$$I = |\mathrm{d}t/\mathrm{d}E|(2\pi)^N \int\int [\delta(\boldsymbol{I}-\boldsymbol{I}_t)/\delta(\boldsymbol{I}-\boldsymbol{I}_t)]\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_t)\,\mathrm{d}^N\boldsymbol{I}\,\mathrm{d}^N\theta \tag{A1.8}$$

$$= |\mathrm{d}t/\mathrm{d}E|(2\pi)^{2N} \int \delta(\boldsymbol{\theta}-\boldsymbol{\theta}_t)\,\mathrm{d}^N\boldsymbol{I} \tag{A1.9}$$

$$= |\mathrm{d}t/\mathrm{d}E|(2\pi)^{2N} |\det \partial(\boldsymbol{\theta}-\boldsymbol{\theta}_t)/\partial \boldsymbol{I}|^{-1} \tag{A1.10}$$

$$= |\mathrm{d}t/\mathrm{d}E|(2\pi)^{2N} t^{-N} |\det(\partial\omega/\partial \boldsymbol{I})|^{-1} \tag{A1.11}$$

$$= |\mathrm{d}t/\mathrm{d}E|(2\pi)^{2N} t^{-N} |\det(\partial^2 H/\partial \boldsymbol{I}^2)|^{-1}. \tag{A1.12}$$

Here the first equality used the $\boldsymbol{\theta}$ independence of the integrand. The third equality used the relation $\boldsymbol{\theta}-\boldsymbol{\theta}_t = \boldsymbol{\omega} t$ where $\boldsymbol{\omega} \equiv \partial H/\partial \boldsymbol{I}$ is the frequency vector of the motion.

If the orbits are mth repetitions rather than primitive there is *no change* in the result (A1.12) provided that, as always, we mean by t the full period of the orbit rather than the primitive one (t/m).

Appendix 2

To bring out the essence of the principle of uniformity we may consider briefly its application in its simplest context—an area preserving mapping of a domain of area A onto itself. We shall, moreover, restrict the discussion to the simplest case of this, where the map is an ergodic one. Consider accumulating successive iterates $\boldsymbol{r}_n$ of some arbitrary initial point $\boldsymbol{r}_0$. The ergodicity means that these iterates will eventually fill out the whole area uniformly. This may be expressed by the equation

$$\langle \delta(\boldsymbol{r}-\boldsymbol{r}_n)\rangle \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \delta(\boldsymbol{r}-\boldsymbol{r}_n) = \frac{1}{A}, \tag{A2.1}$$

independent of $\boldsymbol{r}$ and $\boldsymbol{r}_0$. This is at least true in the sense of arbitrarily small (narrow) smearing of the point $\boldsymbol{r}_0$. That the constant is indeed $1/A$ is easily checked by integrating with respect to $\boldsymbol{r}$ over the whole area A.

Now the principle of uniformity for this especially simple situation follows by taking $\boldsymbol{r}=\boldsymbol{r}_0$ in (A2.1) and examining what the resulting equation means:

$$\langle \delta(\boldsymbol{r}_0-\boldsymbol{r}_n)\rangle = 1/A \qquad \text{a constant, independent of } \boldsymbol{r}_0. \tag{A2.2}$$

The average is over all values of the iteration number n. Each contribution $\delta(\boldsymbol{r}_0-\boldsymbol{r}_n)$ is itself a set of δ spikes in the domain (figure 3). There is a spike for every initial position $\boldsymbol{r}_0$ which returns to itself after n iterations—the 'periodic' points of period n (or any factor of n), or equivalently the fixed points of the nth iterate of the map. Each δ spike in $\delta(\boldsymbol{r}_0-\boldsymbol{r}_n)$ evidently has its own weight—the derivative Jacobian $|\det(\partial \boldsymbol{r}_n/\partial \boldsymbol{r}_0 - 1)|$ (which will generally be different, though periodic points which are iterates of one another are easily shown to have equal weights). So the direct assertion of (B.2) is that these spiked functions average out to uniformity.

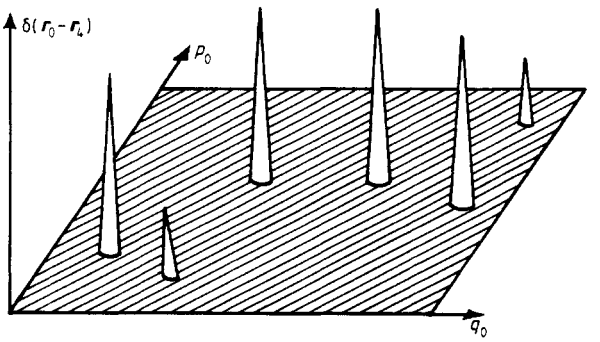

Figure 3. The function $\delta(\boldsymbol{r}_0-\boldsymbol{r}_4)$ for a (hypothetical) area preserving map of a rectangle onto itself. The δ spikes lie on periodic points of period 4 (least period 1, 2 or 4) and their heights indicate the orbit intensities (Jacobian derivatives). If all periods were plotted there would be a uniformly dense 'grass' of δ spikes.

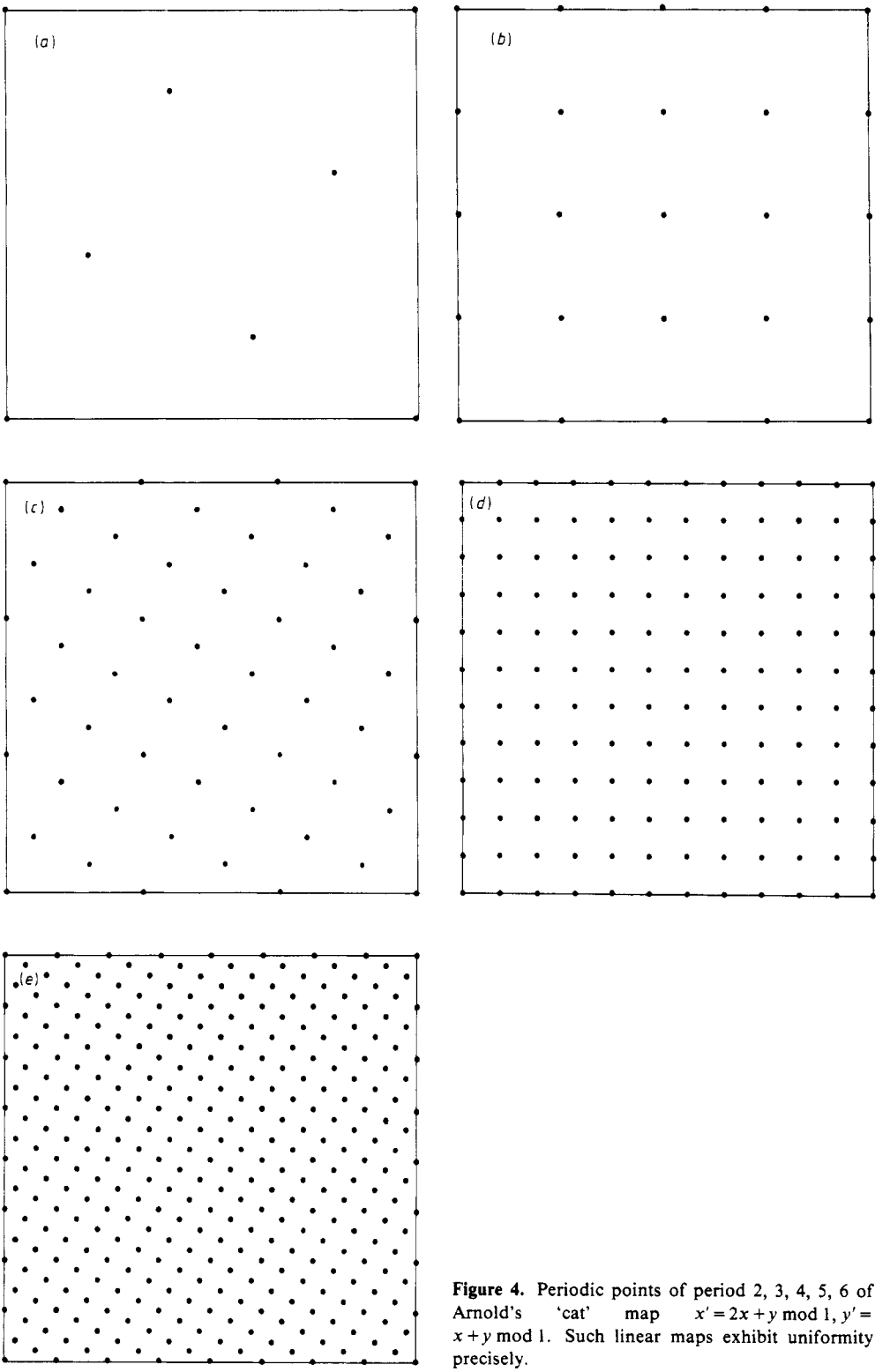

Figure 4. Periodic points of period 2, 3, 4, 5, 6 of Arnold's 'cat' map $x' = 2x + y \bmod 1$, $y' = x + y \bmod 1$. Such linear maps exhibit uniformity precisely.

The analogue of the sum rule (18) is obtained by integrating (A2.2) with respect to r_0. This yields a sum rule for the weights of periodic points

$$\frac{1}{N} \sum_{\substack{\text{Periodic} \\ \text{points of} \\ \text{period} \leqslant N}} \left|\det\left(\frac{\partial r_n}{\partial r_0} - 1\right)\right| \underset{N\to\infty}{=} 1. \tag{A2.3}$$

This emphasises the important point that uniformity owes its truth to high-order periodic points. Because of the division by N, any finite subsection of periodic points, for example low-order ones, has zero net weight. Figure 4 for example shows the periodic points of a simple ergodic map—the linear map on the unit torus defined by $x' = 2x + y \bmod 1$, $y' = x + y \bmod 1$ (Arnold's 'cat' map). The high-order periodic points do not 'avoid' low-order ones to achieve uniformity, they do so simply by being uniformly distributed themselves, and since in this case each order has equal net weight, they just overwhelm the low orders on average.

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