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# Surface integrals and the gravitational action

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**Abstract.** We discuss the modifications needed to free the Einstein–Hilbert action of gravitation from all second derivatives of fields, and give explicitly the resulting action applicable to either metric or vierbein variables. Variation of this action leads to Einstein's equations without boundary conditions. It vanishes for flat space–time and contains one arbitrary real parameter.

## 1. Introduction

The addition of surface integrals to the gravitational action has been used extensively (Gibbons and Hawking 1977, DeWitt 1967, Regge and Teitelboim 1974a, b) to satisfy various criteria. This procedure is equivalent to adding a four-divergence to the gravitational Lagrangian density. It is well known (Lanczos 1949) that the resulting equations of motion remain unchanged, but less well known is the fact that, on variation, there is a remaining boundary term which must be argued away.

We were led to this analysis of surface integrals and the gravitational action during our investigation of the canonical formalism for a purely vierbein (rather than metric) action. By purely vierbein we mean that the action cannot be written in terms of the metric tensor $g_{\mu\nu}$, but only in terms of the vierbein fields $L_{a\mu}$. Such an action may not be invariant under local rotations of the vierbein fields $L_{a\mu}$, but can be made so by implementation of the constraints which act as generators of the local transformations.

In § 2 we discuss the variation of the gravitational Hilbert action in both the metric and vierbein cases and show that it yields not only Einstein's equations but also an integral over the boundary $\partial M$. This integral can be eliminated by imposing boundary conditions, or by adding to the action another surface integral whose variation precisely cancels it. We discuss the possible integrals and, in § 3, derive the most general additional integral which eliminates all second derivatives from the original action. The resulting action takes the value zero for flat space–time.

At all stages we discuss both the metric and the vierbein cases, except when they are entirely equivalent, and restrict ourselves to pure gravity only. That is, we do not discuss the coupling of gravity to other matter fields, since this would not improve our knowledge of the problems we wish to discuss. Our results are indeed independent of any such couplings.

The sign conventions used are those of Misner *et al* (1973) apart from $K$ (equation (2.7)). Other technical data may be found in the appendix.

## 2. Variation of the action

We will take as our starting point the action

$$I(g)=\int_M \sqrt{-{}^4g}\,R(g)\,\mathrm{d}^4x \tag{2.1}$$

treated as a (second-order) functional of the metric tensor $g_{\alpha\beta}$. Under arbitrary variations of $g_{\alpha\beta}$ and its first and second derivatives, the change in $I$ is

$$\delta I=\int_M \sqrt{-{}^4g}\,G_{\mu\nu}\delta g^{\mu\nu}\,\mathrm{d}^4x+\int_M [\sqrt{-{}^4g}(g^{\beta\mu}g^{\alpha\nu}-g^{\mu\nu}g^{\alpha\beta})\delta(g_{\mu\nu,\alpha})]_{,\beta}\,\mathrm{d}^4x \tag{2.2}$$

with the Einstein tensor

$$G_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}R(g)g_{\mu\nu}.$$

The second term in (2.2) can be written as an integral over the boundary $\partial M$ as

$$-\int_{\partial M} \sqrt{\pm{}^3g}\,(g^{\mu\nu}-n^\mu n^\nu/n^2)\delta(g_{\mu\nu,\alpha})n^\alpha\,\mathrm{d}^3x. \tag{2.3a}$$

or more conveniently, as

$$-2\int_{\partial M} \delta(\sqrt{\pm{}^3g})_{,\alpha}n^\alpha\,\mathrm{d}^3x \tag{2.3b}$$

Of course, if $M$ is closed, i.e. $\partial M$ is zero, this term vanishes. The vector $n^\alpha$ occurring in (2.3$a$) and (2.3$b$) is the outward unit normal to $\partial M$, with $n^2=\mp 1$, and $\pm{}^3g$ is the determinant of the induced metric on $\partial M$, the upper signs where $\partial M$ is space-like, and the lower signs where $\partial M$ is time-like. In computing $\delta I$ we have only allowed variations of $g_{\mu\nu}$ with $\delta g_{\mu\nu}=0$ on $\partial M$. This means that on $\partial M$, only the derivatives of $g_{\mu\nu}$ in the normal direction (i.e. normal derivatives) will vary. Thus

$$\delta(g_{\mu\nu,\alpha})=n_\alpha\delta(\dot g_{\mu\nu})$$

with

$$\dot g_{\mu\nu}=g_{\mu\nu,\lambda}n^\lambda/n^2.$$

At this point we note that in the equivalent formalism for vierbein fields $L_{a\mu}$ the change in the action is

$$\delta I=2\int_M \sqrt{-{}^4g}\,G_{a\mu}\delta L^{a\mu}\,\mathrm{d}^4x+\int_M [2\sqrt{-{}^4g}\,(L^{a\alpha}g^{\mu\nu}-L^{a\nu}g^{\mu\alpha})\delta(L_{a\nu,\alpha})]_{,\mu}\,\mathrm{d}^4x \tag{2.4}$$

whose second term can be written as

$$-\int_{\partial M} 2\sqrt{\pm{}^3g}\,(L^{a\nu}-n^a n^\nu/n^2)\delta(L_{a\nu,\alpha})n^\alpha\,\mathrm{d}^3x \tag{2.5}$$

which is equal to (2.3$a$) or (2.3$b$) when the relations

$$g_{\mu\nu}=L_{a\mu}L^a{}_\nu, \qquad n^a=L^{a\nu}n_\nu,$$

are used. Note that

$$G_{a\mu} = L_a{}^\alpha G_{\alpha\mu}$$

and that by $\sqrt{-{}^4g}$ we mean the determinant of $L_{a\mu}$, treated as a $4\times4$ matrix.

In order for the stationarity of the action (2.1) under such variations to follow from Einstein's equations $G_{\mu\nu}=0$ *only*, regardless of the boundary term (2.3*a*) or (2.3*b*), one must insist on one of the following.

(a) $\delta(\sqrt{\pm{}^3g})_{,\alpha}n^\alpha=0$ on $\partial M$, as well as $\delta g_{\alpha\beta}=0$ on $\partial M$. This condition would eliminate a large number of interesting solutions of Einstein's equations (e.g. Schwarzschild, Robertson–Walker) and is unnecessarily restrictive.

(b) The space–time manifold $M$ is closed, i.e. $\partial M=0$. Then the surface integrals (2.3*a*, *b*), (2.5) are zero.

(c) the addition to $I$ of a surface integral to cancel this boundary term (2.3*b*) which will *not* affect the equations of motion.

Our approach here is to determine the most general term that can be added to the action $I$ that satisfies (c) *and* also eliminates second derivatives from $I$. This latter condition is necessary for example in the path integral formalism (Itzykson and Zuber 1980, Gibbons *et al* 1978) and also in canonical formalism, because of the problems with definitions of momenta for second-order actions (Ostragradski 1850, Ellis 1975)†. We wish to eliminate all second derivatives rather than simply those terms containing $g_{\mu\nu,00}$ or $L_{a\mu,00}$ (in an obvious notation) because we wish to include boundaries $\partial M$ (or parts of $\partial M$) more general than $t=$ constant. We also wish as far as possible to treat space and time on the same footing.

Gibbons and Hawking (1977) have shown that the addition to the gravitational action (2.1) of the surface integral ‡

$$2\int_{\partial M} K\,\mathrm{d}\Sigma = 2\int_{\partial M} K\sqrt{\pm{}^3g}\,\mathrm{d}^3x \tag{2.6}$$

† Castellani *et al* (1982) studied the first-order formalism for gravity using vierbein fields (orthonormal frames) as variables. First-order formalism means that the connections (either metric/affine or vierbein/spin) are varied independently of the fields. This leads to extra (second-class) constraints which relate the different components of connections and fields. When these constraints are eliminated by using Dirac rather than Poisson brackets, the remaining constraint algebra is the same as for second-order formalism (Nelson and Teitelboim 1978). The Hamiltonian, however, differs by terms quadratic in constraints.

‡ This surface integral (2.6) was apparently first written down by York (1972) in his analysis of the ADM (Arnowitt, Deser and Misner 1962) decomposition of space-time. He wrote the gravitational Lagrangian $\sqrt{-{}^4g}\,R$ as

$$\sqrt{-{}^4g}\,({}^3R+K_{ij}K^{ij}-K^2)-2[\sqrt{-{}^4g}\,(Ku^\lambda+a^\lambda)]_{,\lambda}$$

where $i,j=1,2,3$; $\lambda$, $\nu=0,1,2,3$; and $u^\lambda$ is the unit time-like normal to the space-like hypersurfaces which foliate the space-time; $a^\lambda$ is the acceleration $u^\lambda{}_{;\nu}u^\nu$. The second term may be transformed to a boundary term in the action by the formula

$$\int_M [\sqrt{-{}^4g}\,A^\lambda]_{,\lambda}\,\mathrm{d}^4x=\int_{\partial M} n^2A^\lambda n_\lambda\sqrt{|{}^3g|}\,\mathrm{d}^3x$$

where $n_\lambda$ is the unit normal to the boundary $\partial M$. When the boundary consists of two closed space-like surfaces $\Sigma_1$ and $\Sigma_2$ ($\Sigma_2$ to the future of $\Sigma_1$), $n_\lambda=u_\lambda$ on $\Sigma_2$, and $n_\lambda=-u_\lambda$ on $\Sigma_1$. Then this term in the action becomes

$$-2\int_{\partial M} K\sqrt{|{}^3g|}\,\mathrm{d}^3x$$

since $a^\lambda u_\lambda=0$, and can be cancelled by the addition of (2.6).

will, on variation, cancel the boundary term (2.3*b*). Here $K$ is the trace (in the metric induced on $M$) of the second fundamental form (or extrinsic curvature) of $\partial M$, defined† by

$$K = (g^{\mu\nu} - n^{\mu}n^{\nu}/n^2)K_{\mu\nu} \tag{2.7a}$$

with

$$K_{\mu\nu} = n_{\mu;\nu} = n_{\mu,\nu} - \Gamma^{\alpha}_{\mu\nu}n_{\alpha}. \tag{2.7b}$$

However, this is not the only possibility because there are other integrals, surface or volume, whose variation is equal to (2.3*b*) when only normal derivatives are allowed to vary. The most obvious is the surface expansion term, defined by

$$2\frac{\partial}{\partial n}\int_{\partial M} \mathrm{d}\Sigma = 2\int_{\partial M} (\sqrt{\pm\,{}^3g})_{,\alpha}n^{\alpha}\,\mathrm{d}^3x, \tag{2.8}$$

but this has the disadvantage that, even when written as a volume integral

$$\int_M [\sqrt{-{}^4g}\,(g^{\mu\nu} - n^{\mu}n^{\nu}/n^2)g^{\alpha\beta}g_{\mu\nu,\alpha}]_{,\beta}\,\mathrm{d}^4x \tag{2.9}$$

its second derivatives (of $g_{\alpha\beta}$) do not cancel all those of the original action, although its second normal derivatives (i.e. $g_{\mu\nu,\alpha\beta}n^{\alpha}n^{\beta}$) do, as can be seen by comparison with (2.2). The same is true for $2\int_{\partial M} K\,\mathrm{d}\Sigma$, which can be written as a volume integral

$$\int_M [\sqrt{-{}^4g}\,(g^{\mu\nu} - n^{\mu}n^{\nu}/n^2)g^{\alpha\beta}(g_{\mu\nu,\alpha} - 2g_{\mu\alpha,\nu})]_{,\beta}\,\mathrm{d}^4x, \tag{2.10}$$

and whose second derivatives also do not cancel those in (2.2), but whose second normal derivatives do. In (2.9) and (2.10) the vector $n^{\alpha}$ is arbitrary inside $M$, but as hitherto agrees with the normal on $\partial M$.

## 3. Removal of second derivatives

The standard method for removing second derivatives from an action is integration by parts. When we use the definition of the four-dimensional affine connection $\Gamma^{\alpha}_{\mu\nu}$ as a function of $g_{\alpha\beta}$ and its first derivatives, obtained from the covariant conservation of $g_{\alpha\beta}$, the gravitational action (2.1) can be written as

$$\begin{aligned} I(g) &= \int_M \sqrt{-{}^4g}\,R(g)\,\mathrm{d}^4x \\ &= \int_M \sqrt{-{}^4g}\,g^{\mu\nu}(\Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\lambda\alpha})\,\mathrm{d}^4x + \int_M [\sqrt{-{}^4g}\,(g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} - g^{\mu\alpha}\Gamma^{\nu}_{\mu\nu})]_{,\alpha}\,\mathrm{d}^4x \end{aligned} \tag{3.1}$$

where the first term in (3.1) is only quadratic in first derivatives of $g_{\mu\nu}$. The second term can be simplified to

$$\int_M [\sqrt{-{}^4g}\,g_{\mu\nu,\alpha}(g^{\mu\lambda}g^{\alpha\nu} - g^{\mu\nu}g^{\alpha\lambda})]_{,\lambda}\,\mathrm{d}^4x \tag{3.2}$$

† Our definition of the second fundamental form agrees with that of Hawking and Ellis (1973) but differs from that of Misner *et al* (1973) and also Arnowitt *et al* (1962) by a sign. Our definition ensures that, for example, for a two-sphere embedded in a flat three-geometry, $K$ is positive and equal to $2/r$.

or written as a surface integral†

$$\int_{\partial M} \sqrt{\pm^3 g}\, g_{\mu\nu,\alpha}(g^{\alpha\nu}n^{\mu} - g^{\mu\nu}n^{\alpha})\, d^3x \tag{3.3}$$

and we check that, if we again allow only normal derivatives of $g_{\mu\nu}$ to vary on $\partial M$, the variation of this term (3.3) is equal to (2.3*b*).

Similarly, in the vierbein case we write the action (2.1)

$$I(L) = \int_M \sqrt{-^4 g}\, B_{\mu ac} B_{\alpha\ b}^{\ c} (L^{a\mu}L^{b\alpha} - L^{a\alpha}L^{b\mu})\, d^4x$$

$$+2\int_{\partial M} \sqrt{\pm^3 g}\, L_{a\nu,\alpha}(L^{a\alpha}n^{\nu} - L^{a\nu}n^{\alpha})\, d^3x \tag{3.4a}$$

and note that the variation of its second term is also equal to (2.3*b*). This second term of (3.4*a*), when written as a four-volume divergence, is

$$2\int_M [\sqrt{-^4 g}\, L_{a\nu,\alpha}(L^{a\alpha}g^{\nu\lambda} - L^{a\nu}g^{\alpha\lambda})]_{,\lambda}\, d^4x. \tag{3.4b}$$

The spin connection $B_{\mu ab}$ is defined as a function of the vierbein fields $L_{a\mu}$ and their first derivatives by their covariant conservation

$$L_{a\mu;\nu} = L_{a\mu,\nu} - \Gamma^{\alpha}_{\mu\nu}L_{a\alpha} + B_{\nu a}^{\ \ b}L_{b\mu} = 0$$

and explicitly

$$B_{\mu ab} = \tfrac{1}{2}L_b^{\ \alpha}L_a^{\ \nu}[L^c_{\ \nu}(L_{c\alpha,\mu} - L_{c\mu,\alpha}) - L^c_{\ \alpha}(L_{c\nu,\mu} - L_{c\mu,\nu}) + L^c_{\ \mu}(L_{c\alpha,\nu} - L_{c\nu,\alpha})]. \tag{3.5}$$

It is important to note here that the two divergences (i.e. metric (3.2) or vierbein (3.4*b*)) obtained by integration by parts are not equal. They differ by the volume integral of a divergence which contains no second derivatives. In fact, our four candidates, namely the metric term (3.2), the vierbein term (3.4*b*), the expansion term (2.8) and the extrinsic curvature term (2.6), all differ by a surface integral of the form

$$\int_{\partial M} f(\phi, \nabla\phi)\, d\Sigma \tag{3.6}$$

where $\phi$ refers to either the metric tensor $g_{\mu\nu}$ or the vierbein fields $L_{a\mu}$, and $\nabla\phi$ means surface derivatives of $\phi$ (i.e. within $\partial M$) only. The addition of terms of the form of (3.6) to $I$ will change the expressions for momenta in terms of fields and velocities since, when written as volume integrals, they do contain velocities (but not accelerations). At the same time the expressions for the potentials as functions of fields will also change, so that the combined effect is to leave the field equations (i.e. Einstein's equations) unchanged.

† It is interesting to compare this expression with equation (1.2) of Regge and Teitelboim (1974a), which is the analogous integral over a two-surface with metric $g$. DeWitt (1967) obtained this term in the Lagrangian (itself being a *three*-dimensional surface integral) by insisting that, for asymptotically flat metrics, the Lagrangian coincides asymptotically with that of the linearised theory of gravity. Regge and Teitelboim obtained this extra term in the *Hamiltonian* rather than the Lagrangian by a completely different route. They insisted that, asymptotically, this term must be added to the canonical Hamiltonian in order that the resulting total Hamiltonian give the correct (Einstein) equations of motion.

We will consider the combined action

$$I'=\int_M \sqrt{-{}^4g}\, g^{\mu\nu}(\Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\alpha\nu}-\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\lambda\alpha})\,\mathrm{d}^4x+\int_M F(\phi,\partial_\mu\phi)\,\mathrm{d}^4x \tag{3.7}$$

which is the original action (2.1) less the metric divergence term (3.2) plus a further volume integral, whose most general form we wish to determine. In (3.7) again $\phi$ represents either the metric or vierbein fields, the two cases to be separately examined. It is interesting to note that the volume integral of any four-divergence can be added to (3.7) provided the corresponding surface integrand is zero. Now, clearly the function $F$ has no second derivatives, as already indicated, since these have already been eliminated from the rest of the action, and we do not want to reintroduce them. The inclusion of second derivatives would permit further terms in the Lagrangian and hence in the action. For a discussion of the metric case see e.g. Rund and Lovelock (1972). Further, since the boundary term (2.3*b*) has already been cancelled out by subtraction of the metric divergence term, we need that $\delta F=0$ which in turn implies that

$$\partial F/\partial\phi=(\partial F/\partial(\partial_\nu\phi))_{,\nu} \tag{3.8}$$

and thus does not contribute to the equations of motion (Einstein's equations) for $\phi$.

For the *vierbein* case, the most general function $F$ that is quadratic† in first derivatives is

$$F(L)=\sqrt{-{}^4g}\,L_{a\mu,\nu}L_{b\rho,\sigma}L_c{}^{\mu}L_d{}^{\nu}L_e{}^{\rho}L_f{}^{\sigma}H^{abcdef} \tag{3.9}$$

where $H^{abcdef}$ is simply a sum of products of Kronecker deltas. The equation of motion

$$\partial F/\partial L_{a\mu}=(\partial F/\partial L_{a\mu,\nu})_{,\nu} \tag{3.10}$$

restricts the form of $H^{abcdef}$, as do the symmetries inherent in the expression (3.9) for $F$. The most general form consistent with these symmetries is

$$\begin{aligned}H^{abcdef}&=\alpha\delta^{ce}(\delta^{ad}\delta^{bf}-\delta^{af}\delta^{bd})+\beta[\delta^{be}(\delta^{ad}\delta^{fc}-\delta^{dc}\delta^{af})+\delta^{ac}(\delta^{bf}\delta^{de}-\delta^{bd}\delta^{fe})]\\&\quad+\gamma[\delta^{bc}(\delta^{ad}\delta^{fe}-\delta^{de}\delta^{af})+\delta^{ae}(\delta^{bf}\delta^{cd}-\delta^{fc}\delta^{bd})]+\delta[\delta^{ab}(\delta^{cd}\delta^{fe}-\delta^{fc}\delta^{de})]\end{aligned} \tag{3.11}$$

for constants $\alpha$, $\beta$, $\gamma$ and $\delta$. Substitution into (3.10) leads easily to $\gamma=\delta=0$, $\alpha+2\beta=0$. Therefore the function $F$ can be written as

$$\begin{aligned}F(L)&=\alpha\sqrt{-{}^4g}\,L_{a\mu,\nu}L_{b\rho,\sigma}[g^{\mu\rho}(L^{a\nu}L^{b\sigma}-L^{a\sigma}L^{b\nu})+L^{b\rho}(g^{\mu\nu}L^{a\sigma}-L^{a\nu}g^{\mu\sigma})]\\&=\alpha[\sqrt{-{}^4g}\,L_{a\mu,\nu}(g^{\mu\nu}L^{a\lambda}-L^{a\nu}g^{\mu\lambda})]_{,\lambda}.\end{aligned} \tag{3.12}$$

In the metric case, a trivial calculation shows that there is no function quadratic in metric first derivatives which will also satisfy (3.8), i.e. in the metric case, the function $F=0$.

† Index contractions limit the function $F$ to even order in first derivatives. We only consider the quadratic case, and suggest that this is not unreasonable since the original action $I$ contains quadratic, but no quartic or higher terms.

## 4. Conclusion

We have arrived at the action

$$I' = \int_M \sqrt{-^4g}\, g^{\mu\nu}(\Gamma^\alpha{}_{\mu\lambda}\Gamma^\lambda{}_{\alpha\nu} - \Gamma^\alpha{}_{\mu\nu}\Gamma^\lambda{}_{\lambda\alpha})\, \mathrm{d}^4x$$
$$+\alpha\int_M [\sqrt{-^4g}\, L_{a\mu,\nu}(g^{\mu\nu}L^{a\lambda} - L^{a\nu}g^{\mu\lambda})]_{,\lambda}\, \mathrm{d}^4x \qquad (4.1)$$

which contains the arbitrary parameter $\alpha$. The case $\alpha = 0$ corresponds to the metric action $\int \Gamma\Gamma - \Gamma\Gamma$ whereas the case $\alpha = 1$ corresponds to the analogous purely vierbein action $\int BB - BB$ (cf equation (3.4$a$)). We insist that the action $I'$ (4.1) is the most general one that is free of second derivatives but which, on variation, produces Einstein's equations without boundary conditions. It can be more succinctly written as

$$I' = \alpha \int_M \sqrt{-^4g}\, B_{\nu ac}B_\alpha{}^c{}_b(L^{a\nu}L^{b\alpha} - L^{a\alpha}L^{b\nu})\, \mathrm{d}^4x$$
$$+(1-\alpha)\int_M \sqrt{-^4g}\, g^{\mu\nu}(\Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\lambda\alpha})\, \mathrm{d}^4x \qquad (4.2)$$

and we have found no restriction on the parameter $\alpha$. It is interesting to note that, since both the spin connection and the affine connection are zero for flat space–time, $I'$ is also then zero.

The surface expansion term (2.8) and $2\int_{\partial M} K\, \mathrm{d}\Sigma$ term (2.6), when added to the original action $I$ (2.1), cannot be written in the form of (4.2), for any value of $\alpha$, because, as shown in § 2, neither one eliminates all the second derivatives from $I$. Further, both of these terms arise as surface integrals, and when expressed as volume integrals ((2.9) and (2.10) respectively), the integrands depend on an arbitrary vector $n^\alpha$ as well as the metric. The metric term (2.2) and the vierbein term (3.4$b$) do not have such ambiguity.

We should point out that of our four candidates, to reiterate, the metric term (2.2), the vierbein term (3.4$b$), the surface expansion term (2.8), and $2\int_{\partial M} K\, \mathrm{d}\Sigma$ (2.6), only $2\int_{\partial M} K\, \mathrm{d}\Sigma$ is invariant under general coordinate transformations. Of course, if $\partial M = 0$ then all four terms vanish. When $\alpha = 0$ the action (4.2) is not general coordinate invariant, whereas when $\alpha = 1$ it is not invariant under local vierbein rotations. When $\alpha$ is neither 0 nor 1, this action has neither invariance. The addition of a non-invariant integral is similar to the addition to the Lagrangian of a gauge-fixing term (like, e.g., the term proportional to $(\partial_\mu A^\mu)^2$ in quantum electrodynamics), where here 'gauge' means coordinates. It is sometimes convenient to use specific coordinates even when the resulting theory will be coordinate invariant. However, the value of the action $I'$ depends on the coordinates used.

Finally, if we define momenta conjugate to the vierbein fields $L_{a\mu}$, from (4.2), by

$$\delta I'/\delta L_{a\mu,\nu} = \pi^{a\mu\nu} = \alpha\tilde{\pi}^{a\mu\nu} + (1-\alpha)\pi'^{a\mu\nu} \qquad (4.3)$$

in an obvious notation, and let

$$\pi^{a\mu} = \pi^{a\mu\lambda}u_\lambda \qquad (4.4)$$

where here $u_\lambda$ is the unit normal to the (arbitrary) three-dimensional space-like foliations of space–time, then the customary vierbein rotation constraint $\pi^{[a|\mu|}L^{b]}{}_\mu$ is proportional to $\alpha$, and not zero unless $\alpha$ is. This means that the action is not invariant under local four-dimensional vierbein rotations unless it is the purely metric action corresponding to $\alpha = 0$. When $\alpha \neq 0$, this invariance must be restored by the imposition of the new rotation constraints, which are

$$J^{ab} = \pi^{[a|\mu|}L^{b]}{}_{\mu} + 2\alpha(\sqrt{-^4g}\, L^{[a|\nu|}L^{b]\lambda})_{,\nu}u_{\lambda} = 0.$$

The importance of vierbein rotation invariance and its full significance for canonical quantisation is currently under investigation.

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## Appendix

Any four-vector $f_\nu$ can be decomposed into components parallel and normal to a given unit four-vector $m^\alpha$ as follows:

$$f_\nu = \bar{f} m_\nu + f_\alpha(\delta^\alpha{}_\nu - m^\alpha m_\nu/m^2)$$

where $\bar{f} = f_\alpha m^\alpha/m^2$. In particular, for partial four-derivatives, we have

$$f_{,\nu} = \dot{f} m_\nu + f_{,\alpha}(\delta^\alpha{}_\nu - m^\alpha m_\nu/m^2)$$

with $\dot{f} = f_{,\nu} m^\nu/m^2$.

All Greek (coordinate) indices $\alpha$, $\beta$ take the values 0, 1, 2, 3, as do the Latin (vierbein) indices. Greek indices are raised and lowered using the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, and Latin indices using $\delta_{ab}$ (Euclidean) or $\eta_{ab}$ (Minkowski). The calculations are done using a Euclidean metric but our conclusions are unaffected by a change in signature.

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