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LETTER TO THE EDITOR

Generalised coherent states and group representations on Hilbert spaces of analytic functions

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Abstract. Using the example for the group SU(1,1), we obtain the basis functions of a Hilbert space of analytic functions from the Perelomov definition of generalised coherent states. The Lie algebra in this space has the form of a Holstein–Primakoff representation appropriate for SU(1, 1).

As pointed out by Bargmann (1970), representations of Lie groups can be constructed by realising the elements of the Lie algebra as bilinear products of the boson annihilation and creation operators a and a^+. With these boson operators represented by operators defined over a Hilbert space of entire analytic functions, the Bargmann–Hilbert space $\mathscr{H}_B$, one can also construct group representations as Hilbert spaces of entire analytic functions. As is well known, the aforementioned Hilbert space $\mathscr{H}_B$ is closely related to the coherent states associated with the canonical algebra of a and a^+ operators and the eigenstates $|n\rangle$ of the number operator $N = a^+a$ (Schweber 1962).

On the other hand, generalised coherent states for an arbitrary Lie group have been considered by Perelomov (1972). One might expect that an alternative way of constructing a representation of a Lie group on a Hilbert space of analytic functions is to exploit the generalised coherent states in a fashion similar to the ordinary coherent states. In this letter we shall indicate how this is possible for the group SU(1, 1) ~ SO(2, 1) ~ Sp(2). We show that in this Hilbert space of analytic functions, which we shall call Perelomov–Hilbert space $\mathscr{H}_P$, the SU(1, 1) group generators have the form of a Holstein–Primakoff (1940) representation.

The SU(1, 1) Lie algebra consists of the three elements $\{K_0, K_+, K_-\}$ which satisfy the commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_-, K_+] = 2K_0. \tag{1}$$

The Casimir invariant is

$$Q = K_0^2 - \tfrac{1}{2}(K_+K_- + K_-K_+) \tag{2}$$

whose eigenvalues we shall denote as $k(k-1)$. We consider here only the representation known as the positive discrete series $\mathscr{D}^+(k)$ for which the operator K_0 is diagonal,

$$K_0|k, n\rangle = (k+n)|k, n\rangle, \tag{3}$$

where $n = 0, 1, 2, \ldots$ and $k > 0$. The basis $\{|k, n\rangle\}$ is complete and satisfies the orthonormality condition $\langle k, m|k, n\rangle = \delta_{mn}$. The state $|k, n\rangle$ is generated from the 'ground state' $|k, 0\rangle$ through the relation

$$|k, n\rangle = [\Gamma(2k)/n!\,\Gamma(n+2k)]^{1/2}(K_+)^n|k, 0\rangle \tag{4}$$

(Barut and Fronsdal 1965).

According to Perelomov (1972), coherent states for the $\mathscr{D}^+(k)$ representation are given by

$$|\xi, k\rangle = D(\alpha)|k, 0\rangle \tag{5}$$

where

$$D(\alpha) = \exp(\alpha K_+ - \alpha^* K_-) \tag{6}$$

$$= \exp(\xi K_+)\exp(\beta K_0)\exp(\gamma K_-) \tag{7}$$

where $\alpha = -(\tau/2)\,\mathrm{e}^{-\mathrm{i}\varphi}$, $\xi = -\tanh(\tau/2)\,\mathrm{e}^{-\mathrm{i}\varphi}$, $\beta = \ln(1-|\xi|^2)$ and $\gamma = -\xi^*$. Expanding the exponential in (7) and using the expression of equation (4), we have from (5)

$$|\xi, k\rangle = (1-|\xi|^2)^k \sum_{n=0}^{\infty}\left(\frac{\Gamma(n+2k)}{n!\,\Gamma(2k)}\right)^{1/2}\xi^n|k, n\rangle. \tag{8}$$

The reproducing kernel for these states is

$$K(\xi', \xi; k) = \langle\xi', k|\xi, k\rangle = (1-|\xi'|^2)^k(1-|\xi|^2)^k(1-\xi'^*\xi)^{-2k} \tag{9}$$

and unity is resolved as

$$I = \frac{2k-1}{\pi}\int\frac{\mathrm{d}^2\xi}{(1-|\xi|^2)^2}|\xi, k\rangle\langle\xi, k|. \tag{10}$$

Now from equation (10) and the orthonormality of the states $\{|k, n\rangle\}$ we have

$$\delta_{mn} = \langle k, m|I|k, n\rangle = \frac{2k-1}{\pi}\int\frac{\mathrm{d}^2\xi}{(1-|\xi|^2)^2}\langle k, m|\xi, k\rangle\langle\xi, k|k, n\rangle. \tag{11}$$

From equation (8) we have

$$\langle k, n|\xi, k\rangle = (1-|\xi|^2)^k[\Gamma(n+2k)/n!\,\Gamma(2k)]^{1/2}\xi^n. \tag{12}$$

We define the functions $u_{nk}(\xi) = A_{nk}\xi^n$, where

$$A_{nk} = [\Gamma(n+2k)/n!\,\Gamma(2k)]^{1/2}$$

such that from equations (11) and (12) we have

$$\delta_{mn} = \int \mathrm{d}\mu_k(\xi)u^*_{nk}(\xi)u_{mk}(\xi) \tag{13}$$

where

$$\mathrm{d}\mu_k(\xi) = [(2k-1)/\pi]\,\mathrm{d}^2\xi(1-|\xi|^2)^{2k-2}. \tag{14}$$

We shall take the functions $u_{nk}(\xi)$ as an orthonormal basis in the space $\mathscr{H}_\mathrm{P}$ with the measure of equation (14).

These functions $u_{nk}(\xi)$ provide a representation of SU(1, 1) such that $\mathscr{H}_\mathrm{P}\sim\mathscr{D}^+(k)$. To see this we write the operator K_0, for a particular k, as

$$K_0^{(k)} = \xi\,\mathrm{d}/\mathrm{d}\xi + k \tag{15}$$

such that

$$K_0^{(k)}u_{nk}(\xi) = (n+k)u_{nk}(\xi). \tag{16}$$

Since $[\mathrm{d}/\mathrm{d}\xi, \xi] = 1$, we have $\mathrm{d}/\mathrm{d}\xi \sim b$, $\xi \sim b^+$ where b and b^+ are boson operators defined in an auxiliary Fock space, $\mathscr{F}_n$. Thus we may write equation (15) in this space as

$$K_0^{(k)} = b^+b + k. \tag{17}$$

The SU(1, 1) Lie algebra may be completed in $\mathscr{F}_n$ by writing

$$K_+^{(k)} = b^+(2k + b^+b)^{1/2}, \tag{18}$$

$$K_-^{(k)} = (2k + b^+b)^{1/2}b. \tag{19}$$

Equations (17)–(19) constitute the Holstein–Primakoff representation of SU(1, 1). Based on the fact that SU(1, 1) for $k = \frac{1}{2}(l + N/2)$ is a dynamical group for the N-dimensional isotropic harmonic oscillator, Mlodinow and Papanicolaou (1980) have used this Holstein–Primakoff representation to develop a large N expansion in quantum mechanics for a class of potentials of the form $V = \alpha r^2 + \beta r^{2\nu}$. Within a given l-sector, fairly accurate results are obtained for the ground states. (On the other hand, from the path integral in the SU(1, 1) coherent state representation (Gerry 1982, Gerry and Silverman 1982) we have recently obtained the large N limit as a semiclassical limit (Gerry *et al* 1982). From this we obtain a phase integral quantisation rule which gives all the energy levels.)

More details will be presented elsewhere.

References

Bargmann V 1970 in *Analytic Methods of Mathematical Physics* ed R Gilbert and R Newton (New York: Gordon and Breach) p 27
Barut A O and Fronsdal C 1965 *Proc. R. Soc.* **287A** 532
Gerry C C 1982 *Phys. Lett.* B to be published
Gerry C C and Silverman S 1982 *J. Math. Phys.* to be published (November)
Gerry C C, Togeas J B and Silverman S 1982 submitted for publication
Holstein T and Primakoff H 1940 *Phys. Rev.* **58** 1098
Mlodinow L D and Papanicolaou N 1980 *Ann. Phys.* **128** 314
Perelomov A M 1972 *Commun. Math. Phys.* **26** 222
Schweber S S 1962 *J. Math. Phys.* **3** 831