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Three-dimensional classical spacetimes

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Abstract. We extend what is known about the structure of (2+1)-dimensional gravitational field theories. The non-existence of any Newtonian limit to these theories is investigated in the presence of Brans-Dicke scalar fields and non-linear curvature terms in the gravitational action. A number of new exact static and non-static solutions of (2+1) general relativity with scalar field, perfect fluid and magnetic field sources are presented and studied in detail. Some of these possess a correspondence with (3+1) solutions of general relativity through a Kaluza-Klein type reduction and exhibit the 'wedge' structure of (3+1)-dimensional solutions describing line sources like vacuum strings. An algebraic classification of (2+1) gravitational fields is derived using the Bach-Weyl tensor. The description of the general cosmological solution is given in terms of arbitrary spatial functions independently specified on a spacelike surface of constant time together with a series approximation to spacetime in the vicinity of a general cosmological singularity. Various results and conjectures regarding spacetime singularities are given. Two exact cosmological solutions containing self-interacting scalar fields that produce inflationary behaviour are also found.

1. Introduction

There have been a number of investigations into the structure of general relativistic gravitation theory in (2+1) spacetime dimensions (Staruszkiewicz 1963, Gott and Alpert 1984, Deser *et al* 1984, Deser and Jackiw 1984, Giddings *et al* 1984, Jackiw 1985, Clement 1976, 1984, 1985, Deser and Mazur 1985, Deser 1985). This interest has been stimulated by the curious character of this field theory. It contains no propagating degrees of freedom (gravitons) and does not reduce to two-dimensional Newtonian gravity in the weak-field limit. Spacetime is flat outside matter and hence there can exist no static interaction between sources. The effects of the sources show up in global aspects of the geometry and we find topology assuming the role played by curvature in the (3+1) theory.

One of the appeals of (2+1) gravity is that it is simple enough to be soluble but yet contains non-trivial features. Some of these features are apparent in the behaviour of cosmic strings and domain walls in (3+1) dimensions (Vilenkin 1981, Gott 1985, Hiscock 1985) and are potentially observable (Kaiser and Stebbins 1984, Gott 1985). In addition, we recall that in the study of quantum field theory it has often been found helpful to look at model systems of low dimension. For example, electromagnetism when formulated in (1+1) dimensions also has no propagating degree of freedom (Schwinger 1962).

In this paper we shall confine our attention to classical gravitational effects and report a series of new static and non-static solutions of the theory. We begin, in § 2, with an examination of the non-existence of any Newtonian limit in (2+1)-dimensional general relativity and other theories of gravity. Section 3 investigates the structure of several new static solutions. From some of these solutions we can obtain (3+1)-dimensional solutions of general relativity by a reversal of the usual Kaluza–Klein procedure. In § 4 we present the Petrov classification of (2+1)-dimensional spacetimes and determine the number of independent pieces of Cauchy data necessary to specify stable initial data in cosmological solutions. This leads to study of the cosmological solutions to the theory in § 5. In particular, we study the general behaviour that can arise in the neighbourhood of a singularity and also present some exactly soluble models containing non-linear scalar field interactions. These have properties similar to inflationary models in (3+1) dimensions.

Our general relativistic conventions are that the metric signature is $(+--)$. The gravitational coupling is defined as κ. Greek indices run $0 \leqslant \alpha, \beta \leqslant 2$ whilst Latin indices run $1 \leqslant a, b \leqslant 2$.

2. The weak-field limit

Newtonian gravity in two-dimensional space is defined by

$$\ddot{x}_i = -\partial_i \Phi_{\rm N} \qquad \nabla^2 \Phi_{\rm N} = \kappa\rho. \tag{2.1}$$

Thus, the acceleration of test particles is the gradient of the Newtonian potential $\Phi_{\rm N}(\boldsymbol{x}, t)$, which is determined by the mass density ρ.

Outside sources the solution is

$$\Phi_{\rm N} = A \ln r \qquad A \text{ constant.} \tag{2.2}$$

In two-dimensional space a circular disc of radius a with mass $M(a)$ has the same external gravitational field as an *equal* point mass if $\Phi_{\rm N}$ has the form

$$\Phi_{\rm N} = \Lambda r^2 + A \ln r \qquad \Lambda, A \text{ constants.} \tag{2.3}$$

Here, Λ plays the role of the cosmological constant. The circular disc can be replaced by some point mass M^* when the potential has the Bessel function form

$$\Phi_{\rm N} = AN_0(\lambda r) + BJ_0(\lambda r) \tag{2.4}$$

but the disc and point masses are unequal:

$$M^* = M(a)J_0(\lambda a) \tag{2.5}$$

where A, B and Λ are real constants and λ a real or complex constant. As $\lambda \to 0$ (2.4) approaches (2.2) and $M^* \to M(a)$.

The (2+1)-dimensional Einstein equations are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \tag{2.6}$$

hence

$$R_{\mu\nu} = \kappa (T_{\mu\nu} - g_{\mu\nu} T). \tag{2.7}$$

However, in three-dimensional spacetime the Riemann tensor is completely determined by the Ricci tensor (Fock 1959) and, via (2.6), is related to the stress tensor by

$$R_{\mu\nu}{}^{\alpha\beta} = \varepsilon_{\mu\nu\lambda}\varepsilon^{\alpha\beta\gamma} G^{\lambda}_{\gamma} = \kappa \varepsilon_{\mu\nu\lambda}\varepsilon^{\alpha\beta\gamma} T^{\lambda}_{\gamma}. \tag{2.8}$$

Hence, all vacuum spacetimes are flat and spacetime is flat outside material sources (see those papers cited in § 1). The Newtonian limit of (2+1)-dimensional general relativity is not (2.1), but $\nabla^2\Phi_N \equiv 0$.

It is interesting to investigate whether there exists another relativistic gravity theory which does have the Newtonian theory (2.1) and (2.2) as its weak-field limit. First, consider a (2+1)-dimensional version of Brans–Dicke theory (Brans and Dicke 1961, Will 1981) where the scalar field Φ will introduce some dynamics. The action is

$$S = \int \left(-\frac{\Phi R}{2\kappa} + \omega \Phi_{,\mu}\Phi^{,\mu} + \mathscr{L}_m \right)\sqrt{|g|}\, d^3x \tag{2.9}$$

where ω is the constant Brans–Dicke parameter and $\mathscr{L}_m$ the Lagrangian of any matter fields. The variation of (2.9) with respect to Φ and the metric gives the field equations

$$\square^2\Phi = \kappa T^{\mu}_{\mu}(M)/2(\kappa\omega + 1) \tag{2.10}$$

$$G_{\mu\nu} = \kappa\Phi^{-1}(T_{\mu\nu}(M) + T_{\mu\nu}(\Phi)) \tag{2.11}$$

where $T_{\alpha\beta}(M)$ is the stress tensor of the matter fields and $T_{\alpha\beta}(\Phi)$ that of the scalar field, which is given by

$$T_{\mu\nu}(\Phi) = \frac{\omega}{\Phi}(2\Phi_{,\mu}\Phi_{,\nu} - g_{\mu\nu}\Phi^{,\lambda}\Phi_{,\lambda}) + \frac{1}{\kappa}(\Phi_{,\mu,\nu} - g_{\mu\nu}\square^2\Phi). \tag{2.12}$$

The weak-field limit of the field equations yields

$$\nabla^2\Phi_N = \kappa\rho/2(\kappa\omega + 1)\Phi \tag{2.13}$$

where $g_{00} \approx 1 + 2\Phi_N$ determines the gravitational potential through the geodesic equation and $T_{00} \approx \rho$ is the only term surviving from $T_{\mu\nu}(M)$ in this non-relativistic limit.

The limit (2.13) is recognisable as Newtonian gravity, (2.1), but with a varying gravitational 'constant' $G \propto \Phi^{-1}$ (equation (2.10)). A truly Newtonian limit with Φ constant requires $\omega \to \infty$, which is the general relativity limit of Brans–Dicke theory, and we revert to the non-Newtonian weak-field limit in which the right-hand side of (2.13) vanishes.

Now, we retain finite ω and search for static solutions of (2.10) and (2.11) in vacuum with circular symmetry, possibly with some source at the origin. Such solutions are constrained by the behaviour at the centre. If we take the trace of (2.11) and eliminate $T^{\alpha}_{\alpha}(M)$ using (2.10) then the resulting equation must, in a distributional sense, exhibit cancellation of all singularities. This requirement is satisfied only by globally flat spacetime. This condition probably also excludes solutions with extended sources.

To conclude our discussion of the Newtonian limit we make some remarks about higher-derivative theories of gravity in (2+1) dimensions where there will exist some dynamics in the vacuum theory. When considering theories with Lagrangians that are quadratic in the curvature invariants we need only consider terms in R^2 and $R_{\mu\nu}R^{\mu\nu}$ because in (2+1) dimensions the combination of R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ which leads to a total divergence (Hawking 1978, Stelle 1978, Barrow and Ottewill 1983) vanishes identically due to (2.8). Hence we take the gravitational action to be

$$S = -\int (\alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2 + R - \mathscr{L}_m)\sqrt{|g|}\, d^N x \tag{2.14}$$

in N-dimensional spacetime. The full field equations are

$$\alpha\Box R_{\mu\nu}+(\tfrac{1}{2}\alpha+2\beta)g_{\mu\nu}\Box R-(\alpha+2\beta)R_{;\mu\nu}+2\alpha R_{\mu\alpha\beta\nu}R^{\alpha\beta}$$
$$-\tfrac{1}{2}\alpha g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}+2\beta RR_{\mu\nu}-\tfrac{1}{2}\beta g_{\mu\nu}R^2+R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=\kappa T_{\mu\nu}. \tag{2.15}$$

The linearised trace of (2.15) is

$$-\nabla^2 R[\tfrac{1}{2}\alpha N+2\beta(N-1)]+(1-\tfrac{1}{2}N)R=\kappa T. \tag{2.16}$$

So, in momentum ($\boldsymbol{p}$) space we have in the Newtonian, slow-motion limit,

$$R=2\kappa\int\frac{T(\boldsymbol{p})\exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x})\,\mathrm{d}^{N-1}p}{[\alpha N+4\beta(N-1)]p^2+2-N} \tag{2.17}$$

and taking the linearised $\binom{0}{0}$ component of (2.15) we have, from (2.17), that

$$(\alpha p^2+1)R_{00}(\boldsymbol{p})=\frac{\kappa T(\boldsymbol{p})[p^2(4\beta+\alpha)(N-1)-4\beta p^2+3-N]}{[\alpha N+4\beta(N-1)]p^2+2-N}. \tag{2.18}$$

For $N\neq 3$ we have a good Newtonian limit with $R_{00}(p))/p^2\sim\Phi_{\mathrm{N}}\sim p^{-2}$ but when $N=3$ the first term vanishes and there is no Newtonian limit.

3. Static solutions

3.1. The relation to (3+1)-dimensional gravity

Many people have noticed that the (3+1)-dimensional metric for a vacuum string (Gott 1985) corresponds, in sections perpendicular to the string, to the conical solutions of a point mass in (2+1) dimensions. To see precisely how such a connection can be made it is useful to consider the Kaluza-Klein procedure. In the simplest case, one spatial direction is, by fiat, compactified onto a circle. The circle is then assumed to be small and we only consider solutions that depend on the zeroth Fourier mode in that coordinate. All that this amounts to is a search for solutions that are independent of one spatial direction. In this case we start with pure gravity in (3+1) dimensions and ignore the A_μ gauge terms (which are the usual point of Kaluza-Klein theory) to write an ansatz for the metric independent of the fourth coordinate

$${}^{(4)}g_{\mu\nu}=\eta^{-1/2}\begin{pmatrix}{}^{(3)}g_{\mu\nu} & 0\\ 0 & -\eta\end{pmatrix}. \tag{3.1}$$

The equations of motion for these fields can be obtained from the reduced (2+1)-dimensional action

$$S=\frac{1}{2\kappa}\int\left(-R+\frac{\eta_{,\mu}\eta^{,\mu}}{8\eta^2}\right)\sqrt{|g|}\,\mathrm{d}^3x. \tag{3.2}$$

Note that without the prefactor $\eta^{-1/2}$ we could obtain part of the Brans-Dicke action (2.9) (Brans 1962, Dicke 1964).

A more familiar kinetic term can be obtained by writing

$$\eta=\eta_0\exp(2\Phi\sqrt{2\kappa}) \tag{3.3}$$

so that

$$S=\int\left(-\frac{R}{2\kappa}+\frac{1}{2}\Phi_{,\mu}\Phi^{,\mu}\right)\sqrt{|g|}\,\mathrm{d}^3x. \tag{3.4}$$

This represents gravity in $(2+1)$ dimensions, but minimally coupled to a scalar field. Solutions of this theory will, in view of the ansatz (3.1), become solutions of pure gravity in $(3+1)$ dimensions that are independent of z. Furthermore, the matter source for that interpretation can be determined by using (3.1) in the definition of the stress tensor

$$^{(4)}T_{\mu\nu}=\frac{2}{\sqrt{^{(4)}g}}\frac{\delta(\mathscr{L}_{\mathrm{m}}\sqrt{^{(4)}g})}{\delta^{(4)}g^{\mu\nu}}. \tag{3.5}$$

For example, the conical solution has Φ constant and the point mass $T^0_0=m\delta^2(\boldsymbol{r})$ becomes a string source, $T^0_0=T^z_z=m\delta^2(\boldsymbol{r})$.

Finally, we note that by reduction of the $(3+1)$-dimensional domain wall solution of Vilenkin (1981) we obtain a $(2+1)$-dimensional line source with the metric

$$\mathrm{d}s^2=\exp(-k|x|)(\mathrm{d}t^2-\mathrm{d}x^2-\exp(kt)\,\mathrm{d}y^2) \qquad k \text{ constant.} \tag{3.6}$$

3.2. *Massless scalar field with minimal coupling*

On the basis of the discussion in the last section on the connection with $(3+1)$-dimensional solutions we are motivated to vary the action (3.4). The field equations that follow are

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=\kappa T_{\mu\nu}=\kappa(\Phi_{,\mu}\Phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}\Phi^{,\lambda}\Phi_{,\lambda}). \tag{3.7}$$

The equation of motion for the scalar field, which amounts to energy-momentum conservation, is

$$\square\Phi=0. \tag{3.8}$$

We search for static solutions in conformal coordinates:

$$\mathrm{d}s^2=N^2(\boldsymbol{r})\,\mathrm{d}t^2-\exp(2\beta(\boldsymbol{r}))\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{3.9}$$

The spatial trace of (3.7) yields

$$-\exp(-2\beta)N^{-1}\nabla^2N=\kappa T^i_i=0. \tag{3.10}$$

We cannot proceed as with the conical solution and take N constant because the trace-free spatial equations cannot then be satisfied with non-trivial Φ. Thus, we are forced to take

$$N=\ln r. \tag{3.11}$$

Note that we might be concerned about points with coordinate value $r=0$, in particular about possible singularities there. As we shall see, such points are an infinite proper distance away from any other point with finite coordinate value. Geodesics are therefore complete and there are no singularities in the ordinary sense at $r=0$. Henceforth, we shall ignore such points in solving the Einstein equations.

If scale factors for t and r are absorbed into the equation of motion, (3.8), it becomes

$$\nabla^2\Phi+\frac{\boldsymbol{r}\cdot\nabla\Phi}{r^2\ln r}=0. \tag{3.12}$$

For a source-free field this gives

$$\Phi'(r)=p/r\ln r \qquad p \text{ constant.} \tag{3.13}$$

The remaining metric function β is determined from the $\binom{0}{0}$ Einstein equation

$$-\nabla^2\beta = \tfrac{1}{2}\kappa\Phi'^2. \tag{3.14}$$

Hence,

$$\beta(r) = \tfrac{1}{2}\kappa p^2 \ln|\ln r| - m \ln \mu r \qquad m \text{ constant.} \tag{3.15}$$

The parameter, m, is required to be 1 if the remaining traceless Einstein equations are to be satisfied.

The complete solution is

$$ds^2 = (\ln r)^2\, dt^2 - |\ln r|^{\kappa p^2} \mu^{-2} r^{-2}(dr^2 + r^2\, d\theta^2) \tag{3.16}$$

$$\Phi(r) = \Phi_0 + p \ln|\ln r|. \tag{3.17}$$

As was noted earlier, the space is non-compact, with $r = 0$ always unreachable by geodesics in finite proper time.

Having obtained the solution in these coordinates, it is simpler to discuss the metric in the coordinates $x = \mu^a \ln r$, $y = \mu^a\theta$ with $a = -2(2+\kappa p^2)^{-1}$; t is also rescaled. We obtain

$$\begin{aligned} ds^2 &= x^2\, dt^2 - |x|^{\kappa p^2}(dx^2 + dy^2) \\ \Phi &= \Phi_0' + p \ln|x|. \end{aligned} \tag{3.18}$$

The symmetry $x \to -x$ is manifest, as is a singularity at $x = 0$. The curvature scalar is given by

$$R = \kappa p^2 |x|^{(\kappa p^2 - 2)} \tag{3.19}$$

and hence $x = 0$ is a curvature singularity if $\kappa p^2 < 2$.

The energy density of the scalar field diverges at $x = 0$, as can be seen from the energy integral for the mass between coordinate distance x_1 and x_2 (Deser *et al* 1984),

$$M(x_1, x_2) = \frac{1}{\kappa}\int \sqrt{^{(2)}g}\; G^0_0\, d^2x = \mu^a p^2 \pi\left(\frac{1}{x_1} - \frac{1}{x_2}\right). \tag{3.20}$$

The proper distance around the cylinder at constant x vanishes at $x = 0$.

The choice of coordinates (t, x, y) of (3.18) is appropriate for establishing the connection with the corresponding (3+1)-dimensional metric as we discussed in § 3.1. We obtain

$$ds^2 = x^{2-p'}\, dt^2 - x^{p'(p'-2)/2}(dx^2 + dy^2) - x^{p'}\, dz^2 \tag{3.21}$$

where

$$p' = p(2\kappa)^{1/2}. \tag{3.22}$$

By a rescaling and change of variable to

$$x' = \frac{4x^{p'(p'-2)/4+1}}{p'(p'-2)+4} \tag{3.23}$$

we may write the (3+1)-dimensional metric as

$$ds^2 = x'^{2p_1}\, dt^2 - dx'^2 - x'^{2p_2}\, dy^2 - x'^{2p_3}\, dz^2 \tag{3.24}$$

with $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$.

This is a Kasner-like solution, but in a spacelike rather than timelike variable. The severe singularities of such metrics are well known.

3.3. *Stiff perfect fluid*

In this subsection we consider static solutions for $(2+1)$-dimensional gravity coupled to a perfect fluid having pressure p and density ρ with energy-momentum tensor

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - p g_{\mu\nu}. \tag{3.25}$$

We shall confine our attention to the 'stiff' equation of state $p=\rho$ (Zeldovich 1961, Barrow 1978) and use coordinates such that the fluid 4-velocity is $u_\mu = \delta^0_\mu$. It is interesting to note that in $(2+1)$ dimensions the stiff fluid has an energy-momentum tensor identical to that of a static magnetic field. The electromagnetic field stress tensor is

$$T_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda - \tfrac{1}{4}g_{\mu\nu}F_{\lambda\alpha}F^{\lambda\alpha} \tag{3.26}$$

and if we set the electric field components, F_{0i}, zero and write $F_i{}^j = \varepsilon_i{}^j(2\rho)^{1/2}$ for the magnetic field components, then (3.26) reduces to (3.25) with $p=\rho$. The equivalence between the $p=\rho$ perfect fluid and the massless scalar field does not hold in the static case. The Einstein equations may be solved to give the following metric when $p=\rho$:

$$\begin{aligned} &\mathrm{d}s^2 = \exp(\kappa\sigma r^2)(\mathrm{d}t^2 - \mathrm{d}r^2) - r^2\,\mathrm{d}\theta^2 \qquad 0 \leqslant \theta \leqslant 2\pi\gamma \leqslant 2\pi \\ &p = \rho = \sigma\exp(-\kappa\sigma r^2) \end{aligned} \tag{3.27}$$

where σ and γ are constants. The restriction of the range of γ, which creates a 'wedge' as in the vacuum solution (compare Deser *et al* 1984) can be elucidated by examining the behaviour of the solution at small r. There, it reduces to a cone and γ is identified with $1 - m\kappa/2\pi$ where m is the strength of a point source at the origin. The existence of this singularity at the origin is not apparent in the form of the solution (3.27). Consider the case with $\gamma = 1$ where there is no singularity at the origin ($m=0$). The mass-energy within a coordinate distance r_0 is given by

$$M(r_0) = \frac{1}{\kappa}\int \sqrt{{}^{(2)}g}\; C^0_0\,\mathrm{d}^2x = \frac{2\pi}{\kappa}[1 - \exp(-\sigma r_0^2/2)]. \tag{3.28}$$

Since for $\sigma > 0$ the matter is localised, we can compute the total mass to be $2\pi/\kappa$, precisely that needed in the cylinder solutions of the last section.

Suppose that we calculate the angle deficit incurred as a vector is parallel-transported around a circuit enclosing the origin. When $\gamma = 1$ there is no deficit for a circuit sufficiently close to the origin, but for a circuit of large radius ($r \gg (\kappa\sigma)^{-1/2}$) the vector rotates by 2π. If $\gamma \neq 1$ there is an additional contribution to all radius circuits from the opening angle of the cone.

3.4. *Exponential potential term*

If we generalise the Kaluza-Klein discussion of § 3.1 by including a cosmological constant, Λ, in $(3+1)$ dimensions we are motivated to consider the $(2+1)$-dimensional action

$$S = \int \left(-\frac{R}{2\kappa} + \frac{\Phi_{,\mu}\Phi^{,\mu}}{2} + \Lambda\exp[-\Phi(2\kappa)^{1/2}]\right)\sqrt{|g|}\,\mathrm{d}^3x. \tag{3.29}$$

We can find a static solution of this theory along the lines of § 3.2. The metric is taken to be of the form (3.9) but with $g_{00} = N^2$ written $\mathrm{e}^{2\alpha}$ and the energy-momentum tensor possesses an extra contribution

$${}^{(\Lambda)}T_{\mu\nu} = -\Lambda\exp[-\Phi(2\kappa)^{1/2}]g_{\mu\nu}. \tag{3.30}$$

We must solve the following equations for $\Phi(r)$, $\alpha(r)$ and $\beta(r)$:

$$\alpha''+\alpha'^2+r^{-1}\alpha'=2\kappa\Lambda\exp[2\beta-\Phi(2\kappa)^{1/2}] \tag{3.31}$$

$$\Phi''+\Phi'\alpha'+r^{-1}\Phi'=\Lambda(2\kappa)^{1/2}\exp[2\beta-\Phi(2\kappa)^{1/2}] \tag{3.32}$$

$$\alpha''+\alpha'^2-r^{-1}\alpha'-2\alpha'\beta'=-\kappa\Phi'^2. \tag{3.33}$$

Provided $\Lambda>0$, which corresponds to anti-de Sitter space, a particular solution can be found in which $\alpha'=\Phi'(2\kappa)^{1/2}$:

$$ds^2=(\mu\ln r)^{-4}\,dt^2-3(\mu r)^{-2}(\kappa\Lambda)^{-1}(\ln r)^{-4}(dr^2+r^2\,d\theta^2) \tag{3.34}$$

$$\Phi=-(\sqrt{2}/\kappa)\ln|\mu\ln r| \tag{3.35}$$

and where μ is an arbitrary constant. Note that the limit of this solution as $\Lambda\to 0$ is not (3.16) and (3.17).

In this case the point $r=0$ is no longer at infinity as in § 3.2. In fact, on inspection of the metric (3.34) it is found that points with $r=1$ are infinitely distant. It is thus more appropriate to use coordinates that invert the space; we choose $x=(\mu\ln r)^{-1}$, $y=\mu\theta$, $\bar{t}=t(\kappa\Lambda/3)^{1/2}$,

$$ds^2=(3/\kappa\Lambda)(x^4\,d\bar{t}^2-dx^2-x^4\,dy^2) \tag{3.36}$$

$$\Phi=(\sqrt{2}/\kappa)\ln|x|. \tag{3.37}$$

The symmetry $x\to -x$ is again manifest. The solution may alternatively be written as conformal to flat spacetime, but the identification of one singularity at $x=0$ is clearest in the above coordinates. This is a scalar-curvature singularity since

$$R=\tfrac{16}{3}\kappa\Lambda x^{-2}. \tag{3.38}$$

In this space the mass within coordinate range x_1 to x_2 is

$$M(x_1,x_2)=\frac{1}{\kappa}\int\sqrt{{}^{(2)}g}\;G^0_0\,d^2x=\frac{4\pi\mu}{\kappa}(x_2-x_1). \tag{3.39}$$

Using the reverse Kaluza–Klein procedure we can generate a $(3+1)$-dimensional solution with cosmological constant ${}^{(4)}\Lambda$. This is best done using coordinates y, $\bar{t}$ and $\bar{x}=x^{-1}$, the resulting $(3+1)$-dimensional metric is conformally flat with conformal factor $3/\kappa\,{}^{(4)}\Lambda\bar{x}^2$.

4. Classification of solutions

4.1. *The Petrov classification of (2+1) spacetimes*

Here we shall pursue a classification of $(2+1)$-dimensional spacetimes in terms of the algebraic properties of the curvature tensor. In $(3+1)$ dimensions this leads to the well known Petrov classification (Kramer *et al* 1980). However, the Weyl tensor, which is the basis of the Petrov classification in $(3+1)$ dimensions, vanishes in lower dimensions. An alternative geometrical object which may be used is the Bach–Weyl tensor (Eisenhart 1926, Deser *et al* 1984)

$$C^{\alpha\beta}=\frac{\varepsilon^{\alpha\lambda\sigma}}{\sqrt{|g|}}\nabla_\lambda(R^\beta_\sigma-\tfrac{1}{4}\delta^\beta_\sigma R). \tag{4.1}$$

$C^{\alpha}{}_{\beta}$ is a traceless 3×3 symmetric matrix which we can classify according to the distribution of eigenvalues and eigenvectors. There exist three distinct equivalence classes.

(a) *Class A.* Three distinct eigenvalues and three linearly independent eigenvectors; $C^{\alpha}{}_{\beta}$ has the form

$$C^{\alpha}{}_{\beta}=\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \tag{4.2}$$

with

$$\lambda_1+\lambda_2+\lambda_3=0. \tag{4.3}$$

(b) *Class B.* Two distinct eigenvalues and three linearly independent eigenvectors; $C^{\alpha}{}_{\beta}$ has the form

$$C^{\alpha}{}_{\beta}=\mathrm{diag}(\lambda_1, \lambda_1, \lambda_2) \tag{4.4}$$

with

$$2\lambda_1+\lambda_2=0. \tag{4.5}$$

(c) *Class C.* One distinct eigenvalue and three linearly independent eigenvectors; $C^{\alpha}{}_{\beta}$ has the form

$$C^{\alpha}{}_{\beta}=\mathrm{diag}(0, 0, 0). \tag{4.6}$$

Since there are no degenerate eigenvectors, there are no preferred directions in $(2+1)$ spacetimes.

We can use this classification scheme on some of the known $(2+1)$ solutions. The solution (3.18) is of class C, as is the Einstein–Maxwell solution of Deser (1984). The Bach–Weyl tensor vanishes identically for the stationary solutions of Clement (1985) and Deser *et al* (1984) showing that the spaces are conformally flat (Deser *et al* give the explicit coordinate transformation of the stationary $(2+1)$ spacetime to $(2+1)$ Minkowski spacetime). We can construct $(2+1)$ metrics which are of, say, class A. If we consider a metric of the form

$$\mathrm{d}s^2=\mathrm{d}t^2-\exp(2B(x^{\alpha}))\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{4.7}$$

then the only non-vanishing components of the Bach–Weyl tensor are $C^0{}_1\equiv a$ and $C^0{}_2\equiv b$, say. Then $C^{\alpha}{}_{\beta}$ possesses eigenvalues $(\lambda_1, \lambda_2, \lambda_3)=(0, (a^2-b^2)^{1/2}, -(a^2-b^2)^{1/2})$. So for $a\neq b$ such a spacetime will be of class A; if $a=b$ it will be of class C.

4.2. *Initial value problem*

In the next section we shall be considering particular cosmological solutions. In order to ascertain their degree of generality a useful guide is to count the number of independent arbitrary functions necessary to specify the solution on a spacelike hypersurface. Suppose we have a spacetime with D spatial dimensions. Then, in the synchronous coordinate system, we must specify $\frac{1}{2}D(D+1)$ components of the symmetric D-dimensional metric g_{ab} and $\frac{1}{2}D(D+1)$ components of $\dot{g}_{ab}$ on a spacelike surface of constant time. From this total of $D(D+1)$ functions we can subtract $(D+1)$ by using the Bianchi identities and a further $(D+1)$ by using coordinate transformations. Hence, in vacuum the number of arbitrary D-dimensional functions is

$$V(D)=(D+1)(D-2). \tag{4.8}$$

If a perfect fluid is present with an equation of state $p(\rho)$ and a normalised $(D+1)$ velocity, u_α, then we require a further $1+D$ functions to specify the solution. The number of functions of D variables, $F(D)$, required to specify the perfect fluid solution is thus

$$F(D)=D^2-1. \tag{4.9}$$

If a massless scalar field source was admitted with minimal coupling then the number of D-dimensional functions, $S(D)$, is just

$$S(D)=D(D-1). \tag{4.10}$$

We see that $V(2)$ is zero in accord with the fact that all vacuum solutions are flat and that gravitons do not exist when $D=2$.

5. Cosmological models

5.1. Singularities

The conditions under which geodesic incompleteness arises in $(3+1)$-dimensional spacetimes have been established using techniques of differential topology (Hawking and Ellis 1973, Tipler *et al* 1978). A typical example is the Hawking–Penrose theorem (Hawking and Penrose 1969). This postulates physically reasonable restrictions on the causal structure of spacetime and demands that gravity be an attractive force. From these assumptions it is possible to prove that gravitational focusing will inevitably create a focal point along the congruence of causal geodesics in the spacetime. Since the development of such a focal point would contradict the very assumptions about the causal structure assumed to derive it, one must conclude that geodesics never reach these focal points. There must exist at least one incomplete geodesic and the endpoints of all such incomplete geodesics form the singularity boundary of spacetime. It has been shown under quite general conditions (Clarke 1975, Clarke and Isenberg 1982) that geodesic incompleteness is accompanied by divergences in the values of curvature invariants, tidal forces or other physically observable quantities. However, examples are known where the curvature invariants remain finite at singularities but it has yet to be proved that these are of measure zero in some appropriate sense.

Singularity theorems can be established, *mutatis mutandis*, for $(2+1)$-dimensional cosmological models. The essential difference will be in the energy conditions that must be imposed on the material content of spacetime in order to guarantee geodesic convergence under gravity. Either of two energy conditions are employed in essentially all singularity theorems (Hawking and Ellis 1973). The *weak energy condition* demands only that $T_{\mu\nu}u^\mu u^\nu > 0$ for all causal vectors u_μ and, regardless of the spatial dimension, this corresponds to the requirement that comoving observers see a non-negative material density (that is, $\rho > 0$ for the stress tensor (3.25)). The more restrictive *strong energy condition* ensures that the net effect of gravity is to converge neighbouring bundles of geodesics; that is, $R_{\mu\nu}u^\mu u^\nu > 0$ for all causal vectors u_μ. Hence, by (2.7), it requires

$$(T_{\mu\nu}-Tg_{\mu\nu})u^\mu u^\nu > 0. \tag{5.1}$$

In a $(D+1)$-dimensional general relativistic spacetime with perfect fluid stress tensor, (3.25), the energy condition (5.1) is

$$(D-2)\rho+Dp>0. \tag{5.2}$$

In $(3+1)$ spacetime this corresponds to $\rho+3p>0$, but in $(2+1)$ spacetime it requires $p>0$. In $(2+1)$ dimensions the weak and strong energy conditions therefore place independent positivity conditions on the density and pressure.

Consider first the $(2+1)$ homogeneous and isotropic cosmological models (Collas 1977, Giddings *et al* 1984). The metric is determined by a time-dependent scale factor $a(t)$

$$ds^2 = dt^2 - a^2(t)\delta_{ab}\, dx^a\, dx^b. \tag{5.3}$$

The scale factor is determined by a Friedman-like equation

$$\frac{\dot{a}^2}{a^2} = \frac{M}{a^{2\gamma}} - \frac{k}{a^2} \qquad M \geqslant 0,\ k \text{ constants} \tag{5.4}$$

when the material content is a perfect fluid with equation of state

$$p = (\gamma - 1)\rho \tag{5.5}$$

where γ is constant.

For $1<\gamma\leqslant 2$, the solutions of (5.4) are 'closed', 'open' or 'flat' cosmological models according to whether the constant k is positive, negative or zero, respectively. They all possess initial curvature singularities since the fluid obeys both the energy conditions. Note that if $\gamma=1$ the situation is anomalous: the solutions of (5.4) all expand as $a(t)\propto t$ regardless of the sign of k. This is a reflection of the vanishing of the effect of gravity when $p=0$; see (5.1). Hence, the resulting solutions must be flat spacetime or a conformal transformation of it. An identical phenomenon occurs in the $(3+1)$ theory when $\rho+3p$ is zero. By analogy with the situation in $(3+1)$ dimensions (Barrow and Tipler 1985) we can investigate the conditions under which closed (that is, those possessing compact Cauchy surfaces) $(2+1)$ universes recollapse to a final curvature singularity. We conjecture that all closed, globally hyperbolic cosmological solutions of the $(2+1)$-dimensional Einstein equations with S^2 spatial topology will recollapse to a future curvature singularity if $\rho>0$ and $p>0$. Presumably, closed universes not possessing 2-sphere topology, for example a 2-torus, cannot recollapse to a future singularity. It would be interesting to try and prove this conjecture rigorously. The corresponding conjecture in $(3+1)$ dimensions is more complicated and is also unproved (Barrow and Tipler 1985). In general, the absence of gravitational radiation in $(2+1)$ gravity makes the classification of singularities considerably easier than in $(3+1)$ dimensions. We conjecture that in $(2+1)$ dimensions almost every spacetime singularity is a curvature singularity as in the simplest cosmological models above, or quasi-regular as in the static conical examples of § 3. In order to support the first half of this conjecture we shall now consider the behaviour of $(2+1)$ cosmologies that are more general than the homogeneous and isotropic metric (5.3).

Since any 2-space is locally conformally flat we can choose spatial coordinates so that, in a synchronous coordinate system on any surface of constant time, the $(2+1)$ metric can be written

$$ds^2 = dt^2 - \exp(2B(x^\alpha))\delta_{ab}\, dx^a\, dx^b. \tag{5.6}$$

Other discussions (Giddings *et al* 1984) have assumed that this representation holds for all time. However, at different times a different coordinate choice will in general be necessary to express the metric in the form (5.6). It is necessary that the spatial stresses in the energy-momentum tensor be isotropic in order that the form (5.6) is preserved under time evolution with the same x^i. An example with an anisotropic stress tensor is the solution (3.6). Conversely, the assumption that the metric always has the form (5.6) ensures that perfect fluid velocities will be comoving. To see the effect of the assumption (5.6) explicitly, reconsider the question of the number of arbitrary functions of the spatial variables needed to specify the general solution of the (2+1) Einstein equations on a spacelike surface of constant time. According to equation (4.9) this number is three in the presence of a perfect fluid stress tensor. If the metric is chosen as (5.6) then we must specify B, $\dot{B}$, ρ, u_1 and u_2 independently: that is, five functions of two space variables. However, we can eliminate three of these using the three constraint field equations and another by employing the one remaining coordinate transformation (the other two coordinate freedoms have already been employed to specify the form of (5.6)). This leaves the solution specified by just one arbitrary function of two variables, two lower than the general case.

We shall now examine the behaviour of a general (2+1) cosmological model in the vicinity of a curvature singularity of the type described by the homogeneous model (5.3). The metric is

$$ds^2 = dt^2 - g_{ab}\, dx^a\, dx^b. \tag{5.7}$$

We shall, for algebraic simplicity, take the fluid source to be a perfect fluid with equation of state $p = \rho$ but we shall not assume the fluid to be comoving. We assume that locally the metric can be expanded as a power series in time as $t \to 0$, with the leading order time dependence fixed as $g_{ab} \propto t$ by the homogeneous and isotropic solution to (5.4) when $\gamma = 2$ (Lifshitz and Khalatnikov 1963); so we assume

$$g_{ab} = a_{ab}t + b_{ab}t^2 + O(t^3) \tag{5.8}$$

where the symmetric tensor functions a_{ab} and b_{ab} are functions only of the spatial coordinates x^i. Using (5.8), the Einstein constraint equations yield to second order

$$R_{00} = 1/2t^2 - b/2t = 2\rho\kappa u_0^2 \tag{5.9}$$

$$R_{0a} = \tfrac{1}{2}(b^b_{a;b} - b_{,a}) = 2\rho\kappa u_0 u_a \tag{5.10}$$

where $b \equiv b^a_a$ and all index gymnastics are carried out with the first term of (5.8). Using the velocity normalisation $u_\alpha u^\alpha = 1$, equations (5.8) and (5.9) allow us to determine ρ and u_a to leading order

$$\rho = \frac{1}{4\kappa t^2} - \frac{b}{4\kappa t} \tag{5.11}$$

$$u_a = t^2(b^b_{a;b} - b_{,a}). \tag{5.12}$$

To order t^{-1} the space-space Einstein equations are

$$a^{gb}a_{ab}b + 2a^{gb}b_{ab} + 4R^g_a + 4b_{ab}a^{gb} = 0 \tag{5.13}$$

where R_{ab} is the two-dimensional Ricci tensor defined by g_{ab}. Therefore, we have

$$b^g_a = -\tfrac{2}{3}R^g_a - \tfrac{1}{6}b\delta^g_a \tag{5.14}$$

and

$$R^a_a = -2b. \tag{5.15}$$

Hence

$$b^g_{a;g} = -\tfrac{2}{3}R^g_{a;g} + \tfrac{1}{12}\delta^g_a R^a_{a,g}. \tag{5.16}$$

Now, for any 2-surface defined by g_{ab} there is only one non-zero component of the Riemann tensor, R_{1212}, so we can write

$$R^a_{b;a} = R^a_{a,b} \tag{5.17}$$

and therefore (5.15) and (5.16) give

$$b^g_{a;g} = \tfrac{2}{3} b_{,a} \tag{5.18}$$

and (5.12) becomes

$$u_a = \tfrac{1}{3} t^2 b_{,a}. \tag{5.19}$$

From (5.19) we see that the velocity field is curl-free to leading order as $t \to 0$ and the velocity field also becomes comoving in this limit as the singularity is approached. The solution (5.11) and (5.19) leaves the three independent spatial functions a_{ab} arbitrary. The three spatial functions b_{ab} are completely determined in terms of the a_{ab} via the relations (5.15). Thus a solution with the leading order metric behaviour (5.8) possesses the necessary three-function arbitrariness of the general cosmological solution of the $(2+1)$ Einstein equations. This means that (5.8) could, if the series decomposition (5.8) really is admissible as $t \to 0$ (see Barrow and Tipler 1978), represent an approximation to part of the general solution of the $(2+1)$ Einstein equations in the neighbourhood of a cosmological singularity. If so, it demonstrates that part of the general solution describes a quasi-isotropic curvature singularity that is a small perturbation of the exact homogeneous and isotropic example (5.3). As the spacetime expands away from the singularity at $t = 0$ we see from (5.11) and (5.19) that deviations from isotropy and homogeneity will grow. The growth of the inhomogeneous density term in (5.11) simply reflects the presence of the Jeans gravitational instability in self-gravitating fluids and the velocity growth (5.19) reflects the conservation of linear momentum for the fluid motion. Note also that if we had supposed that the spatial metric preserved a conformally Euclidean form, (5.6), during its time evolution then in our approximation we would have been able to write $a_{ab} = a(x^i)\delta_{ab}$ where $a(x^i)$ is a scalar function. Hence, in this case the solution would be completely specified by the single spatial function $a(x^i)$ on a surface of constant time and would not be part of the general solution.

We have taken an equation of state $p = \rho$ in the above calculation but the conclusions are found to be essentially the same for any other perfect fluid equation of state with $0 < p \leqslant \rho$. The $p = 0$ case is obviously different. The vanishing of the gravitational stresses in this case means, as discussed above, that the metric is conformally flat and hence the only non-zero spatial functions in an expansion of the form (5.8) are the a_{ab}, in that case they can be removed by a redefinition of coordinates.

5.2. *Exact inflationary solutions*

In this subsection we give two exact solutions for the evolution of self-interacting scalar fields with physically interesting potentials, $V(\Phi)$, in the zero-curvature homogeneous and isotropic cosmological metric (5.3).

The two field equations to be solved for $\Phi(t)$ and $a(t)$ are the Friedman-like equation

$$\dot{a}^2/a^2 = \tfrac{1}{2}\kappa\dot{\Phi}^2 + \kappa V(\Phi) \tag{5.20}$$

and the equation of motion of the Φ field

$$\ddot{\Phi} + 2(\dot{a}/a)\dot{\Phi} + V'(\Phi) = 0. \tag{5.21}$$

First we look for expanding universe solutions of the model introduced in § 3.4 by Kaluza–Klein reduction from $(3+1)$ dimensions with

$$V(\Phi) = \Lambda \exp[-\Phi(2\kappa)^{1/2}] \qquad \Lambda \text{ constant.} \tag{5.22}$$

We would expect the Φ field to roll slowly down this fairly flat exponential potential, so causing a period of very rapid inflationary expansion (Guth 1981, Sato 1981, Hawking and Moss 1982, Linde 1982, Albrecht and Steinhardt 1982). This inflationary phase would be followed by slower expansion when the effective cosmological constant induced by the Φ field has decreased. In order to have this interpretation we must choose Λ to have a sign appropriate to de Sitter spacetime, which is opposite to that of § 3.4. The equations (5.20)-(5.22) may then be solved exactly by a lengthy series of transformations to give

$$a(t) = t^2(1+A/t^3)^{1/2} \qquad A \text{ constant} \tag{5.23}$$

$$\Phi(t) = \frac{1}{(2\kappa)^{1/2}} \ln\left[\frac{\kappa\Lambda t^2}{3}\left(1+\frac{A}{t^3}\right)\right]. \tag{5.24}$$

Plots of $\dot{a}/a$ and $\Phi(t)$ are shown in figure 1 where we have chosen initial conditions $\dot{\Phi}=0$ at $t=(A/2)^{1/3}$. In this case $\dot{a}/a$ is initially equal to $(2/A)^{1/3}$ but as $t\to\infty$ it approaches $2t^{-1}$. This is clearly suggestive of inflation, although it is power law rather than exponentially rapid expansion that occurs. The exponential expansion would result from a constant potential stress in (5.22). This indicates that the potential (5.22) is not quite flat enough to give true inflation and furthermore the Φ field cannot dissipate its kinetic energy into other forms by particle production as in the conventional (3+1) scenario. Nevertheless, the gross features are similar and an exact solution of this type might be useful in a quantum mechanical analysis. Unfortunately the (3+1)-dimensional analogue of the system (5.20)-(5.22) does not admit of an exact analytic solution.

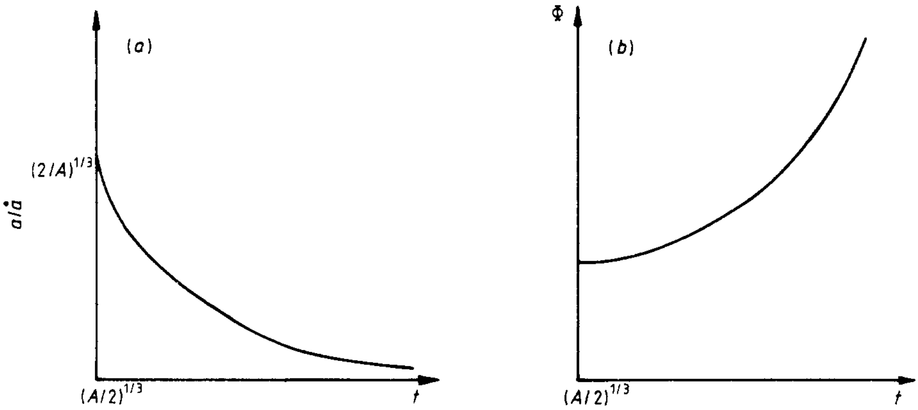

Figure 1. (*a*) The expansion rate of the isotropic cosmological model (5.23) which contains a scalar field with potential energy (5.22). There is quasi-inflationary behaviour. (*b*) The time dependence of the scalar field $\Phi(t)$ in the solution (5.24).

A second solution of (5.20)-(5.22), which admits symmetry breaking, can be found if we take the potential to be a sum of quadratic and quartic parts. The complete solution is

$$V(\Phi) = \mu(\tfrac{1}{2}\kappa\Phi^4 - \Phi^2) \tag{5.25}$$

$$\Phi(t) = A\exp[\pm(2\mu)^{1/2}t] \tag{5.26}$$

$$\ln(a(t)/a_0) = -\tfrac{1}{4}\kappa A^2 \exp[\pm(8\mu)^{1/2}t] \tag{5.27}$$

where A, μ and a_0 are constants. The expanding cosmological model corresponds to the negative sign in the exponents of (5.26) and (5.27). The scale factor approaches $a=0$ asymptotically as $t \to -\infty$ and approaches the asymptote $a=a_0$ monotonically from below as $t\to\infty$. The double exponential evolution of the scale factor $a(t)$ in (5.27) is a dramatic type of 'super-inflation' but as a consequence there is a scalar curvature singularity at $t=-\infty$. Plots of $\dot{a}/a$ and $\Phi(t)$ are shown in figure 2.

The (3+1)-dimensional version of this solution was found by Madsen (1986).

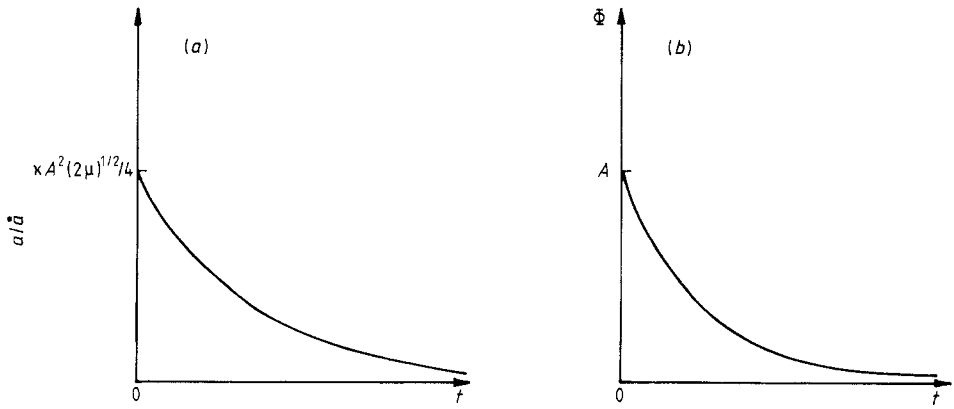

Figure 2. (a) The expansion rate of the isotropic cosmological model (5.27) (where the negative sign is taken). (b) The time dependence of the scalar field $\Phi(t)$ in the solution (5.26). The behaviour of this solution is very peculiar. The Φ field begins high up the potential (5.25) with a large downward velocity but at late times it does not reside in the symmetry breaking minimum.

6. Conclusions

In this paper we have extended the earlier studies of the structure of (2+1)-dimensional classical spacetimes made by those authors cited in § 1. We have analysed the non-existence of a Newtonian weak-field limit in (2+1) general relativity and have found this property to persist in related theories containing Brans-Dicke scalar fields and non-linear gravitational Lagrangians. Motivated by the Kaluza-Klein reduction procedure from (3+1) dimensions we have studied in detail a number of static spacetimes containing scalar field and magnetic field sources (we point out that the latter corresponds to a perfect fluid with pressure equal to density in (2+1) dimensions). These solutions also illustrate aspects of the 'wedge' structure found in (2+1) spacetimes as well as in (3+1)-dimensional line sources used as descriptions of the spacetime exterior to static vacuum strings. In § 4 we gave a classification of (2+1)-dimensional gravitational fields analogous to the Petrov classification in (3+1) dimensions but in this case based upon the algebraic structure of the Bach-Weyl tensor since the Weyl tensor used in Petrov's classification vanishes in (2+1) dimensions. In the last section we studied the cosmological (2+1) spacetimes in detail, determining the number of independent arbitrary functions of the two spatial variables that must be specified on a spacelike surface of constant time in order to determine part of the general solution of the Einstein equations in the presence of various material sources. This enables some investigations to be made of singularities in $(2+1)$ spacetimes and we show that there exists a quasi-isotropic solution containing non-comoving perfect fluid prescribed by the requisite number of undetermined parameters and which becomes comoving and isotropic as the singularity is approached. Finally, we give two exact solutions exhibiting the evolution of cosmological models containing self-interacting scalar fields. These are exact models of inflation and in the absence of $(3+1)$-dimensional examples may be interesting for quantum mechanical studies of this phenomenon.

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