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Analytic description of singularities in Gowdy spacetimes

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Abstract. We use the Fuchsian algorithm to construct singular solutions of Einstein's equations which belong to the class of Gowdy spacetimes. The solutions have the maximum number of arbitrary functions. Special cases correspond to polarized or other known solutions. The method provides precise asymptotics at the singularity, which is Kasner-like. All of these solutions are asymptotically velocity-dominated. The results account for the fact that solutions with velocity parameter uniformly greater than one are not observed numerically. They also provide a justification of formal expansions proposed by Grubišić and Moncrief.

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1. Introduction

The singularity theorems of Hawking and Penrose (see [7]) show that solutions of Einstein's equations are 'non-continuable' under rather general conditions, but do not provide very specific information about the structure of singularities. This motivated several attempts to try and provide an analytical description of singularities of solutions of Einstein's equations. Our approach in this paper is to try and determine how to perturb known exact solutions and to decide whether or not the type of singularity they possess is representative of the behaviour of more general solutions.

There is a technique which provides precisely this type of information for rather general classes of partial differential equations: the Fuchsian algorithm. It consists in constructing singular solutions with a large number of arbitrary functions by considering the equation satisfied by a rescaled unknown, which represents in fact the 'regular part' of the solution. This new unknown satisfies a *Fuchsian PDE*, i.e. a system of the form

$$t\frac{\partial \boldsymbol{u}}{\partial t} + A\boldsymbol{u} = f(t, x_1, \ldots, x_n, \boldsymbol{u}, \boldsymbol{u}_x),$$

where A is a square matrix and f vanishes like some power of t as $t \to 0$. A general introduction to this algorithm with several applications can be found in [8, 9], and a brief presentation is given in section 2 below. We just note here that non-singular solutions can also be constructed by the Fuchsian algorithm. In fact, the Cauchy problem itself reduces to a very special case of the method.

We prove in this paper that the Fuchsian algorithm applies to Einstein's vacuum equations for Gowdy spacetimes, and establishes the existence of a family of solutions

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depending on the maximal number of arbitrary functions, namely four, in the 'low-velocity' case, whose definition is recalled below. When one of these functions is constant, the solution actually extends to the 'high-velocity' case as well. We will refer to the former solutions as 'generic' and to the latter as 'non-generic'. Earlier exact solutions are obtained by specializing the arbitrary functions in the solutions of this paper.

In both cases, the solutions are 'asymptotically velocity-dominated' (AVD) in the sense of Eardley *et al* [4], and precise asymptotics at the singularity are given. The reduction to Fuchsian form actually provides a mechanism whereby inhomogeneous solutions can become AVD in the neighbourhood of such a singularity. The results explain the paradoxical features of numerical computations described next.

1.1. Earlier results

$\mathbb{T}^3 \times \mathbb{R}$ Gowdy spacetimes [5] have spacelike slices, homeomorphic to the 3-torus, on which a $U(1) \times U(1)$ isometry group acts. It is convenient to take as time coordinate the area t of the orbits of this two-dimensional group; the spacetime corresponds to the region $t > 0$. The metric then takes the form

$$\mathrm{d}s^2 = \mathrm{e}^{\lambda/2} t^{-1/2}(-\mathrm{d}t^2 + \mathrm{d}x^2) + t\big[\mathrm{e}^{-Z}(\mathrm{d}y + X\,\mathrm{d}z)^2 + \mathrm{e}^{Z}\,\mathrm{d}z^2\big],$$

where λ, X and Z are functions of t and x only, and are periodic of period 2π with respect to x. We also let

$$D = t\partial_t.$$

1.1.1. Form of the equations. With the above conventions, the equations take the form:

$$D^2X - t^2X_{xx} = 2(DX\,DZ - t^2X_xZ_x) \tag{1}$$

$$D^2Z - t^2Z_{xx} = -\mathrm{e}^{-2Z}((DX)^2 - t^2X_x^2) \tag{2}$$

$$\lambda_x = 2(Z_xDZ + \mathrm{e}^{-2Z}X_xDX)$$

$$D\lambda = (DZ)^2 + t^2Z_x^2 + \mathrm{e}^{-2Z}\big((DX)^2 + t^2X_x^2\big),$$

where subscripts denote derivatives.

The last two equations arise, respectively, from the momentum and Hamiltonian constraints. It suffices to solve the first two equations, and we therefore focus on them from now on. Of course, one should also ensure that the integral of λ_x from 0 to 2π vanishes.

If $X = Z = 0$, we recover a metric equivalent to the $(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ Kasner solution. Other Kasner solutions are recovered for $X = 0$ and $Z = k\ln t$; the corresponding Kasner exponents are $(k^2-1)/(k^2+3)$, $2(1-k)/(k^2+3)$ and $2(1+k)/(k^2+3)$. The equations for X and Z are often interpreted as expressing that (X, Z) generates a 'harmonic-like' map from $1+1$ Minkowski space with values in hyperbolic space with the metric

$$\mathrm{d}Z^2 + \mathrm{e}^{-2Z}\,\mathrm{d}X^2.$$

The usual Cartesian coordinates on the Poincaré model of hyperbolic space are X and $Y = \mathrm{e}^Z$, so that the metric coincides with the familiar expression $(\mathrm{d}X^2 + \mathrm{d}Y^2)/Y^2$.

It is occasionally useful to use polar coordinates (w, ϕ) on the hyperbolic space, so that the metric on the target space is

$$\mathrm{d}w^2 + \sinh^2 w\,\mathrm{d}\phi^2.$$

The equations for w and ϕ then take the form

$$D^2 w - t^2 \partial_{xx} w = \tfrac{1}{2} \sinh 2w[(Dw)^2 - t^2 w_x^2] \tag{3}$$

$$D^2 \phi - t^2 \partial_{xx} \phi = -2 \coth w[Dw D\phi - t^2 w_x \phi_x]. \tag{4}$$

For fixed t, the solution represents a *loop* in hyperbolic space.

For extensive references on Gowdy spacetimes, see [1–3, 6].

1.1.2. Exact solutions. Both sets of equations can be solved exactly if we seek solutions independent of x. In terms of the (X, Z) variables, for example, these solutions have leading behaviour of the form

$$Z \sim k \ln t + \mathrm{O}(1), \qquad X = \mathrm{O}(1),$$

where k is a positive constant, and represent solutions in which the loop degenerates to a point which follows a geodesic and tends to a point at infinity in hyperbolic space.

Motivated by this, it was suggested that a general solution corresponds to a loop, each point of which asymptotically follows a geodesic and tends to some point at infinity in hyperbolic space. This regime is called the 'geodesic loop approximation'.

This is borne out in the case of the 'circular loop' which corresponds to $\phi(x) = nx$, $n = 1, 2, \ldots$, and has w independent of x.

However, the only case in which this behaviour could be established for solutions containing an arbitrary function of x was the 'polarized' case, defined by the condition $X \equiv 0$ (see [3] and references therein). The equation for Z is then a linear Euler–Poisson–Darboux equation, the general solution of which can be represented explicitly in terms of Bessel functions; this fact does not necessarily make the investigation of singularities straightforward, see [3]. This provides a family of solutions involving two arbitrary functions. These solutions have

$$Z \sim k \ln t + \mathrm{O}(1),$$

where k now depends on x and can be arbitrary.

Numerical computations suggest more complicated behaviour in the full nonlinear system for X and Z [1]. Indeed, if one monitors the 'velocity' $v(x, t) = [(DZ)^2 + \exp(-2Z)(DX)^2]^{1/2}$, which should tend to $|k(x)|$, one finds that it is not possible to find solutions which satisfy $v > 1$ on any interval as $t \to 0$. Even if one starts out with $v > 1$ and solves towards $t = 0$, the parameter v dwindles to values less than 1, except for some sharp spikes located near places where $X_x = 0$, and which eventually disappear at any fixed resolution. They may persist longer at higher resolutions. Solutions such that $v < 1$ are called 'low velocity', and others are called 'high velocity'. A formal asymptotic computation, proposed in [6], also suggests that the low-velocity case allows asymptotics that would not be available in the high-velocity case. This expansion is obtained by introducing a parameter η in front of the spatial derivative terms in the equations, and expanding the solution in powers of η.

Note that solutions with k positive and negative are qualitatively quite different, even though they will have the same value for v.

Since the numerical computations we wish to account for were performed in the (X, Z) variables, we will focus on them. However, we will briefly mention what happens in the (w, ϕ) variables, since the circular loop is then more simply described.

The problem can be summarized as follows: if the geodesic loop approximation is valid, v approaches $|k|$. We therefore need a mechanism which forces $|k| < 1$; but if v must be smaller than 1, how do we account for the polarized solutions? Also, should we restrict ourselves to $k > 0$, given that the numerics do not give information on the sign of k?

1.1.3. Results. Our results account for the various types of behaviour observed in numerical and special solutions by exhibiting a solution with the maximum number of 'degrees of freedom', and which, under specialization, reproduces the main features listed above. We describe these results first for $k > 0$.

When k is positive, we first define new unknowns $u(x, t)$ and $v(x, t)$ by the relations

$$Z(x, t) = k(x) \ln t + \varphi(x) + t^{\varepsilon} u(x, t), \tag{5}$$

$$X(x, t) = X_0(x) + t^{2k(x)}(\psi(x) + v(x, t)), \tag{6}$$

where ε is a small positive constant to be chosen later. The objective is to construct solutions of the form (5), (6), where u and v tend to zero as t tends to zero. If $0 < k < 1$, the periodicity condition $\oint \lambda_x \, dx = 0$ is equivalent to

$$\int_0^{2\pi} k(\varphi_x + 2X_{0x}\psi e^{-2\varphi}) \, dx = 0, \tag{7}$$

which we assume from now on. If $k > 1$, we will require in addition that $X_{0x} \equiv 0$, for reasons described later. In both cases, we find that $\lambda = k^2 \ln t + O(1)$ as $t \to 0$.

We then prove that, upon substitution of (5), (6) into (1), (2), we obtain a Fuchsian equation for (u, v), in which the right-hand side may contain positive and negative powers of t, as well as logarithmic terms. If there are only positive powers of t, possibly multiplied by powers of $\ln t$, we prove an existence theorem which ensures that there are actual solutions of this form in which u and v tend to zero. In fact, one can derive iteratively a full expansion of the solution near the singularity at $t = 0$. We prove that there are only positive powers of t in two cases:

(i) if k lies strictly between 0 and 1; this provides a 'generic' solution involving four arbitrary functions of x, namely k, X_0, φ and ψ;
(ii) if $k > 0$ and X_0 is independent of x; this provides a solution involving only three functions of x and one constant. This case includes both the x-independent solutions and the polarized solutions, and explains why these cases do not lead to a restriction on k.

The fact that high velocity is allowed when X_0 is constant is to be compared with the numerical results which show spikes when $X_x = 0$.

If k is negative, one can proceed in a similar manner, except that one should start with

$$Z = k(x) \ln t + \varphi(x) + t^{\varepsilon} u(x, t) \tag{8}$$

$$X = X_0(x) + t^{\varepsilon} v(x, t), \tag{9}$$

where k, φ and X_0 are arbitrary functions. In fact, one can generate solutions with negative k from solutions with positive k. Indeed, if (X, Z) is any solution of the Gowdy equations, so is $(\tilde{X}, \tilde{Z})$, where

$$\tilde{X} = \frac{X}{X^2 + Y^2}, \qquad \tilde{Z} = \ln \frac{Y}{X^2 + Y^2},$$

with $Y = e^Z$ as before. This corresponds to an inversion in the Poincaré half-plane.

Our existence results can actually be applied in two different ways to the problem. One is to assume the arbitrary functions to be analytic and 2π-periodic, and to produce solutions which are periodic in x. One can also use the results to produce solutions which are only defined near some value of x. This is useful for cases when the solution is not conveniently represented in the (X, Z) coordinates, in which one of the points at infinity in hyperbolic space plays a distinguished role. In such cases, one can patch local solutions obtained from several local charts in hyperbolic space.

1.2. Organization of the paper

Section 2 presents a brief introduction to Fuchsian techniques. Section 3 is devoted to the reduction of the basic equations to Fuchsian form, and shows how the distinction between low and high velocity arises naturally from the Fuchsian algorithm (theorems 1 and 2). Section 4 proves the existence result (theorem 3) which produces the above solutions. It also shows the impact of the rigorous results on formal asymptotics.

2. Introduction to Fuchsian techniques

We briefly review the main features of Fuchsian methods that are relevant to our results. The main advantages of these techniques are:

(i) Fuchsian reduction provides an asymptotic representation of singular solutions of fairly general partial differential equations.
(ii) The arbitrary functions in this representation generalize the Cauchy data, in the sense that knowledge of them is equivalent to knowledge of the full solution. The Cauchy problem is itself a special case of the Fuchsian algorithm.
(iii) The reduction of a PDE to Fuchsian form explains why solutions should become AVD, i.e. how the spatial derivative terms can become less important than the temporal derivatives near singularities, even though the solution is genuinely inhomogeneous.

The starting point is a reinterpretation of the solution of the Cauchy problem for, say, a second-order equation

$$F[u] = 0.$$

The geometric nature of the unknown is not important for the following discussion. Solving the Cauchy problem amounts to showing that the solution is determined by the first two terms of its Taylor expansion:

$$u = u^{(0)} + tu^{(1)} + \cdots.$$

One can think of $u^{(0)}$ and $u^{(1)}$ as prescribed on the initial surface $\{t = 0\}$. This statement does not require any information about the geometric meaning of the unknown u, which may be a scalar or a tensor, for instance.

However, this representation may fail if the solution presents singularities. The Fuchsian approach seeks an alternative representation near singularities, in a form such as

$$u = t^{\nu}(u^{(0)} + tu^{(1)} + \cdots).$$

There are several issues that need to be dealt with if one seeks such a solution:

(i) How do we construct such a series formally to all orders? This question is far from trivial because any amount of inhomogeneity, for example, can force the appearance of logarithmic terms at arbitrarily high orders. Furthermore, the arbitrary terms in the series can occur at very high orders even if the equation is only of second order.

(ii) How do we know there is one solution corresponding to this expansion, rather than infinitely many solutions differing by exponentially small corrections?
(iii) How restrictive is it to start with power behaviour: in particular, is logarithmic behaviour allowed?

Once these issues have been addressed, the formal series can be used much in the same way as an exact solution would.

It turns out that all of these issues can be addressed simultaneously by reducing the given equation to a Fuchsian PDE in the following way.

First, identify the leading terms. This requires being able to find an expression $a(x^q)$ in the coordinates x^q such that, upon substitution of a into the equation, the most singular terms cancel each other.

Second, define a renormalized regular part v by setting, typically,

$$u = a + t^m v.$$

If a is a formal solution up to order k, it is reasonable to set $m = k + \varepsilon$. If the structure of logarithmic terms is made explicit, one can also specify the dependence of v on logarithmic variables, as in [8, 12]. Examples display considerable flexibility in the form of the renormalized part v, and the list of possible cases where these ideas apply seems to be growing.

Third, obtain the equation for v. It is important to ensure, by introducing derivatives of v as additional variables if necessary, that one is left with a *Fuchsian system*, that is, one of the form

$$t\frac{\partial v}{\partial t} + Av = t^{\varepsilon} f(t, x, v, \partial_\alpha v),$$

where A is a matrix, which could depend on spatial variables, but should be independent of t (otherwise we could incorporate the time dependence into f). $\partial_\alpha v$ stands for first-order spatial derivatives; a second-order equation is converted to such a form by adding derivatives of the unknowns as additional unknowns. In general, f can be assumed to be analytic in all of its arguments *except* t, because a may contain logarithms or other more complicated expressions.

Fuchsian PDEs are a generalization of linear ordinary differential equations with a regular or Fuchsian singularity at $t = 0$, such as the Bessel or hypergeometric equations.

Once this reduction has been accomplished, general results on Fuchsian equations give us the desired results, intuitively because the equation can be thought of as a perturbation of the case when $f = 0$. The initial-value problem for such equations can be solved in the non-analytic as well as the analytic case [10].

The Fuchsian form has several advantages, in addition to being the one which allows one to construct and validate the expansions in the first place:

(i) It makes AVD behaviour natural, because the spatial derivative terms appear only in f, which is preceded by a positive power of t. We therefore expect spatial derivative terms to be switched off at leading order, but to contribute at higher order. By contrast, the term $t\partial_t v$ behaves like a term of order zero, because it transforms any power t^j into a multiple of itself (namely jt^j).
(ii) It is invariant under restricted changes of coordinates which preserve the set $t = 0$: if we change (t, x^α) into (t', x'^α), it suffices to require that t'/t be bounded away from zero and independent of x^α near $t = 0$. One can even allow non-smooth changes of coordinates such as $t' = t^r$. Further generalizations are possible.

(iii) It is invariant under 'peel-off': for instance, if we write $v = v^{(0)} + tw$, and assume for simplicity that $\varepsilon = 1$, we find that w solves a Fuchsian system with A replaced by $A + 1$. A more general property of this kind can be found in [12]. This explains why the Fuchsian form is adapted to the construction of formal solutions as well as to their justification.

(iv) It can be used to generate the formal expansion systematically: assume the solution is known to some order k. Substitute into f, and call g the result; now solve the resulting equation $t\partial_t v + Av = t^{\varepsilon} g$ for v. It is easy to see that the result will contain corrections of order higher than k. This method is useful if the exact form of the solution is unknown, or if it is very complicated.

Let us now turn to examples.

2.1. *The Cauchy problem*

The Cauchy problem can always be thought of as an initial-value problem for a first-order system

$$\frac{\partial u}{\partial t} = f(t, x, u, \partial_x u),$$

where $x = (x^{\alpha})$ stands for several space variables, and, to be definite, f is analytic in all its arguments. For instance, in the case of Einstein's equations in harmonic coordinates, u represents the list of all the components of the metric as well as their first time derivatives.

Let us now take as the principal part a the initial condition $u^{(0)}$, and write

$$u = u^{(0)} + tv.$$

If we insert this into f, we find that all of the v-dependent terms must contain a positive power of t. In other words,

$$f = f^{(0)} + tg(t, x, v, \partial_x v),$$

where $f^{(0)} = f(0, x, u^{(0)}, \partial_x u^{(0)})$. The equation for v is therefore

$$t\frac{\partial v}{\partial t} + v = f^{(0)} + tg,$$

which is a Fuchsian equation for $(v - f^{(0)})$, with $A = 1$. The existence of solutions of Fuchsian systems ensures in this case that one can solve the initial-value problem. To recover a solution of Einstein's equations, one needs to handle the propagation of the constraints separately, as usual.

2.2. *A nonlinear ODE*

Consider the equation

$$u_{tt} = u^2,$$

where subscripts denote derivatives, and $u = u(t)$ is a scalar.

Let us try to find a leading part of the form $u \sim at^s$ with $a \neq 0$. The left-hand side is then $\sim as(s-1)t^{s-2}$ and the right-hand side is $\sim a^2 t^{2s}$. If $s(s-1) = 0$, it means that we are dealing with a Cauchy problem: $u \approx a + u_1 t + \cdots$ if $s = 0$, and $u \approx at + u_2 t^2 + \cdots$ if $s = 1$. We therefore assume $s(s-1) \neq 0$. It is then necessary for the two sides to balance each other as $t \to 0$, which means that we need

$$s - 2 = 2s \qquad \text{and} \qquad s(s-1) = a.$$

This means that $s = -2$ and $a = 6$. The principal part is $6/t^2$, and the first step is complete.

For the second step, let us define the renormalized unknown v by

$$u = t^{-2}(6 + vt).$$

Finally, let us write the equation for v. We find

$$(D - 5)(D + 2)v = tv^2,$$

where $D = t\,\mathrm{d}/\mathrm{d}t$. This is a Fuchsian equation of second order, which can be converted into a first-order Fuchsian system by introducing (v, Dv) as a two-component unknown. This would lead to an equation where the eigenvalues of A are 2 and -5.

The knowledge of the eigenvalue -5 combined with general properties of Fuchsian systems ensures that there is a complete formal solution for v where the coefficient of t^5 in the expansion of v is arbitrary. One can convince oneself of this fact by direct substitution, but this is often cumbersome, because of the need to compute a formal solution to sixth order in this case. In general, the expansion of v also contains powers of $t \ln t$, but they are not necessary for this simple example.

The same method applies to any equation of the form

$$u_{tt} = u^2 + c_1(t)u + c_0(t) + c_{-1}(t)u^{-1} + \cdots$$

and yields a convergent series solution

$$u(t) = t^{-2}\sum_{j,k} u_{j,k}t^j(t\ln t)^k,$$

which is entirely determined once the coefficient $u_{6,0}$ is prescribed. The translates $u(t - t_0)$ of this solution form a two-parameter family of solutions, parametrized by $(u_{6,0}, t_0)$, which is stable under perturbations (i.e. 'generic'). It is possible to show that the other eigenvalue of A, namely 2, is related to the variation of the parameter t_0, although we do not dwell on this point.

Logarithmic terms are *not* due to logarithms in the equation itself. For instance, the equation $u_{tt} = u^2 + t^2$ has no solution which is free of logarithms.

2.3. *The Euler–Poisson–Darboux equation*

As an example of a linear Fuchsian PDE, let us consider the Euler–Poisson–Darboux (EPD) equation

$$u_{tt} - \frac{\lambda - 1}{t}u_t = u_{xx} + u_{yy},$$

in two space variables. This equation has a variety of uses, from the solution of the wave equation in Minkowski space to computer vision. In particular, the Einstein equations in the 'polarized' Gowdy spacetime (i.e. when $X = 0$) reduce to the above equation with only one space variable, and with $\lambda = 0$.

To reduce it to Fuchsian form, one may introduce new unknowns: $v = u$, $v_0 = tu_t$, $v_1 = tu_x$ and $v_2 = tu_y$ (numerical subscripts do not denote derivatives). One then finds the system

$$t\partial_t v - v_0 = 0$$

$$t\partial_t v_0 - \lambda v_0 = t\partial_x v_1 + t\partial_y v_2$$

$$t\partial_t v_1 = t\partial_x(v + v_0)$$

$$t\partial_t v_2 = t\partial_y(v + v_0).$$

The general solution can in this case be computed explicitly using the Fourier transform (or Fourier series in a finite domain) in terms of Bessel functions. The solution has the form $U + V \ln t$, where U and V are series in t and do not involve logarithms.

Fuchsian reduction applies directly to nonlinear perturbations of the EPD equation. However, the nonlinearity causes the appearance of products of logarithms. The Fuchsian algorithm, by ensuring that the solutions are actually functions of t and $t \ln t$, guarantees that the singularity of the logarithm is always compensated by powers of t.

Remark. There are cases when it is useful to make a change of time variable. Consider an example such as

$$(t\partial_t)^2 u - t u_{xx} = 0.$$

If we let $(v, v_0, v_1) = (u, tu_t, tu_x)$, we obtain the system

$$t\partial_t v = v_0$$

$$t\partial_t v_0 = \partial_x v_1$$

$$t\partial_t v_1 = v_1 + t\partial_x v_0,$$

in which the term $\partial_x v_1$ does not have a factor of t. We can nevertheless obviate this problem by letting $t = s^2$. The original equation then becomes

$$(s\partial_s)^2 u - 4s^2 u_{xx} = 0,$$

expanding and dividing through by s^2, we recover the Euler–Poisson–Darboux equation, up to the harmless factor of 4.

2.4. Leading logarithms

The first case to be treated by Fuchsian PDE methods actually required a logarithmic leading term. We merely state the result, as it is developed extensively elsewhere [10, 11]. Consider the equation

$$\eta^{ab}\partial_{ab} u = \mathrm{e}^u,$$

in Minkowski space. This equation admits a Fuchsian reduction with a singularity on any spacelike hypersurface $t = \psi(x)$, which is obtained by applying the above ideas to the equation satisfied by e^u. This generates a family of stable singularities which do not propagate on characteristic surfaces, since the singularity locus is spacelike. There is a complete expansion of the solution at the singularity, and it is free of logarithms if and only if the singularity surface has vanishing scalar curvature (i.e. ${}^{(3)}R = 0$).

To summarize, the Fuchsian approach to singularity formation consists of three steps: (i) identification of the leading part; (ii) identification of a convenient renormalized unknown; and (iii) solution of the Fuchsian system for the new unknown. This technique is now applied to the Gowdy problem.

3. Reduction to Fuchsian form

3.1. General results

In this section we first reduce the Gowdy equations to a second-order system for u and v, which is then converted to a first-order Fuchsian system. The subscripts 0, 1 and 2 in this section do *not* denote derivatives. The equations now become:

$$
\begin{aligned}
(D+\varepsilon)^2 u = {}& t^{2-\varepsilon}[k_{xx}\ln t+\varphi_{xx}+t^{\varepsilon}u_{xx}]-\exp(-2\varphi-2t^{\varepsilon}u)\big\{t^{2k-\varepsilon}((D+2k)(v+\psi))^2\\
&-t^{2-2k-\varepsilon}[X_{0x}+t^{2k}(v_x+\psi_x+k_x(v+\psi)\ln t)]^2\big\}
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
D(D+2k)v = {}& t^{2-2k}X_{0xx}+2t^{\varepsilon}(D+\varepsilon)u(D+2k)(v+\psi)\\
&+t^2[(v+\psi)_{xx}+4k_x(v_x+\psi_x)\ln t+(2k_{xx}\ln t+4k_x^2(\ln t)^2)(v+\psi)]\\
&-2t^{2-2k}[X_{0x}+t^{2k}(v_x+\psi_x+k_x(v+\psi)\ln t)][k_x\ln t+\varphi_x+t^{\varepsilon}u_x].
\end{aligned}
\tag{11}
$$

This second-order system will now be reduced to a first-order system. To this end, let us introduce the new variables

$$\boldsymbol{u}=(u_0,u_1,u_2,v_0,v_1,v_2)=(u,Du,tu_x,v,Dv,tv_x).$$

We then find

$$Du_0=u_1$$

$$
\begin{aligned}
Du_1 = {}& -2\varepsilon u_1-\varepsilon^2u_0+t^{2-\varepsilon}(k_{xx}\ln t+\varphi_{xx})+t\partial_x u_2\\
&-\exp(-2\varphi-2t^{\varepsilon}u_0)\big\{t^{2k-\varepsilon}(v_1+2kv_0+2k\psi)^2-t^{2-2k-\varepsilon}X_{0x}^2\\
&-2t^{1-\varepsilon}X_{0x}(v_2+t\psi_x+k_x(v_0+\psi)t\ln t)-t^{2k-\varepsilon}(v_2+t\psi_x+k_xv_0t\ln t)^2\big\}
\end{aligned}
$$

$$Du_2=t\partial_x(u_0+u_1)$$

$$Dv_0=v_1$$

$$
\begin{aligned}
Dv_1 = {}& -2kv_1+t^{2-2k}X_{0xx}+t\partial_x(v_2+t\psi_x)+4k_x(v_2+t\psi_x)t\ln t\\
&+(v+\psi)[2k_{xx}t^2\ln t+4(k_xt\ln t)^2]+2t^{\varepsilon}(v_1+2kv_0+2k\psi)(u_1+\varepsilon u_0)\\
&-2X_{0x}t^{2-2k}(k_x\ln t+\varphi_x+t^{\varepsilon}\partial_x u_0)\\
&-2t(\partial_x(v_0+\psi)+k_x(v_0+\psi)\ln t)(k_xt\ln t+t\varphi_x+t^{\varepsilon}u_2)
\end{aligned}
$$

$$Dv_2=t\partial_x(v_0+v_1).$$

This system therefore has the form

$$(D+A)\boldsymbol{u}=g(t,x,\boldsymbol{u},\boldsymbol{u}_x),\tag{12}$$

where the right-hand side g involves various powers of t, possibly multiplied by logarithms. We will choose ε so that all of these terms nevertheless tend to zero as t goes to zero. The low-velocity case is precisely the one in which it is possible to achieve this without making any assumptions about the arbitrary functions in the system, namely k, X_0, φ and ψ.

In fact, the high- and low-velocity cases are now distinguished by the absence or presence of the terms involving t^{2-2k} (and $t^{2-2k-\varepsilon}$). As is clear from the above equations, these terms disappear precisely if X_0 is a constant (i.e. $X_{0x}=0$).

For any positive number σ, we define the matrix

$$\sigma^A=\exp(A\ln\sigma):=\sum_{r=0}^{\infty}\frac{(A\ln\sigma)^r}{r!}.$$

One checks by inspection that the matrix A has eigenvalues ε, 0, and $2k$, and that there is a constant C such that $|\sigma^A|\leqslant C$ for any $\sigma\in(0,1)$ if $\varepsilon>0$. This can be seen, for instance, by reducing A and explicitly computing the matrix exponential.

Note that this system is of Cauchy–Kowalewska type for $t > 0$, and that the solutions will in fact be analytic in all variables for $t > 0$. The issue is to construct solutions with controlled behaviour as $t \to 0$.

We are interested in solutions of (12) which satisfy $\boldsymbol{u} = 0$ for $t = 0$. Let us check that these solutions have the property that u_0 and v_0 solve the original Gowdy system. Since the second and fifth equations of the system satisfied by $\boldsymbol{u}$ are obtained directly from the second-order system, it suffices to check that $u_1 = Du_0$, $v_1 = Dv_0$, $u_2 = tu_{0x}$ and $v_2 = tv_{0x}$. The first two statements are identical with the first and fourth equations, respectively. As for the last two, we note that the first and third equations imply

$$D(u_2 - t\partial_x u_0) = t\partial_x(u_0 + u_1 - Du_0 - u_0) = 0.$$

Since $u_2 - t\partial_x u_0$ tends to zero as $t \to 0$, it must be identically zero for all time, as desired. The same argument applies to v.

The computations for the case $k < 0$ are entirely analogous, and are therefore omitted.

We now study the low- and high-velocity cases separately.

3.2. Low-velocity case

The following theorem gives the existence of a solution depending on four arbitrary functions in the case when k lies between zero and one:

Theorem 1. Let $k(x)$, $X_0(x)$, $\phi(x)$ and $\psi(x)$ be real analytic, and assume $0 < k(x) < 1$ for $0 \leqslant x \leqslant 2\pi$. Then there exists a unique solution of the form (5), (6), where u and v tend to zero as $t \to 0$.

Proof. By inspection, the vector $\boldsymbol{u}$ satisfies a system of the form (12), where g can be written as $t^\alpha f$, provided that we take α and ε to be small enough. Letting $t = s^m$, we obtain a new system of the same form, but with α replaced by $m\alpha$. By taking α large enough, we may therefore assume that we have a system to which theorem 3 below applies. The result follows.

3.3. High-velocity case

The following theorem gives the existence of a solution depending on three arbitrary functions in the case when k is only assumed to be positive, and may take values greater than one. If k is less than one, we recover the solutions obtained above, but with $X_{0x} = 0$:

Theorem 2. Let $k(x)$, $\phi(x)$ and $\psi(x)$ be real analytic, and assume $X_{0x} = 0$ and $k(x) > 0$ for $0 \leqslant x \leqslant 2\pi$. Then there exists a unique solution of the form (5), (6), where u and v tend to zero as $t \to 0$.

Proof. Since X_{0x} is now zero, we find that $\boldsymbol{u}$ satisfies, if $\varepsilon > 0$, a Fuchsian system of the form (12), where g can be written as $t^\alpha f$, provided that we take α and ε to be small enough. Letting as before $t = s^m$, we obtain a new system of the same form, but with α replaced by $m\alpha$. By taking m large enough, we may therefore assume that we have a system to which theorem 3 applies. The result follows.

4. Existence of solutions of Fuchsian systems

Consider, quite generally, a Fuchsian system, for a 'vector' unknown $u(x, t)$, of the form

$$(D + A)u = F[u] := tf(t, x, u, u_x). \tag{13}$$

In this equation, A is an analytic matrix near $x = 0$, such that $\|\sigma^A\| \leqslant C$ for $0 < \sigma < 1$. The number of space dimensions is n ($n = 1$ for the application to the Gowdy problem). It suffices that the nonlinearity f preserve analyticity in space and continuity in time, and depend in a locally Lipschitz manner on u and u_x, i.e. that its partial derivatives with respect to these arguments be bounded when these arguments are. To be definite, one may assume that f is a sum of products of analytic functions of x, u and u_x by powers of t, $t^{k(x)}$ and $\ln t$. In fact, all one needs is to ensure the estimate in step 2 below. In this section, the number of space variables is arbitrary. We are only interested in positive values of t.

Theorem 3. The system (13) has exactly one solution which is defined near $x = 0$ and $t = 0$, and which is analytic in x, continuous in t, and tends to zero as $t \downarrow 0$.

Remark 1. The solutions are constructed as the uniform limit of a sequence of continuous functions which are analytic in x. They are classical solutions as well, by construction. However, by the Cauchy–Kowalewska theorem, they are also analytic in t away from $t = 0$.

Remark 2. The solution provided by the theorem will be defined for x in a complex neighbourhood of a subset Ω of $\mathbb{R}^n$. This can be applied to the Gowdy problem in two different ways: one can take Ω to be an interval of length greater than 2π, and note that the solution will be 2π-periodic if the right-hand side is, since it is given as a limit of a sequence all of whose terms are periodic. It is this solution which shows that the 'geodesic loop approximation' corresponds to a generic family of exact solutions in the low-velocity case, and a non-generic family otherwise. However, one could also take Ω to be a small interval of length less than 2π, and generate solutions which are defined only locally. This second application can itself be useful in two contexts: (i) for generalizations of Gowdy spacetimes where the space variable is unbounded, or compactified in a different fashion; (ii) for the description of 'circular loop' type solutions, which correspond to a solution which depends linearly on the angular coordinate in terms of polar coordinates on the Poincaré half-plane.

Proof. Let us begin by defining an operator H which corresponds to the inversion of $(D + A)$. The proof will consist in showing that the operator $v \mapsto G[v] := F[H[v]]$ is a contraction for a suitable norm. Its fixed point generates a solution $u = H[v]$ to our problem.

Before we jump into the details, let us first motivate the strategy by examining some of the possible difficulties. For more details on the history of existence theorems in the complex domain, see ch 2 in [8].

The basic difficulty in achieving a successful iteration is that it is not clear at all how to build a measure of the size of u (that is, a function space norm) which remains finite after even one step of the iteration. The problem is that in order to control $G[v]$, we need to estimate the spatial derivative of v in terms of a norm which only involves v. This cannot be remedied by adding information on the derivatives of v in the definition of the norm: we would then need to estimate both $H[v]$ and its derivative, in order to have a well defined iteration. In fact, this is an essential problem because the result would be false if the right-hand side involved second as well as first derivatives of u. Majorant methods are not appropriate because the nonlinearity f does not have an expansion in powers of t—only in powers of x for fixed t. It is not possible to estimate the derivative of an analytic function by its values on the same domain: think of the function $\sqrt{1 - z}$ on the unit disk, which is bounded on $(-1, 1)$ even though its derivative is not. However, by going into the complex domain, it is possible to estimate the derivatives of an analytic function in terms of its values on the boundary of a *slightly larger domain*†. This is given by Cauchy's theorem, which expresses the value of an analytic function ϕ at any point as a weighted average of its values on any curve γ circling that point once:

$$\phi(z) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\phi(\zeta)\,\mathrm{d}\zeta}{\zeta - z}.$$

Differentiating with respect to z and taking absolute values, we see that we have a means of estimating the derivatives of ϕ from its values on a larger domain. However, we must move into the complex domain to achieve this. The transition to several variables offers no difficulty, because an analytic function of several variables is separately analytic in each of its arguments, and it therefore suffices to apply the above to each variable separately to obtain some estimate of derivatives—which is all we need. For instance, the relevant Cauchy integral formula in two variables is simply

$$\phi(z_1, z_2) = \frac{1}{(2\pi \mathrm{i})^2} \oint\oint \frac{\phi(\zeta_1, \zeta_2)\,\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)},$$

where the integration extends over a product of circles:

$$|\zeta_1 - z_1| = r_1 \qquad \text{and} \qquad |\zeta_2 - z_2| = r_2.$$

The proof below differs from the existence result used in [8, 12] by the fact that the nonlinearity is no longer analytic with respect to time. It is therefore necessary to check carefully that the estimates on f can still be carried out.

We now present the proof of the result.

Throughout the proof, the meaning of the letter C will change from line to line: it denotes various constants, the specific values of which are not needed.

We let

$$H[v] = \int_0^1 \sigma^{A(x)-1} v(\sigma t)\,\mathrm{d}\sigma.$$

It is easily checked that this provides the solution of

$$(D + A)u = v,$$

with $u(0) = 0$, provided that $v = \mathrm{O}(t)$ near $t = 0$.

We are ultimately interested in real values of x in some open set Ω, so that we work in a small complex neighbourhood of the real set Ω. The proof in fact does not depend on the nature or size of this set. We also define two norms which will be useful. The s-norm of a function of x is

$$\|u\|_s = \sup\{|u(x)| : x \in \mathbb{C}^n \text{ and } \mathrm{d}(x, \Omega) < s\}.$$

The a-norm of a function of x and t is defined by

$$|u|_a = \sup\left\{\frac{s_0 - s}{t}\|u(t)\|_s\sqrt{1 - \frac{t}{a(s_0 - s)}} : t < a(s_0 - s)\right\}.$$

Note that this norm allows functions to become unbounded when $t = a(s_0 - s)$. This can be thought of basically as the boundary of the domain of dependence of the solution. For the reasons indicated earlier, the iteration would not be well defined if we had worked simply with the supremum of the s-norm over some time interval.

† In the non-analytic case, this problem is avoided thanks to the additional assumption of hyperbolicity, by showing that there are expressions which can be estimated *as though the right-hand side did not involve derivatives of* v *at all*, see [10], or ch 2 of [8] for a broader introduction.

The objective is to prove that the iteration $u_0 = 0$, $u_{n+1} = G[u_n]$ is well defined and converges to a fixed point of G, which gives us the desired solution. This will be achieved by exhibiting a set of functions which contains zero and on which G is a contraction in the a-norm. Since a contraction has a unique fixed point, we also obtain uniqueness.

We choose $R > 0$ and s_0 such that $\|F[0](t)\|_{s_0} \leqslant Rt$. This can always be achieved since we are allowed to take R very large.

Step 1. Estimating H. Using the definition of $|u|_a$, we find, with the notation $\rho = \sigma t/a(s_0 - s)$,

$$\begin{aligned}\|H[u](t)\|_s &\leqslant \frac{|u|_a}{s_0 - s}\int_0^1 |\sigma^A|\frac{\sigma t}{\sigma}\left(1 - \frac{\sigma t}{a(s_0 - s)}\right)^{-1/2} \mathrm{d}\sigma \\ &= \frac{C|u|_a}{s_0 - s}\int_0^{t/a(s_0-s)} \frac{a(s_0 - s)\,\mathrm{d}\rho}{\sqrt{1-\rho}} \\ &\leqslant C_0 a|u|_a.\end{aligned}$$

Step 2. Estimating F. Using Cauchy's integral representation, and the fact that f contains a factor of t, we claim that there is a constant C_1 such that

$$\|F[p] - F[q]\|_s(t) \leqslant \frac{C_1 t}{s' - s}\|p - q\|_{s'}$$

if $s' > s$ and $\|p\|_s$ and $\|q\|_s$ are both less than R; this constraint will be ensured in step 3 thanks to the argument of the previous step.

Indeed, $F[p]$ is the product of t by a linear expression in the gradient of p, with coefficients which are Lipschitz functions of p; it is in fact, in the Gowdy case, an analytic function of these variables, x, and positive powers of t multiplied by logarithms. If the dependence of F on u_x had been nonlinear, one would have considered the Fuchsian system satisfied by (u, u_x). The bound on the s-norm ensures that all the partial derivatives of F with respect to p and $\nabla_x p$ are bounded by some constant C. Therefore, we have

$$|F[p] - F[q]| \leqslant Ct(|p - q| + |\nabla_x p - \nabla_x q|).$$

We want to estimate the supremum of this expression as x varies so as to satisfy $\mathrm{dist}(x, \Omega) < s$. The first term is clearly less than or equal to $\|p - q\|_s$, and *a fortiori* no bigger than $\|p-q\|_{s'}$. The second is estimated by Cauchy's inequality on each component. Thus, for the first component, we write

$$p(x,t) - q(x,t) = \frac{1}{2\pi \mathrm{i}}\int_{|z-x_1|=s'-s} \frac{(p(z, x_2, \ldots, t) - q(z, x_2, \ldots, t))\,\mathrm{d}z}{z - x_1}.$$

Differentiating with respect to x_1, we find

$$\begin{aligned}|\partial_1(p - q)| &= \left|\frac{1}{2\pi \mathrm{i}}\int_{|z-x_1|=s'-s} \frac{(p(z, x_2, \ldots, t) - q(z, x_2, \ldots, t))\,\mathrm{d}z}{(z - x_1)^2}\right| \\ &\leqslant \frac{1}{2\pi}\int_{|z-x_1|=s'-s} \frac{|p(z, x_2, \ldots, t) - q(z, x_2, \ldots, t)|\,|\mathrm{d}z|}{(s' - s)^2} \\ &\leqslant \frac{1}{2\pi}\|p - q\|_{s'}\frac{2\pi(s' - s)}{(s' - s)^2},\end{aligned}$$

which provides the desired estimate for the second term as well.

Step 3. G is contractive. Let us assume in this section that $|u|_a$ and $|v|_a$ are both less than $R/2C_0a$. We prove that

$$|G[u]-G[v]|_a \leqslant C_2 a|u-v|_a.$$

One should think of u and v as two successive terms u_n and u_{n+1} in the iterative procedure. To obtain this inequality, we first write

$$G[u]-G[v] = \sum_{j=1}^{n} F[w_j]-F[w_{j-1}],$$

where

$$w_j = \int_0^{j/n} \sigma^{A-1}u(\sigma t)\,\mathrm{d}\sigma + \int_{j/n}^{1} \sigma^{A-1}v(\sigma t)\,\mathrm{d}\sigma.$$

By the arguments of step 1, we have $\|w_j\|_s < \frac{1}{2}R$ for $t < a(s_0-s)$.

We therefore have, using step 2 with $p = w_j$ and $q = w_{j-1}$,

$$\|G[u]-G[v]\|_s(t) \leqslant \sum_{j=1}^{n} \frac{Ct}{s_j - s}\|w_j - w_{j-1}\|_{s_j}.$$

Let us choose a sequence of numbers, $s_j = s(j/n)$, where

$$s(\sigma) = \frac{1}{2}\left(s + s_0 - \frac{\sigma t}{a}\right).$$

We now find

$$\begin{aligned}
\|w_j - w_{j-1}\|_{s_j} &= \left\| \int_{(j-1)/n}^{j/n} \sigma^{A-1}[u(\sigma t)-v(\sigma t)]\,\mathrm{d}\sigma \right\|_{s_j} \\
&\leqslant \int_{(j-1)/n}^{j/n} C\|u-v\|_{s(\sigma)}(\sigma t)\,\mathrm{d}\sigma/\sigma \\
&\leqslant \int_{(j-1)/n}^{j/n} \frac{Ct}{s_0 - s(\sigma)} \frac{|u-v|_a\,\mathrm{d}\sigma}{\sqrt{1-\sigma t/a(s_0 - s(\sigma))}}.
\end{aligned}$$

Letting n tend to infinity, we find the estimate

$$\|G[u]-G[v]\|_s(t) \leqslant \int_0^1 C\frac{t^2|u-v|_a}{(s(\sigma)-s)(s_0-s(\sigma))}\frac{\mathrm{d}\sigma}{\sqrt{1-\sigma t/a(s_0-s(\sigma))}}.$$

We now make the change of variables $\rho = \sigma t/a(s_0-s)$. Note that

$$(s(\sigma)-s)(s_0-s(\sigma)) = \frac{(s_0-s)^2}{4}(1-\rho^2), \qquad 1 - \frac{\sigma t}{a(s_0-s(\sigma))} = \frac{1-\rho}{1+\rho}.$$

We therefore find

$$\begin{aligned}
\|G[u]-G[v]\|_s(t) &\leqslant \frac{Cat|u-v|_a}{s_0-s}\int_0^{t/a(s_0-s)} \frac{\mathrm{d}\rho}{(1-\rho)^{3/2}} \\
&\leqslant \frac{Cat|u-v|_a}{s_0-s}\left(1-\frac{t}{a(s_0-s)}\right)^{-1/2}.
\end{aligned}$$

Using the definition of the a-norm, we see that we have obtained the desired estimate.

Step 4. End of proof. Let $u_0 = 0$ and define inductively u_n by $u_{n+1} = G[u_n]$. We show that this sequence converges in the a-norm if a is small. Since $\|u_1\|_{s_0} \leqslant Rt$, we have $|u_1|_a < R/4C_0a$ if a is small. We may assume $C_2a < \frac{1}{2}$. It follows by induction that $|u_{n+1} - u_n|_a \leqslant 2^{-n}|u_1|_a$, and $|u_{n+1}|_a < R/2C_0a$, which implies in particular $\|Hu_n\|_s < \frac{1}{2}R$. Therefore all the iterates are well defined and lie in the domain in which G is contractive. As a result, the iteration converges, as desired.

Impact on formal expansions. The expansion of [6] amounts to seeking X and Z as functions of $(t, \varepsilon x)$, expanding in ε, and then letting $\varepsilon = 1$. Its convergence can therefore be derived from the analyticity of the solutions in x. Note that the reference solution in that paper is slightly more restrictive than those considered here: they are geodesic loops travelling 'to the right' in the Poincaré half-plane.

The Fuchsian algorithm provides a different way of generating formal solutions: by following the existence proof itself. Thus, starting with $u = 0$, we can compute $F[0]$, then solve $(D + A)u_1 = F[0]$, which is a linear ODE in t, compute $F[u_1]$, etc. The higher-order corrections are automatically generated even if their order is not known in advance.

Remarks on the nature of the singularity. One could check that AVD Gowdy spacetimes with $0 < k < 1$ or $k > 1$ do have a curvature singularity at $t = 0$ by directly computing the Kretschmann scalar $B := R_{ijkl}R^{ijkl}$ (for large classes of such spacetimes, see [2], where symbolic manipulation is used; see also a brief remark in this direction at the end of [6]). We give a simpler argument which reduces the issue to the corresponding problem for Kasner spacetimes, where the answer is classical.

Indeed, consider the orthonormal coframe

$$(\mathrm{e}^{\lambda/4}t^{-1/4}\,\mathrm{d}t, \mathrm{e}^{\lambda/4}t^{-1/4}\,\mathrm{d}x, t^{1/2}\mathrm{e}^{-Z/2}(\mathrm{d}y + X\,\mathrm{d}z), t^{1/2}\mathrm{e}^{Z/2}\,\mathrm{d}z),$$

and the dual frame $\{\boldsymbol{e}_a = \boldsymbol{e}_a^k\partial_k\}$. One finds, by direct computation, that the Ricci rotation coefficients of this frame all have the form:

$$\gamma^a{}_{bc} = C^a{}_{bc}t^{-3(k^2+1)/4}(1 + \mathrm{o}(1)),$$

where the leading-order coefficients $C^a{}_{bc}$ are t-independent quantities which involve only k: its derivatives, or the functions X_0, φ and ψ do not affect the value of these coefficients. A similar property holds for the coefficients $b^c{}_{ab}$ defined by $[\boldsymbol{e}_a, \boldsymbol{e}_b] = b^c{}_{ab}\boldsymbol{e}_c$. It follows that the product terms in the expression of the frame components of the curvature tensor are at most $\mathrm{O}(t^{-3(k^2+1)/2})$. As for the Pfaffian derivative terms, it turns out that they are not worse, because they are coordinate derivatives multiplied by appropriate frame components. There are still no x derivatives at leading order. It follows that the most singular term in B as $t \to 0$ is in fact the same as the one corresponding to $X_0 = \varphi = \psi = 0$, and $k =$ constant, which is the Kasner case.

In extrinsic terms, we may express the result as follows: if h is the mean curvature of the slices $t =$ constant, then B/h^4 tends to a non-zero constant for $0 < k < 1$ or $k > 1$, which has the same expression as in the Kasner case. In particular, B blows up like $t^{-3(k^2+1)}$, so that we have a curvature singularity. □

Remark 3. It is easy to check that this singularity is reached in finite proper time by observers with $x =$ constant, so that this space is indeed (past) geodesically incomplete.

Remark 4. There is no change in the leading power of B as k goes through 1: only the coefficient of the leading term in B vanishes.

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References

[1] Berger B and Moncrief V 1994 Numerical investigation of cosmological singularities *Phys. Rev.* D **48** 4676–87
[2] Chruściel P T 1991 On uniqueness in the large of solutions of Einstein's equations ('Strong Cosmic Censorship') *Proc. CMA* **27** ANU
[3] Chruściel P T, Isenberg J and Moncrief V 1990 Strong cosmic censorship in polarized Gowdy spacetimes *Class. Quantum Grav.* **7** 1671–80
[4] Eardley D, Liang E and Sachs R 1972 Velocity-dominated singularities in irrotational dust cosmologies *J. Math. Phys.* **13** 99–106
[5] Gowdy R H 1974 Vacuum spacetimes and compact invariant hypersurfaces: topologies and boundary conditions *Ann. Phys., NY* **83** 203–41
[6] Grubišić B and Moncrief V 1993 Asymptotic behavior of the $\mathbf{T}^3 \times \mathbf{R}$ Gowdy space-times *Phys. Rev.* D **47** 2371–82
[7] Hawking S W and Ellis G F R 1973 *The Large-Scale Structure of Spacetime* (Cambridge: Cambridge University Press)
[8] Kichenassamy S 1996 *Nonlinear Wave Equations* (New York: Marcel Dekker)
[9] Kichenassamy S 1998 WTC expansions and non-integrable equations *Stud. Appl. Math.* to appear
[10] Kichenassamy S 1996 Fuchsian equations in Sobolev spaces and blow-up *J. Diff. Eqns.* **125** 299–327
[11] Kichenassamy S 1996 The blow-up problem for exponential nonlinearities *Commun. PDE* **21** 125–62
[12] Kichenassamy S and Srinivasan G K 1995 The structure of WTC expansions and applications *J. Phys. A: Math. Gen.* **28** 1977–2004