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LETTER TO THE EDITOR

Entropy of a dressed black hole and properties of the Hartle–Hawking state

O B Zaslavskii

Department of Physics, Kharkov University, Svobody Sq 4, Kharkov 310077, Ukraine

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Abstract. The entropy S_φ of quantum radiation in equilibrium with a black hole is obtained in a one-loop approximation without recourse to the general first law. The approach developed does not need information about quantum corrections to the Hawking temperature and metric. It follows from properties of the Hartle–Hawking state that S_φ is finite, which confirms the universality of the Bekenstein–Hawking entropy.

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The entropy S of a black hole in thermal equilibrium with its Hawking radiation consists of two contributions—the Bekenstein–Hawking entropy S^{BH} [1, 2] of a hole itself and that S_φ of quantum radiation. Universality of $S^{\mathrm{BH}} = A/4$ in general relativity (A is the area of the event horizon, I use units with $G = \hbar = c = 1$) implies that S_φ is finite. Otherwise the divergent part of S_φ in the sum $S = S^{\mathrm{BH}} + S_\varphi$ would renormalize the entropy of a black hole [3]. Therefore, for understanding the concept of black-hole entropy it is very important to trace in detail why S_φ proves to be finite and to compare different approaches to calculating S_φ.

Quite recently an interesting paper appeared [4] in which the key role of the connection between thermodynamics and geometry was elucidated in the context under discussion. In particular, this connection is to be taken into account in obtaining entropy by differentiating a free energy that enables one to understand why S_φ is finite in spite of divergences in the statistical-mechanical entropy S^{SM}.

However, the approach of [4] leaves some issues unresolved. It relies on relations $\delta F = -S^{\mathrm{SM}}\delta T_0 + \Lambda(T_0, r_+)\delta r_+$ and $\delta F = -S\delta T$ (the simplified version of equations (1) and (2) of [4]). Here F is a free energy, T is the local Tolman temperature on a boundary which defines a canonical ensemble [5], r_+ is the radius of a horizon (the system is assumed to be spherically symmetrical), T_0 is the temperature at infinity, Λ is some function of T_0 and r_+. Both these relations are used in [4] separately for gravitation and the quantum field, but such use conflicts with the essential non-additivity of free energies (in contrast to additivity of entropies [6–8]). Careful treatment of a canonical ensemble [5] also shows that one cannot, in general, neglect the difference between T and T_0 as was done in [4]. It seems plausible that these subtleties do not change the main ideas of [4] qualitatively; nevertheless, neglecting a number of essentials of gravitational thermodynamics of finite-size systems leaves the matter not quite clear.

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On the other hand, the entropy S_φ of the massless field has been found in a general form in [9] by direct integration of the general first law. The corresponding result is quite sufficient for practical purposes but is not enough for the problem we are dealing with—to elucidate the origin of finiteness of S_φ. The point is that the first law assumes the validity of Einstein equations [10] for the combined system ‘a hole plus matter’. However, S_φ in the one-loop approximation is expressed in terms of quantities calculated with respect to the Schwarzschild background and is insensitive to successive corrections either to the metric or to the Hawking temperature T_{H}. For this reason basing the finiteness of S_φ and deriving the explicit expression for it seem to contain ‘superfluous’ information connected with properties of the geometry corrected due to backreaction.

The aim of the present paper is to give a very simple and clear derivation of S_φ without using the first law. Finiteness of S_φ and universality of S^{BH} follow directly from properties of the Hartle–Hawking state [11].

There are also two more reasons why deriving S_φ without the first law seems to be very important. First, recently the permissible range of the parameter ξ between the quantum field and curvature has been found from positivity of entropy [12] that gives a new insight into the link between thermodynamics and field theory. However, the entropy in [12] is derived from the first law that operates with the notion of quasilocal energy [13] which is not defined yet for arbitrary ξ when the field is coupled to a metric neither minimally nor conformally. In this sense our derivation of S_φ without the first law puts this direction of investigations on a firm basis.

Second, a rigorous treatment of S_φ in the framework of the first law should be faced with the deficit angle as a free parameter [14] which adds new complexity and obscures the whole picture whereas this problem does not arise in our approach.

Consider a quantum field φ at finite temperature $T_0 = \beta_0^{-1}$. Its Euclidean action takes the standard form [15]

$$I_\varphi = -\beta_0 \int \mathrm{d}^3x \, \sqrt{g} T_0^0 - S_\varphi \, . \tag{1}$$

Here T_μ^ν is the renormalized stress–energy tensor calculated with respect to the metric $g_{\mu\nu}$ with the determinant g. We consider the class of Euclidean metrics which have the form

$$\mathrm{d}s^2 = f(r_+/r)\,\mathrm{d}\tau^2 + f^{-1}(r_+/r)\,\mathrm{d}r^2 + r^2(\sin^2\theta\,\mathrm{d}\varphi^2 + \mathrm{d}\theta^2)\, . \tag{2}$$

This metric is assumed to describe a black hole, so $f(1) = 0$ but otherwise f is an arbitrary function of the ratio r_+/r. For the particular case of the Schwarzschild metric $f(x) = 1 - x$.

From the variation of (1) we have

$$\delta|_\varphi = -\delta\beta_0 \int \mathrm{d}^3x \, \sqrt{g} T_0^0 + \frac{\beta_0}{2} \int \mathrm{d}^3x \, \sqrt{g} T_{\mu r} \delta g^{\mu r} + 4\pi T_0^0(r_+) r_+^2 \delta r_+ \, . \tag{3}$$

The first term in (3) describes the response of I_φ to the change of temperature T_0 in the fixed geometry, the second term follows from the definition of the stress–energy tensor, the third one is connected with the change of the integration region when r_+ is allowed to vary (note that there is no sense in writing down a similar term in δS_φ since it can always be removed by a simple redefinition of S_φ).

It is worth noting that in contrast to (1) of [4], T_0 and r_+ are not independent parameters since $T_0 = T_{\mathrm{H}}$. On the other hand, we are dealing with the variation of the field action only (not with that of the total system), so the problem of a deficit angle [14] does not arise.

We assume that as usual T_μ^ν is calculated in the one-loop approximation in the background (2) for which $\beta_0 = -4\pi r_+/f'(1)$, $f'(1) < 0$.

The key idea consists in the consideration of such variations of the metric in which the background retains its form (2) and only the parameter r_+ changes.

Then the second term in (2) turns into

$$\frac{(4\pi)^2\delta r_+}{2|f'(1)|}\int_{r_+}^{R}\frac{\mathrm{d}r\, r}{f}f'(T_r^r - T_0^0), \qquad f' = \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=r_+/r}, \tag{4}$$

where R is the radius of a cavity into which a hole is enclosed.

From the conservation law $T^{\mu}_{r;\mu} = 0$ in the background (2) it follows that

$$\frac{r}{2}\frac{(T_r^r - T_0^0)f'}{f} = \frac{1}{r_+}[(r^3T_r^r)_{,r} - r^2T_i^i] \tag{5}$$

where i denotes spatial indices.

Then from (2)–(5) we have

$$(4\pi)^{-2}f'(1)\frac{\mathrm{d}I_\varphi}{\mathrm{d}r_+} = \int_{r_+}^{R}\mathrm{d}r\, r^2T_i^i - R^3T_r^r(R) + r_+^3[T_r^r(r_+) - T_0^0(r_+)]\,. \tag{6}$$

The regularity of T_μ^ν at the event horizon in the Hartle–Hawking state demands that according to (5) $T_r^r(r_+) = T_0^0(r_+)$, so the last term in (6) in square brackets cancels.

The above formulae are applicable to any field with a tensor T_μ^ν. If a field is massless, there exists only one relevant parameter of length r_+, and from dimensional grounds

$$T_\mu^\nu = r_+^{-4}f_\mu^\nu(r_+/r)\,. \tag{7}$$

Substitute (7) into (6). Then

$$(4\pi)^{-2}\frac{\mathrm{d}I_\varphi}{\mathrm{d}w}|f'(1)| = \frac{f_r^r(w)}{w^4} - \frac{1}{w}\int_w^1\frac{\mathrm{d}u}{u^4}f_\mu^\mu(u) \tag{8}$$

where $w = r_+/r$.

Integrate this equality with the natural boundary condition $I_\varphi(r_+ = R) = 0$. It is convenient to use the relation (for any $g(u)$)

$$\int_w^1\frac{\mathrm{d}u'}{u'}\int_u^1\mathrm{d}u\, g(u) = \int_w^1\mathrm{d}u\, g(u)\ln\frac{u}{w}\,. \tag{9}$$

As a result we obtain in terms of T_μ^ν:

$$I_\varphi = 16\pi^2r_+|f'(1)|^{-1}\int_{r_+}^{R}\mathrm{d}r\, r^2\left(-T_r^r + T_\mu^\mu\ln\frac{R}{r}\right) \tag{10}$$

$$S_\varphi = 16\pi^2r_+|f'(1)|^{-1}\int_{r_+}^{R}\mathrm{d}r\, r^2\left(T_r^r - T_0^0 - T_\mu^\mu\ln\frac{R}{r}\right). \tag{11}$$

In equations (10) and (11) the energy–momentum tensor corresponds to the Hartle–Hawking state describing equilibrium between a black hole and radiation, $T_\mu^\nu = T_{\mu_{\mathrm{HH}}}^\nu$. Now use the remarkable quality [16]

$$T_{\mu_{\mathrm{HH}}}^\nu = T_{\nu_{\mathrm{B}}}^\nu + T_{\mu_{\mathrm{th}}}^\nu\,. \tag{12}$$

Here $T_{\mu_{\mathrm{B}}}^\nu$ corresponds to the Boulware vacuum, $T_{\mu_{\mathrm{th}}}^\nu$ describes the black-body radiation with Hawking temperature: $T_{\mu_{\mathrm{th}}}^\nu = \alpha T^4\,\mathrm{diag}(-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ (α is constant). As the trace $T_{\mu_{\mathrm{th}}}^\nu = 0$ we obtain from (11) that

$$S_\varphi = S_{\mathrm{th}} + S_{\mathrm{B}}\,. \tag{13}$$

Here S_{th} coincides with S_1^{SM} of [4] and represents the entropy of thermal radiation with the density $\frac{4}{3}\alpha T^3$ which diverges near the horizon. The quantity S_{B} is the infinite renormalization constant which makes the sum (13) finite. It corresponds to ΔS_1 of [4] and is obtained by replacing $T^{\nu}_{\mu_{\rm HH}}$ by $T^{\nu}_{\mu_{\rm B}}$ in (11). In the approach under discussion it is clear that $S_{\rm B}$ is connected with vacuum polarization described by $T^{\nu}_{\mu_{\rm B}}$. We see that not only the energy but the entropy of the quantum field also arise in a finite form as a result of thermal excitations of field modes over the Boulware state at Hawking temperature. In so doing, the infinite parts of thermal and Boulware contributions to S_φ cancel exactly. If the temperature of excitations $T_0 \neq T_{\rm H}$ compensation in (11), (12) near the horizon does not occur and the entropy diverges. For the thermal atmosphere of a black hole this gives one direct justification of the renormalization procedure for the entropy proposed in [17] (cf also [4]).

If the field is massive, the relation (7) fails and one cannot obtain the expression for S_φ as simply as in (11). However, it is important that (6) still holds, so the entropy and action are again linear functionals of $T^{\nu}_{\mu_{\rm HH}}$. It explains the finiteness of S_φ and universality of $S^{\rm BH}$ from properties of the Hartle–Hawking state in the general case.

It is worth stressing that the general first law refers to the total system 'gravitation plus field' and assumes the validity of the Einstein equations for it; whereas our approach operates only with the field action and does not need information about quantum corrections to the geometry. In this sense one can say that in the one-loop approximation we found the entropy of off-shell black holes. Moreover, for the arbitrary function $f(r_+/r)$ in (2), when not only corrections to it but f itself does not obey the Einstein equations, the result obtained gives one the entropy of the particular class of such holes described by the metric (2), i.e. it is more general than the Schwarzschild case. (It is only required that T^{ν}_{μ} be conserved in the metric (2).)

This gives one the one-loop correction to the entropy of $1+1$ black holes in the fixed background in the closed form. Investigation of its properties will be reported elsewhere.

After this work had been completed we became aware of a number of other papers on related topics. In [18, 19] the authors discuss new features introduced by the presence of the event horizon into the general scheme of renormalization of quantum theory in curved spacetime. In contrast to [18, 19] we took for granted the existence of well defined renormalized action and entropy from the very beginning and concentrated our attention on obtaining explicit formulae for them. The results for the entropy in [20] seem to give only the term $S_{\rm th}$ in (13) while our approach enables one to obtain the total quantity (11). In [21] quantum corrections to temperature and entropy of a black hole were derived from conformal properties of a field. The results are in disagreement with (11). One can show [22] that the discrepancy can be removed if the finiteness of a system and boundary conditions (which were neglected in [21]) are taken into account properly.

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