You may also like

# Gravitons from loops: non-perturbative loop-space quantum gravity contains the graviton-physics approximation 

To cite this article: Junichi Iwasaki and Carlo Rovelli 1994 Class. Quantum Grav. 111653

Hermitian operators on quantum general relativity loop space D Rayner

Towards a loop space description of nonlinear sigma model Partha Mukhopadhyay

Monopoles in superloop space Mir Faizal and Tsou Sheung Tsun

View the article online for updates and enhancements.

# Gravitons from loops: non-perturbative loop-space quantum gravity contains the graviton-physics approximation 

Junichi Iwasaki $\dagger \S$ and Carlo Rovelli $\dagger \ddagger \|$<br>$\dagger$ Physics Department, University of Pittsburgh, Pittsburgh PA 15260, USA<br>$\ddagger$ Dipartimento di Fisica, Universitá di Trento and INFN sez. Padova, Italia

Received 30 April 1993, in final form 23 September 1993


#### Abstract

We investigate the physical interpretation of the loop states of non-perturbative quantum general relativity in the regime of graviton physics, namely the regime of first order excitations around the Poincare-invariant vacuum. We construct the general form of the loop state functionals invariant under the linearized constraints. We present explicitly the loop state functionals that represent the Poincare-invariant vacuum and the graviton states. We find that physical information emerges entirely from intersections of loops. We obtain these results by utilizing the recently introduced 'map $\mathcal{M}$ ', which relates the loop-space states of non-perturbative quantum general relativity to the state space of the linearized theory. The general picture of the linearization of the loop-space quantum general relativity is discussed.


PACS numbers: 0420,0460

## 1. Introduction

The loop representation of quantum general relativity [1] is a candidate for a quantum theory of gravity. An important open problem in this approach is the difficulty of recovering simple 'low energy' gravitational physics from the full theory. This is the problem we discuss in this paper.

The physical quantum states of the loop representation (from now on loop rep) of quantum general relativity (from now on QGR) are represented by linear combinations of knots. We would like to recover the Riemannian geometry of spacetime from these knot states, in some sort of 'high quantum numbers' limit: somehow in the same way in which one recovers the electron trajectories from the $|n, l, m\rangle$ states of the hydrogen atom in the 'high quantum numbers' limit. Riemannian geometry should emerge (in approximate form) from the purely topological world of the knots.

We do not fully solve this problem in the present paper, but we make a substantial step toward its solution. The idea that we follow is to focus on the 'low energy' regime given by flat spacetime and small oscillations around flat spacetime. In this regime, the gravitational field can be described by means of linearized GR. The corresponding quantum theory, namely quantum linearized GR (from now on QLGR), describes free gravitons on a Poincare-invariant vacuum state. The physics of QLGR is familiar, easy to be interpreted, and its classical limit is well understood (weak gravitational waves on Minkowski space). Our aim is to find a sector of the loop rep of QGR that describes the same physics as QLGR;

[^0]namely, to identify the Poincare-invariant vacuum and the $n$-graviton states within QGR. Of course, this identification makes sense only within a certain approximation, because the physics described by QLGR and QGR are genuinely different at high energy.

Working within this approximation we construct here loop states of QGR that represent the low energy physics. In particular, we explicitly construct the loop functional $\Psi_{0}$ that describes the Poincare-invariant vacuum, that is, the 'quantum flat spacetime', and the related $n$-graviton loop functionals $\Psi_{k_{1} . . k_{n}}$.

We use three main tools in our construction. The first is a version of QLGR developed in [2]. This version of QLGR is fully equivalent to standard ones (QLGR is a free quantum field theory, and therefore can be realized in a variety of equivalent ways), but has the advantage that it is a loop rep; namely, quantum states are represented by loop functionals. This makes the comparison with the loop rep of QGR easier. The second tool is the weave, introduced in [3]. The weave is a particular loop state in QGR that approximates a flat Euclidean metric at large distances. The third and the main tool that we use is the map $\mathcal{M}$, introduced in [4]. This is a linear map from (a sector of) the state space of QGR to (a sector of) the state space of QLGR. It is defined by using the weave and it has the essential property of intertwining between operators of the two theories that have the same physical interpretation within the approximation in which we work. In this paper the map $\mathcal{M}$ will be explicitly utilized in order to find the states $\Psi_{0}$ and $\Psi_{k_{1} \ldots k_{n}}$. In the next subsection, we provide an outline of the strategy that we follow.

Our main result is the explicit construction of a loop state $\Psi_{0}$ that represents the Poincare-invariant vacuum or the 'quantum flat spacetime', and the corresponding $n$-graviton states $\Psi_{k_{1} \ldots k_{n}}$, within QGR. This result indicates that QGR in the loop rep contains a sector that describes the known low energy physics, and may provide a basis for the physical interpretation of all loop states. Perhaps surprisingly, we find that the physical information is coded entirely in the intersections between loops and the weave; these are 'magnified' by the map $\mathcal{M}$ and become the low-energy information relevant in graviton physics. This may be seen as a result supporting the claims on the importance of the intersections in the loop rep.

The result we present is not yet a fully satisfactory solution of the problem of recovering low energy physics from the knot states, for the following reason. Ideally, we would like to construct exact physical states of QGR that represent the low energy physics approximately. These states should solve the QGR constraints exactly, and therefore be knot states. The states $\Psi_{0}$ and $\Psi_{k_{1} \ldots k_{n}}$ that we construct, on the other side, are only approximate solutions of the QGR constraints. They are, in a sense, approximate knot states. Indeed, we will see that they are not fully invariant under all diffeomorphisms, but only under the 'low frequency' diffeomorphisms (that will be defined below). Thus, the present work should be completed by finding exact knot states that 'differ from $\Psi_{0}$ and $\Psi_{k_{1} \ldots k_{k}}$ at high energies'. We will discuss these remaining open issues in the conclusion.

### 1.1. Outline of our strategy

The present paper is a continuation of the line of research developed in [1] where the loop rep of QGR was defined, [2] where the loop rep of QLGR was defined, [3] where the weave state was constructed, and [4] where the map $\mathcal{M}$ was introduced. In particular, this paper may be seen as a follow up of [4]. We made an effort to make the paper self-contained by summarizing previous works in section 2 ; however, its comprehension would be much enhanced by the knowledge of those references, in particullar [4].

Our general philosophy is to fix a 'small' positive number $\varepsilon$ that determines the accuracy to which we want to reproduce the expectation values of the linear theory (QLGR) from
expectation values of the exact theory (QGR). As shown in [4], we have then to restrict our considerations only to a sector of states and operators, this sector being dependedent on $\varepsilon$. Intuitively, these are the states close enough to the vacuum and the operators that do not probe the state at too short scale. The precise technical characterization of this sector of states and operators is given in [4] and recalled below in section 2.4. It is important to keep those restrictions in mind in order to follow the calculations in the paper. We can then consider $\varepsilon$ as an expansion parameter: to first order in $\varepsilon$ the predictions of the two theories should agree. Thus, we may disregard terms of higher order in $\varepsilon$ in all calculations, since we are only interested in the results of first order in $\varepsilon$.
$\mathcal{M}$ is a map from the state space of QGR to the state space of QLGR. It intertwines between operators of the two theories that represent the same physical variable to first order in $\varepsilon$. The notion of representing the same physical variable is well defined since classical GR and LGR are defined on the same (unconstrained) phase space. However, the map $\mathcal{M}$ turns out to be highly non-trivial, because of a number of reasons (that will be recalled in section 2.4). The main one of these reasons is the fact that the state space of QLGR and the state space of QGR are different spaces (unlike, for instance, the Schrödinger quantization of the harmonic oscillator and the Schrödinger quantization of the anharmonic oscillator, which are both defined on the same state space of wavefunctions $\psi(x)$ ). It also turns out that the map $\mathcal{M}$ is defined (and makes sense) only up to terms of higher order in $\varepsilon$. These terms represent the short-scale indeterminacy which is left open in the identification of the low energy free theory with the exact interacting theory.

At first sight, one could think that the map $\mathcal{M}$ can be used to directly map the vacuum and the $n$-graviton states from QLGR to QGR. However, this is not possible, because the quantum states contain more information than just the one given by the first order in $\varepsilon$. (The linearization in $x$ of many dynamical systems produces a harmonic oscillator, but the harmonic oscillator vacuum state $\psi_{o}(x)=\exp \left[-x^{2}\right]$ is not linear in $x$.) Thus, the strategy we adopt is to use the map $\mathcal{M}$ not for transferring states from QLGR to QGR, but rather to transfer the most relevant operators of QLGR into the state space of QGR. We denote the set of operators that we obtain, which are defined on the state space of QGR but define the same physics as QLGR, as 'low frequency operators', for a reason that will be clarified later. We also denote as the 'low frequency theory' this theory defined by the low frequency operators, namely QLGR on the state space of QGR.

The relevant operators are the constraints and the Hamiltonian. The transfer of the linearized constraints is relatively straightforward and will be performed in sections 3.1 and 3.2. The low frequency vector constraint that we obtain turns out to be the generator of 'low frequency diffeomorphisms'. The meaning of this result will be discussed below. A geometrical interpretation of the low frequency scalar constraint is more difficult. As far as the Hamiltonian is concerned, we again run into the problem that the map $\mathcal{M}$ makes sense only to first order, while the Hamiltonian is quadratic in $\varepsilon$. We circumvent this problem by using the creation and annihilation operators (which are first order), instead of the Hamiltonian.

Even transferring the creation and annihilation operators is far from straightforward. The reason is that these depend on the linearized metric field, but there is no metric field operator $\hat{g}_{a b}(x)$ in QGR. There is only its 'smeared' version $\hat{Q}(\omega)$, which corresponds to $Q(\omega)=\int \mathrm{d}^{3} x\left[(\operatorname{det} g) g^{a b} \omega_{a} \omega_{b}\right]^{1 / 2}$. The creation and annihilation operators of the two polarizations depend on the linearized metric field $h^{a b}(x)$, via terms which in momentum space have the form $m_{a}(k) m_{b}(k) h^{a b}(k)$. We therefore have the technical difficulty of expressing these quantities in terms of $Q(\omega)$. We achieve this by introducing, in section 3.3, a triplet of covectors $\omega_{a}^{i}(x) i=1,2,3$, that represent, in a sense, 'polarization vectors in
position space', and are roughly defined by Fourier transforming (couples of) polarization vectors as $m_{a}(k)$. By using these, we can write the creation and annihilation operators in terms of $\hat{Q}(\omega)$, therefore transferring them to the QGR state space. This is done in section 3.4.

Section 4 is then dedicated to solving the low frequency theory. In its original (loop rep) form the main ingredients for solving QLGR are the form factors [2]. In section 4.1 we find the loop functionals that play the very same role in the low-frequency theory, and we denote them as $G^{ \pm}[y, \alpha]$. These are first found by using again the map $\mathcal{M}$ as a hint, and then checking directly their behaviour under the constraints and the dynamical operators. The calculations are quite involved, but the final result is simple: the states that solve the low frequency constraints are given by arbitrary functions of $G^{ \pm}[y, \Delta \# \alpha]$. Finally, in section 4.2 we find the Poincare-invariant state and the graviton states.

## 2. Review of the exact and linearized theories, weave and map $\mathcal{M}$

We briefly review the non-perturbative (QGR) and linearized (QLGR) theories in the loop representation and the definitions and properties of the weave and map $\mathcal{M}$, in order to fix our notations and conventions. For more detail the reader may consult references [1-4,6].

## 2.I. QGR, the non-pertubative theory

The classical $T$ variables are defined by

$$
\begin{aligned}
& T[\alpha]=\frac{1}{2} \operatorname{Tr} P \exp \left[G \oint \mathrm{~d} t \dot{\alpha}^{b}(t) A_{b}(\alpha(t))\right] \\
& T^{a}[\alpha](s)=\frac{1}{2} \operatorname{Tr} P \exp \left[G \oint \mathrm{~d} t \dot{\alpha}^{b}(t) A_{b}(\alpha(t))\right] \widetilde{E}^{a}(\alpha(s))
\end{aligned}
$$

where $A_{a}(x)=A_{a}^{i}(x) \tau_{i}$ and $\widetilde{E}^{a}(x)=4 \widetilde{E}^{a i}(x) \tau_{i}$ are the Ashtekar connection and its conjugate momentum field ( $\tau_{i}$ is a Pauli matrix divided by $2 i$ ). Indices $a, b, \ldots$ are space indices and $i, j, \ldots$ are internal $S O(3)$ indices. Note the $\frac{1}{2}$ that we use in our conventions. A variable that we need is

$$
Q(\omega)=\int \mathrm{d}^{3} x\left[(\operatorname{det} g) g^{a b} \omega_{a} \omega_{b}\right]^{1 / 2}
$$

where $g^{a b}$ is the inverse of the 3 -metric and $\omega_{a}(x)$ is a test 1 -form. In the quantum theory the corresponding operations are defined on a space of functionals $\Psi(\alpha)=\langle\alpha \mid \Psi\rangle$ of multiple loops. (A multiple loop is an unordered set of loops. We identify a single loop $\alpha$ with the multiple loop that contain only the single loop $\alpha$; and we indicate single as well as multiple loops by greek letters.) These operators are

$$
\begin{align*}
& \langle\beta| \widehat{T}[\alpha]=\langle\beta \cup \alpha| \\
& \langle\beta| \widehat{T}^{a}[\alpha](s)=\frac{1}{2} l_{p}^{2} \oint \mathrm{~d} t \dot{\beta}^{a}(t) \delta^{3}(\alpha(s), \beta(t))\left(\langle\beta \# \alpha|-\left\langle\beta \# \alpha^{-1}\right|\right)  \tag{2.1.1}\\
& \langle\beta| \widehat{Q}(\omega)=\frac{1}{2} l_{p}^{2} \int \mathrm{~d}^{3} x\left|\oint \mathrm{~d} s \dot{\beta}^{a}(s) \omega_{a}(\beta(s)) \delta^{3}(x, \beta(s))\right|\langle\beta| .
\end{align*}
$$

The symbol $\cup$ indicates union of multiple loops; thus $\alpha \cup \beta$ is the multiple loop formed by the single loops in (or the loop) $\alpha$ and the single loops in (or the loop) $\beta$. The symbol \#
indicates the joining of two loops at an intersection; thus if $P$ is the point where the loops $\alpha$ and $\beta$ intersect, $\alpha \# \beta$ is the loop obtained by starting from $P$, going around $\alpha$, then along $\beta$, and then closing. Note that the last equation holds not only for non-intersecting loops but also for intersecting loops as long as intersections are isolated ones. We do not consider in this paper loops which have a segment overlapping with another segment.

The state functionals satisfy some conditions, one of which is the spinor relation

$$
\begin{equation*}
\langle\alpha \cup \beta|=\frac{1}{2}\left(\langle\alpha \# \beta|+\left\langle\alpha \# \beta^{-1} 1\right)\right. \tag{2.1.2}
\end{equation*}
$$

if $\alpha$ and $\beta$ have a common point.

### 2.2. QLGR, the linearized theory

The classical variables are defined by

$$
\begin{aligned}
t_{r}^{i}(\alpha) & =\exp \left[G \int \mathrm{~d}^{3} x F_{r}^{a}[x, \alpha] A_{a}^{i}(x)\right] \\
h^{a b}(x) & =2 \delta_{i}^{(a} \mathrm{e}^{b) \mathrm{i}}(x)
\end{aligned}
$$

with the form factor

$$
F_{r}^{a}[x, \alpha]=\oint \mathrm{d} s \dot{\alpha}^{a}(s) f_{r}(x-\alpha(s))
$$

where $\mathrm{e}^{a \mathrm{i}}(x)$ is the deviation of the triad $\widetilde{E}^{a \mathrm{i}}(x)$ from the flat $\delta^{a \mathrm{i}}$ and $f_{r}(x)$ is a smearing function. We choose

$$
f_{r}(x)=\left(2 \pi r^{2}\right)^{-3 / 2} \mathrm{e}^{-x^{2} / 2 r^{2}}
$$

The corresponding quantum operators are defined on a space of functionals $\psi_{r}(\vec{\alpha})=\left\langle\vec{\alpha} \mid \psi_{r}\right\rangle$ of triplets of multiple loops. Those are

$$
\langle\vec{\beta}| \hat{t}_{r}^{\hat{i}}(\alpha)=\left\langle\vec{\beta} \cup_{i} \alpha\right| \quad\langle\vec{\beta}| \hat{h}^{a b}(x)=-2 l_{p}^{2} F_{r}^{(a}\left[x, \beta^{b)}\right]\langle\vec{\beta}|
$$

The notation $\vec{\beta} \bigcup_{i} \alpha$ indicates the union of the loop $\alpha$ to the $i$ component of the triplet of multiple loops $\vec{\beta}:\langle\vec{\beta}| \equiv\left\langle\beta^{1}, \beta^{2}, \beta^{3}\right\rangle$ and $\left\langle\vec{\beta} \cup_{1} \alpha\right| \equiv\left\langle\beta^{1} \cup \alpha, \beta^{2}, \beta^{3}\right\rangle$ for example.

An operator which we will frequently use is the linearized magnetic field operator ('differential' operator),

$$
G \widehat{B}^{a i}(x)=\lim _{\delta \rightarrow 0} \frac{1}{\pi \delta^{2}}\left(\hat{t}_{r}^{i}\left(\gamma_{x, \delta}^{a}\right)-1\right)
$$

where $\gamma_{x, \delta}^{a}$ is a loop of radius $\delta$ centered at $x$ and normal to the $a$ direction. In terms of these operators, we can write several operators we need. The linearized vector and scalar constraint operators are

$$
\hat{\mathcal{V}}_{a}^{L}(x)=-\epsilon_{a b c} G \widehat{B}^{b c}(x) \quad \hat{\mathcal{S}}^{L}(x)=-\delta_{a b} G \widehat{B}^{a b}(x)
$$

It turns out that symmetric, traceless, transverse components of $F_{r}^{a}\left[x, \alpha^{i}\right]$ (seen as a twotensor in the indices $a$ and $i$ ) solve these constraints. In Fourier transform, these component are denoted as

$$
\begin{aligned}
& F_{r}^{+}[k, \vec{\alpha}]=\bar{m}_{a}(k) \bar{m}_{i}(k) F_{r}^{a}\left[k, \alpha^{i}\right] \\
& F_{r}^{-}[k, \vec{\alpha}]=m_{a}(k) m_{i}(k) F_{r}^{a}\left[k, \alpha^{i}\right]
\end{aligned}
$$

and hence physical states $\psi(\vec{\alpha})$ are defined by analytic functions of $F_{r}^{ \pm}[k, \vec{\alpha}]$; namely, they have the form $\psi(\vec{\alpha})=\psi\left(F_{r}^{ \pm}[k, \vec{\alpha}]\right)$. Here $m_{a}(k)$ and its complex conjugate $\bar{m}_{a}(k)$ are polarization vectors defined up to a phase by

$$
\begin{array}{lr}
m_{a}(k) \bar{m}^{a}(k)=1 & m_{a}(k) m^{a}(k)=m_{a}(k) k^{a}=0 \\
\bar{m}_{a}(k)=-m_{a}(-k) & \epsilon^{a b c} k_{a} m_{b}(k) \bar{m}_{c}(k)=-\mathrm{i}|k| .
\end{array}
$$

The annihilation and creation operators are

$$
\begin{align*}
& \hat{a}_{+}(k)=-\left(\frac{|k|}{2 \hbar G}\right)^{1 / 2}\left(\hat{h}^{+}(-k)+\frac{G}{|k|^{2}} \widehat{B}^{+}(-k)\right) \\
& \hat{a}_{-}(k)=\left(\frac{G}{2 \hbar|k|^{3}}\right)^{1 / 2} \widehat{B}^{-}(k) \\
& \hat{a}_{+}^{\dagger}(k)=\left(\frac{G}{2 \hbar|k|^{3}}\right)^{1 / 2} \widehat{B}^{+}(k)  \tag{2.2.1}\\
& \hat{a}_{-}^{\dagger}(k)=-\left(\frac{|k|}{2 \hbar G}\right)^{1 / 2}\left(\hat{h}^{-}(-k)+\frac{G}{|k|^{2}} \widehat{B}^{-}(-k)\right)
\end{align*}
$$

where $\hat{h}^{ \pm}(k)$ and $\widehat{B}^{ \pm}(k)$ are the transverse components of the Fourier transforms of $\hat{h}^{a b}(x)$ and $\widehat{B}^{a b}(x)$ respectively. For example $\hat{h}^{ \pm}(k)$ are

$$
\begin{aligned}
& \hat{h}^{+}(k)=\bar{m}_{a}(k) \bar{m}_{b}(k) \hat{h}^{a b}(k) \\
& \hat{h}^{-}(k)=m_{a}(k) m_{b}(k) \hat{h}^{a b}(k) .
\end{aligned}
$$

By computing the action of $\hat{h}^{ \pm}(k)$ and $G \widehat{B}^{ \pm}(k)$ on the physical state space described above, we obtain

$$
\begin{aligned}
& \left(\hat{h}^{ \pm}(k) \psi\right)(\vec{\alpha})=-2 l_{p}^{2} F_{r}^{ \pm}[k, \vec{\alpha}] \psi(\vec{\alpha}) \\
& \left(G \widehat{B}^{ \pm}(k) \psi\right)(\vec{\alpha})=\left.\mp|k| \mathrm{e}^{-r^{2} k^{2} / 2} \frac{\partial}{\partial F_{r}^{ \pm}(-k)} \psi\left(F_{r}^{ \pm}(k)\right)\right|_{F_{r}^{ \pm}(k)=F_{r}^{ \pm}[k, \vec{\alpha}]^{2}}
\end{aligned}
$$

The vacuum state, which is annihilated by the annihilation operators, turns out to be

$$
\psi_{0}(\vec{\alpha})=\exp \left(-l_{p}^{2} \int \mathrm{~d}^{3} k|k| \mathrm{e}^{2^{2} k^{2} / 2}\left|F_{r}^{+}[k, \vec{\alpha}]\right|^{2}\right)
$$

The excitation states are obtained by applying the creation operators on the vacuum state. For example, one-graviton states with positive and negative helicities are

$$
\begin{aligned}
& \psi_{k+}(\vec{\alpha})=\left(2 l_{p}^{2}|k|\right)^{1 / 2} F_{r}^{+}[k, \vec{\alpha}] \psi_{0}(\vec{\alpha}) \\
& \left.\psi_{k-}(\vec{\alpha})=\left(2 l_{p}^{2} \mid k\right]\right)^{1 / 2} F_{r}^{-}[-k, \vec{\alpha}] \psi_{0}(\vec{\alpha})
\end{aligned}
$$

Note that $\psi_{0}(\vec{\alpha})$ and $G \widehat{B}^{ \pm}(k)$ here are different from ones given in [2]. In [2] the presence of the smearing function was taken into account by simply replacing $F^{ \pm}$by $F_{r}^{ \pm}$; this procedure is not always correct.

### 2.3. The weave

The weave $\Delta$ is a set of randomly oriented circles of radius $a$ whose centres are randomly and uniformly distributed in the 3 -space with an average number density $n=1 / a^{3} ; a$ is of order of the Planck length $l_{p}$. Some of these circles may be linked to one another. There is an infinite number of different weaves $\Delta$, and each one determines a different loop state $\langle\Delta|$, namely, a different point in the state space of loops; these states are denoted as 'weave states'. If $\omega_{a}$ varies at a scale $L$ large compared to $l_{p}$, the eigenvalue of the metric operator on a weave state is a flat metric, that is,

$$
\begin{equation*}
\langle\Delta| \widehat{Q}(\omega)=\frac{1}{2} l_{p}^{2} \oint \mathrm{~d} s\left|\dot{\Delta}^{a} \omega_{a}\right|\langle\Delta|=\int \mathrm{d}^{3} x|\omega|\langle\Delta|+\mathrm{O}\left(l_{p} / L\right) \tag{2.3.1}
\end{equation*}
$$

In this sense all weave states approximate equally well the same flat metric for large enough $L$. To define the 'map $\mathcal{M}$,' we choose one particular weave state. In the process of linearizing the theory, we introduce errors; therefore we must fix a precision within which we want the approximation to hold. The errors in the eigenvalues of the quantities relevant in the linearized theory, including the eigenvalue of the metric above, should be smaller than this required precision.

### 2.4. The map $\mathcal{M}$

In the 'metric representation', quantum states of the gravitational field are formally represented by functionals $\Psi[g]$ of the three-dimensional metric. In QLGR, one may represent quantum states by functionals $\psi[h]$ of the linearized metric field $h_{a b}=g_{a b}-\delta_{a b}$ (this representation is equivalent to the common Fock basis representation). The relation between these two representations is straightforward: a state $\psi$ of QLGR physically corresponds to a state $\Psi$ of QGR, where $\Psi[g]=\psi[g-\delta]$. This relation establishes a linear map, which we may denote as map $\mathcal{M}$, between the state space of QGR and the state space of QLGR. The expectation value of a linearized variable, as for instance $h_{a b}$, in the state $\psi$, and the expectation value of an exact theory variable, as $g_{a b}$, in the corresponding state $\Psi$, are related by the correct classical relation ( $h=g-\eta$ ) that relates the two variables in the classical theory.

In the loop formalism used here, on the other side, the relation between QGR and QLGR is more complicated, owing to a number of reasons. The first reason is that the loop rep is not obtained by means of functions on a configuration space. Imagine we want to establish the linear relation between the state space of an anharmonic oscillator, expressed in the energy basis $|N\rangle$, and the state space of a harmonic oscillator (seen as an approximation to the anharmonic oscillator dynamics), again expressed in the energy basis $|n\rangle$. The relation between the two state spaces $\left(|N\rangle=\sum_{n} C_{N}^{n}|n\rangle\right.$ ) is non-trivial: we must be able to solve the dynamics in order to find its explicit form. Similarly, in the loop rep we need to control the eigenvalue problem of the $T$ operators in order to find the explicit relation between the state spaces of QLGR and QGR.

The second reason is that the loop formalism is invariant under the internal gauge transformations of the Ashtekar variables, but the internal gauge group of QLGR, which is $U(1)^{3}$, is different from the internal gauge group of QGR, which is $S O(3)$. The variables of QLGR that are invariant under $U(1)^{3}$ are not invariant under $S O(3)$. More precisely, they are invariant under $S O$ (3) only to first order in the expansion for small fields. Therefore the very quantities on which QLGR is constructed lose their meaning as variables in the loop rep of $Q G R$ beyond first order in a small fields expansion. If we set a small positive number $\varepsilon$ as the precision to which we want our approximation to hold, and we restrict the range
of the dynamical variables by $\left|h^{a b}(x)\right|<\varepsilon$ and $\left|t_{r}^{i}(\alpha)-1\right|<\varepsilon$, then the classical variables of QLGR and QGR are related by

$$
\begin{align*}
& T[\alpha]=1 \\
& \left(1-\delta_{a}^{i} \oint \mathrm{~d} s \int \mathrm{~d}^{3} x f_{r}(x) T^{a}[\alpha+x](s)\right)=t_{r}^{i}(\alpha)  \tag{2.4.1}\\
& Q(\omega)=\left(\int \mathrm{d}^{3} x|\omega|+\frac{1}{2} \int \mathrm{~d}^{3} x h^{a b} \omega_{a} \omega_{b}|\omega|^{-1}\right)
\end{align*}
$$

up to an error of order $\varepsilon^{2}$. This error cannot be avoided within the present formalism, because the two theories are invariant under different gauge groups. Since the relation between the state spaces of the two theories is determined by the relation between the respective variables, it follows that we can determine the relation between the two theories only to first order in $\varepsilon$. This fact, however, does not prevent us from looking at QLGR as an approximation of QGR, since QLGR can be a good approximation of QGR only to linear order in a small field expansion.

In the quantum theory the magnitude of the quantum fluctuations $\Delta t_{r}^{i}(\alpha)$ and $\Delta h^{a b}(x)$ around the flat metric is determined by the Heisenberg indeterminacy relations. If we want to say that the metric is flat within a precision $\varepsilon$, then we need $\Delta t_{r}^{i}(\alpha)<\varepsilon$ and $\Delta h^{a b}(x)<\varepsilon$. This requirement forces us to restrict the parameters that determine the variables in the theory, as follows: $|\alpha|<l_{p} / \varepsilon$ and $r>l_{p} / \varepsilon$. Here $|\alpha|$ is the length of a loop $\alpha$ and $r$ is the scale of a smearing function involved in QLGR. The restriction of the lengths of loops does not mean any restriction of physical information coded in state functionals, since the relevant information is already contained in the limit in which the loops shrink to zero. The restriction of $|\alpha|$ is, however, necessary to impose one of the linearization conditions, $\left|t_{r}^{i}(\alpha)-1\right|<\varepsilon$, in the loop formalism and to define the map $\mathcal{M}$ unambiguously such that the low-energy sector of the state space of QGR is transformed to the low-energy sector of the state space of QLGR, and not to violate the Heisenberg indeterminacy relations. The restriction of $r$ does mean a restriction to the low-energy sector of physical informations determined by the precision $\varepsilon$ imposed. The low-energy sector of operators and state functionals may include only such loops and scales except the weave $\Delta$. The low-energy sector of the state space of QGR, denoted by $H^{0}$, is defined by

$$
\|(\widehat{T}[\alpha]-1) \Psi\|<\varepsilon^{2} \quad \text { and } \quad\left\|\int \mathrm{d}^{3} x f_{r}(x) \widehat{T}^{a}[\alpha+x](s) \Psi\right\|<\varepsilon
$$

The low-energy sector of the state space of QLGR, denoted by $h^{0}$, is defined by $\dagger$

$$
\left\|\left(\hat{t}_{r}^{i}(\alpha)-1\right) \psi\right\|<\varepsilon \text { and }\left\|\hat{h}^{a \mathrm{i}}(x) \psi\right\|<\varepsilon
$$

(A discussion of the norms we use is given in [4].) Physically, these represent the sectors of the two theories that give the same predictions, to order $\varepsilon$. The map $\mathcal{M}$ sends states in $H^{0}$ to states in $h^{0}$; it is given by [4]

$$
\begin{equation*}
\langle\vec{\alpha}| \mathcal{M}=\langle\Delta|-\frac{1}{2} l_{p}^{2} \sum_{\substack{i=1,2,3 \\ \sigma x i,-1}} \sigma \oint \mathrm{~d} s \oint \mathrm{~d} t \dot{\Delta}^{i}(t) f_{r}\left(\Delta(t)-\alpha^{i}(s)\right)\left\langle\Delta \#\left(\alpha_{s, t}^{i}\right)^{\sigma}\right|+\mathrm{O}\left(\varepsilon^{2}\right) \tag{2.4.2}
\end{equation*}
$$

$\dagger$ Equation (4.4.1) in [4] contains a (quite unfortunate) typing mistake. The corrected version is the one given here.
where $\alpha_{s, t}^{i}:=\alpha^{i}+\Delta(t)-\alpha^{i}(s)$. In this paper, all loops are parameterized so that $\oint \mathrm{d} s=1$. The map is defined only up to terms of order $\varepsilon^{2}$. Note that the map depends on the weave $\Delta$ and a different choice of weave, on which another map is based, makes a difference only within an error of order $\varepsilon^{2}$. The essential property of the map $\mathcal{M}$ (from which it has been determined) is that it relates corresponding operators in QGR and QLGR, up to an error of order $\varepsilon^{2}$; that is,

$$
\begin{align*}
& \mathcal{M} \widehat{T}[\alpha]=1 \mathcal{M} \\
& \mathcal{M}\left(1-\delta_{a}^{i} \oint \mathrm{~d} s \int \mathrm{~d}^{3} x f_{r}(x) \widehat{T}^{a}[\alpha+x](s)\right)=\hat{t}_{r}^{i}(\alpha) \mathcal{M}  \tag{2.4.3}\\
& \mathcal{M} \widehat{Q}(\omega)=\left(\int \mathrm{d}^{3} x|\omega|+\frac{1}{2} \int \mathrm{~d}^{3} x \hat{h}^{a b} \omega_{a} \omega_{b}|\omega|^{-1}\right) \mathcal{M}
\end{align*}
$$

to be compared with equations (2.4.1). Here $\alpha$ and $r$ satisfy the conditions described above and $\omega_{a}$ is a one-form slowly varying over a scale $l_{p} / \varepsilon^{2}$ so that equation (2.3.1) is used without bothering calculations of terms of order $\varepsilon$ of quantum fluctuations. In general, given an operator $\hat{o}$ in QLGR, we may find the corresponding operator $\hat{O}$ in QGR using

$$
\begin{equation*}
\mathcal{M} \hat{O}=\hat{o} \mathcal{M} \tag{2.4.4}
\end{equation*}
$$

We will use this relation to translate between the two theories.

## 3. Operators

We now begin using the map $\mathcal{M}$, or, more precisely, equation (2.4.4), to transfer the linearized operators from QLGR to QGR. First, we find the expressions for the linearized constraints in QGR, namely we transfer the linearized vector and scalar constraints $\hat{V}_{a}^{L}$ and $\hat{\mathcal{S}}^{L}$, to the QGR state space. We indicate the resulting two operators as $\hat{\mathcal{V}}_{a}$ and $\hat{\mathcal{S}}$, and we refer to them as 'low frequency vector and scalar constraints'. The reason for this denomination will be clear below. They are defined by

$$
\begin{equation*}
\mathcal{M} \hat{V}_{a}=\hat{V}_{a}^{L} \mathcal{M} \quad \text { and } \quad \mathcal{M} \hat{S}=\hat{\mathcal{S}}^{L} \mathcal{M} \tag{3.1}
\end{equation*}
$$

These low frequency constraints are defined on the state space of QGR, but they describe (together with the Hamiltonian) the low energy physics.

Next, we consider the dynamics. Here, however, we find a difficulty, owing to the Hamiltonian structure of the quantum theory: while the linearized equations of motion are, by definition, linear in the dynamical variables, and therefore in $\varepsilon$, the 'linearized' Hamiltonian $H_{0}$, on the other hand, is of course quadratic in the dynamical variables; and therefore it is of order $\varepsilon^{2}$. The linearized Hamiltonian operator $\widehat{H}_{0}$ in QGR cannot be obtained from the Hamiltonian operator of QLGR $\hat{h}$, by using $\mathcal{M} \widehat{H}_{0}=\hat{h} \mathcal{M}$, because the Hamiltonian is of second order in $\varepsilon$, while $\mathcal{M}$ is determined only to first order in $\varepsilon$.

The very same problem arises if we try to map directly the eigenstates of the linearized Hamiltonian, namely the vacuum and the graviton states, from one state space to the other. To see this by means of an analogy, consider the linearization of an anharmonic oscillator: to first order in the position $x$, and under suitable conditions, the classical as well as the quantum dynamics are approximated by means of the harmonic oscillator dynamics.

However, we clearly cannot take the harmonic oscillator ground state, $\psi(x)=C \exp \left[-\sigma x^{2}\right]$, expand in $x$, keep only terms of first order in $x$, and assume that these give a good approximation to the ground state of the exact theory.

In order to circumvent this problem, we need some way of bringing the information about the linearized dynamics from QLGR to QGR, using only operators of order $\varepsilon$. The idea that we shall follow is to use the creation and annihilation operators. These, unlike the Hamiltonian, are of first order in $\varepsilon$. Therefore we may use the map $\mathcal{M}$ to transport them from QLGR to QGR, without losing relevant information. In section 3.4, we find the expressions for the creation and annihilation operators in the QGR state space. These operators will allow us, in section 4, to find the states annihilated by the low-frequency constraints, the Poincare-invariant vacuum and the graviton states.

### 3.1. The low-frequency vector constraint

As a preliminary step, we derive the operator $\widehat{D}^{a b}(x)$ that corresponds to the operator $G \widehat{B}^{a b}(x)$ of QLGR. From $\mathcal{M} \widehat{D}^{a b}(x)=G \widehat{B} \widehat{B}^{a b}(x) \mathcal{M}$, it is easy to verify that

$$
\begin{equation*}
\widehat{D}^{a b}(x)=-\lim _{\delta \rightarrow 0} \frac{1}{\pi \delta^{2}} \oint \mathrm{~d} s \int \mathrm{~d}^{3} y f_{r}(y) \widehat{T}^{b}\left[\gamma_{x, \delta}^{a}+y\right](s) \tag{3.1.1}
\end{equation*}
$$

This is a kind of 'differential' operator.
The low-frequency vector constraint operator $\widehat{\mathcal{V}}_{a}$ in QGR, which corresponds to the one in QLGR and is determined by equation (3.1), is then

$$
\begin{equation*}
\widehat{\mathcal{V}}_{a}(x)=-\epsilon_{a b c} \widehat{D}^{b c}(x) \tag{3.1.2}
\end{equation*}
$$

Its action is

$$
\begin{align*}
\langle\beta| \widehat{\mathcal{V}}_{a}(x)=-l_{P}^{2} \epsilon_{a b c} \oint \mathrm{~d} s \oint \mathrm{~d} t \dot{\beta}^{b}(t) \lim _{\delta \rightarrow 0} \frac{1}{\pi \delta^{2}} f_{r}\left(\beta(t)-\gamma_{x, \delta}^{c}(s)\right) \\
\times\left(\left\langle\beta \#\left(\gamma_{x, \delta}^{c}\right)_{s, \beta(t)}\right|-\left\langle\beta \cup\left(\gamma_{x, \delta}^{c}\right)_{s, \beta(t)}\right|\right) \tag{3.1.3}
\end{align*}
$$

where $\gamma_{s, \beta(t)}:=\gamma+\beta(t)-\gamma(s)$.
From equations (3.1.1) and (3.1.2), we obtain

$$
\begin{equation*}
\widehat{V}_{a}(x)=\int \mathrm{d}^{3} y f_{r}(x-y) \widehat{C}_{a}(y) \tag{3.1.4}
\end{equation*}
$$

where $\widehat{C}_{a}(x)$ is the exact (non-linearized) vector constraint [1]. Thus, the 'low frequency constraint' that we have constructed differs from the exact vector constraint of QGR only by virtue of the smearing at the scale $r$ produced by the smearing function $f_{r}$. At first sight, this result is surprising, because it implies that the exact quantum vector constraint equation

$$
\begin{equation*}
\widehat{C}_{a}(x) \Psi=0 \tag{3.1.5}
\end{equation*}
$$

and the equation

$$
\begin{equation*}
\widehat{V}_{a}(x) \Psi=0 \tag{3.1.6}
\end{equation*}
$$

are equivalent, as can be easily verified by smearing these two equations with arbitrary vector fields. However, we must recall that the low-frequency theory operators and equations that
we are obtaining make sense only to order $\varepsilon$; the map $\mathcal{M}$ itself, indeed, makes sense only to this order. Thus the low-frequency quantum vector constraint equation is not equation (3.1.6), but rather

$$
\begin{equation*}
\left\|\widehat{\mathcal{V}}_{a}(x) \Psi\right\|<\varepsilon^{2} \tag{3.1.7}
\end{equation*}
$$

(we refer to [4] for a discussion of the norm we use). Indeed, we will discard terms small in $\varepsilon^{2}$ in computing the solutions of this equation in section 4.1. Thanks to this $\varepsilon$ approximation, the effect of the smearing function $f_{r}$ in $\widehat{V}_{a}(x)$ is then essentially to cut off the Fourier components of $\widehat{C}_{a}(x)$ that have frequency much higher than $1 / r$. (We recall that we have preferred Euclidean coordinates determined by a background flat metric.) More precisely, if we Fourier transform equation (3.1.7) we obtain

$$
\begin{equation*}
\left\|\widehat{C}_{n}(k) \Psi\right\|<\mathrm{e}^{+r^{2} k^{2}} \varepsilon^{2} \tag{3.1.8}
\end{equation*}
$$

where $l_{p}$ is the Planck length and $k$ is the momentum variable. This equation shows that to first order in $\varepsilon$ the low frequency constraint corresponds to the exact vector constraints with the high frequencies (with respect to the assumed background metric) cutoff.

It follows that a knot functional, namely a solution of the exact equation (3.1.5), also solves the low frequency constraint, equation (3.1.6); but the converse is not true. Thus, the low-frequency theory that we are defining on the QGR state space allows also states that are not knot functionals, but differ from a knot functional 'at high frequency'. This means that a loop functional $\Psi(\alpha)$ which solves the low frequency vector constraint does not change for a 'very smooth' diffeomorphism on $\alpha$, that is a diffeomorphism generated by a vector field without high frequencies (with respect to the background metric), but may change for a 'very rough' diffeomorphism on $\alpha$.

Since the scale $r$ is much larger than the weave scale, we may visualize these low frequency diffeomorphisms as the ones that move the weave around without 'destroying' it: loops that are close to each other at the weave scale remain close to each other under low frequency diffeomorphisms. $\dagger$

We also note that a functional depending on a loop shorter than $r$ is not in general (the functionals $G^{ \pm}$, which will be introduced in section 4.1, are special cases) low-frequency diffeomorphism invariant in the same sense that a sharply peaked function in the space (e.g. a delta function) has a variety of momentum components in its Fourier spectrum. To be invariant, the loop must be uniformly distributed over the space so that its low-frequency components can be clarified and ignored by the state functional. Such structure is allowed only in the weave in this construction.

We leave the study of the precise geometrical meaning of this result for future investigations; for the moment, we simply note that the result is consistent with the fact that QLGR becomes a bad description of the gravitational field at high frequencies.

### 3.2. The low-frequency scalar constraint

The low-frequency scalar constraint operator $\widehat{\mathcal{S}}$ in QGR, which corresponds to the one in QLGR and is determined by equation (3.1), is

$$
\begin{equation*}
\widehat{\mathcal{S}}(x)=-\delta_{a b} \widehat{D}^{a b}(x) \tag{3.2.1}
\end{equation*}
$$

[^1]Its action is

$$
\begin{align*}
\langle\beta| \widehat{S}(x)=l_{p}^{2} \delta_{a b} & \oint \mathrm{~d} s \oint \mathrm{~d} t \dot{\beta}^{a}(t) \lim _{\delta \rightarrow 0} \frac{1}{\pi \delta^{2}} f_{r}\left(\beta(t)-\gamma_{x, \delta}^{b}(s)\right) \\
& \times\left(\left\langle\beta \#\left(\gamma_{x, \delta}^{b}\right)_{s, \beta(t)}\right|-\left\langle\beta \cup\left(\gamma_{x, \delta}^{b}\right\rangle_{s, \beta(t)}\right|\right) \tag{3.2.2}
\end{align*}
$$

Note that the low-frequency scalar constraint cannot be recovered by simply smearing the exact scalar constraint by the smearing function $f_{r}$, in contrast to the case of the vector constraint. At present, the relation to the exact scalar constraint is not clear. Some light is partially obtained by considering the following fact: The classical constraint $\mathcal{S}(x)$ corresponding to equation (3.2.1) is equivalent (up to terms of order $\varepsilon^{2}$ ) to

$$
\begin{gather*}
\mathcal{S}^{\prime}(x)=\lim _{\delta \rightarrow 0} \frac{1}{\pi \delta^{2}} \frac{1}{4} \epsilon_{a b c} \oint \mathrm{~d} s \int \mathrm{~d}^{3} y f_{r}(y) T^{[a b]}\left[\gamma_{x, \delta}^{c}+y\right]\left(s, s+\delta^{2}\right) \\
=\int \mathrm{d}^{3} y f_{r}(x-y) C(y) \tag{3.2.3}
\end{gather*}
$$

where $C(x)$ is the exact (classical) scalar constraint. Thus, $\mathcal{S}(x)$ would be seen as a lowfrequency scalar constraint in the spirit described in the previous subsection. To show the equivalence as quantum operators, more careful considerations are still needed.

In this paper, we shall consider states annihilated by the low-frequency vector and scalar constraints. These are relevant in graviton physics, even though they are unphysical at the Planck scale. In a sense, they are 'physical states at large distances'. We expect that they could be suitably modified at short distance in order to become exact solutions of the exact constraints.

### 3.3. The 'polarization' vectors in position space

Before discussing the creation and annihilation operators, we introduce in this subsection a triplet of covectors $\omega_{a}^{i}(x)(i=1,2,3)$ in position space that will play an important role in the following. They are, in a sense, 'polarization vectors in position space', as $m_{a}(k)$ are polarization vectors in momentum space. First, we consider the integral

$$
\begin{equation*}
I_{a b}(x)=\frac{1}{(2 \pi)^{3}} \int \mathrm{~d}^{3} k|k|^{n} m_{a}(k) m_{b}(k) \mathrm{e}^{\mathrm{i} k \cdot x} \tag{3.3.1}
\end{equation*}
$$

where $n$ is determined in such a way that the integral exists. We recall that $m_{a}(k)$ can be determined up to a phase factor $\mathrm{e}^{\mathrm{i} \lambda(k)}$, where $\lambda(k)$ is a real scalar odd function of $k$. Thus the integral is not unique. For a particular choice of cartesian coordinates and $m_{a}(k)$, which is

$$
m_{a}(k)=\frac{1}{\sqrt{2}}\left(\frac{\partial \hat{k}_{a}}{\partial \theta}+\frac{\mathrm{i}}{\sin \theta} \frac{\partial \hat{k}_{a}}{\partial \phi}\right)
$$

with

$$
\hat{k}=(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)
$$

it turns out that the integral exists if $n$ is such that $-3<n<-\frac{3}{2}$. We choose $n=-2$ to show the existence of the integral. Straightforward calculations show that the components of $I_{a b}(x)$ are

$$
\begin{align*}
& I_{11}(x)=\frac{1}{4 \pi^{2}|x|}\left(-\frac{\pi}{8}\left(1+\cos ^{2} \theta\right)-\frac{\cos 2 \phi}{\sin ^{2} \theta} A(\theta)+\frac{\sin 2 \phi}{\sin ^{2} \theta} B(\theta)\right) \\
& I_{22}(x)=\frac{1}{4 \pi^{2}|x|}\left(-\frac{\pi}{8}\left(1+\cos ^{2} \theta\right)+\frac{\cos 2 \phi}{\sin ^{2} \theta} A(\theta)-\frac{\sin 2 \phi}{\sin ^{2} \theta} B(\theta)\right) \\
& I_{33}(x)=\frac{1}{4 \pi^{2}|x|}\left(\frac{\pi}{4}\left(1+\cos ^{2} \theta\right)\right)  \tag{3.3.2}\\
& I_{12}(x)=I_{21}(x)=\frac{1}{4 \pi^{2}|x|}\left(-\frac{\sin 2 \phi}{\sin ^{2} \theta} A(\theta)-\frac{\cos 2 \phi}{\sin ^{2} \theta} B(\theta)\right) \\
& I_{13}(x)=I_{31}(x)=\frac{1}{4 \pi^{2}|x|}\left(\frac{\pi}{8} \sin 2 \theta \cos \phi-\sin \theta \sin \phi\right) \\
& I_{23}(x)=I_{32}(x)=\frac{1}{4 \pi^{2}|x|}\left(\frac{\pi}{8} \sin 2 \theta \sin \phi+\sin \theta \cos \phi\right)
\end{align*}
$$

where $A(\theta)$ and $B(\theta)$ are given by

$$
\begin{align*}
& A(\theta)=\frac{\pi}{8}\left(3+6 \cos ^{2} \theta-\cos ^{4} \theta-8|\cos \theta|\right)  \tag{3.3.3}\\
& B(\theta)=\cos \theta\left[\sin ^{2} \theta+\ln \left(\cos ^{2} \theta\right)\right]
\end{align*}
$$

they have the properties

$$
\begin{array}{llllll}
\frac{A(\theta)}{\sin ^{2} \theta} \rightarrow 0 & \text { and } & \frac{B(\theta)}{\sin ^{2} \theta} \rightarrow 0 & \text { as } & \theta \rightarrow 0 & \text { or } \quad \theta \rightarrow \pi \\
A(\theta) \rightarrow \frac{3 \pi}{8} & \text { and } & B(\theta) \rightarrow 0 & \text { as } & \theta \rightarrow \frac{\pi}{2} .
\end{array}
$$

Here $x$ is denoted by $x=(|x| \sin \theta \cos \phi,|x| \sin \theta \sin \phi,|x| \cos \theta)$, and should not be confused with $\hat{k}$, which was fixed above. These components are finite and well defined at all points $x$ except the origin. Note that $I_{a b}(x)$ is real, symmetric, traceless, divergenceless, consistently with its integral expression (3.3.1), and its determinant is always positive. The positiveness of the determinant guarantees that the sign of the eigenvalues of $I_{a b}(x)$ do not change by changing the position $x$, which is crucial in defining the $x$-independent $v_{i}$ below.

Next, we define three real orthogonal eigenvectors $\omega_{a}^{i}(x)(i=1,2,3)$ of $I_{a b}(x)$ with $\sum_{i=1}^{3} v_{i}\left|\omega^{i}(x)\right|=0$, where $v_{i}$ is the sign of the eigenvalue corresponding to the $i$ th eigenvector, in other words, $\frac{1}{2} v_{i}\left|\omega^{i}(x)\right|$ is the corresponding eigenvalue, such that

$$
\begin{equation*}
I_{a b}(x)=\frac{1}{2} \sum_{i=1}^{3} v_{i} \frac{\omega_{a}^{i}(x) \omega_{b}^{i}(x)}{\left|\omega^{i}(x)\right|} \tag{3.3.4}
\end{equation*}
$$

Here $\left|\omega^{i}(x)\right|:=\left[\omega_{a}^{i}(x) \omega^{i a}(x)\right]^{1 / 2}$ is the norm of $\omega_{a}^{i}(x)$ and $\frac{1}{2}$ is set for later convenience. Note that $v_{i}$ are independent of $x$ as mentioned above and $v_{1} v_{2} v_{3}=+1$. Note also that
$\omega_{a}^{i}(x)$ are not unique in the same sense that $m_{a}(k)$ is not unique as mentioned above. The three vectors, for the specific choice of $I_{a b}$ above, are given by
$\omega_{a}^{i}(x)=\frac{C_{i}(\theta)}{4 \pi^{2}|x|}\left(X_{i}(\theta) \cos \phi-Y_{i}(\theta) \sin \phi, X_{i}(\theta) \sin \phi+Y_{i}(\theta) \cos \phi, Z_{i}(\theta)\right)$
where $C_{i}(\theta)$ is a function of $\theta$ such that $\frac{1}{2} v_{i} C_{i}(\theta)$ divided by $4 \pi^{2}|x|$ is the $i$ th eigenvalue of $I_{a b}$ (therefore $\sum_{i=1}^{3} v_{i} C_{i}(\theta)=0$ ), and $X_{i}(\theta), Y_{i}(\theta)$ and $Z_{i}(\theta)$ are functions of $\theta$ satisfying

$$
\begin{aligned}
& X_{i} X_{j}+Y_{i} Y_{j}+Z_{i} Z_{j}=\delta_{i j} \\
& \sum_{i=1}^{3} \frac{1}{2} v_{i} C_{i}(\theta) X_{i}^{2}(\theta)=-\frac{\pi}{8}\left(1+\cos ^{2} \theta\right)-\frac{A(\theta)}{\sin ^{2} \theta} \\
& \sum_{i=1}^{3} \frac{1}{2} v_{i} C_{i}(\theta) Y_{i}^{2}(\theta)=-\frac{\pi}{8}\left(1+\cos ^{2} \theta\right)+\frac{A(\theta)}{\sin ^{2} \theta} \\
& \sum_{i=1}^{3} \frac{1}{2} v_{i} C_{i}(\theta) Z_{i}^{2}(\theta)=\frac{\pi}{4}\left(1+\cos ^{2} \theta\right) \\
& \sum_{i=1}^{3} \frac{1}{2} v_{i} C_{i}(\theta) X_{i}(\theta) Y_{i}(\theta)=-\frac{B(\theta)}{\sin ^{2} \theta} \\
& \sum_{i=1}^{3} \frac{1}{2} v_{i} C_{i}(\theta) X_{i}(\theta) Z_{i}(\theta)=\frac{\pi}{8} \sin 2 \theta \\
& \sum_{i=1}^{3} \frac{1}{2} v_{i} C_{i}(\theta) Y_{i}(\theta) Z_{i}(\theta)=\sin \theta
\end{aligned}
$$

For example, at $x=(|x| \cos \phi,|x| \sin \phi, 0)$

$$
\begin{array}{ll}
\omega_{a}^{1}(x)=\frac{1}{4 \pi|x|}(\cos \phi, \sin \phi, 0) & \text { with } \nu_{1}=-1 \\
\omega_{a}^{2}(x)=\frac{1}{8 \pi^{2}|x|} \frac{4-\pi}{\sqrt{2}}(-\sin \phi, \cos \phi,-1) & \text { with } \nu_{2}=-1 \\
\omega_{a}^{3}(x)=\frac{1}{8 \pi^{2}|x|} \frac{4+\pi}{\sqrt{2}}(-\sin \phi, \cos \phi,+1) & \text { with } \nu_{3}=+1
\end{array}
$$

and, at $x=(0,0, \pm|x|)$

$$
\begin{array}{lll}
\omega_{a}^{1}(x)=\frac{1}{8 \pi|x|}(1,0,0) & \text { with } & v_{1}=-1 \\
\omega_{a}^{2}(x)=\frac{1}{8 \pi|x|}(0,1,0) & \text { with } & v_{2}=-1 \\
\omega_{a}^{3}(x)=\frac{1}{4 \pi|x|}(0,0,1) & \text { with } & v_{3}=+1
\end{array}
$$

Using $\omega_{a}^{i}$, we now define our main technical tool. We want functions that vary slowly and approximate $\omega_{\alpha}^{i}$ on a scale $R=l_{p} / \varepsilon^{2}$. To this purpose we define

$$
\begin{equation*}
\omega_{a}^{ \pm \mathrm{i}}(y)=\int \mathrm{d}^{3} x f_{R}(x) \omega_{a}^{i}( \pm(y-x)) \tag{3.3.6}
\end{equation*}
$$

Below we always use $\omega_{a}^{ \pm \mathrm{i}}$ in place of $\omega_{a}^{i}$.

### 3.4. The creation and annihilation operators and the equations for the Poincare-invariant vacuum

We denote by $\widehat{A}_{ \pm}(k)$ and $\widehat{A}_{ \pm}^{\dagger}(k)$ the QGR operators that represent the annihilation and creation operators for the two polarizations. These are defined by $\mathcal{M} \widehat{A}_{ \pm}(k)=\hat{a}_{ \pm}(k) \mathcal{M}$ and $\mathcal{M} \widehat{A}_{ \pm}^{\dagger}(k)=\hat{a}_{ \pm}^{\dagger}(k) \mathcal{M}$, where the QLGR annihilation and creation operators, $\hat{a}_{ \pm}(k)$ and $\hat{a}_{ \pm}^{\dagger}(k)$, are given in equations (2.2.1).

To find $\widehat{A}_{ \pm}(k)$ and $\widehat{A}_{ \pm}^{\dagger}(k)$, we have to find the QGR operators that correspond both to $\widehat{B}^{ \pm}$and to $\hat{h}^{ \pm}$. The operator corresponding to $\widehat{B}^{ \pm}$is given by the transverse components of the Fourier transform of the operator $\widehat{D}^{a b}(x)$, defined in equation (3.1.1), that is,

$$
\begin{align*}
& \widehat{D}^{+}(k)=\bar{m}_{a}(k) \bar{m}_{b}(k)(2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} x \mathrm{e}^{-\mathrm{i} k \cdot x} \widehat{D}^{a b}(x) \\
& \widehat{D}^{-}(k)=m_{a}(k) m_{b}(k)(2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} x \mathrm{e}^{-\mathrm{i} k \cdot x} \widehat{D}^{a b}(x) \tag{3.4.1}
\end{align*}
$$

Thus, the problem is to find the operator corresponding to $\hat{h}^{ \pm}$. We denote it as $\widehat{H}^{ \pm}$; namely: $\mathcal{M} \widehat{H}^{ \pm}(x)=\hat{h}^{ \pm}(x) \mathcal{M}$. The difficulty is caused by the fact that we do not have an operator corresponding directly to the metric in QGR. In QGR, we have the operator $\widehat{Q}$ (section 2.1), which is obtained by smearing the metric in position space with the square of a one-form. On the other side, $\hat{h}^{ \pm}$are obtained by smearing the metric, in momentum space, with the polarization vectors. To go around this difficulty, we use the 'position space polarization vectors' introduced in the previous subsection. We define $\widehat{H}^{ \pm}(x)$ as the Fourier transform of $|k|^{n} \widehat{H}^{ \pm}(k)$ (note the insertion of $|k|^{n}$ ), and we write:

$$
\begin{aligned}
\mathcal{M} \widehat{H}^{ \pm}(-y)= & (2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} k \mathrm{e}^{\mathrm{i} k \cdot y} \mathcal{M}|k|^{n} \widehat{H}^{ \pm}(-k) \\
& =(2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} k \mathrm{e}^{\mathrm{i} k \cdot y}|k|^{n} \hat{h}^{ \pm}(-k) \mathcal{M} \\
& =(2 \pi)^{-3} \int \mathrm{~d}^{3} k \mathrm{e}^{\mathrm{i} k \cdot y}|k|^{n} m_{a}( \pm k) m_{b}( \pm k) \int \mathrm{d}^{3} x \mathrm{e}^{-\mathrm{i} k \cdot x} \hat{h}^{a b}(-x) \mathcal{M} \\
& =\int \mathrm{d}^{3} x\left((2 \pi)^{-3} \int \mathrm{~d}^{3} k|k|^{n} m_{a}(k) m_{b}(k) \mathrm{e}^{\mathrm{i} k \cdot( \pm(y+x))}\right) \hat{h}^{a b}(x) \mathcal{M}
\end{aligned}
$$

Now we make use of the technology developed in the previous section: using equations (3.3.1) and (3.3.4), we have

$$
\mathcal{M} \widehat{H}^{ \pm}(-y)=\sum_{i=1}^{3} \nu_{i} \int \mathrm{~d}^{3} x \frac{1}{2} \frac{\omega_{a}^{i}( \pm(y+x)) \omega_{b}^{i}( \pm(y+x))}{\left|\omega^{i}( \pm(y+x))\right|} \hat{h}^{a b}(x) \mathcal{M}
$$

To utilize equation (2.4.3), replace $\omega_{a}^{i}( \pm(y+x))$ by $\omega_{a}^{ \pm i}(y+x)$. This process does not change large scale structures coded in $\omega_{a}^{i}$. Then, by using the last of equations (2.4.3), we find

$$
\widehat{H}^{ \pm}(-y)=\sum_{i=1}^{3} v_{i}\left(\widehat{Q}\left(\omega^{ \pm i}(y+\ldots)\right)-\int \mathrm{d}^{3} x\left|\omega^{ \pm i}(x)\right|\right)
$$

where $\omega^{ \pm i}(y+\ldots)$ means $\omega^{ \pm i}(y+x)$ as a function of $x$, and hence $\widehat{Q}\left(\omega^{ \pm i}(y+\ldots)\right)$ is a functional of $\omega^{ \pm i}(y+\ldots)$. Noticing that the second term vanishes because of $\sum_{i=1}^{3} \nu_{i}\left|\omega^{ \pm \mathrm{i}}(x)\right|=0$, we have

$$
\begin{equation*}
\widehat{H}^{ \pm}(-y)=\sum_{i=1}^{3} \nu_{i} \widehat{Q}\left(\omega^{ \pm i}(y+\ldots)\right) \tag{3.4.2}
\end{equation*}
$$

This is the operator that we were searching for. Its explicit action on a state close to the weave is given as follows. Appiying $\widehat{H}^{ \pm}(-y)$ on $\langle\Delta \# \alpha|$ and using the last of equations (2.1.1), we have

$$
\begin{equation*}
\langle\Delta \# \alpha| \widehat{H}^{ \pm}(-y)=-2 l_{p}^{2} \mathcal{F}^{ \pm}[-y, \Delta \# \alpha]\langle\Delta \# \alpha| \tag{3.4.3}
\end{equation*}
$$

where

$$
\begin{align*}
\mathcal{F}^{ \pm}[-y, \Delta \# \alpha] & \left.=-\frac{1}{4} \sum_{i=1}^{3} v_{i} \int \mathrm{~d}^{3} x \right\rvert\, \oint \mathrm{d} u \dot{\Delta}^{b}(u) \delta^{3}(x, \Delta(u)) \omega_{b}^{ \pm i}(y+x) \\
& +\oint \mathrm{d} s \dot{\alpha}^{a}(s) \delta^{3}(x, \alpha(s)) \omega_{a}^{ \pm i}(y+x) \mid \tag{3.4.4}
\end{align*}
$$

Now we have all the ingredients for writing the creation operators:

$$
\begin{align*}
& \widehat{A}_{+}^{\dagger}(k)=\left(2 l_{p}^{2}\right)^{-1 / 2}|k|^{-3 / 2} \widehat{D}^{+}(k) \\
& \widehat{A}_{-}^{\dagger}(k)=-\left(2 l_{p}^{2}\right)^{-1 / 2}\left(|k|^{1 / 2} \widehat{H}^{-}(-k)+|k|^{-3 / 2} \widehat{D}^{-}(-k)\right) \tag{3.4.5}
\end{align*}
$$

and the annihilation operators:

$$
\begin{align*}
& \widehat{A}_{+}(k)=-\left(2 l_{p}^{2}\right)^{-1 / 2}\left(|k|^{1 / 2} \widehat{H}^{+}(-k)+|k|^{-3 / 2} \widehat{D}^{+}(-k)\right)  \tag{3.4.6}\\
& \widehat{A}_{-}(k)=\left(2 l_{p}^{2}\right)^{-1 / 2}|k|^{-3 / 2} \widehat{D}^{-}(k)
\end{align*}
$$

The differential equations for the Poincaré-invariant vacuum state $\left|\Psi_{0}\right\rangle$ can be obtained by applying the annihilation operators times $|k|^{n-\frac{1}{2}}$ on the vacuum state. They are

$$
\begin{align*}
& (2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} k \mathrm{e}^{\mathrm{i} k \cdot y}|k|^{n-2}\langle\Delta \# \alpha| \widehat{D}^{+}(-k)\left|\Psi_{0}\right\rangle=2 t_{p}^{2} \mathcal{F}^{+}[-y, \Delta \# \alpha]\left\langle\Delta \# \alpha \mid \Psi_{0}\right\rangle  \tag{3.4.7}\\
& \langle\Delta \# \alpha| \widehat{D}^{-}(k)\left|\Psi_{0}\right\rangle=0 .
\end{align*}
$$

We shall solve these equations in section 4.2.

## 4. States

In the previous section, we have found the operators that represent the linearized constraints and the creation and annihilation operators on the state space of the non-perturbative quantum theory ( $Q G R$ ). Here we solve the constraint equations and we find the Poincare invariant vacuum and the graviton states.

### 4.1. Low-frequency constraint-invariant states

In QLGR, the form factor loop functions $F_{r}^{ \pm}[k, \vec{\alpha}]$, are the solutions of the constraint equations. We now construct a corresponding function $G^{ \pm}[k, \beta]$ in the QGR state space. Since $F_{r}^{ \pm}[k, \vec{\alpha}]$ is of the same order as $\hat{h}^{ \pm}(k)$, that is, linear in $\varepsilon, G^{ \pm}[k, \beta]$ must be so. Therefore we can utilize the map $\mathcal{M}$ to find $G^{ \pm}[k, \beta]$. Using the map $\mathcal{M}, F_{r}^{ \pm}[k, \vec{\alpha}]$ can be expressed in terms of $G^{ \pm}[k, \beta]$, by $F_{r}^{ \pm}=\mathcal{M} G^{ \pm}$. From this, we obtain

$$
\begin{align*}
& F_{r}^{ \pm}[k, \vec{\alpha}]=G^{ \pm}[k, \Delta] \\
&-\frac{1}{2} l_{p}^{2} \sum_{i, \sigma} \sigma \oint \mathrm{~d} s \oint \mathrm{~d} t \dot{\Delta}^{i}(t) f_{r}\left(\Delta(t)-\alpha^{i}(s)\right) G^{ \pm}\left[k, \Delta \#\left(\alpha_{s, t}^{i}\right)^{\sigma}\right] . \tag{4.1.1}
\end{align*}
$$

In order to find the explicit form of $G^{ \pm}[k, \beta]$, we apply $|k|^{n} \hat{h}^{ \pm}(k) \mathcal{M}=\mathcal{M}|k|^{n} \widehat{H}^{ \pm}(k)$ to an arbitrary state $\langle\vec{\alpha}|$ :

$$
\begin{aligned}
\langle\vec{\alpha}||k|^{n} \hat{h}^{ \pm}(k) \mathcal{M} & =\langle\Delta||k|^{n} \widehat{H}^{ \pm}(k) \\
& -\frac{1}{2} l_{p}^{2} \sum_{i, \sigma} \sigma \oint \mathrm{~d} s \oint \mathrm{~d} t \dot{\Delta}^{i}(t) f_{r}\left(\Delta(t)-\alpha^{i}(s)\right)\left\langle\Delta \#\left(\alpha_{s, t}^{i}\right)^{\sigma} \|\left. k\right|^{n} \widehat{H}^{ \pm}(k)\right.
\end{aligned}
$$

or, using equation (3.4.3),

$$
\begin{aligned}
& |k|^{n} F_{r}^{ \pm}[k, \vec{\alpha}]\langle\vec{\alpha}| \mathcal{M}=(2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} y \mathrm{e}^{-i k \cdot y}\left(\mathcal{F}^{ \pm}[y, \Delta]\langle\Delta|\right. \\
& \left.\quad-\frac{1}{2} l_{p}^{2} \sum_{i, \sigma} \sigma \oint \mathrm{~d} s \oint \mathrm{~d} t \dot{\Delta}^{i}(t) f_{r}\left(\Delta(t)-\alpha^{i}(s)\right) \mathcal{F}^{ \pm}\left[y, \Delta \#\left(\alpha_{s, t}^{i}\right)^{\sigma}\right]\left\langle\Delta \#\left(\alpha_{s, t}^{i}\right)^{\sigma}\right|\right) .
\end{aligned}
$$

By applying both sides to a ket state which always gives 1 for any loop bra in QGR, we find

$$
\begin{aligned}
|k|^{n} F_{r}^{ \pm}[k, \vec{\alpha}]= & (2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} y \mathrm{e}^{-\mathrm{i} k \cdot y} \\
& \times\left(\mathcal{F}^{ \pm}[y, \Delta]-\frac{1}{2} l_{p}^{2} \sum_{i, \sigma} \sigma \oint \mathrm{~d} s \oint \mathrm{~d} t \dot{\Delta}^{i}(t) f_{r}\left(\Delta(t)-\alpha^{i}(s)\right)\right. \\
& \left.\times \mathcal{F}^{ \pm}\left[y, \Delta \#\left(\alpha_{s, t}^{i}\right)^{\sigma}\right]\right)
\end{aligned}
$$

Comparing this with equation (4.1.1), we find that $|k|^{n} G^{ \pm}[k, \Delta \# \alpha]$ is the Fourier transform of $\mathcal{F}^{ \pm}[y, \Delta \# \alpha]$, that is,
$|k|^{n} G^{ \pm}[k, \Delta \# \alpha]=(2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} y \mathrm{e}^{-\mathrm{i} k \cdot y} \mathcal{F}^{ \pm}[y, \Delta \# \alpha]$.
Therefore, we define the Fourier transform of $|k|^{n} G^{ \pm}[k, \Delta \# \alpha]$ as

$$
\begin{equation*}
G^{ \pm}[y, \Delta \# \alpha]:=\mathcal{F}^{ \pm}[y, \Delta \# \alpha] \quad \text { for } \quad \Delta \# \alpha . \tag{4.1.3a}
\end{equation*}
$$

However, $G^{ \pm}[y, \Delta \cup \alpha]$ can not be defined in this construction, because $\mathcal{F}^{ \pm}[y, \Delta \cup \alpha]$ are of order $\varepsilon^{2}$, and they do not approximate any aspect of $F_{r}^{ \pm}$through the map $\mathcal{M}$. Therefore, we assume that

$$
\begin{equation*}
G^{ \pm}[y, \Delta \cup \alpha]:=\frac{1}{2}\left(\mathcal{F}^{ \pm}[y, \Delta \# \alpha]+\mathcal{F}^{ \pm}\left[y, \Delta \# \alpha^{-I}\right]\right) \tag{4.1.3b}
\end{equation*}
$$

so that $G^{ \pm}$satisfy the spinor relation, equation (2.1.2). As we show below, the functions $G^{ \pm}[k, \Delta \# \alpha]$, and hence $G^{ \pm}[y, \Delta \# \alpha]$ are annihilated by the low-frequency constraints (3.1.2) and (3.2.1). In this sense these are 'low-frequency physical states.' These states determine the kinematical linear gravity states in QGR. If we interpret $G^{ \pm}$as a non-Abelian version of form factors, then equation (3.4.3) (replacing $\mathcal{F}^{ \pm}$by $G^{ \pm}$) is analogous to $\langle\vec{\alpha}| \hat{h}^{ \pm}(k)=-2 l_{p}^{2} F_{r}^{ \pm}[k, \vec{\alpha}]\langle\vec{\alpha}|$ in QLGR $\{2]$.

Note that in the transformation (4.1.1), only information on the intersection(s) in $G^{ \pm}[k, \Delta \# \alpha]$ is transferred to $F_{r}^{ \pm}[k, \vec{\alpha}]$. Therefore, since the physical states in QLGR consist of polynomials of $F_{r}^{ \pm}[k, \vec{\alpha}]$ [2], all physical information about gravitons should be coded entirely in the intersections in $G^{ \pm}[k, \Delta \# \alpha]$. Let us prove this statement. From equations (3.4.4) and (4.1.3a), we have

$$
\begin{align*}
G^{ \pm}[y, \Delta \# \alpha]= & \left.-\frac{1}{4} \sum_{i=1}^{3} v_{i} \int \mathrm{~d}^{3} x \right\rvert\, \oint \mathrm{d} u \dot{\Delta}^{b}(u) \delta^{3}(x, \Delta(u)) \omega_{b}^{ \pm i}(x-y) \\
& +\oint \mathrm{d} s \dot{\alpha}^{a}(s) \delta^{3}(x, \alpha(s)) \omega_{a}^{ \pm i}(x-y) \mid \\
= & -\frac{1}{4} \sum_{i=1}^{3} v_{i}\left(\frac{\dot{\Delta}_{l}^{b} \omega_{b}^{ \pm i}\left(\Delta_{I}-y\right)}{\left|\dot{\Delta}_{l}^{c} \omega_{c}^{ \pm i}\left(\Delta_{I}-y\right)\right|} \dot{\alpha}_{I}^{a} \omega_{a}^{ \pm \mathrm{i}}\left(\alpha_{I}-y\right)\right. \\
& \left.+\oint \mathrm{d} s\left|\dot{\alpha}^{a}(s) \omega_{a}^{ \pm \mathrm{i}}(\alpha(s)-y)\right|\right) \tag{4.1.4}
\end{align*}
$$

The non-intersection part of the weave has vanished because of equation (2.3.1), up to an error of order $\varepsilon^{2}$, and $\sum_{i=1}^{3} \nu_{i}\left|\omega^{+i}(x)\right|=0$; the first term in the last line is the expansion of the intersection part $\left|\dot{\Delta}_{I}^{b} \omega_{b}^{ \pm i}\left(\Delta_{I}-y\right)+\dot{\alpha}_{I}^{a} \omega_{a}^{ \pm \mathrm{i}}\left(\alpha_{I}-y\right)\right|$ ( $I$ stands for an intersection point); and the second term is the non-intersection part of loop $\alpha$, which is not transferred through the map $\mathcal{M}$ and hence does not contribute to $F_{r}^{ \pm}[k, \vec{\alpha}]$. We define the Fourier transform of $|k|^{n} F_{r}^{ \pm}[k, \vec{\alpha}]$ and express it in terms of $\omega_{a}^{ \pm i}(x)$ using equations (3.3.1) and (3.3.4):

$$
\begin{align*}
F_{r}^{ \pm}[y, \vec{\alpha}]= & (2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} k \mathrm{e}^{\mathrm{i} k \cdot y}|k|^{n} F_{r}^{ \pm}[k, \vec{\alpha}] \\
& =\int \mathrm{d}^{3} x\left[(2 \pi)^{-3} \int \mathrm{~d}^{3} k|k|^{n} m_{a}(k) m_{b}(k) \mathrm{e}^{\mathrm{i} k \cdot( \pm(x-y))}\right] F_{r}^{a}\left[x, \alpha^{b}\right] \\
& =\int \mathrm{d}^{3} x\left[\sum_{i=1}^{3} v_{i} \frac{1}{2} \frac{\omega_{a}^{\mp i}(y-x) \omega_{b}^{\mp i}(y-x)}{\left|\omega^{\mp i}(y-x)\right|}\right] F_{r}^{a}\left[x, \alpha^{b}\right] . \tag{4.1.5}
\end{align*}
$$

On the other hand, from equation (4.1.1), realizing $G^{ \pm}[y, \Delta]=0$ and using the spinor relation, we have

$$
F_{r}^{ \pm}[y, \vec{\alpha}]=-l_{p}^{2} \sum_{i=1}^{3} \oint \mathrm{~d} s \oint \mathrm{~d} t \dot{\Delta}^{i}(t) f_{r}\left(\Delta(t)-\alpha^{i}(s)\right) G^{ \pm}\left[y, \Delta \# \alpha_{s, t}^{i}\right]
$$

By substituting the first term of the right hand side of equation (4.1.4) into the last equation, we have
$F_{r}^{ \pm}[y, \vec{\alpha}]=l_{p}^{2} \sum_{i=1}^{3} \oint \mathrm{~d} s \oint \mathrm{~d} t \dot{\Delta}^{i}(t) f_{r}\left(\Delta(t)-\alpha^{i}(s)\right)$

$$
\begin{align*}
& \quad \times \frac{1}{4} \sum_{j=1}^{3} v_{j} \frac{\dot{\Delta}^{b}(t) \omega_{b}^{ \pm j}(\Delta(t)-y)}{\left|\dot{\Delta}^{c}(t) \omega_{c}^{ \pm j}(\Delta(t)-y)\right|}\left(\dot{\alpha}^{i}\right)^{a}(s) \omega_{a}^{ \pm j}(\Delta(t)-y) \\
& =\frac{1}{4} l_{p}^{2} \sum_{i, j=1}^{3} v_{j} \oint \mathrm{~d} t \dot{\Delta}^{i}(t) \frac{\dot{\Delta}^{b}(t) \omega_{b}^{ \pm j}(\Delta(t)-y)}{\left|\dot{\Delta}^{c}(t) \omega_{c}^{ \pm j}(\Delta(t)-y)\right|} \omega_{a}^{ \pm j}(\Delta(t)-y) \\
& \quad \times \oint \mathrm{d} s\left(\dot{\alpha}^{i}\right)^{a}(s) f_{r}\left(\Delta(t)-\alpha^{i}(s)\right) \tag{4.1.6}
\end{align*}
$$

Now, consider the line integral of $\Delta$. Noticing that $\omega_{a}^{ \pm j}(x)$ is a slowly varying function, we have

$$
K^{i}=\oint \mathrm{d} t \dot{\Delta}^{i}(t) \frac{\dot{\Delta}^{b}(t) \omega_{b}^{ \pm j}(\Delta(t)-y)}{\left|\dot{\Delta}^{c}(t) \omega_{c}^{ \pm j}(\Delta(t)-y)\right|}
$$

Since $F_{r}^{ \pm}[y, \vec{\alpha}]$ is already of order $\varepsilon$, by assuming $\omega_{i}^{ \pm j}$ is constant in space, we derive

$$
\begin{aligned}
K^{i} \omega_{i}^{ \pm j}= & \oint \mathrm{d} t\left|\dot{\Delta}^{b}(t) \omega_{b}^{ \pm j}(\Delta(t)-y)\right| \\
& =2 l_{p}^{-2} \int \mathrm{~d}^{3} x\left|\omega^{ \pm j}\right|=2 l_{p}^{-2} \delta_{b}^{i} \int \mathrm{~d}^{3} x \frac{\omega_{b}^{ \pm j} \omega_{i}^{ \pm j}}{\left|\omega^{ \pm j}\right|}
\end{aligned}
$$

or

$$
K^{i}=2 l_{p}^{-2} \delta_{b}^{i} \int \mathrm{~d}^{3} x \frac{\omega_{b}^{ \pm j}}{\left|\omega^{ \pm j}\right|}
$$

where we have used the properties of the weave. By substituting $K^{i}$ into equation (4.1.6) we obtain

$$
\begin{aligned}
& F_{r}^{ \pm}[y, \vec{\alpha}]=\frac{1}{2} \sum_{j=1}^{3} \nu_{j} \int \mathrm{~d}^{3} x \frac{\omega_{b}^{ \pm j}(x-y)}{\left|\omega^{ \pm j}(x-y)\right|} \omega_{a}^{ \pm j}(x-y) F_{r}^{a}\left[x, \alpha^{b}\right] \\
= & \int \mathrm{d}^{3} x\left[\sum_{j=1}^{3} \nu_{j} \frac{1}{2} \frac{\omega_{a}^{\mp j}(y-x) \omega_{b}^{\mp j}(y-x)}{\left|\omega^{\mp j}(y-x)\right|}\right] F_{r}^{a}\left[x, \alpha^{b}\right] .
\end{aligned}
$$

Comparing the last line with equation (4.1.5), we conclude that $F_{r}^{ \pm}[y, \vec{\alpha}]$ are determined entirely by the intersection part in $G^{ \pm}[y, \Delta \# \alpha]$.

Next, we show that $G^{ \pm}[y, \Delta \# \alpha]$ are low-frequency constraint invariant. For simplicity of notation, we set $\beta:=\Delta \# \alpha$ and define $\gamma_{s, \beta(t)}:=\gamma+\beta(t)-\gamma(s)$. By applying the differential operator given by equation (3.1.1) to $G^{ \pm}[y, \Delta \# \alpha]$, we have

$$
\begin{aligned}
&\left(\widehat{D}^{a b}(x) G^{ \pm}\right)[y, \beta] \\
&=-\lim _{\delta \rightarrow 0} \frac{1}{\pi \delta^{2}} \oint \mathrm{~d} s \int \mathrm{~d}^{3} z f_{r}(z)\left(\widehat{T}^{b}\left[\gamma_{x, \delta}^{a}+z\right](s) G^{ \pm}\right)[y, \beta] \\
&=-\lim _{\delta \rightarrow 0} \frac{1}{\pi \delta^{2}} \oint \mathrm{~d} s \int \mathrm{~d}^{3} z f_{r}(z) \frac{1}{2} l_{p}^{2} \oint \mathrm{~d} t \dot{\beta}^{b}(t) \delta^{3}\left(\gamma_{x, \delta}^{a}(s)+z, \beta(t)\right) \\
& \times \sum_{\sigma} \sigma G^{ \pm}\left[y, \beta \#\left(\gamma_{x, \delta}^{a}+z\right)^{\sigma}\right]
\end{aligned}
$$

$$
\begin{aligned}
= & -\frac{1}{2} l_{p}^{2} \lim _{\delta \rightarrow 0} \frac{1}{\pi \delta^{2}} \oint \mathrm{~d} s \oint \mathrm{~d} t \dot{\beta}^{b}(t) f_{r}\left(\beta(t)-\gamma_{x, \delta}^{a}(s)\right) \sum_{\sigma} \sigma G^{ \pm}\left[y, \beta \#\left(\gamma_{x, \delta}^{a}\right)_{s, \beta(t)}^{\sigma}\right] \\
= & -l_{p}^{2} \lim _{\delta \rightarrow 0} \frac{1}{\pi \delta^{2}} \oint \mathrm{~d} s \oint \mathrm{~d} t \dot{\beta}^{b}(t) f_{r}\left(\beta(t)-\gamma_{x, \delta}^{a}(s)\right) \\
& \times\left(G^{ \pm}\left[y, \beta \#\left(\gamma_{x, \delta}^{a}\right)_{s, \beta(t)}\right]-G^{ \pm}\left[y, \beta \cup\left(\gamma_{x, \delta}^{a}\right)_{s, \beta(t)}\right]\right) \\
= & l_{p}^{2} \lim _{\delta \rightarrow 0} \frac{1}{\pi \delta^{2}} \oint \mathrm{~d} s \oint \mathrm{~d} t \dot{\beta}^{b}(t) f_{r}\left(\beta(t)-\gamma_{x, \delta}^{a}(s)\right) \\
& \times \frac{1}{4} \sum_{i=1}^{3} v_{i} \frac{\dot{\beta}^{c}(t) \omega_{c}^{ \pm \mathrm{i}}(\beta(t)-y)}{\left|\dot{\beta}^{d}(t) \omega_{d}^{ \pm \mathrm{i}}(\beta(t)-y)\right|}\left(\dot{\gamma}_{x, \delta}^{a}\right)^{e}(s) \omega_{z}^{ \pm \mathrm{i}}(\beta(t)-y) .
\end{aligned}
$$

To derive the last line we realized that the difference between $G^{ \pm}[y, \beta \# \gamma]$ and $G^{ \pm}[y, \beta \cup \gamma]$ is simply the intersection part of $\beta$ and $\gamma$, in the same way as we derived equation (4.1.4). Then

$$
\begin{aligned}
\left(\widehat{D}^{a b}(x) G^{ \pm}\right) & {[y, \beta] } \\
= & \frac{1}{4} l_{p}^{2} \sum_{i=1}^{3} v_{i} \lim _{\delta \rightarrow 0} \frac{1}{\pi \delta^{2}} \oint \mathrm{~d} t \dot{\beta}^{b}(t) \frac{\dot{\beta}^{c}(t) \omega_{c}^{ \pm i}(\beta(t)-y)}{\left|\dot{\beta}^{d}(t) \omega_{d}^{ \pm i}(\beta(t)-y)\right|} \omega_{e}^{ \pm i}(\beta(t)-y) \\
& \times \oint \mathrm{d} s\left(\dot{\gamma}_{x, \delta}^{a}\right)^{e}(s) f_{r}\left(\beta(t)-\gamma_{x, \delta}^{a}(s)\right) \\
= & \frac{1}{4} l_{p}^{2} \sum_{i=1}^{3} v_{i} \oint \mathrm{~d} t \dot{\beta}^{b}(t) \frac{\dot{\beta}^{c}(t) \omega_{c}^{ \pm i}(\beta(t)-y)}{\left|\dot{\beta}^{d}(t) \omega_{d}^{ \pm i}(\beta(t)-y)\right|} \omega_{e}^{ \pm i}(\beta(t)-y) \epsilon^{a e s} \frac{\partial}{\partial x^{f}} f_{r}(\beta(t)-x) .
\end{aligned}
$$

Since the contribution from $\alpha$ is of order $\varepsilon^{2}$, we make an approximation $\beta=\Delta \# \alpha \approx \Delta$, and obtain
$\left(\widehat{D}^{a b}(x) G^{ \pm}\right)[y, \Delta \# \alpha]$

$$
\begin{aligned}
= & \frac{1}{4} l_{p}^{2} \sum_{i=1}^{3} v_{i} \oint \mathrm{~d} t \dot{\Delta}^{b}(t) \frac{\dot{\Delta}^{c}(t) \omega_{c}^{ \pm i}(\Delta(t)-y)}{\left|\dot{\Delta}^{d}(t) \omega_{d}^{ \pm i}(\Delta(t)-y)\right|} \\
& \times \omega_{e}^{ \pm i}(\Delta(t)-y) \epsilon^{a e f} \frac{\partial}{\partial x^{f}} f_{r}(\Delta(t)-x) \\
= & \frac{1}{2} \sum_{i=1}^{3} v_{i} \int \mathrm{~d}^{3} z \frac{\omega_{b}^{ \pm i}(z-y)}{\left|\omega^{ \pm i}(z-y)\right|} \omega_{e}^{ \pm i}(z-y) \epsilon^{a e f} \frac{\partial}{\partial x^{f}} f_{r}(z-x) \\
= & \int \mathrm{d}^{3} z\left[\sum_{i=1}^{3} v_{i} \frac{1}{2} \frac{\omega_{b}^{ \pm i}(z-y) \omega_{e}^{ \pm i}(z-y)}{\left|\omega^{ \pm i}(z-y)\right|}\right] \epsilon^{a e f} \frac{\partial}{\partial x^{f}} f_{r}(z-x) .
\end{aligned}
$$

By using equations (3.3.1) and (3.3.4) again and the properties of $m_{a}(k)$, we obtain $\left(\widehat{D}^{a b}(x) G^{ \pm}\right)[y, \Delta \# \alpha]$

$$
\begin{align*}
& =\int \mathrm{d}^{3} z\left[(2 \pi)^{-3} \int \mathrm{~d}^{3} k|k|^{n} m^{b}(k) m_{e}(k) \mathrm{e}^{\mathrm{i} k \cdot( \pm(z-y))}\right] \epsilon^{a e f} \frac{\partial}{\partial x^{f}} f_{r}(z-x) \\
& =\mp \int \mathrm{d}^{3} z\left[(2 \pi)^{-3} \int \mathrm{~d}^{3} k|k|^{n+1} m^{a}(k) m^{b}(k) \mathrm{e}^{\mathrm{i} k \cdot( \pm(z-y))}\right] f_{r}(z-x) \\
& =\mp(2 \pi)^{-3} \int \mathrm{~d}^{3} k|k|^{n+1} m^{a}(k) m^{b}(k) \mathrm{e}^{\mathrm{i} \cdot \cdot( \pm(x-y))} \mathrm{e}^{-r^{2} k^{2} / 2} . \tag{4.1.7}
\end{align*}
$$

Since this is a symmetric and traceless matrix, applying the constraint operators, we immediately find

$$
\begin{aligned}
& \left(\widehat{V}_{c}(x) G^{ \pm}\right)[y, \Delta \# \alpha]=-\epsilon_{c a b}\left(\widehat{D}^{a b}(x) G^{ \pm}\right)[y, \Delta \# \alpha]=0 \\
& \left(\widehat{\mathcal{S}}(x) G^{ \pm}\right)[y, \Delta \# \alpha]=-\delta_{a b}\left(\widehat{D}^{a b}(x) G^{ \pm}\right)[y, \Delta \# \alpha]=0 .
\end{aligned}
$$

Therefore, $G^{ \pm}[y, \Delta \# \alpha]$ are low-frequency constraint invariant states. The solution of the linearized constraints is given in the QGR state space by arbitrary functions of $G^{ \pm}[y, \Delta \# \alpha]$.

For later convenience we compute here $\widehat{D}^{ \pm} G^{ \pm}$and $\widehat{D}^{\mp} G^{ \pm}$. By Fourier transformation of equation (4.1.7), we have

$$
\begin{gathered}
\left(\widehat{D}^{a b}(k) G^{ \pm}\right)[y, \Delta \# \alpha]=(2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} x \mathrm{e}^{-\mathrm{i} k \cdot x}\left(\widehat{D}^{a b}(x) G^{ \pm}\right)[y, \Delta \# \alpha] \\
=\mp(2 \pi)^{-3 / 2}|k|^{n+1} m^{a}( \pm k) m^{b}( \pm k) \mathrm{e}^{-\mathrm{i} k \cdot y} \mathrm{e}^{-r^{2} k^{2} / 2}
\end{gathered}
$$

Therefore, we find

$$
\begin{align*}
& \left(\widehat{D}^{ \pm}(k) G^{ \pm}\right)[y, \Delta \# \alpha]=\mp(2 \pi)^{-3 / 2}|k|^{n+1} \mathrm{e}^{-\mathrm{i} k \cdot y} \mathrm{e}^{-r^{2} k^{2} / 2} \\
& \left(\widehat{D}^{\mp}(k) G^{ \pm}\right)[y, \Delta \# \alpha]=0 . \tag{4.1.8}
\end{align*}
$$

### 4.2. The Poincare-invariant vacuum state

To determine the Poincare-invariant vacuum state, we have to solve equations (3.4.7). We claim that the solution is

$$
\begin{align*}
\Psi_{0}(\Delta \# \alpha)= & \exp \left(-l_{p}^{2} \int \frac{\mathrm{~d}^{3} k}{(2 \pi)^{3}} \mathrm{e}^{\mathrm{r}^{2} k^{2} / 2}|k|^{1-2 n}\right. \\
& \left.\times \int \mathrm{d}^{3} x \mathrm{~d}^{3} y \mathrm{e}^{\mathrm{i} k \cdot(x-y)} G^{+}[x, \Delta \# \alpha] G^{+}[y, \Delta \# \alpha]\right) \tag{4.2.1}
\end{align*}
$$

This can be verified by a straightforward calculation, using equations (4.1.8).
Note that since $G^{+}[y, \Delta \cup \alpha]$ does not have any physical meaning as explained in Sec 4.1, $\Psi_{0}$ is defined only for loops $\Delta \# \alpha$ (including $\Delta$ ) and undefined for $\Delta U \alpha$ in this construction. Equation (4.2.1) is our first main result. $\Psi_{0}$ is the state in the state space of the exact non-perturbative theory, that represents the physical vacuum, formed by virtual gravitons on Minkowski space.

The quantities $\omega_{a}^{+\mathrm{i}}(x)$ introduced in section 3.3 are defined up to the arbitrary phase of $m_{a}(k)$. As such they are, in a sense, unphysical. An important check for the consistency of this result is the independence of $\Psi_{0}(\Delta \# \alpha)$ from the arbitrariness of $\omega_{a}^{+\mathrm{i}}(x)$. The proof of this independence follows.
$m_{a}(k)$ is determined up to a phase factor $\mathrm{e}^{\mathrm{i} \lambda(k)}$, where $\lambda(k)$ is a real scalar function of $k$ such that $\lambda(-k)=-\lambda(k)$. The last condition guarantees that $\bar{m}_{a}(k)=-m_{a}(-k)$. First we show the dependence of $G^{+}[y, \Delta \# \alpha]$ on $\lambda(k)$ and then show that the Poincare-invariant vacuum $\Psi_{0}(\Delta \# \alpha)$ does not depend on $\lambda(k)$. Let us denote for a particular choice $m_{a}^{(0)}(k)$

$$
\begin{aligned}
& I_{a b}^{(0)}(k):=(2 \pi)^{-3 / 2}|k|^{r} m_{a}^{(0)}(k) m_{b}^{(0)}(k) \\
& I_{a b}^{(0)}(x):=(2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} k I_{a b}^{(0)}(k) \mathrm{e}^{\mathrm{i} k \cdot x} .
\end{aligned}
$$

For another choice $m_{a}(k):=m_{a}^{(0)}(k) \mathrm{e}^{\mathrm{j} \lambda(k)}$, we denote

$$
I_{a b}(k):=I_{a b}^{(0)}(k) \mathrm{e}^{2 i \lambda(k)}=(2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} x \mathrm{e}^{-\mathrm{i} k \cdot x} \mathrm{e}^{2 i \lambda(k)} I_{a b}^{(0)}(x)
$$

The Fourier transform of the last line is

$$
\begin{aligned}
I_{a b}(y)= & (2 \pi)^{-3 / 2} \int \mathrm{~d}^{3} k \mathrm{e}^{\mathrm{i} k \cdot y} I_{a b}(k) \\
& =(2 \pi)^{-3} \int \mathrm{~d}^{3} x I_{a b}^{(0)}(x) \int \mathrm{d}^{3} k \mathrm{e}^{\mathrm{i} k \cdot(y-x)} \mathrm{e}^{2 i \lambda(k)}
\end{aligned}
$$

or shifting variable $y$ by $z$ and changing the integration variable $x$ appropriately we find
$I_{a b}(y+z)=(2 \pi)^{-3} \int \mathrm{~d}^{3} x I_{a b}^{(0)}(x+z) \int \mathrm{d}^{3} k \mathrm{e}^{\mathrm{i} k \cdot(y-x)} \mathrm{e}^{2 i \lambda(k)}$.
Multiplying operator $\hat{h}^{a b}(z)$ on both sides, we have

$$
\begin{aligned}
\int \mathrm{d}^{3} z I_{a b}(y & +z) \hat{h}^{a b}(z) \\
& =(2 \pi)^{-3} \int \mathrm{~d}^{3} x\left(\int \mathrm{~d}^{3} z I_{a b}^{(0)}(x+z) \hat{h}^{a b}(z)\right) \int \mathrm{d}^{3} k \mathrm{e}^{\mathrm{i} k \cdot(y-x)} \mathrm{e}^{2 i \lambda(x)} .
\end{aligned}
$$

After replacing $\omega_{a}^{i}$ involved in both sides by $\omega_{a}^{+i}$, transforming to operators in the exact theory and applying the resulting equation on a loop state $\langle\Delta \# \alpha|$, we deduce the equality of eigenvalues that appear on both sides, namely

$$
G^{+}[y, \Delta \# \alpha]=(2 \pi)^{-3} \int \mathrm{~d}^{3} x G^{(0) \dagger}[x, \Delta \# \alpha] \int \mathrm{d}^{3} k \mathrm{e}^{\mathrm{i} k \cdot(y-x)} \mathrm{e}^{2 i \lambda(k)} .
$$

This is the dependence of $G^{+}[y, \Delta \# \alpha]$ on $\lambda(k)$.
Next, let us compute the exponent of the Poincare-invariant vacuum functional. This is a straightforward calculation of integrals with delta functions, and finally we find

$$
\begin{aligned}
& \int \frac{\mathrm{d}^{3} k}{(2 \pi)^{3}} \mathrm{e}^{r^{2} k^{2} / 2}|k|^{1-2 n} \int \mathrm{~d}^{3} x \mathrm{~d}^{3} y \mathrm{e}^{\mathrm{i} k \cdot(x-y)} G^{+}[x, \Delta \# \alpha] G^{+}[y, \Delta \# \alpha] \\
& \quad=\int \frac{\mathrm{d}^{3} k}{(2 \pi)^{3}} \mathrm{e}^{r^{2} k^{2} / 2}|k|^{1-2 n} \int \mathrm{~d}^{3} x \mathrm{~d}^{3} y \mathrm{e}^{\mathrm{i} k \cdot(x-y)} G^{(0)+}[x, \Delta \# \alpha] G^{(0)+}[y, \Delta \# \alpha]
\end{aligned}
$$

This implies that the Poincare-invariant vacuum functional does not depend on the arbitrary phase factor of $m_{a}(k)$.

### 4.3. Graviton states

Applying the annihilation and creation operators on the Poincare-invariant vacuum state, we have

$$
\begin{align*}
& -\left(2 l_{p}^{2}\right)^{-1 / 2}\left[\left(|k|^{1 / 2} \widehat{H}^{+}(-k)+|k|^{-3 / 2} \widehat{D}^{+}(-k)\right) \Psi_{0}\right](\Delta \# \alpha)=0 \\
& \quad\left(2 l_{p}^{2}\right)^{-1 / 2}|k|^{-3 / 2}\left(\widehat{D}^{-}(k) \Psi_{0}\right)(\Delta \# \alpha)=0 \\
& \quad\left(2 l_{p}^{2}\right)^{-1 / 2}|k|^{-3 / 2}\left(\widehat{D}^{+}(k) \Psi_{0}\right)(\Delta \# \alpha)=\Psi_{k+}(\Delta \# \alpha)  \tag{4.3.1}\\
& -\left(2 l_{p}^{2}\right)^{-1 / 2}\left[\left(|k|^{1 / 2} \widehat{H}^{-}(-k)+|k|^{-3 / 2} \widehat{D}^{-}(-k)\right) \Psi_{0}\right](\Delta \# \alpha)=\Psi_{k-}(\Delta \# \alpha) .
\end{align*}
$$

From these equations, the one-graviton states with positive and negative helicities are

$$
\begin{aligned}
& \Psi_{k+}(\Delta \# \alpha)=-\left(2 l_{p}^{2}\right)^{-1 / 2}|k|^{1 / 2}\left(\widehat{H}^{+}(+k) \Psi_{0}\right)(\Delta \# \alpha) \\
& \Psi_{k-}(\Delta \# \alpha)=-\left(2 l_{p}^{2}\right)^{-1 / 2}|k|^{1 / 2}\left(\widehat{H}^{-}(-k) \Psi_{0}\right)(\Delta \# \alpha)
\end{aligned}
$$

or, using equations (3.4.3) and (4.1.2),

$$
\begin{align*}
& \Psi_{k+}(\Delta \# \alpha)=\left(2 l_{p}^{2}|k|\right)^{1 / 2} G^{+}[+k, \Delta \# \alpha] \Psi_{0}(\Delta \# \alpha)  \tag{4.3.2}\\
& \Psi_{k-}(\Delta \# \alpha)=\left(2 l_{p}^{2}|k|\right)^{1 / 2} G^{-}[-k, \Delta \# \alpha] \Psi_{0}(\Delta \# \alpha)
\end{align*}
$$

These graviton states are also undefined for loops $\Delta \cup \alpha$.

## 5. The picture of the linearization of quantum gravity in the loop representation.

We discuss here the overall picture of the linearization of the loop-space quantum general relativity. We choose a background weave state which approximates a flat-space metric. The notion of flatness of the background in the presence of quantum fluctuations is meaningful only if it is measured on a scale $L>l_{p} / \varepsilon^{2}$ [4]. By using this metric, we define the orthogonal vectors $\omega_{a}^{ \pm i}(i=1,2,3)$ described in Sec 3.3 and a smearing function $f_{r}$ with a scale $r>l_{p} / \varepsilon$, which is required by the Heisenberg indeterminacy relations (see [4]). The need of this smearing function in the formalism reflects the existence of the quantum fluctuations of order $\varepsilon$ around the background metric. The emergence of the smearing function from the weave state can be qualitatively understood in the following way. Consider an open surface on the 3 -space. Given the weave state, we can determine the area of this surface in terms of the area operator [3], which is available at the unconstrained level to measure the geometry of loop states. If we consider only scales much larger than the Planck length, then a change in the perimeter of the surface produces a change in the area of the surface. However, if we want a sensitivity comparable to the Planck scale, then a Planck-scale change of the perimeter will not affect the area of the surface, which is the quantity that we can measure. Hence, there is an uncertainty in the information about the position of the perimeter that is of order of the Planck scale. This same uncertainty is present in the information of the position of any loop on the weave (the perimeter is a loop). It is this uncertainty that depends on the fixed background weave, and determines the smearing function and its scale $r$.

Given the smearing function $f_{r}$, we define the differential operator $\widehat{D}^{a b}$, equation (3.1.1), the low-frequency vector and scalar constraint operators, equations (3.1.2) and (3.2.1), and the linear $\operatorname{map} \mathcal{M}$, equation (2.4.2). Given the vectors $\omega_{a}^{ \pm i}$, we define the linear metric operators $\widehat{H}^{ \pm}$, equation (3.4.2), the annihilation and creation operators, equations (3.4.5) and (3.4.6), and the low-frequency constraint-invariant state functionals $G^{ \pm}$, equations (4.1.3). $G^{ \pm}$are non-zero around $\Delta$ over 'deviation' loops $\alpha$. By utilizing the annihilation and creation operators, the Poincare-invariant vacuum and the graviton states, equations (4.2.1) and (4.3.2), are determined.

All these operators and loop states are mapped to the corresponding ones in the linearized theory by the map $\mathcal{M}$. Since the latter operators and states describe free graviton physics, and since the map preserves the eigenvalues, it follows that all these operators and loop states describe graviton physics on a flat background geometry.

A curious aspect of this construction is that only intersection points in $G^{ \pm}[x, \Delta \# \alpha]$ are essential for approximating the physical states $F_{r}^{ \pm}[x, \vec{\alpha}]$ in the linearized theory. This
fact suggests that information about graviton physics is coded into the intersections of loops, and to understand the actions of operators on the intersections of loops is crucial for understanding the physics. However, this is not related to the fact that $G^{ \pm}$are lowfrequency constraint invariant. Rather, this is a consequence of symmetry of the operator $\hat{h}^{a b}(x)$. The reason is the following.

Given an arbitrary symmetric matrix (Sym) ${ }_{a b}(k)$, we can define
$\mathcal{M} \widehat{H}^{\text {sym }}(y)=\int \mathrm{d}^{3} x\left[(2 \pi)^{-3} \int \mathrm{~d}^{3} k|k|^{n}(\operatorname{Sym})_{a b}(k) \mathrm{e}^{\mathrm{i} k \cdot(x-y)}\right] \hat{h}^{a b}(x) \mathcal{M}$.
We also define three orthogonal vectors $\theta_{a}^{i}(x)(i=1,2,3)$ such that

$$
(2 \pi)^{-3} \int \mathrm{~d}^{3} k|k|^{n}(\mathrm{Sym})_{a b}(k) \mathrm{e}^{\mathrm{i} k \cdot x}=\frac{1}{2} \sum_{i=1}^{3} \frac{\theta_{a}^{i}(x) \theta_{b}^{i}(x)}{\left|\theta^{i}(x)\right|}
$$

From these two definitions we have

$$
\mathcal{M} \widehat{H}^{\mathrm{sym}}(y)=\sum_{i=1}^{3} \int \mathrm{~d}^{3} x \frac{1}{2} \frac{\theta_{a}^{l}(x-y) \theta_{b}^{i}(x-y)}{\left|\theta^{i}(x-y)\right|} \hat{h}^{a b}(x) \mathcal{M}
$$

and this leads to

$$
\widehat{H}^{\mathrm{sym}}(y)=\sum_{i=1}^{3}\left[\widehat{Q}\left(\theta^{i}(-y+\ldots)\right)-\int \mathrm{d}^{3} x\left|\theta^{i}(x)\right|\right]
$$

This operator creates 'physics' at intersections only, regardless of $\theta^{i}$. The low-frequency constraint invariance of $G^{ \pm}$is due to the tracelessness and transversality encoded into the specific functions $\omega_{a}^{ \pm i}$, which are derived from the traceless and transverse component $m_{a}(k) m_{b}(k)$ (and its complex conjugate). Because of that, $F_{r}^{ \pm}$solve the linearized constraints.

We close with a speculative remark. $m_{a}(k)$ and the Fourier transformations are defined and meaningful only on the flat space and distance scales compared with the Planck length, and so are $\omega_{a}^{ \pm i}$. If we had an extension of $\omega_{a}^{ \pm i}$, which can be defined by the weave, without background metrics, and which would approximate $\omega_{a}^{ \pm i}$ at large scales, then this extension could provide an insight for understanding the emergence of the single spin2 particle representation from the triplet of spin-1 particle representation due to the vector and scalar constraints (see [2]). By replacing $\omega_{a}^{ \pm i}$ by background-independent quantities that at large scales approximate their definition based on $m_{a}(k)$ on the flat space, but such that at the Planck scale, $G^{ \pm}$become exact-constraint-invariant, we would find expressions for the Poincare-invariant vacuum and the graviton states, which are exact-constraint-invariant. In other words, we would have the exact constraint-invariant states which allow physical interpretation in terms of gravitons. Work is in progress along these lines.

## References

[^2]
[^0]:    § E-mail address: iwasaki@phyast.pitt.edu.
    $\|$ E-mail address: rovelli@vms.cis.pitt.edu.

[^1]:    $\dagger$ Consider the metaphor of the weave as an elastic fabric: diffeomorphisms at scales large compared with the fabric scale do not destroy the structure of the fabric; while diffeomorphism at the scale of the fabric can break the structure of the fabric apart.

[^2]:    Rovelli C and Smolin L 1988 Phys. Rev, Lett. 611155 ; 1990 Nucl. Phys. B 33180
    [2] Ashtekar A, Rovelli C and Smolin L 1991 Phys. Rev. D 441740
    [3] Ashtekar A, Rovelli C and Smolin L 1992 Phys. Rev. Lett. 69237
    [4] Iwasaki J and Rovelli C 1993 Gravitons as embroidery on the weave Int. J. Mod. Phys. D 1 533-57
    [5] Zegwaard J 1992 Nucl. Phys. B 378288
    [6] Rovelli C 1991 Class. Quantum Grav. 81613

