GENERAL

Self-Similar Solutions of Variable-Coefficient Cubic-Quintic Nonlinear Schrödinger Equation with an External Potential

Commun. Theor. Phys. (Beijing, China) **54** (2010) pp. 55–59
Vol. 54, No. 1, July 15, 2010

Self-Similar Solutions of Variable-Coefficient Cubic-Quintic Nonlinear Schrödinger Equation with an External Potential*

WU Hong-Yu (吴红玉),[1,†] FEI Jin-Xi (费金喜),[1] and ZHENG Chun-Long (郑春龙)[2,3]

[1]College of Mathematics and Physics, Lishui University, Lishui 323000, China

[2]School of Physics and Electromechanical Engineering, Shaoguan University, Guangdong 512005, China

[3]Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, China

(Received September 16, 2009; revised manuscript received February 2, 2010)

Abstract *An improved homogeneous balance principle and an F-expansion technique are used to construct exact self-similar solutions to the cubic-quintic nonlinear Schrödinger equation. Such solutions exist under certain conditions, and impose constraints on the functions describing dispersion, nonlinearity, and the external potential. Some simple self-similar waves are presented.*

PACS numbers: 03.65.Ge, 05.45.Yv
Key words: F-expansion technique, the cubic-quintic nonlinear Schrödinger equation, self-similar solutions, propagate self-similarly

1 Introduction

The cubic-quintic nonlinear Schrödinger equation (CQNLSE) with nonlinearity management presents practical interest since it appears in many branches of physics such as nonlinear optics, nuclear physics, and Bose–Einstein condensate (BEC). In nonlinear optics it describes the propagation of pulses in double-doped optics fibers.[1] Efimov resonances, which are responsible for three-body interactions, have been observed in an ultracold gas of cesium atoms.[2] In BEC it models the condensate with two and three body interactions.[3−4] In optical fibers periodic variation of the nonlinearity can be achieved by varying the type of dopants along the fiber. In BEC the variation of the atomic scattering length by the Feshbach resonance technique leads to the oscillations of the mean field cubic nonlinearity.[5] The CQNLSE when the cubic term is equal to zero, is the critical quintic NLSE. The quintic Townes soliton is an unstable solution of the quintic NLSE.[6]

Self-similarity has been intensively explored in many areas of physics such as hydrodynamics and quantum field theory.[7] Also in nonlinear optics, there has been a significant surge of research activities on self-similarity. As examples, the self-similar behaviors in stimulated Raman scattering,[8] the evolution of self-written wave-guides,[9] the formation of Cantor set fractals in materials that support spatial solitons,[10] the evolution of optical wave collapse,[11] and the nonlinear propagation of parabolic pulses in optical fibers with normal dispersion[12] were investigated. Recent attention has been riveted on the self-similar propagation of parabolic optical pulses in an optical fiber amplifier,[13] a dispersion-decreasing optical fiber,[14] and a laser resonator,[15] opening prospects for studying the self-similar phenomena in dispersion and nonlinearity management systems.

In nonlinear science, the construction of exact solutions for nonlinear partial differential equations (NLPDEs) is one of the most important and essential tasks. With the help of exact solutions, the phenomena modelled by these NLPDEs such as the stability of optical soliton propagation can be well understood. In recent years, many powerful methods to construct exact analytical solutions have been proposed, such as the inverse scattering method, the Bäcklund transformation and Darboux transformation, the Painlevé truncation expansion, the homogeneous balance method, the sine-cosine function method, the tanh-function method, and the Jacobian elliptic function method.[16−24] Very recently, an F-expansion technique has been developed to obtain the new exact self-similar solutions.[25−26]

In this paper we present the exact self-similar solutions to the nonlinear Schrödinger equation with an external potential, which describes the propagation of pulses in the optic fibers where all parameters are functions of the time variable. This class also encloses the set of solitary wave solutions which describes, for example, such physically important applications as the amplification and compression of pulses in optical fiber amplifiers.[27] The importance of the results reported here is twofold: first, the approach leads to a broad class of exact solutions to the nonlin-

*Supported by Natural Science Foundation of Zhejiang Province of China under Grant Nos. Y604106 and Y606182, the Special Foundation of "University Talent Indraught Engineering" of Guangdong Province of China under Grant No. GDU2009109, and the Key Academic Discipline Foundation of Guangdong Shaoguan University under Gant No. KZ2009001
†E-mail: why160@126.com

ear differential equation in a systematic way. Some of these solutions have been obtained serendipitously in the past, but we emphasize the importance of the use of self-similarity techniques, which are broadly application for finding solutions to a range of nonlinear partial differential equations, having applications in a variety of other physical situations. These equations are not integrable by the inverse scattering method, and, therefore, they do not have soliton solutions;[27] however, they do have solitary wave solutions, which have often been called solitons. The second and more specific significance of these results lies in their potential application to the design of fiber optic amplifiers, optical pulse compressors, and solitary wave based communications links.

2 Exact Self-Similar Solutions of Variable Coefficient Cubic-Quintic Nonlinear Schrödinger Equation with an External Potential

The variable coefficient cubic-quintic nonlinear Schrödinger equation with an external potential can be written as

$$i\psi_t + \beta(t)\psi_{zz} + \gamma(t)|\psi|^2\psi + \alpha(t)|\psi|^4\psi = iV(t)\psi\,, \tag{1}$$

where $\psi(z,t)$ is the complex envelope of the electrical field in a comoving frame, $V(t)$ is an external potential, $\beta(t)$ is the second dispersive effects, $\gamma(t)$ is the nonlinearity parameter and $\alpha(t)$ is the saturation of the nonlinear refractive index (i.e. higher-order nonlinearity). They are real functions of the normalized propagation distance z, and t is the retarded time.

In order to make Eq. (1) an ordinary differential equation, we take the following transformation:

$$\psi(z,t) = u(z,t)\exp(iv(z,t))\,, \tag{2}$$

where $u(z,t)$ and $v(z,t)$ are real functions. Substituting $\psi(z,t)$ into Eq. (1), we find the following coupled equations for the phase $v(z,t)$ and $u(z,t)$:

$$u_t + 2\beta u_z v_z + \beta u v_{zz} - uV = 0\,, \tag{3}$$

$$u_{zz} + \beta u v_z^2 - \gamma u^3 - \alpha u^5 = 0\,. \tag{4}$$

For the sake of simplicity, we take $u(z,t) = \sqrt{w(z,t)}$. It follows from Eqs. (3) and (4).

$$w_t + 2\beta w_z v_z + 2\beta w v_{zz} - 2wV = 0\,, \tag{5}$$

$$4w^2 v_t + \beta w_z^2 - 2\beta w w_{zz} - 4\gamma w^3 - 4\alpha w^4 + 4\beta w^2 v_z^2 = 0\,. \tag{6}$$

According to the balance principle and F-expansion technique. The main idea of the algorithm is: for a general nonlinear physical system

$$P(v) \equiv P(x_0 = t, x_1, x_2, \ldots, x_n, v, v_{xixj}, \ldots)\,, \tag{7}$$

where

$$v = v(v_1, v_2, \ldots, v_q)^{\mathrm{T}}\,,$$
$$P(v) = (P_1(v), P_2(v), \ldots, P_q(v))^{\mathrm{T}}\,,$$

$P_i(v)$ are polynomials of v_i and their derivatives (T indicates the transposition of a matrix). We assume its solution in an extended symmetric form

$$v_i = \sum_{j=-N}^{N} \alpha_{ij}(x) F^j(\theta(x))\,, \quad x \equiv (t, x_1, x_2, \ldots, x_n)\,,$$
$$i = 1, 2, \ldots, q\,, \tag{8}$$

where $\alpha_{ij}(x)$, $\theta(x)$ are arbitrary functions to be determined, F is a solution of the JEFs function. N is determined by balancing the highest nonlinear terms and the highest-order partial terms in the given nonlinear system.

Substituting the ansatz in Eq. (8) together with the JEFs function into Eq. (7), collecting coefficients of polynomials of F, then setting each coefficient to zero, yields a set of partial differential equations concerning $\alpha_{ij}(x)$ and $\theta(x)$. Solving the system of partial differential equations to obtain $\alpha_{ij}(x)$ and $\theta(x)$ substituting the derived results and the solutions of JEFs function into Eq. (8), one can derive exact solutions to the given nonlinear system.

According to the above idea and by the balancing procedure, the ansatz in Eq. (8) becomes

$$w(z,t) = f_0(t) + f_1(t)F(\theta) + f_2(t)F(\theta)^{-1}\,, \tag{9}$$

$$\theta = n(t)z + k(t)\,, \tag{10}$$

$$v(z,t) = a(t)z^2 + b(t)z + e(t)\,, \tag{11}$$

where f_0, f_1, f_2, n, k, a, b, and e are the parameters to be determined. The parameter $a(z)$ is related to the wave front curvature; it is also a measure of the phase chirp imposed on the solitary wave. The function $F(\theta)$ is one of JEFs, which in general satisfy the following general first- and second-order nonlinear ordinary differential equations:

$$\Big(\frac{dF}{d\theta}\Big)^2 = c_0 + c_2F^2 + c_4F^4\,, \tag{12}$$

$$\frac{d^2F}{d\theta^2} = c_2F + 2c_4F^3\,, \tag{13}$$

where c_0, c_2, and c_4 are real constants related to the elliptic modulus of the JEFs (see Table 1). Substituting Eqs. (9), (10), and (11) into Eqs. 5) and (6), and requiring that z^qF^l ($q = 0,1,2; l = 0,1,2,3,4,5,6,7,8$), and $\sqrt{c_0 + c_2F^2 + c_4F^4}$ of each term be separately equal to zero, we obtain a system of algebraic or first-order ordinary differential equations for f_p, n, k, a, b, and e:

$$\frac{df_p}{dt} + (4\beta a - 2V)f_p = 0\,, \tag{14}$$

$$f_j\Big(\frac{dk}{dt} + 2\beta nb\Big) = 0\,, \tag{15}$$

$$f_j\Big(\frac{dn}{dt} + 4\beta na\Big) = 0\,, \tag{16}$$

$$\frac{da}{dt} + 4\beta a^2 = 0\,, \tag{17}$$

$$\frac{db}{dt} + 4\beta ab = 0\,, \tag{18}$$

$$\beta n^2 f_0 c_0 + 4\alpha f_0 f_2^2 + \gamma f_2^2 = 0\,, \tag{19}$$

$$\beta n^2 f_0 c_4 + 4\alpha f_0 f_1^2 + \gamma f_1^2 = 0\,, \tag{20}$$

$$4\alpha f_1^2 + 3\beta n^2 c_4 = 0\,, \tag{21}$$

$$4\alpha f_2^2 + 3\beta n^2 c_0 = 0\,, \tag{22}$$

$$4f_1\frac{de}{dt} - \beta f_1 n^2 c_2 - 6\beta n^2 f_2 c_4 + 4\beta f_1 b^2 - 12\gamma f_0 f_1 - 16\alpha f_1^2 f_2 - 24\alpha f_0^2 f_1 = 0\,, \tag{23}$$

$$4f_2\frac{de}{dt} - \beta f_2 n^2 c_2 - 6\beta n^2 f_1 c_0 + 4\beta f_2 b^2 - 12\gamma f_0 f_2 - 16\alpha f_1^2 f_2 - 24\alpha f_0^2 f_2 = 0\,, \tag{24}$$

$$4f_0\frac{de}{dt} - \beta f_0 n^2 c_2 - 6\gamma f_1 f_2 + 4\beta f_0 b^2 - 6\gamma f_0^2 - 8\alpha f_0^3 - 24\alpha f_0 f_1 f_2 = 0\,, \tag{25}$$

where $p = 0, 1, 2$ and $j = 1, 2$. By solving self-consistently, one can obtain a set of conditions on the coefficients and parameters, necessary for Eq. (1) to have exact periodic wave solutions.

Table 1 Jacobi elliptic functions.

Solution	c_0	c_2	c_4	F	$M=0$	$M=1$
1	1	$-(1+M^2)$	M^2	sn	sin	tanh
2	$1-M^2$	$2M^2-1$	$-M^2$	cn	cos	sech
3	M^2-1	$2-M^2$	-1	dn	1	sech
4	M^2	$-(1+M^2)$	1	ns	cosec	coth
5	$-M^2$	$2M^2-1$	$1-M^2$	nc	sec	cosh
6	-1	$2-M^2$	M^2-1	nd	1	cosh
7	1	$2-M^2$	$1-M^2$	sc	tan	sinh
8	$1-M^2$	$2-M^2$	1	cs	cot	cosech
9	1	$-(1+M^2)$	M^2	cd	cos	1
10	M^2	$-1+M^2$	1	dc	sec	1

We consider the most generic case, in which f_1 and f_2 are assumed nonzero and β and V are arbitrary. The following set of exact solutions is found:

$$f_0 = 2\Big(\frac{c_0}{c_4}\Big)^{1/4} f_1\,,\quad f_1 = f_{10}\delta\exp\Big(2\int_0^t V dt\Big)\,,$$

$$f_2 = \sqrt{\frac{c_0}{c_4}}\, f_1\,, \tag{26}$$

$$a = a_0\delta\,,\quad b = b_0\delta\,, \tag{27}$$

$$n = n_0\delta\,,\quad k = k_0 - 2n_0 b_0\delta\int_0^t \beta dt\,, \tag{28}$$

$$e = e_0 - \frac{\alpha}{4}(4b_0^2 - c_2 n_0^2 - 18n_0^2\sqrt{c_0 c_4})\int_0^t \beta dt\,, \tag{29}$$

$$\gamma = \frac{4c_4(c_0/c_4)^{1/4} n_0^2\beta\delta}{f_{10}\exp(2\int_0^t V dt)}\,,\quad \alpha = -\frac{3\beta n_0^2 c_4}{4f_{10}\exp(4\int_0^t V dt)}\,, \tag{30}$$

where $\delta = (1 + 4a_0\int_0^t \beta dt)^{-1}$ is the chirp function. It is related to the wave front curvature and presents a measure the phase chirp imposed on the wave. the subscript 0 denotes the value of the given function at $z = 0$.

Incorporating these solutions back into Eq. (2) we obtain the general periodic travelling wave solutions to the generalized CQNLSE:

$$\psi = \Big\{f_{10}\delta\exp\Big(2\int_0^t V dt\Big) \times \Big[2\Big(\frac{c_0}{c_4}\Big)^{1/4} + F(\theta) + \sqrt{\frac{c_0}{c_4}}F^{-1}(\theta)\Big]\Big\}^{1/2} \times \exp[i(az^2 + bz + e)]\,, \tag{31}$$

where $\theta = nz + k$. Apart from the solutions given in Eqs. (26)–(29).

3 Wave Self-Similar Propagation — Few Simple Examples

In this part, as few characteristic examples of the solution (31), when $c_0 = M^2 - 1, c_2 = 2 - M^2$, and $c_4 = -1$, rewrite Eq. (31) in a simple form, namely

$$A \equiv |\psi|^2 = f_{10}^2\delta\exp\Big(2\int_0^t V dt\Big) \times \Big[-2(1-M^2)^{1/4} + dn(\theta) + \frac{\sqrt{1-M^2}}{dn(\theta)}\Big]\,, \tag{32}$$

we present some of the periodic waves and propagating self-similarly soliton solutions, taking the dispersion coefficient β to be of the form $\beta = \beta_0\cos k_b z$ and the external potential $V = V_0\cos t/2(1 + V_0\sin t)$. This choice leads to alternating regions of positive and negative values of both α and γ, which is required for an eventual stability of localized wave solutions. The wave shape remains stable for subsequent times as the running program for a rather long time ($t = 10^6$). If β and V are considered to be $\beta = \cos z$ and $V = 0$ ($V_0 = 0$), then, in Fig. 1 we depict periodic wave solutions expressed by Eq. (32). If β and V are considered to be $\beta = \cos z$ and $V_0 = 0.5$, then, in Fig. 2 we show periodic wave solutions expressed by Eq. (32).

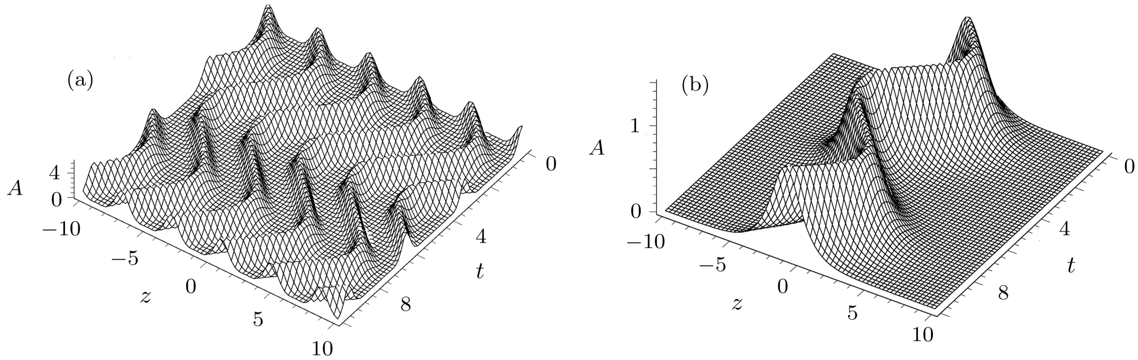

Fig. 1 Periodic travelling wave solutions, as functions of the propagation time. (a) Shows the intensity $A \equiv |\psi|^2$ of solutions expressed by Eq. (32). Coefficients and parameters: $\beta(z) = \cos z$, $V_0 = 0$, $a_0 = 0.01$, $f_{10} = 1$, $k_0 = 0$, $n_0 = 0$, $\beta_0 = 1$, $k_b = 1$, and $b_0 = 1$. (a) $M = 0.8$; (b) $M \to 1$.

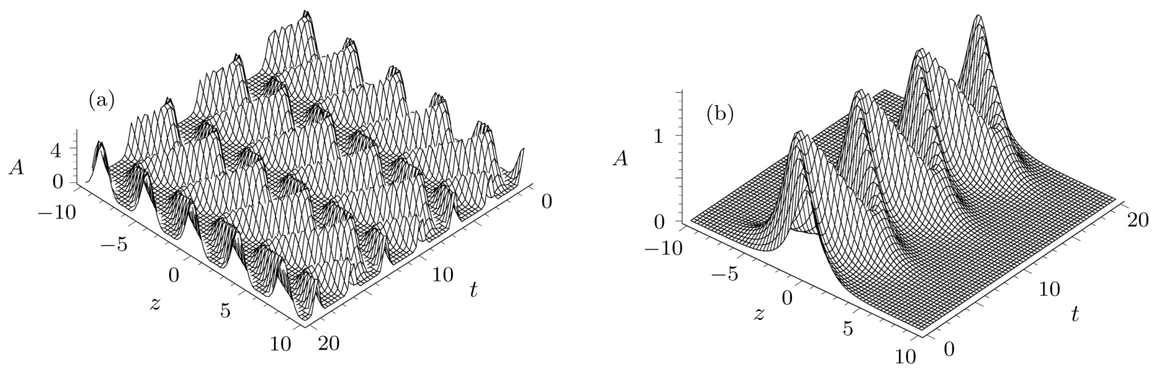

Fig. 2 Travelling solitary wave solutions, as functions of the propagation time. Intensity $A \equiv |\psi|^2$ of solution expressed by Eq. (32). Coefficients and parameters: $\beta(z) = \cos z$, $V_0 = 0, 5$, $M = 1$, $a_0 = 0.01$, $f_{10} = 1$, $k_0 = 0$, $n_0 = 0$, $\beta_0 = 1$, $k_b = 1$, and $b_0 = 1$. (a) $M = 0.8$; (b) $M \to 1$.

From these characteristic examples listed here, one can find that the optical wave types, wave scalings, and their propagating behaviors are artificially controlled via prescribing appropriate system coefficients and parameters, which suggest many potential applications in areas such as optical fiber amplifiers, optical fiber compressors, nonlinear optical switches, and optical communications.

4 Summary and Conclusion

In summary, an improved homogeneous balance principle and an F-expansion technique are applied to the variable coefficient cubic-quintic nonlinear Schrödinger equation with an external potential. Abundant exact self-similar periodic wave solutions are obtained. In some limited cases, different types of soliton solutions are found. A simple and valid procedure is presented for control behavior of solitons, in which one may select the dispersion and the external potential, to control propagation behavior of solitons. The present solution method provides a reliable technique that is more transparent and less tedious than the Jacobi elliptic function ansatz, or other expansion and variational methods. The technique is also applicable to other multidimensional nonlinear partial differential equation systems.

References

[1] C.D. Angelis, IEEE J. Quant. Electron **30** (1994) 818.

[2] C. Chin, T. Kraemer, and M. Mark, Phys. Rev. Lett. **94** (2005) 123201.

[3] F.K. Abdullaev, A. Gammal, L. Tomio, and T. Frederico, Phys. Rev. A **63** (2001) 043604.

[4] W. Zhang, E.M. Wright, H. Pu, and P. Meystre, Phys. Rev. A **68** (2003) 023605.

[5] S. Inouye, *et al.*, Nature (London) **392** (1998) 151.

[6] C. Sulem and P.L. Sulem, *The Nonlinear Schröinger Equation*, Springer, New York (1999).

[7] G.I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics*, Cambridge University Press, Cam-

bridge, England (1996).

[8] C.R. Menyuk, D. Levi, and P. Winternitz, Phys. Rev. Lett. **69** (1992) 3048.

[9] T.M. Monro, D. Moss, M. Bazylenko, C. Martin de Sterke, and L. Poladian, Phys. Rev. Lett. **80** (1998) 4072; T.M. Monro, P.D. Millar, L. Poladian, and C.M. de Sterke, Opt. Lett. **23** (1998) 268.

[10] M. Soljacic, M. Segev, and C.R. Menyuk, Phys. Rev. E **61** (2000) R1048.

[11] K.D. Moll, A.L. Gaeta, and G. Fibich, Phys. Rev. Lett. **90** (2003) 203902.

[12] D. Anderson, M. Desaix, M. Karlsson, M. Lisak, and M.L. Quiroga-Teixeiro, J. Opt. Soc. Am. B **10** (1993) 1185.

[13] M.E. Fermann, V.I. Kruglov, B.C. Thomsen, J.M. Dudley, and J.D. Harvey, Phys. Rev. Lett. **84** (2000) 6010; V.I. Kruglov, A.C. Peacock, J.M. Dudley, and J.D. Harvey, Opt. Lett. **25** (2000) 1753.

[14] T. Hirooka and M. Nakazawa, Opt. Lett. **29** (2004) 498.

[15] F.Ö. Ilday, J.R. Buckley, W.G. Clark, and F.W. Wise, Phys. Rev. Lett. **92** (2004) 213902.

[16] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University, Cambridge (1991).

[17] M. Wadati, H. Sanuki, and K. Konno, Prog. Theor. Phys. **53** (1975) 419.

[18] R. Hirota, Phys. Rev. Lett. **27** (1971) 1192.

[19] M.L. Wang, Phys. Lett. A **199** (1995) 169.

[20] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Phys. Lett. A **289** (1995) 69.

[21] E. Yomba, Chaos, Solitons and Fractals **21** (1995) 1135.

[22] J.M. Zhu and Z.Y. Ma, Commun. Theor. Phys. **46** (1995) 393.

[23] Q. Zhao, S.K. Liu, and Z.T. Fu, Commun. Theor. Phys. **43** (2005) 615.

[24] X.L. Yang and J.S. Tang, Commun. Theor. Phys. **48** (2005) 1.

[25] Milivoj Belić, Nikola Petrović, Wei-Ping Zhong, Rui-Hua Xie, and Goong Chen, Phys. Rev. Lett. **101** (2008) 123904.

[26] Wei-Ping Zhong, Rui-Hua Xie, Milivoj Belić Nikola Petrović, Goong Chen, and Lin Yi, Phys. Rev. A **78** (2008) 023821.

[27] Govind P. Agrawal, *Nonlinear Fiber Optics*, Academic, San Diego, CA (1995).