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Quantum systems with finite Hilbert space

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Abstract

Quantum systems with finite Hilbert space are considered, and phase-space methods like the Heisenberg–Weyl group, symplectic transformations and Wigner and Weyl functions are discussed. A factorization of such systems in terms of smaller subsystems, based on the Chinese remainder theorem, is studied. The general formalism is applied to the case of angular momentum. In this context, $SU(2)$ coherent states are used for analytic representations. Links between the theory of finite quantum systems and other fields of research are discussed.

Contents

1. Introduction

The quantum mechanical formalism and related quantum phase-space methods are usually presented in the context of the harmonic oscillator where both position and momentum take values in $\mathcal{R}$ (real numbers). An analogous formalism can also be developed in the context of quantum systems with a d-dimensional Hilbert space where the dual variables that we call 'position' and 'momentum' take values in $\mathcal{Z}(d)$ (the integers modulo d). Such systems with finite Hilbert space were studied originally by Weyl [1] and also by Schwinger [2]. More recently they have been studied by many authors [3–19] both as a subject in its own right and also in the context of various applications. Related is also the mathematical work of [20]. In this paper, we review the work on quantum systems with finite Hilbert space.

The subject is quantum mechanics but in a position space that consists of a finite lattice with periodicity. In this context, we discuss quantum mechanical topics like position and momentum states and their relation through a Fourier transform; displacements in the position–momentum phase-space; symplectic transformations; Wigner functions; general unitary transformations; composite systems; etc. Due to the finite nature of the space, there are no mathematical difficulties related to convergence. However, the fact that the position and momentum are integers introduces other difficulties, and in some cases number theory can be used to derive interesting results. This blending of quantum mechanics with number theory [21] makes the formalism very exciting from an intellectual point of view and also in terms of potential applications in the emerging area of quantum technologies. So there is close analogy between finite quantum systems and the usual quantum mechanics in a real line; but there are also differences, which we discuss in this article.

At first sight the subject might appear to be highly specialized, but this is not the case. There is a wide variety of applications: quantum optics; quantum computing; two-dimensional electron systems in magnetic fields and the magnetic translation group; the quantum Hall effect; hydrodynamics; mathematical physics; applied mathematics; etc. Sometimes, specialized scientific communities 'rediscover' in their own context ideas that are known to another community (often with a different terminology). The aim of this article is to provide the formalism of finite quantum systems, so that researchers in various areas can use it in their own context.

There are strong links between quantum mechanics and signal processing that have been emphasized by Gabor and Ville [22] a long time ago. In the present context, there is overlap between the theory of finite quantum systems and the subject of signal processing and fast Fourier transforms. The Fourier operator, the Wigner and Weyl functions, various analytic representations, etc, are examples of tools that have been studied by both communities.

In section 2, we discuss the finite Fourier transform and its use to define two dual orthonormal bases that we call 'position' and 'momentum'. We also discuss the uncertainty principle in this context. It is expressed quantitatively with the entropic uncertainty relation.

The phase-space in finite quantum systems is the toroidal lattice $\mathcal{Z}(d) \times \mathcal{Z}(d)$. In section 3 we study displacement operators in this phase-space and the corresponding Heisenberg–Weyl group.

Many of the phase-space methods in the harmonic oscillator context are intimately connected with the fact that the phase-space is the Euclidean plane $\mathcal{R} \times \mathcal{R}$, which has a geometrical structure. In finite systems, the phase-space is in general a set of d^2 points with no geometrical structure. However, when the dimension, d, of the Hilbert space is the power of a prime number ($d = p^n$), $\mathcal{Z}(d)$ is a field (the Galois field $GF(p^n)$). In this case the phase-space $GF(p^n) \times GF(p^n)$ is a finite geometry (e.g. [23]) and has very powerful geometrical properties (e.g. there are well defined translations and rotations, and they form groups). We call the corresponding systems Galois quantum systems. For those systems we can develop phase-space methods that are equally powerful as the harmonic oscillator ones. In section 4, we explain this point and study in detail the group of symplectic transformations $Sp(2, GF(p^n))$ (the analogue of $Sp(2, R)$ symplectic transformations in the harmonic oscillator). The symplectic operator S depends on three integer parameters (in $GF(p^n)$) and is constructed explicitly both analytically and numerically. Various examples elucidate the physical meaning of these transformations in our discrete phase-space.

The displaced parity operators and the displacement operators are related to each other through a Fourier transform and play an important role in phase-space methods. In section 5, we discuss these operators in the context of finite systems and present their properties. Some of these properties are general for all systems; but if we make extra assumptions, we can prove much stronger properties. For example, in the case of Galois quantum systems where the phase-space is a finite geometry, we study Radon transforms of the displaced parity operators and the displacement operators.

In section 6, we discuss the Wigner and Weyl functions for finite quantum systems. They are related to each other through a Fourier transform. The Wigner and Weyl functions are intimately related to the displaced parity operators and the displacement operators, correspondingly. Consequently, their properties are analogous to the properties of the displaced parity operators and the displacement operators, correspondingly.

In section 7, we consider general transformations and show that they can be written as a sum of displacement operators with the Weyl functions as coefficients and also as a sum of displaced parity operators with the Wigner functions as coefficients. In this sense the displacement operators (and also the displaced parity operators) are the 'building blocks' of general transformations. An important class of transformations are the unitary transformations, and we explain that the displacement operators can be used as generators of unitary $SU(d)$ transformations.

Calculations in large Hilbert spaces can be tedious. For example, numerical calculation of the Fourier transform in a large Hilbert space can be very expensive in terms of computer time. The 'fast Fourier transform' method [24] addressed this problem by factorizing the large Hilbert space in terms of smaller spaces. The Fourier transform is performed in the smaller spaces, and the results are 'appropriately' combined to produce the Fourier transform in the large space. In section 8, we use such a factorization based on the Chinese remainder theorem, to show that all unitary transformations and more generally the whole quantum mechanical formalism in the large Hilbert space reduce to calculations in the smaller spaces, which should be performed and combined appropriately to produce the results in the large Hilbert space.

In section 9, we consider transformations in composite finite quantum systems. In a bi-partite system comprising two d-dimensional subsystems, we study local $SU(d) \times SU(d)$ unitary transformations and more general $SU(d^2)$ unitary transformations that can entangle the two subsystems. We also discuss in more detail local and entangling symplectic transformations in these systems and calculate numerically the symplectic operators. More general symplectic transformations in multi-partite systems are also studied. This section provides a connection between finite quantum systems and their transformations discussed in this article and the problem of entanglement. However, it is outside the scope of this article to review entanglement, e.g. to discuss quantities that measure the amount of entanglement, or to discuss mixed states and their characterization as factorizable, separable or entangled ones, etc.

In the next two sections, the above general discussion about finite systems is applied to the case of angular momentum. In section 10, we start with the usual angular momentum states and operators and perform a Fourier transform to get angle states and angle operators.

The $SU(2)$ group becomes involved, and we get some additional features (in comparison with the case of general finite quantum systems) that we discuss. For example, we make the distinction between the Bose sector (with odd dimension) and the Fermi sector (with even dimension) because the formulae are slightly different in these two cases.

In section 11, we study $SU(2)$ coherent states. There exists already a lot of literature on coherent states, and the purpose of this section is not to review the subject of coherent states but to make the connection between $SU(2)$ coherent states and the theory of finite quantum systems presented in the previous sections. Intimately related to coherent states are the analytic representations that exploit the powerful theory of analytic functions to derive strong results in a quantum mechanical context. In section 12, we study the analytic representation in the extended complex plane; and the Dirac contour representation in the extended complex plane. They are both related to $SU(2)$ coherent states.

In section 13, we consider systems with an infinite dimensional Hilbert space, which are 'naturally' expressed as the direct sum of finite Hilbert spaces because a certain class of transformations leaves these finite Hilbert spaces invariant. In such problems we can apply the formalism of finite quantum systems. We discuss two examples: functions on a sphere and spherical harmonics; and the Schwinger $SU(2)$ formalism for two-mode systems.

There is a wide variety of applications of the theory of finite quantum systems, and section 14 is a brief guide to the relevant literature. We conclude in section 15 with an overall discussion of the topics covered in this article.

2. Quantum mechanics of finite systems

2.1. Fourier transform

We consider a quantum system with a d-dimensional Hilbert space $\mathcal{H}$. In this space, we consider an orthonormal basis of 'position states', which we denote as $|X; m\rangle$. Here, X is not a variable, but it simply indicates position states. m belongs to $\mathcal{Z}(d)$ (the integers modulo d). Clearly, the states $|X; m\rangle$ obey the relations

$$\langle X, m|X, n\rangle = \delta(m, n), \qquad \sum_m |X; m\rangle\langle X, m| = \mathbf{1}, \tag{1}$$

where $\delta(n, m)$ is the Kronecker delta, which is equal to 1 when $n = m(\mathrm{mod}(d))$.

The finite Fourier transform plays an important role in the formalism and is defined as

$$F = d^{-1/2} \sum_{m,n} \omega(mn)|X; m\rangle\langle X; n|, \qquad \omega(\alpha) \equiv \omega^{\alpha} = \exp\left[\mathrm{i}\frac{2\pi\alpha}{d}\right]. \tag{2}$$

An identity that is easily proved and that is very useful later is

$$\frac{1}{d}\sum_n \omega^{n(m-\ell)} = \delta(m, \ell). \tag{3}$$

Using it, we prove that

$$FF^{\dagger} = F^{\dagger}F = \mathbf{1}, \qquad F^4 = \mathbf{1} \tag{4}$$

The fact that $F^4 = \mathbf{1}$ implies that the Fourier operator has four eigenvalues: 1, -1, i, $-$i. The multiplicity of these eigenvalues is given in table 1 for the four possible cases where $d = 4m$, $d = 4m + 1$, $d = 4m + 2$ and $d = 4m + 3$ [3]. Using this table, we conclude that $\mathrm{Tr}F = 1 + \mathrm{i}$ when $d = 4m$; $\mathrm{Tr}F = 1$ when $d = 4m + 1$; $\mathrm{Tr}F = 0$ when $d = 4m + 2$; and $\mathrm{Tr}F = \mathrm{i}$ when $d = 4m + 3$.

Table 1. The multiplicity of the eigenvalues 1, −1, i, −i of the Fourier operator in a d-dimensional Hilbert space.

	1	−1	i	−i
$d = 4m$	$m + 1$	m	m	$m - 1$
$d = 4m + 1$	$m + 1$	m	m	m
$d = 4m + 2$	$m + 1$	$m + 1$	m	m
$d = 4m + 3$	$m + 1$	$m + 1$	$m + 1$	m

The eigenvectors of F have been studied in [6, 25]. We note that there is a lot of work by the signal processing and fast Fourier transform community on the Fourier operator and its properties, and clearly these results are useful for the theory of finite quantum systems and its applications.

2.2. *Position and momentum bases*

Using the Fourier transform, we define another orthonormal basis, the 'momentum states', as

$$|P; m\rangle = F|X; m\rangle = d^{-1/2} \sum_n \omega^{mn} |X; n\rangle. \tag{5}$$

Here, P is not a variable, but it simply indicates momentum states. It is now clear that an arbitrary state $|s\rangle$ in $\mathcal{H}$ can be expanded as

$$|s\rangle = \sum_n \lambda_n |X; n\rangle = \sum_m \mu_m |P; m\rangle, \qquad \lambda_n = d^{-1/2} \sum_m \mu_m \omega^{mn}. \tag{6}$$

$\{\lambda_n\}$ and $\{\mu_n\}$ are 'wavefunctions' for the state $|s\rangle$ in the position and momentum representations, correspondingly.

We also define the 'position and momentum operators', x and p, as

$$x = \sum_{n=0}^{d-1} n|X; n\rangle\langle X; n|, \qquad p = \sum_{n=0}^{d-1} n|P; n\rangle\langle P; n|, \qquad p = FxF^\dagger. \tag{7}$$

We note that n are integers modulo d and consequently the x and p are defined modulo $d\mathbf{1}$. However, below we will use exponentials of these operators and they are single-valued.

It is easily seen that

$$FxF^\dagger = p, \qquad FpF^\dagger = -x. \tag{8}$$

The x and p are finite matrices, and consequently their powers are not all independent. Using the Cayley–Hamilton theorem of the theory of matrices, we can express the d-power of these matrices as linear combinations of lower powers. The eigenvalues of both of these matrices are the integers from 0 to $(d - 1)$, and therefore the corresponding characteristic polynomial is

$$Q(y) = y \prod_{n=0}^{d-1} (y - n) \equiv y^d + \mu_{d-1} y^{d-1} + \cdots + \mu_1 y, \tag{9}$$

where the above equation defines the integers μ_n. The Cayley–Hamilton theorem states that

$$Q(x) = Q(p) = 0. \tag{10}$$

This implies than an arbitrary function $f(x)$ is defined modulo the polynomial $Q(x)$ and therefore it can be reduced to the remainder polynomial $R(x)$ of order $d - 1$, as follows:

$$f(x) = Q(x)S(x) + R(x), \qquad R(x) = \sum_{n=0}^{d-1} \sigma_n x^n. \tag{11}$$

The coefficients σ_n are calculated if we insert in this equation the roots of $Q(x)$ and get a system of d equations with d unknowns:

$$\sum_{n=0}^{d-1} \sigma_n m^n = f(m), \qquad m = 0, \ldots, (d-1). \tag{12}$$

The same comment applies to an arbitrary function of p also.

2.3. Δ_m *functions*

We introduce the Δ_m functions [16], which are the analogues of the delta function and its derivatives in the harmonic oscillator case. The Δ_m functions are of course well defined because all sums are finite. We first define

$$\Delta_0(x) = d^{-1} \sum_{n=0}^{d-1} \omega(nx), \tag{13}$$

where x is a real number. It is easily seen that

$$\Delta_0(x+d) = \Delta_0(x) \tag{14}$$

and that

$$x \neq 0 \to \Delta_0(x) = \frac{\omega(xd) - 1}{d(\omega - 1)}, \tag{15}$$

$$x = 0 \to \Delta_0(0) = 1. \tag{16}$$

For integer values of x we get

$$\Delta_0(n) = \delta(n, 0). \tag{17}$$

We next introduce the function

$$\Delta_m(x) = \partial_x^m \Delta_0(x) = d^{-1} \sum_{n=0}^{d-1} \left(\mathrm{i} \frac{2\pi n}{d} \right)^m \omega(nx). \tag{18}$$

We note that n are integers modulo d, and extra care is required when we do calculations with powers of n.

These functions are useful in the calculations of matrix elements. For example,

$$\langle X, n | p^k | X, m \rangle = \frac{1}{d} \sum_{\ell} \ell^k \omega[\ell(n-m)] = \left(\frac{d}{2\pi \mathrm{i}} \right)^k \Delta_k(n-m). \tag{19}$$

A direct consequence of equation (10) is that

$$\left(\frac{d}{2\pi \mathrm{i}} \right)^d \Delta_d(n-m) + \mu_{d-1} \left(\frac{d}{2\pi \mathrm{i}} \right)^{d-1} \Delta_{d-1}(n-m) + \cdots + \mu_1 \left(\frac{d}{2\pi \mathrm{i}} \right) \Delta_1(n-m) = 0. \tag{20}$$

Therefore for x in $\mathcal{Z}(d)$ only the $\Delta_m(x)$ with $0 \leqslant m \leqslant d-1$ are independent of each other.

2.4. *Uncertainty principle and entropic uncertainty relations*

A Fourier transform (in any context) is intimately related to an uncertainty principle. It states that the two distributions, of a function of a variable and its Fourier transform that is a function of the dual variable, cannot both be narrow. In our case the two distributions associated with the state $|s\rangle$ of equation (6) are $|\lambda_n|^2$, and $|\mu_m|^2$, and the uncertainty relation states that they cannot both be narrow. In order to quantify the uncertainty relation, we introduce the entropies

$$S_X = -\sum_n |\lambda_n|^2 \ln |\lambda_n|^2, \qquad S_P = -\sum_m |\mu_m|^2 \ln |\mu_m|^2. \tag{21}$$

It has been proved in [26] that they obey the inequality

$$S_X + S_P \geqslant \ln d. \tag{22}$$

This is the entropic uncertainty relation for finite quantum systems. We note that for the states $|X; n\rangle$ we get $S_X = 0$ and $S_P = \ln d$ and the above relation holds as the equality. Similarly for the states $|P; n\rangle$ we get $S_X = \ln d$, and $S_P = 0$, and the above relation holds as the equality. For another discussion of the uncertainties in this context, see [27].

3. Displacements in phase-space

3.1. The phase-space: a toroidal lattice

The position–momentum phase-space of the harmonic oscillator is the plane $R \times R$. Let $\mathcal{X}$ and $\mathcal{P}$ be the harmonic oscillator position and momentum operators, correspondingly. Infinitesimal displacements in the phase-space are performed with the operators $\mathbf{1} + \mathrm{i}(\delta A)\mathcal{X}$ and $\mathbf{1} + \mathrm{i}(\delta B)\mathcal{P}$, and their non-commutativity is described with the commutation relation $[\mathcal{X}, \mathcal{P}] = \mathrm{i}\mathbf{1}$. Finite displacements are performed with the operators $\exp(\mathrm{i}A\mathcal{X})$ and $\exp(\mathrm{i}B\mathcal{P})$, which obey the relation

$$\exp(\mathrm{i}A\mathcal{X}) \exp(\mathrm{i}B\mathcal{P}) = \exp(\mathrm{i}B\mathcal{P}) \exp(\mathrm{i}A\mathcal{X}) \exp(-\mathrm{i}AB). \tag{23}$$

They form the Heisenberg–Weyl group (e.g. [28–30]).

In our finite quantum system, both the position and momentum are integers modulo d. Therefore the position–momentum phase-space is the toroidal lattice $\mathcal{Z}(d) \times \mathcal{Z}(d)$. In this phase-space we define the displacement operators

$$Z = \exp\left[\mathrm{i}\frac{2\pi}{d}x\right], \qquad X = \exp\left[-\mathrm{i}\frac{2\pi}{d}p\right]. \tag{24}$$

They are unitary operators and perform displacements along the P and X axes in the phase-space. Indeed we can show that

$$Z^\alpha|P; m\rangle = |P; m + \alpha\rangle, \qquad Z^\alpha|X; m\rangle = \omega(\alpha m)|X; m\rangle, \tag{25}$$

$$X^\beta|P; m\rangle = \omega(-m\beta)|P; m\rangle, \qquad X^\beta|X; m\rangle = |X; m + \beta\rangle. \tag{26}$$

The displacement operators obey the relations

$$X^d = Z^d = \mathbf{1}, \qquad X^\beta Z^\alpha = Z^\alpha X^\beta \omega^{-\alpha\beta}, \tag{27}$$

where α, β are integers in $\mathcal{Z}(d)$. These relations are easily proved if we calculate the matrix elements of both sides in the $|X; m\rangle$ basis, taking into account equations (25), (26). Equations $X^d = Z^d = \mathbf{1}$ are related to the toroidal nature of the phase-space. We note that in the special case $d = 2$ the $\langle X; m|X|X; n\rangle$ and $\langle X; m|Z|X; n\rangle$ become the Pauli matrices σ_x and σ_z correspondingly (and the notation for the operators X and Z has been inspired by that).

3.2. The Heisenberg–Weyl group

The general displacement operators are defined as

$$D(\alpha,\beta)=Z^{\alpha}X^{\beta}\omega(-2^{-1}\alpha\beta),\qquad [D(\alpha,\beta)]^{\dagger}=D(-\alpha,-\beta). \tag{28}$$

The phase-factor $\omega(-2^{-1}\alpha\beta)$ is not essential, but if it is not included the equation $[D(\alpha,\beta)]^{\dagger}=D(-\alpha,-\beta)$ and many other equations below, it will require additional phase factors. The $D(\alpha,\beta)$ are unitary operators and are associated with the Heisenberg–Weyl group in the context of finite quantum systems. Using equation (27) we can prove the multiplication rule

$$D(\alpha_1,\beta_1)D(\alpha_2,\beta_2)=D(\alpha_1+\alpha_2,\beta_1+\beta_2)\omega[2^{-1}(\alpha_1\beta_2-\alpha_2\beta_1)]. \tag{29}$$

We can also prove that

$$D(\alpha,\beta)x[D(\alpha,\beta)]^{\dagger}=x-\beta\mathbf{1},\qquad D(\alpha,\beta)p[D(\alpha,\beta)]^{\dagger}=p-\alpha\mathbf{1} \tag{30}$$

and that

$$\begin{aligned}D(\alpha,\beta)|X;m\rangle&=\omega(2^{-1}\alpha\beta+\alpha m)|X;m+\beta\rangle\\ D(\alpha,\beta)|P;m\rangle&=\omega(-2^{-1}\alpha\beta-\beta m)|P;m+\alpha\rangle.\end{aligned} \tag{31}$$

Acting with the Fourier operator on the displacement operators we get

$$FXF^{\dagger}=Z,\qquad FZF^{\dagger}=X^{-1},\qquad FD(\alpha,\beta)F^{\dagger}=D(\beta,-\alpha). \tag{32}$$

We note that in finite quantum systems the Heisenberg–Weyl group is discrete, there is no Lie algebra (there are no infinitesimal displacements) and the role of the position and momentum operators, x, p, is limited. For this reason our formalism is based mainly on the displacement operators X, Z. Consequently, the commutator $[x,p]$ plays no important role. For completeness, we easily calculate it to be

$$[x,p]=-\mathrm{i}\left(\frac{d}{2\pi}\right)(n-m)\Delta_1(n-m)|X;n\rangle\langle X;m|. \tag{33}$$

It is seen that the commutator $[x,p]$ is not equal to $\mathrm{i}\mathbf{1}$.

3.3. Sinusoidal functions of the position and momentum operators

The operator X is an 'exponential' of the momentum operator, p. We can also define 'cosine' and 'sine' operators as

$$C_X=\frac{1}{2}(X+X^{\dagger})=\cos\left(\frac{2\pi}{d}p\right),\qquad S_X=\frac{1}{2\mathrm{i}}(X^{\dagger}-X)=\sin\left(\frac{2\pi}{d}p\right), \tag{34}$$

$$[C_X,S_X]=0,\qquad C_X^2+S_X^2=\mathbf{1}. \tag{35}$$

They are Hermitian operators, and their eigenvectors are the $|P;m\rangle$. In a similar way, we can define the operators C_Z, S_Z.

$$C_Z=\frac{1}{2}(Z+Z^{\dagger})=\cos\left(\frac{2\pi}{d}x\right),\qquad S_Z=\frac{1}{2\mathrm{i}}(Z-Z^{\dagger})=\sin\left(\frac{2\pi}{d}x\right), \tag{36}$$

$$[C_Z,S_Z]=0,\qquad C_Z^2+S_Z^2=\mathbf{1}. \tag{37}$$

The commutator of the operator C_Z with the operator C_X can be found using equation (27).

4. Symplectic transformations

4.1. Symplectic transformations in the harmonic oscillator phase-space

An important class of transformations in the harmonic oscillator case is the Bogoliubov transformations

$$\mathcal{X}' = \kappa\mathcal{X} + \lambda\mathcal{P}, \qquad \mathcal{P}' = \mu\mathcal{X} + \nu\mathcal{P}, \qquad \kappa\nu - \lambda\mu = 1, \tag{38}$$

which preserve the commutation relations

$$[\mathcal{X}', \mathcal{P}'] = [\mathcal{X}, \mathcal{P}] = \mathrm{i}\mathbf{1}. \tag{39}$$

They are associated with the symplectic group $Sp(2, R)$ (e.g. [30–32]) which has three generators, $\mathcal{X}^2$, $\mathcal{P}^2$ and $\mathcal{X}\mathcal{P}$. These transformations play an important role in quantum optics, where they are related to the concept of squeezing (e.g. [33]), in superconductivity, in the theory of accelerated observers (e.g. [34]), etc.

The above transformations have been expressed in terms of the position and momentum operators, $\mathcal{X}$, $\mathcal{P}$, which are generators of infinitesimal displacements. They can also be expressed in terms of the displacements operators that perform finite displacements as

$$\begin{aligned} \exp(\mathrm{i}\mathcal{X}') &= \exp[\mathrm{i}(\kappa\mathcal{X} + \lambda\mathcal{P})] = \exp(\mathrm{i}\kappa\mathcal{X})\exp(\mathrm{i}\lambda\mathcal{P})\exp\left(\frac{\mathrm{i}\kappa\lambda}{2}\right), \\ \exp(\mathrm{i}\mathcal{P}') &= \exp[\mathrm{i}(\mu\mathcal{X} + \nu\mathcal{P})] = \exp(\mathrm{i}\mu\mathcal{X})\exp(\mathrm{i}\nu\mathcal{P})\exp\left(\frac{\mathrm{i}\mu\nu}{2}\right). \end{aligned} \tag{40}$$

Using the relation $\kappa\nu-\lambda\mu = 1$, we can show that these transformations preserve equation (23). It is this form of the symplectic transformations that will be extended to the finite quantum systems below.

4.2. Symplectic transformations in Galois quantum systems

In the $\mathcal{Z}(d) \times \mathcal{Z}(d)$ phase-space of a finite quantum system, we consider the unitary transformations

$$\begin{aligned} &X' = SXS^\dagger = X^\kappa Z^\lambda \omega(2^{-1}\kappa\lambda) = D(\lambda, \kappa), \\ &Z' = SZS^\dagger = X^\mu Z^\nu \omega(2^{-1}\mu\nu) = D(\nu, \mu), \\ &\kappa\nu - \lambda\mu = 1 \ (\mathrm{mod}(d)), \end{aligned} \tag{41}$$

where $\kappa, \lambda, \mu, \nu$ are integers in $\mathcal{Z}(d)$. S is a unitary operator that will be constructed explicitly below. It is easily seen that these transformations preserve equation (27). Therefore, the X', Z' can also be used as displacement operators. In comparison to X, Z, they displace in different directions. The phase-factors $\omega(2^{-1}\kappa\lambda)$ and $\omega(2^{-1}\mu\nu)$ in the above equations are not essential, but if they are omitted, many of the equations below will require additional phase-factors.

The transformations of equation (41) contain three independent variables (the fourth is defined by the constraint). The important question here is whether for a given triplet κ, λ, μ, there exists ν that satisfies the constraint. And this is intimately connected to the existence of 'inverses' of the elements of $\mathcal{Z}(d)$ because if they exist then $\nu = \kappa^{-1}(\lambda\mu + 1)$.

When d is a power of a prime p ($d = p^n$), $\mathcal{Z}(p^n)$ is a Galois field [35] (the notation $GF(p^n)$ is also used in the literature) and all non-zero elements have an inverse. For $n = 1$ it is easily seen that $\mathcal{Z}(p)$ is a field. For $n \geqslant 2$ the concept of field extension of $\mathcal{Z}(p)$ of degree n is required. The elements are written as polynomials of an indeterminate x with coefficients in $\mathcal{Z}(p)$. These polynomials are defined modulo an irreducible polynomial of degree n. Different irreducible polynomials of the same degree n lead to isomorphic finite fields, and in this sense there is only one finite field which we denote as $GF(p^n)$. Addition and multiplication rules in $GF(p^n)$ are for $n \geqslant 2$ different from the 'normal' ones and are not easy to construct, but for practical purposes they can be found in computer libraries (e.g. MATLAB). When d is not a power of a prime, $\mathcal{Z}(d)$ is a commutative ring with a unity, and inverses do not necessarily exist.

We call Galois quantum systems those with a dimension that is a power of a prime. The name Galois quantum system aims to remind the reader that in the $n \geqslant 2$ case, the Galois addition and multiplication rules should be used. In this case the phase-space $GF(p^n) \times GF(p^n)$ is a finite geometry (e.g. [23]). This is a geometrical structure with strong mathematical properties. For example, transformations like dilations, contractions, discrete rotations, etc. are well defined and form groups.

In the case of non-Galois quantum systems (with a dimension that is not a power of a prime), the phase-space is a set of points with no geometrical structure. Consequently, symplectic transformations are 'accidental' in the sense that for a given triplet (κ, λ, μ) we might or we might not find ν such that $\kappa\nu - \lambda\mu = 1\ (\mathrm{mod}(d))$.

More generally, the harmonic oscillator phase-space formalism, which is a set of very powerful techniques, can be transferred to other quantum systems, provided that the corresponding phase-space has some geometrical structure. In Galois quantum systems the phase-space has a geometrical structure, and as we will see later, most of the harmonic-oscillator phase-space formalism can be transfered in this context. Of course many results are valid for all finite systems. In our discussion we make clear which assumptions are needed for each result.

Below we will study in more detail the symplectic transformations $S(\kappa, \lambda, \mu)$ for Galois quantum systems [14, 16]. For simplicity, all our examples involve systems with a dimension that is the first power of a prime number. In this case, the addition and multiplication rules are simply the 'normal' ones. [36–38] have studied symplectic transformations over a Galois field from a pure mathematics point of view. Here, we study these transformations at an applied level and in our particular context.

We can easily show that they form a group. First, we show that combining two of the transformations (41) we get another transformation of the same type:

$$\begin{aligned}
&S(\kappa_2, \lambda_2, \mu_2)S(\kappa_1, \lambda_1, \mu_1) = S(\epsilon, \zeta, \eta),\\
&\epsilon = \kappa_1\kappa_2 + \lambda_1\mu_2,\\
&\zeta = \kappa_1\lambda_2 + \lambda_1\kappa_2^{-1}(1 + \lambda_2\mu_2),\\
&\eta = \kappa_2\mu_1 + \mu_2\kappa_1^{-1}(1 + \lambda_1\mu_1).
\end{aligned} \tag{42}$$

We can also show that associativity holds, that the identity element exists and that inverses exist. We call this group $Sp(2, GF(p^n))$ (in analogy with $Sp(2, R)$ in the harmonic oscillator).

4.3. Example: the $S(\xi, 0, 0)$ dilation/contraction (squeezing) transformations

In this subsection, we give an example that will elucidate the physical meaning of an important special case of symplectic transformations.

We consider the unitary operators $S(\xi, 0, 0)$, which (by definition) lead to the transformations

$$SXS^\dagger = X^\xi, \qquad SZS^\dagger = Z^{\xi^{-1}}. \tag{43}$$

It is easily seen that

$$S(\xi, 0, 0)|X; n\rangle = |X; \xi n\rangle, \qquad S(\xi, 0, 0)|P; n\rangle = |P; \xi^{-1} n\rangle, \tag{44}$$

and consequently the operator S can be written as

$$S(\xi, 0, 0) = \sum_n |X; n\rangle\langle X; \xi n| = \sum_n |P; \xi^{-1} n\rangle\langle P; n|. \tag{45}$$

We note that as n takes all values in $\mathcal{Z}(p)$, ξn takes all values in $\mathcal{Z}(p)$, and this is important for the proof that S is a unitary operator.

Equation (44) shows that S stretches the X-axis and simultaneously contracts the P-axis by the same factor (similar to squeezing in the harmonic oscillator). It should be appreciated that stretching and contraction in a space of finite points is a one-to-one mapping from $\mathcal{Z}(p)$ to $\mathcal{Z}(p)$ (reordering of the points on a line). For example, for $p = 3$ and $\xi = 2$ ($\xi^{-1} = 2$)

$$\begin{aligned} &S|X; 0\rangle = |X; 0\rangle, \qquad &&S|X; 1\rangle = |X; 2\rangle, \qquad &&S|X; 2\rangle = |X; 1\rangle, \\ &S|P; 0\rangle = |P; 0\rangle, \qquad &&S|P; 1\rangle = |P; 2\rangle, \qquad &&S|P; 2\rangle = |P; 1\rangle. \end{aligned} \tag{46}$$

The operators $S(\xi, 0, 0)$ (for the various non-zero values of ξ in $\mathcal{Z}(p)$) form a subgroup of $Sp(2, \mathcal{Z}(p))$. Indeed, using equation (42), we can show that $S(\xi_1, 0, 0)S(\xi_2, 0, 0) = S(\xi_1\xi_2, 0, 0)$.

4.4. *Example: the $S(1, \xi, 0)$ and $S(1, 0, \xi)$ symplectic transformations*

We consider the operators

$$S(1, \xi, 0) = \sum_m \omega(2^{-1}\xi m^2)|X; m\rangle\langle X; m| \tag{47}$$

and prove that

$$SXS^\dagger = XZ^\xi\omega(2^{-1}\xi), \qquad [S, Z] = 0. \tag{48}$$

These operators (for the various values of ξ in $\mathcal{Z}(p)$) form a subgroup of $Sp(2, \mathcal{Z}(p))$. Indeed, using equation (42), we can show that $S(1, \xi_1, 0)S(1, \xi_2, 0) = S(1, \xi_1 + \xi_2, 0)$.

The corresponding symplectically transformed states are

$$\begin{aligned} |X'; n\rangle &= S(1, \xi, 0)|X; n\rangle = \omega(2^{-1}\xi n^2)|X; n\rangle, \\ |P'; n\rangle &= S(1, \xi, 0)|P; n\rangle = d^{-1/2}\sum_m \omega(2^{-1}\xi m^2 + mn)|X; m\rangle. \end{aligned} \tag{49}$$

It is seen that the X'-basis is the same as the X-basis, up to phase factors; but the P'-basis is very different from the P-basis. Extra care is required with the phases, as is already obvious from the Fourier transform of equation (5).

Another set of operators very similar to those in equation (47) are the $S(1, 0, \xi)$ given by

$$S(1, 0, \xi) = \sum_m \omega(-2^{-1}\xi m^2)|P; m\rangle\langle P; m|. \tag{50}$$

They also form a subgroup of $Sp(2, \mathcal{Z}(p))$. The corresponding symplectically transformed states are

$$\begin{aligned} |X'; n\rangle &= S(1, 0, \xi)|X; n\rangle = d^{-1/2}\sum_m \omega(-2^{-1}\xi m^2 - mn)|P; m\rangle, \\ |P'; n\rangle &= S(1, 0, \xi)|P; n\rangle = \omega(-2^{-1}\xi n^2)|P; n\rangle. \end{aligned} \tag{51}$$

It is seen that the P'-basis is the same as the P-basis, up to phase factors; but the X'-basis is very different from the X-basis.

4.5. Analytical calculation of the symplectic operator $S(\kappa, \lambda, \mu)$

Using equation (42), we can express the general symplectic operator $S(\kappa, \lambda, \mu)$ as

$$\begin{aligned} &S(\kappa, \lambda, \mu) = S(1, 0, \xi_1)S(1, \xi_2, 0)S(\xi_3, 0, 0), \\ &\xi_1 = \mu\kappa(1 + \lambda\mu)^{-1}, \\ &\xi_2 = \lambda\kappa^{-1}(1 + \lambda\mu), \\ &\xi_3 = \kappa(1 + \lambda\mu)^{-1}. \end{aligned} \tag{52}$$

We have given analytic expressions for the operators $S(1, 0, \xi_1)$, $S(1, \xi_2, 0)$, $S(\xi_3, 0, 0)$ in equations (50), (47) and (45), correspondingly; and through equation (52) we get an analytical expression for any symplectic operator.

4.6. Numerical calculation of the symplectic operator $S(\kappa, \lambda, \mu)$

In this section, we discuss how to calculate the operator $S(\kappa, \lambda, \mu)$ numerically. We consider the matrix $\langle X; m|Z'|X; n\rangle = \omega(2^{-1/2}\nu\mu + n\nu)\delta(m, n + \mu)$ and calculate numerically its (normalized) eigenvectors, which are the states $|X'; m\rangle$ (up to phase factors). The phases are important in order to obey the relations $X^{\beta}|X; m\rangle = |X; m + \lambda\rangle$. To calculate the phases, we start with the lowest eigenvector, $|X'; 0\rangle$ (for which we can choose any phase), and we use numerically the equation

$$(X')^m|X'; 0\rangle = |X'; m\rangle, \tag{53}$$

where X' is the matrix $\omega(2^{-1/2}\kappa\lambda + n\lambda)\delta(m, n + \kappa)$. The $|X'; m\rangle$ calculated through this equation should differ from the ones calculated earlier as eigenvectors of the matrix Z' only by a phase factor. This is a test that the numerical work is correct and at the same time it provides the required phases. The operator S is now given in a matrix form as $S(n, m) \equiv \langle X; n|S|X; m\rangle$.

As an example, we consider a five-dimensional Hilbert space ($d = 5$) and $S(1, -1, -1)$, which leads (by definition) to the transformations

$$X' = SXS^{\dagger} = XZ^{-1}\omega\left(-\tfrac{1}{2}\right), \qquad Z' = SZS^{\dagger} = X^{-1}Z^2\omega(-1). \tag{54}$$

The matrix elements $S(n, m)$ for this case are given in table 2.

4.7. Symplectically transformed bases and displacement operators

We have started with the orthonormal bases $|X; n\rangle$ and $|P; n\rangle$, and with respect to them we have defined the displacement operators X and Z. The symplectic operators $S(\kappa, \lambda, \mu)$ transform this basis into a new one

$$|X'; n\rangle = S(\kappa, \lambda, \mu)|X; n\rangle, \qquad |P'; n\rangle = S(\kappa, \lambda, \mu)|P; n\rangle \tag{55}$$

Table 2. The coefficients $S(n, m)$ for the transformations of equation (54). Here, $z_1 = 0.4472$, $z_2 = z_1\omega^{-1}$ and $z_3 = \omega^{-2}$.

	$m = -2$	$m = -1$	$m = 0$	$m = 1$	$m = 2$
$n = -2$	z_2	$-z_2^*$	z_3^*	$-z_3^*$	z_2^*
$n = -1$	z_1	$-z_3$	z_1	$-z_2^*$	z_2^*
$n = 0$	z_2	$-z_3$	z_2^*	$-z_3$	z_2
$n = 1$	z_2^*	$-z_2^*$	z_1	$-z_3$	z_1
$n = 2$	z_2^*	$-z_3^*$	z_3^*	$-z_2^*$	z_2

and the displacement operators X and Z into $X' = SXS^\dagger = D(\lambda, \kappa)$ and $Z' = SZS^\dagger = D(\nu, \mu)$. More generally, the displacement operators $D(\alpha, \beta)$ are transformed into

$$D'(\alpha, \beta) \equiv SD(\alpha, \beta)S^\dagger = D(\alpha\nu + \beta\lambda, \alpha\mu + \beta\kappa). \tag{56}$$

It is seen that $D'(\alpha, \beta)$ perform displacements by (α, β) in the $P' - X'$ bases or equivalently displacements by $(\alpha\nu + \beta\lambda, \alpha\mu + \beta\kappa)$ in the $P - X$ bases.

5. Parity and displacement operators

5.1. Displaced parity operators

We define the parity operator around the origin as

$$P(0, 0) = F^2, \qquad [P(0, 0)]^2 = \mathbf{1}. \tag{57}$$

Its name is justified from the fact that

$$\begin{aligned} &P(0, 0)|X; m\rangle = |X; -m\rangle, &&P(0, 0)|P; m\rangle = |P; -m\rangle,\\ &P(0, 0)x[P(0, 0)]^\dagger = -x, &&P(0, 0)p[P(0, 0)]^\dagger = -p,\\ &P(0, 0)Z[P(0, 0)]^\dagger = Z^\dagger, &&P(0, 0)X[P(0, 0)]^\dagger = X^\dagger. \end{aligned} \tag{58}$$

The fact that $[P(0, 0)]^2 = \mathbf{1}$ implies that the parity operator has two eigenvalues, $1, -1$. The multiplicity of these eigenvalues can be found using table 1, which contains the multiplicities of the eigenvalues of the Fourier operator. For even $d = 2n$, the multiplicity of 1 is $n + 1$, the multiplicity of -1 is $n - 1$ and $\mathrm{Tr}\, P(0, 0) = 2$. For odd $d = 2n + 1$, the multiplicity of 1 is $n + 1$, the multiplicity of -1 is n and the $\mathrm{Tr}\, P(0, 0) = 1$. We call $\mathcal{H}_1(0, 0)$ and $\mathcal{H}_{-1}(0, 0)$ the subspaces spanned by the eigenvectors corresponding to the eigenvalues 1 and -1, respectively. It is clear that $\mathcal{H}$ is the direct sum of $\mathcal{H}_1(0, 0)$ and $\mathcal{H}_{-1}(0, 0)$. We call $\pi_1(0, 0)$ and $\pi_{-1}(0, 0)$ the projection operators into these subspaces.

$$\pi_i(0, 0)\pi_j(0, 0) = \delta_{ij}\pi_i(0, 0), \qquad (i, j = 1, -1), \qquad \pi_1(0, 0) + \pi_{-1}(0, 0) = \mathbf{1}. \tag{59}$$

Then

$$P(0, 0) = \pi_1(0, 0) - \pi_{-1}(0, 0). \tag{60}$$

The displaced parity operator (parity operator around the point (α, β) in phase-space) has been studied in the context of the harmonic oscillator in [39–41]. In the present context it is defined as

$$P(\alpha, \beta) = D(\alpha, \beta)P(0, 0)[D(\alpha, \beta)]^\dagger = D(2\alpha, 2\beta)P(0, 0) = P(0, 0)[D(2\alpha, 2\beta)]^\dagger, \tag{61}$$

$$[P(\alpha, \beta)]^2 = \mathbf{1}. \tag{62}$$

All our comments above about the multiplicities of the eigenvalues and about the trace of $P(0, 0)$ are also valid for $P(\alpha, \beta)$. We also introduce here the subspaces $\mathcal{H}_1(\alpha, \beta)$ and $\mathcal{H}_{-1}(\alpha, \beta)$ spanned by the eigenvectors of $P(\alpha, \beta)$ with eigenvalues 1 and -1, correspondingly. We call $\pi_1(\alpha, \beta)$ and $\pi_{-1}(\alpha, \beta)$ the projection operators into these subspaces and prove that

$$P(\alpha, \beta) = \pi_1(\alpha, \beta) - \pi_{-1}(\alpha, \beta), \qquad \pi_i(\alpha, \beta) = D(\alpha, \beta)\pi_i(0, 0)[D(\alpha, \beta)]^\dagger. \tag{63}$$

We note that the product of two displaced parity operators is not a displaced parity operator (it is a displacement operator):

$$P(\alpha,\beta)P(\gamma,\delta) = D(2\alpha-2\gamma, 2\beta-2\delta)\omega(2\beta\gamma-2\alpha\delta). \tag{64}$$

5.2. Marginal properties of the displacement operators

We now consider systems with odd dimension, $d = 2j+1$, where j is an integer. In this case the 2 and d are coprime and the 2^{-1} exists within $\mathcal{Z}(d)$ (which is in general a ring). It is easily seen that $2^{-1} = j+1$ (since $2(j+1) = 1$ modulo $2j+1$). We prove that in systems with odd dimension the displacement operators obey the relations

$$\begin{aligned}\frac{1}{d}\sum_{\beta=0}^{d-1} D(\alpha,\beta) &= |P;2^{-1}\alpha\rangle\langle P;-2^{-1}\alpha|,\\ \frac{1}{d}\sum_{\alpha=0}^{d-1} D(\alpha,\beta) &= |X;2^{-1}\beta\rangle\langle X;-2^{-1}\beta|\end{aligned} \tag{65}$$

and also

$$\frac{1}{d}\sum_{\alpha,\beta} D(\alpha,\beta) = P(0,0). \tag{66}$$

There are analogous relations for the harmonic oscillator [41] (with the summations replaced by integrals), and the use of 2^{-1} above has been chosen in order to make this analogy clear. But as we explained, $2^{-1}\alpha$ is here the integer $(j+1)\alpha$ (otherwise the state $|P;2^{-1}\alpha\rangle$ makes no sense). In order to prove these equations, we take the matrix elements of both sides with regard to the states $\langle X;n|$ and $|X;m\rangle$ and use equation (3) (which is valid for integers, and here again we need the fact that 2^{-1} is the integer $j+1$).

We next consider Galois quantum systems and act with the symplectic operator S on the left and with $S^\dagger$ on the right of equation (65). Taking into account equations (55) and (56), we can prove

$$\begin{aligned}\frac{1}{d}\sum_{\beta=0}^{d-1} D'(\alpha,\beta) &= |P';2^{-1}\alpha\rangle\langle P';-2^{-1}\alpha|,\\ \frac{1}{d}\sum_{\alpha=0}^{d-1} D'(\alpha,\beta) &= |X';2^{-1}\beta\rangle\langle X';-2^{-1}\beta|,\\ \frac{1}{d}\sum_{\alpha,\beta} D'(\alpha,\beta) &= SP(0,0)S^\dagger \equiv P'(0,0),\end{aligned} \tag{67}$$

where the 'prime states' on the right-hand side have been defined in equation (55). They show that in Galois quantum systems we have relations analogous to (65) not only in the original X–P frame but in all frames. Analogous results are known for the harmonic oscillator displacement operators. Of course, in order to be able to talk about frames and validity of properties in all frames, we need to have a phase-space with a geometrical structure; and this is the case in Galois quantum systems where the phase-space is a finite geometry.

5.3. Marginal properties of the displaced parity operators

Acting with the parity operator $P(0,0)$ on the right of equations (65) and using equation (61), we prove that in systems with odd dimension,

$$\begin{aligned}
\frac{1}{d}\sum_{\beta=0}^{d-1} P(\alpha,\beta) &= |P;\alpha\rangle\langle P;\alpha|,\\
\frac{1}{d}\sum_{\alpha=0}^{d-1} P(\alpha,\beta) &= |X;\beta\rangle\langle X;\beta|,\\
\frac{1}{d}\sum_{\alpha,\beta} P(\alpha,\beta) &= \mathbf{1}.
\end{aligned} \tag{68}$$

We note that the last of these equations is consistent with the fact that in systems with odd dimension $\mathrm{Tr}\,P(\alpha,\beta)=1$.

We next consider Galois quantum systems, and acting with the parity operator $P(0,0)$ on the right of equations (67) we prove that analogous relations to (68) hold in other frames also:

$$\begin{aligned}
\frac{1}{d}\sum_{\beta=0}^{d-1} P'(\alpha,\beta) &= |P';\alpha\rangle\langle P';\alpha|,\\
\frac{1}{d}\sum_{\alpha=0}^{d-1} P'(\alpha,\beta) &= |X';\beta\rangle\langle X';\beta|,\\
\frac{1}{d}\sum_{\alpha,\beta} P'(\alpha,\beta) &= \mathbf{1}.
\end{aligned} \tag{69}$$

where the ‘prime states’ on the right-hand side have been defined in equation (55).

5.4. Fourier transform between the displacement operators and the displaced parity operators

We consider systems with odd dimension, and multiplying equation (66) by $D(\gamma,\delta)$ on the left and by $[D(\gamma,\delta)]^\dagger$ on the right and using equation (61) we prove

$$\frac{1}{d}\sum_{\alpha,\beta} D(\alpha,\beta)\omega(\beta\gamma-\alpha\delta) = P(\gamma,\delta). \tag{70}$$

This shows that the displaced parity operators are related to the displacement operators through a two-dimensional Fourier transform. The inverse Fourier transform gives

$$\frac{1}{d}\sum_{\gamma,\delta} P(\gamma,\delta)\omega(-\beta\gamma+\alpha\delta) = D(\alpha,\beta). \tag{71}$$

5.5. Radon transforms

We consider Galois quantum systems and substitute equation (56) into equations (67). We introduce the variables $\epsilon=\alpha\nu+\beta\lambda$ and $\zeta=\alpha\mu+\beta\kappa$ and point out that in Galois fields as β (or α) takes all values in $GF(p^n)$, ϵ and ζ also take all values in $GF(p^n)$. Therefore we can rewrite equations (67) as

$$\begin{aligned}
&\frac{1}{d}\sum_{\epsilon,\zeta} D(\epsilon,\zeta)\delta(\kappa\epsilon-\lambda\zeta,\alpha) = |P';2^{-1}\alpha\rangle\langle P';-2^{-1}\alpha|,\\
&\frac{1}{d}\sum_{\epsilon,\zeta} D(\epsilon,\zeta)\delta(-\mu\epsilon+\nu\zeta,\beta) = |X';2^{-1}\beta\rangle\langle X';-2^{-1}\beta|.
\end{aligned} \tag{72}$$

In these equations we sum over all points on the lines $\kappa\epsilon-\lambda\zeta=\alpha$ and $-\mu\epsilon+\nu\zeta=\beta$. On the right-hand side we have the 'prime states' defined in equation (55).

The summation along one of the lines is the analogue in our context of the integration of a two-dimensional function $f(x,y)$ along a line in the Euclidean plane x–y which is the Radon transform [42]. Therefore, equations (72) are the Radon transform in a finite geometry.

We note that the phase factors $\omega(2^{-1}\kappa\lambda)$ and $\omega(2^{-1}\mu\nu)$ in the symplectic transformations of equations (41) have been chosen so that they lead to Radon transforms in the simple form of equations (72). Any other choice of phase factors is also acceptable, but the corresponding equations (56) and (72), will be more complicated.

In a similar way we can rewrite equations (69) in terms of the Radon transform:

$$\begin{aligned}
&\frac{1}{d}\sum_{\epsilon,\zeta} P(\epsilon,\zeta)\delta(\kappa\epsilon-\lambda\zeta,\alpha) = |P';\alpha\rangle\langle P';\alpha|,\\
&\frac{1}{d}\sum_{\epsilon,\zeta} P(\epsilon,\zeta)\delta(-\mu\epsilon+\nu\zeta,\beta) = |X';\beta\rangle\langle X';\beta|.
\end{aligned} \tag{73}$$

This is proved using equations (72).

The inverse Radon transform of equation (73) is performed in two steps. In the first step, we Fourier transform equations (73) and using equation (71) we get

$$\begin{aligned}
&D(\lambda\beta,\kappa\beta) = \sum_{\alpha} |P';\alpha\rangle\langle P';\alpha|\omega(-\alpha\beta),\\
&D(\nu\alpha,\mu\alpha) = \sum_{\beta} |X';\beta\rangle\langle X';\beta|\omega(\alpha\beta).
\end{aligned} \tag{74}$$

In the second step, we perform the two-dimensional Fourier transform of equation (70) to get the displaced parity operators.

The inverse Radon transform of equation (72) can also be writen in a similar way. We gave explicitly equation (74) because from a physical point of view it is much more useful as it involves projectors. Later in the study of Wigner and Weyl functions, we will take the trace of these projectors with respect to density matrices and get probabilities that are measurable quantities.

6. Wigner and Weyl functions

The Wigner and Weyl (or characteristic) functions play an important role in fundamental problems in quantum mechanics [43] and also in more applied problems in quantum optics. Recently Wigner tomography [44–47] provided a method for constructing the Wigner function from its Radon transforms that can be measured experimentally.

The equations of quantum mechanics are usually expressed in terms of the wavefunction (or the density matrix). They can also be expressed in terms of the Wigner (or Weyl) function because all the information in the density matrix is also contained in the corresponding Wigner function. However, the Wigner function can be reconstructed from all its Radon transforms (for all angles in the phase plane) which are probabilities. Consequently, the equations of quantum mechanics can be expressed in terms of tomographic probabilities only. This has been studied by V Man'ko, M Man'ko, O Man'ko and their collaborators [48].

In this section, we discuss these functions in our context of finite quantum systems. The Wigner functions are intimately related to the displaced parity operators. The Weyl functions are intimately related to the displacement operators. Therefore, most of the results in this section are a direct consequence of the results in the previous section. We will see that there is one-to-one mapping between the operators, Wigner functions and Weyl functions. And if one of them is given (e.g. the Wigner function) then we can find the other two (i.e. the corresponding operator and its Weyl function).

Related to Wigner functions are the P and Q functions and the Moyal formalism [49,50]. The P and Q functions are intimately related to coherent states that have not been introduced at this stage. Later we will apply the theory of finite systems to systems with angular momentum j, and in that context we have $SU(2)$ coherent states and we will discuss briefly the P and Q representations.

The Wigner and Weyl functions also play an important role in signal processing (e.g. [51]) where the Weyl function is known as an ambiguity function.

Recently, the so-called extended phase-space has been introduced, in the context of the harmonic oscillator [52,53]. This work considers simultaneously in a four-dimensional phase-space the Wigner function describing the uncertainties (noise) in the system and the Weyl function describing the correlations in the system. This formalism has not yet been applied to finite systems and will not be discussed here.

6.1. Wigner functions

Let Θ be an arbitrary operator. The Wigner function corresponding to the operator Θ is defined as

$$W(\Theta;\alpha,\beta) = \mathrm{Tr}[\Theta P(\alpha,\beta)]. \tag{75}$$

If Θ is a Hermitian operator, then the Wigner function is real; but for non-Hermitian operators it is complex. The Wigner function is the Fourier transform of the matrix elements of the operator Θ:

$$\begin{aligned} W(\Theta;\alpha,\beta) &= \omega(2\alpha\beta)\sum_{\ell}\omega(-2\alpha\ell)\Theta_X(\ell,2\beta-\ell) \\ &= \omega(-2\alpha\beta)\sum_{\ell}\omega(2\beta\ell)\Theta_P(\ell,2\alpha-\ell), \\ \Theta_X(m,\ell) &\equiv \langle X;m|\Theta|X;\ell\rangle, \qquad \Theta_P(m,\ell) \equiv \langle P;m|\Theta|P;\ell\rangle. \end{aligned} \tag{76}$$

The inverse Fourier transform can be used to express the matrix elements of the operator Θ in terms of the Wigner function.

Multiplying equations (68) with the operator Θ and taking the trace, we can prove that in the case of systems with odd dimension, the Wigner function obeys the following 'marginal properties':

$$\begin{aligned} &\frac{1}{d}\sum_{\beta=0}^{d-1} W(\Theta;\alpha,\beta) = \Theta_P(\alpha,\alpha), \\ &\frac{1}{d}\sum_{\alpha=0}^{d-1} W(\Theta;\alpha,\beta) = \Theta_X(\beta,\beta), \\ &\frac{1}{d}\sum_{\alpha,\beta} W(\Theta;\alpha,\beta) = \mathrm{Tr}\Theta. \end{aligned} \tag{77}$$

Density matrices ρ are Hermitian, and the corresponding Wigner functions are real. In this case the Wigner function can be interpreted as a pseudoprobability distribution of the particle in the position–momentum phase-space, which is consistent with quantum mechanics. It is not a real probability distribution because it can take negative values. But the marginal properties show that the 'integral' (in our context finite sum) of the Wigner function along the momentum axis gives the probability distribution $\langle X;\beta|\rho|X;\beta\rangle$ along the position axis; and the integral of the Wigner function along the position axis gives the probability distribution $\langle P;\beta|\rho|P;\beta\rangle$ along the momentum axis. This strengthens the interpretation of the Wigner function as a pseudoprobability distribution of the particle in the position–momentum phase-space.

Another set of marginal properties that is less known involves the absolute value of the Wigner function squared [52]. Again, we consider systems with odd dimension, and using equation (76) we can prove

$$\begin{aligned}
&\frac{1}{d}\sum_{\beta=0}^{d-1}|W(\Theta;\alpha,\beta)|^2=\sum_{k=0}^{d-1}|\Theta_P(k,2\alpha-k)|^2,\\
&\frac{1}{d}\sum_{\alpha=0}^{d-1}|W(\Theta;\alpha,\beta)|^2=\sum_{k=0}^{d-1}|\Theta_X(k,2\beta-k)|^2,\\
&\frac{1}{d}\sum_{\alpha,\beta}|W(\Theta;\alpha,\beta)|^2=\mathrm{Tr}[\Theta\Theta^\dagger].
\end{aligned}\tag{78}$$

We have seen in equation (76) that the matrix elements of Θ are related to the Wigner function through a Fourier transform and the first two equations are Parseval's theorem in this context.

We next use equation (63) to express the Wigner function of a density matrix ρ as the difference between two probabilities:

$$\begin{aligned}
&W(\rho;\alpha,\beta)=Q_1(\rho;\alpha,\beta)-Q_{-1}(\rho;\alpha,\beta),\\
&Q_i(\rho;\alpha,\beta)\equiv\mathrm{Tr}[\rho\pi_i(\alpha,\beta)],\qquad Q_1(\rho;\alpha,\beta)+Q_{-1}(\rho;\alpha,\beta)=1.
\end{aligned}\tag{79}$$

This clarifies the fact that the Wigner function can take negative values. It also shows that $-1\leqslant W(\rho;\alpha,\beta)\leqslant 1$.

The Wigner function of the sum of two operators is simply the sum the Wigner functions corresponding to these operators. With regard to the product of two operators, the corresponding Wigner function is given by the Moyal star product, which is written in the context of finite systems as

$$\begin{aligned}
W(\Theta_1)\star W(\Theta_2)\equiv W(\Theta_1\Theta_2;\alpha,\beta)=\frac{1}{d^2}\sum_{\alpha_1,\beta_1,\alpha_2,\beta_2}\omega(2\alpha_2\beta_1-2\alpha_1\beta_2)\\
\times W(\Theta_1;\alpha+\alpha_1,\beta+\beta_1)W(\Theta_2;\alpha+\alpha_2,\beta+\beta_2).
\end{aligned}\tag{80}$$

Since the mapping between operators and Wigner functions is one-to-one, the Moyal star product has properties similar to the product of two operators. For example, the Moyal star product of two operators is (in general) non-commutative and the Moyal star product of three operators is associative. A full discussion of the Moyal star product and the related topic of deformations [54] is beyond the scope of this article.

6.2. Weyl functions

The Weyl function corresponding to the operator Θ is defined in terms of the displacement operators as

$$\tilde{W}(\Theta;\alpha,\beta)\equiv\mathrm{Tr}[\Theta D(\alpha,\beta)].\tag{81}$$

The Weyl function is the Fourier transform of the matrix elements of the operator Θ:

$$\begin{aligned}\tilde{W}(\Theta;\alpha,\beta) &= \omega(2^{-1}\alpha\beta)\sum_{\ell}\omega(\alpha\ell)\Theta_X(\ell,\beta+\ell)\\ &= \omega(-2^{-1}\alpha\beta)\sum_{\ell}\omega(-\beta\ell)\Theta_P(\ell,\alpha+\ell),\end{aligned} \tag{82}$$

where $\Theta_X(m,\ell)$ and $\Theta_P(m,\ell)$ have been defined in equation (76). The inverse Fourier transform can be used to express the matrix elements of the operator Θ in terms of the Weyl function.

The Weyl function is related to the Wigner function through a 'two-dimensional' Fourier transform (indicated with the tilde in the notation). Indeed, using equations (76), (82) we can prove that

$$\tilde{W}(\Theta;\alpha,\beta) = \frac{1}{d}\sum_{\gamma,\delta} W(\Theta;\gamma,\delta)\omega(\alpha\delta-\beta\gamma). \tag{83}$$

We recall here that the displaced parity operators (which are intimately related to the Wigner functions) are connected to the displacement operators (which are intimately related to the Weyl functions) through a two-dimensional Fourier transform and equation (70) can be used for an alternative derivation of equation (83).

The Weyl function is a generalized correlation function. In order to see this we consider the density matrix $\rho = |s\rangle\langle s|$ of a pure state $|s\rangle$. The corresponding Weyl function is $\langle s|D(\alpha,\beta)|s\rangle$. In order to calculate the correlation function of some wavefunction $f(x)$, we displace it from x to $x+\alpha$ and calculate the overlap with the original wavefunction. In the Weyl function we do exactly the same, but we displace the state in both position and momentum. In this sense the parameters α, β entering in the Weyl function are position and momentum increments.

Using equations (65) we can prove that in the case of systems with odd dimension, the Weyl function obeys the following 'marginal properties':

$$\begin{aligned}&\frac{1}{d}\sum_{\beta=0}^{d-1}\tilde{W}(\Theta;\alpha,\beta) = \Theta_P(-2^{-1}\alpha, 2^{-1}\alpha),\\ &\frac{1}{d}\sum_{\alpha=0}^{d-1}\tilde{W}(\Theta;\alpha,\beta) = \Theta_X(-2^{-1}\beta, 2^{-1}\beta),\\ &\frac{1}{d}\sum_{\alpha,\beta}\tilde{W}(\Theta;\alpha,\beta) = W(\Theta;0,0).\end{aligned} \tag{84}$$

Another set of marginal properties that is less known involves the absolute value of the Weyl function squared [52]. Using equation (82) we can prove that

$$\begin{aligned}&\frac{1}{d}\sum_{\beta=0}^{d-1}|\tilde{W}(\Theta;\alpha,\beta)|^2 = \sum_{\ell=0}^{d-1}|\Theta_P(\ell,\alpha+\ell)|^2,\\ &\frac{1}{d}\sum_{\alpha=0}^{d-1}|\tilde{W}(\Theta;\alpha,\beta)|^2 = \sum_{\ell=0}^{d-1}|\Theta_X(\ell,\beta+\ell)|^2,\\ &\frac{1}{d}\sum_{\alpha,\beta}|\tilde{W}(\Theta;\alpha,\beta)|^2 = \mathrm{Tr}[\Theta\Theta^\dagger].\end{aligned} \tag{85}$$

We have seen in equation (82) that the matrix elements of Θ are related to the Weyl function through a Fourier transform and the first two equations are Parseval's theorem in this context.

The Weyl function of the sum of two operators is simply the sum of the Weyl functions corresponding to these operators. With regard to the product of two operators we can prove that

$$\tilde{W}(U_1U_2;\alpha,\beta) = \frac{1}{d}\sum_{\alpha_1,\beta_1}\omega(2^{-1}\alpha_1\beta - 2^{-1}\alpha\beta_1) \times \tilde{W}(U_1;\alpha+\alpha_1,\beta+\beta_1)\tilde{W}(U_2;-\alpha_1,-\beta_1). \tag{86}$$

Since the mapping between operators and Weyl functions is one-to-one, the above product has properties similar to the product of two operators (e.g. it is associative, non-commutative in general, etc.).

6.3. Radon transforms

In Galois quantum systems we can prove stronger marginal properties for the Wigner functions than equation (77). Multiplying equation (73) with the operator Θ and taking the trace, we get

$$\begin{aligned}
&\frac{1}{d}\sum_{\epsilon,\zeta} W(\Theta;\epsilon,\zeta)\delta(\kappa\epsilon - \lambda\zeta,\alpha) = \langle P';\alpha|\Theta|P';\alpha\rangle,\\
&\frac{1}{d}\sum_{\epsilon,\zeta} W(\Theta;\epsilon,\zeta)\delta(-\mu\epsilon + \nu\zeta,\beta) = \langle X';\beta|\Theta|X';\beta\rangle,
\end{aligned} \tag{87}$$

where the 'prime states' on the right-hand side have been defined in equation (55). This is the Radon transform in the present context and says that the sum of the Wigner function of an operator Θ on a line is equal to the matrix elements of this operator with the appropriate states. If the operator Θ is a density matrix, then these matrix elements are probabilities.

The inverse Radon transform is performed in two steps. In the first step, we Fourier transform equation (87), and using equation (83) we get

$$\begin{aligned}
&\tilde{W}(\Theta;\lambda\beta,\kappa\beta) = \sum_{\alpha}\langle P';\alpha|\Theta|P';\alpha\rangle\omega(-\alpha\beta),\\
&\tilde{W}(\Theta;\nu\alpha,\mu\alpha) = \sum_{\beta}\langle X';\beta|\Theta|X';\beta\rangle\omega(\alpha\beta).
\end{aligned} \tag{88}$$

We note that these relations are intimately related to equations (74). They give the Weyl function and the second step involves the inverse of the two-dimensional Fourier transform of equation (83), in order to get the Wigner function.

In the case that the operator Θ is a density matrix, the matrix elements on the right-hand side of equation (88) are probabilities (measurable quantities) and these equations can be used to construct the Wigner function from measurable quantities.

7. General transformations

In this section we consider general transformations and show that they can be expanded in terms of the displacement operators with the Weyl functions as coefficients and also in terms of the displaced parity operators with the Wigner functions as coefficients. In this sense both the displacement operators and the displaced parity operators can be viewed as building blocks of general transformations. The results are valid for any finite dimension d.

We also consider unitary transformations and show that the displacement operators are generators of the $SU(d)$ group.

7.1. Expansion of an arbitrary operator in terms of displacement operators with the Weyl functions as coefficients

We consider an arbitrary transformation Θ and show that it can be expanded in terms of the displacement operators with the Weyl functions as coefficients:

$$\Theta = \frac{1}{d}\sum_{\alpha,\beta}\tilde{W}(\Theta; -\alpha, -\beta)D(\alpha, \beta). \tag{89}$$

This is proved by taking the matrix elements of both sides with regard to the states $\langle X; n|$ and $|X; m\rangle$ and using equation (82).

We have seen earlier that the displacement operators displace states and operators in phase-space. We see here that they also play another important role, as building blocks of general transformations.

As an example, we consider the symplectic operator of equation (52), for a system with dimension a prime number p. Using equation (52) in conjunction with equations (45), (49), (51), we calculate its Weyl function,

$$\tilde{W}(S; \alpha, \beta) = \frac{1}{d}\sum_{\ell,n}\omega[2^{-1}\xi_2 n^2 - 2^{-1}\xi_1\ell^2 + (\xi_3 - 1)\ell n - \ell\beta + \alpha\xi_3 n - 2^{-1}\alpha\beta], \tag{90}$$

where all variables are integers in $\mathcal{Z}(p)$. This sum is the product of two Gauss sums and a phase factor. Indeed, for $\xi_2 \neq 0 \pmod p$, it can be written as

$$\tilde{W}(S; \alpha, \beta) = \frac{1}{d}\sum_{\ell,n}\omega[\gamma_1^{-1}(\gamma_1 n + \gamma_2\ell + \gamma_3)^2 + \gamma_4(\ell + \gamma_5)^2 + \gamma_6], \tag{91}$$

where

$$\begin{aligned}
&\gamma_1 = 2^{-1}\xi_2, \qquad \gamma_2 = 2^{-1}(\xi_3 - 1), \qquad \gamma_3 = 2^{-1}\alpha\xi_3,\\
&\gamma_4 = -2^{-1}\xi_1 - (2\xi_2)^{-1}(\xi_3 - 1)^2, \qquad \gamma_5 = [\beta\xi_2 + \alpha\xi_3(\xi_3 - 1)][\xi_1\xi_2 + (\xi_3 - 1)^2]^{-1},\\
&\gamma_6 = -2^{-1}\alpha\beta - (2\xi_2)^{-1}(\alpha\xi_3)^2 - \gamma_4^{-1}\gamma_5^2,
\end{aligned} \tag{92}$$

are integers in $\mathcal{Z}(p)$. We change variables into $n' = \gamma_1 n + \gamma_2\ell + \gamma_3$ and $\ell' = \ell + \gamma_5$. As n, ℓ take all values in $\mathcal{Z}(p)$, the variables n', ℓ' also take all values in $\mathcal{Z}(p)$. Therefore,

$$\begin{aligned}
\tilde{W}(S; \alpha, \beta) &= d^{-1}\omega(\gamma_6)\left[\sum_{n'}\omega(\gamma_1^{-1}n'^2)\right]\left[\sum_{\ell'}\omega(\gamma_4\ell'^2)\right]\\
&= d^{-1}\omega(\gamma_6)G(\gamma_1^{-1})G(\gamma_4),
\end{aligned} \tag{93}$$

where

$$G(\alpha) \equiv \sum_n \omega(\alpha n^2) \tag{94}$$

is the Gauss sum. The symplectic operator S can be expanded in terms of the displacement operators as

$$S = \frac{G(\gamma_1^{-1})G(\gamma_4)}{d^2}\sum_{\alpha,\beta}\omega[-2^{-1}\alpha\beta - (2\xi_2)^{-1}(\alpha\xi_3)^2 - \gamma_4^{-1}\gamma_5^2]D(\alpha, \beta). \tag{95}$$

7.2. Expansion of an arbitrary operator in terms of displaced parity operators with the Wigner functions as coefficients

We consider an arbitrary transformation Θ and show that it can be expanded in terms of the displaced parity operators with the Wigner functions as coefficients:

$$\Theta = \frac{1}{d}\sum_{\alpha,\beta} W(\Theta;\alpha,\beta)P(\alpha,\beta). \tag{96}$$

This is proved by taking the matrix elements of both sides with regard to the states $\langle X; n|$ and $|X; m\rangle$ and using equation (76).

We note here that as we explained in equation (64) the product of two displaced parity operators is not a displaced parity operator and this might be inconvenient if we want to multiply two operators expressed in the form of equation (96). However the Moyal star product of equation (80) evaluates the Wigner function of the product of two operators, and this can be used to provide the product of two operators in the form of equation (96).

7.3. The displacement operators as generators of unitary transformations

An important class of transformations are the unitary transformations that form the group $SU(d)$. This group has d^2-1 generators. It has been shown in [55] that the d^2-1 displacement operators $D(\alpha,\beta)$ (with $(\alpha,\beta)\neq(0,0)$) are generators of $SU(d)$ transformations in the Hilbert space $\mathcal{H}$. They are an alternative to the usual Cartan–Weyl generators.

Their commutator (which describes the corresponding $SU(d)$ algebra) is given by

$$\begin{aligned}[][D(\alpha_1,\beta_1), D(\alpha_2,\beta_2)] &\equiv D(\alpha_1,\beta_1)D(\alpha_2,\beta_2) - D(\alpha_2,\beta_2)D(\alpha_1,\beta_1)\\ &= 2\mathrm{i}\sin\left[\frac{\pi}{d}(\alpha_1\beta_2-\alpha_2\beta_1)\right]D(\alpha_1+\alpha_2,\beta_1+\beta_2).\end{aligned} \tag{97}$$

Infinitesimal $SU(d)$ transformations can be written as

$$g = \mathbf{1} + \sum_{\alpha,\beta}\epsilon(\alpha,\beta)D(\alpha,\beta), \tag{98}$$

where $\epsilon(\alpha,\beta)$ are infinitesimal coefficients.

Finite $SU(d)$ transformations involve the exponentials of the generators, which are here finite matrices. Since the exponential of a finite matrix is a polynomial, we expect that a finite $SU(d)$ transformation U can be written as the finite sum:

$$U = \sum_{\alpha,\beta}\mu(\alpha,\beta)D(\alpha,\beta). \tag{99}$$

Comparison of this with equation (95) shows that this is indeed the case and in fact the coefficients $\mu(\alpha,\beta)$ are simply the Weyl functions:

$$\mu(\alpha,\beta) = \frac{1}{d}\tilde{W}(U;-\alpha,-\beta). \tag{100}$$

As an application of this general result we consider a system with a Hamiltonian H that at $t=0$ is in a state $|s(0)\rangle = \sum_m s_m|X; m\rangle$. Then the evolution operator can be expanded in terms of the displacement operators as

$$\exp(\mathrm{i}Ht) = \sum_{\alpha,\beta}\tau(\alpha,\beta;t)D(\alpha,\beta), \qquad \tau(\alpha,\beta;t) = \frac{1}{d}\mathrm{Tr}[\exp(\mathrm{i}Ht)D(-\alpha,-\beta)] \tag{101}$$

The displacement operators act on the state of the system as described in equation (31). Therefore, the state of the system at time t is

$$|s(t)\rangle = \sum_{\alpha,\beta,m}\tau(\alpha,\beta;t)s_m\omega(2^{-1}\alpha\beta+\alpha m)|X; m+\beta\rangle. \tag{102}$$

8. Factorization of large systems in terms of smaller ones

The formalism described above is from a practical point of view easily implemented in systems with small dimension d, but for large d it can be very tedious. Especially with unitary $SU(d)$ transformations, explicit calculations for large d can be very cumbersome. As an example, we mention the Fourier transform where the computational time increases quickly with d.

The 'fast Fourier transform' method [24] addressed this problem by factorizing 'appropriately' the large Hilbert space in terms of smaller spaces. The Fourier transform is performed in the smaller spaces, and the results are 'appropriately' combined to produce the Fourier transform in the large space. There are many fast Fourier transform schemes, and here we will use the scheme by Good, which is based on a factorization of d as $d = d_1 \times \cdots \times d_N$, where the factors $d_1, \ldots, d_n$ are coprime with respect to each other. This scheme is based on the Chinese remainder theorem.

In this section, based on [15], we use the scheme by Good for fast Fourier transforms, in a large quantum system. In our context we are interested not only in Fourier transforms but in all unitary transformations and more generally in the whole quantum mechanical formalism. For example, we show how the phase-space $\mathcal{Z}(d) \times \mathcal{Z}(d)$ factorizes into the 'multidimensional' phase-space $(\mathcal{Z}(d_1) \times \mathcal{Z}(d_1)) \times \cdots \times (\mathcal{Z}(d_N) \times \mathcal{Z}(d_N))$, and we study the correspondence between displacement operators and Wigner and Weyl functions in these two phase-spaces. We also show how unitary transformations and other quantum mechanical calculations in the large quantum system can be reduced to calculations in many small quantum systems. In a sense the whole quantum mechanics in the large system decomposes to quantum mechanics in many small systems.

Related factorizations based on the Chinese remainder theorem have been used in a different context in [56].

8.1. One-to-one mappings between $\mathcal{Z}(d)$ and $\mathcal{Z}(d_1) \times \cdots \times \mathcal{Z}(d_N)$

We factorize a number d as $d = d_1 \times \cdots \times d_N$, where the factors $d_1, \ldots, d_n$ are coprime with respect to each other. In this subsection, we introduce two one-to-one mappings between $\mathcal{Z}(d)$ and $\mathcal{Z}(d_1) \times \cdots \times \mathcal{Z}(d_N)$. In order to do this we first introduce the integers r_i and t_i:

$$r_i = \frac{d}{d_i}, \qquad t_i r_i = 1 \ (\mathrm{mod}\ d_i). \tag{103}$$

Here, t_i is the 'inverse' of r_i within $\mathcal{Z}(d_i)$. Its existence is guaranteed by the fact that r_i and d_i are coprime. We also introduce $s_i = t_i r_i$ in $\mathcal{Z}(d)$. We note that since t_i is the inverse of r_i in $\mathcal{Z}(d_i)$, $s_i = t_i r_i$, defined in $\mathcal{Z}(d)$ is an integer multiple of d_i plus 1.

The first one-to-one mapping between $\mathcal{Z}(d)$ and $\mathcal{Z}(d_1) \times \cdots \times \mathcal{Z}(d_N)$ is

$$m \leftrightarrow (m_1, \ldots, m_N), \qquad m_i = m(\mathrm{mod}\ d_i), \qquad m = \sum_i m_i s_i. \tag{104}$$

The second one-to-one mapping between $\mathcal{Z}(d)$ and $\mathcal{Z}(d_1) \times \cdots \times \mathcal{Z}(d_N)$ (which we call the 'dual mapping') is

$$m \leftrightarrow (\bar{m}_1, \ldots, \bar{m}_N), \qquad \bar{m}_i = m t_i(\mathrm{mod}\ d_i), \qquad m = \sum_i \bar{m}_i r_i(\mathrm{mod}\ d). \tag{105}$$

The proof that these mappings are indeed one-to-one is based on the Chinese remainder theorem and will not be discussed here.

An important formula that is needed for many proofs in this section is

$$\omega(r_i s_i) = \omega_i \equiv \exp\left(\mathrm{i}\frac{2\pi}{d_i}\right), \qquad i \neq j \to \omega(r_i s_j) = 1. \tag{106}$$

Using this, we see easily that

$$\omega(mn) = \prod_{i=1}^{N} \omega_i(m_i \bar{n}_i). \tag{107}$$

As an example, we consider the case where $d = 15$, $d_1 = 3$ and $d_2 = 5$. In this case we find that $r_1 = 5$, $t_1 = 2$, $s_1 = 10$ and $r_2 = 3$, $t_2 = 2$, $s_2 = 6$. Therefore, $m = 10m_1 + 6m_2 = 5\bar{m}_1 + 3\bar{m}_2$. For example, $m = 7$ in $\mathcal{Z}(15)$ corresponds to $(m_1 = 1, m_2 = 2)$ according to the mapping of equation (104) and $(\bar{m}_1 = 2, \bar{m}_2 = 4)$ according to the mapping of equation (105).

8.2. *Quantum states*

We consider a quantum system with Hilbert space $\mathcal{H}(d)$ (here the d in the notation indicates the dimension of the space). As above, we factorize d as $d = d_1 \times \cdots \times d_N$, where the factors $d_1, \ldots, d_N$ are coprime with respect to each other.

In this subsection we introduce an isomorphism between the Hilbert space $\mathcal{H}(d)$ and a product of Hilbert spaces $\mathcal{H}(d_1) \otimes \cdots \otimes \mathcal{H}(d_N)$. We use the previous notation with an extra index i to indicate states and operators in the Hilbert space $\mathcal{H}(d_i)$. We first introduce the following mapping between the position and momentum bases in $\mathcal{H}(d)$ and in $\mathcal{H}(d_1) \otimes \cdots \otimes \mathcal{H}(d_N)$:

$$|X; m\rangle \leftrightarrow |X_1; \bar{m}_1\rangle \otimes \cdots \otimes |X_N; \bar{m}_N\rangle, \qquad |P; m\rangle \leftrightarrow |P_1; m_1\rangle \otimes \cdots \otimes |P_N; m_N\rangle. \tag{108}$$

Here, we use equation (104) to map the momentum basis in $\mathcal{H}(d)$ into the momentum basis in $\mathcal{H}(d_1) \otimes \cdots \otimes \mathcal{H}(d_N)$. We then use equations (106) and (107) to prove that the position basis in $\mathcal{H}(d)$ is mapped into the position basis in $\mathcal{H}(d_1) \otimes \cdots \otimes \mathcal{H}(d_N)$ as described in equation (108) (which uses the 'dual map' of equation (105)).

We next consider an arbitrary operator Θ and its matrix elements with regard to the bases $|X; m\rangle$ and also $|P; m\rangle$:

$$\sigma(n, m) \equiv \langle X; n|\Theta|X; m\rangle, \qquad \tau(\ell, k) \equiv \langle P; \ell|\Theta|P; k\rangle. \tag{109}$$

$\sigma(n, m)$ are related to $\tau(\ell, k)$ through a double Fourier transform:

$$\sigma(n, m) = d^{-1} \sum_{\ell,k} \tau(\ell, k)\omega(\ell n - mk). \tag{110}$$

We also consider the matrix elements of the same operator with regard to the bases $|X_1; \bar{m}_1\rangle \otimes \cdots \otimes |X_N; \bar{m}_N\rangle$ and also $|P_1; m_1\rangle \otimes \cdots \otimes |P_N; m_N\rangle$, which we denote as $\sigma(\{\bar{n}_i\}, \{\bar{m}_j\})$ and $\tau(\{\ell_i\}, \{k_j\})$, correspondingly. The matrix elements $\sigma(n, m)$ are here relabelled as $\sigma(\{\bar{n}_i\}, \{\bar{m}_j\})$; and similarly the matrix elements $\tau(\ell, k)$ are relabelled $\tau(\{\ell_i\}, \{k_j\})$. $\sigma(\{\bar{n}_i\}, \{\bar{m}_j\})$ are related to $\tau(\{\ell_i\}, \{k_j\})$ through a multi-dimensional Fourier transform:

$$\sigma(\{\bar{n}_i\}, \{\bar{m}_j\}) = \left(\prod_{i=1}^{N} d_i^{-1}\right) \sum_{\ell_i,k_i} \tau(\{\ell_i\}, \{k_j\}) \prod_{i=1}^{N} \omega_i(\ell_i \bar{n}_i - \bar{m}_i k_i). \tag{111}$$

We can check explicitly that the 'large' Fourier transform of equation (110) is equivalent to the combination of 'small' Fourier transforms of equation (111). It is this equivalence that is exploited in 'fast Fourier transforms'.

A special class of operators are the ones that can be factorized as

$$\Theta = \Theta_1 \otimes \cdots \otimes \Theta_N, \tag{112}$$

where Θ_i are operators acting on the Hilbert spaces $\mathcal{H}(d_i)$. In this special case, the matrix elements factorize:

$$\sigma(\{\bar{n}_i\}\{\bar{m}_i\}) = \prod_{i=1}^{N} \sigma_i(\bar{n}_i, \bar{m}_i), \qquad \sigma_i(\bar{n}_i, \bar{m}_i) \equiv \langle X_i; \bar{n}_i|\Theta_i|X_i; \bar{m}_i\rangle,$$
$$\tau(\{\ell_i\}, \{k_i\}) = \prod_{i=1}^{N} \tau_i(\ell_i, k_i), \qquad \tau_i(\ell_i, k_i) \equiv \langle P_i; \ell_i|\Theta_i|P_i; k_i\rangle. \tag{113}$$

8.3. Displacements in phase-space

The toroidal lattice phase-space factorizes into the 'multi-dimensional' toroidal lattice phase-space $(\mathcal{Z}(d_1) \times \mathcal{Z}(d_1)) \times \cdots \times (\mathcal{Z}(d_N) \times \mathcal{Z}(d_N))$. Here we show that the displacement operators in $\mathcal{Z}(d) \times \mathcal{Z}(d)$ can be expressed as products of the displacement operators in the various 'factor phase-spaces' as

$$D(\alpha, \beta) = \prod_{i=1}^{N} D_i(\alpha_i, \bar{\beta}_i), \tag{114}$$

where as explained in equations (104) and (105) $\alpha_i = \alpha(\mathrm{mod}\ d_i)$ and $\bar{\beta}_i = \beta t_i(\mathrm{mod}\ d_i)$. In order to prove this, we first prove that

$$Z = \sum_m |P;\ m+1\rangle\langle P;\ m| = \sum_m \prod_{i=1}^{N} |P_i; m_i+1\rangle\langle P_i; m_i| = \prod_{i=1}^{N} Z_i,$$
$$X = \sum_m |X; m+1\rangle\langle X; m| = \sum_m \prod_{i=1}^{N} |X_i; \bar{m}_i + t_i\rangle\langle X_i; \bar{m}_i| = \prod_{i=1}^{N} X_i^{t_i}. \tag{115}$$

For simplicity we use the same symbol $\prod$ for both ordinary product and tensor product of operators. Using this we prove that

$$\left(\prod_{i=1}^{N} Z_i\right)^{\alpha} \left(\prod_{i=1}^{N} X_i^{t_i}\right)^{\beta} = \prod_{i=1}^{N} \left(Z_i^{\alpha_i} X_i^{\bar{\beta}_i}\right), \qquad \omega(-2^{-1}\alpha\beta) = \prod_{i=1}^{N} \omega_i(-2^{-1}\alpha_i\bar{\beta}_i), \tag{116}$$

which leads to equation (114).

We can also prove a relation analogous to equation (114) for the displaced parity operators. We first see easily that

$$P(0,0) = \prod_{i=1}^{N} P_i(0,0) \tag{117}$$

and then use equations (61) and (114) to prove that

$$P(\alpha, \beta) = \prod_{i=1}^{N} P_i(\alpha_i, \bar{\beta}_i). \tag{118}$$

A direct consequence of equations (114) and (118) is that the Wigner and Weyl functions of an operator Θ can be relabelled as

$$W(\Theta; \alpha, \beta) = W(\Theta; \{\alpha_i\}, \{\bar{\beta}_i\}), \qquad \tilde{W}(\Theta; \alpha, \beta) = \tilde{W}(\Theta; \{\alpha_i\}, \{\bar{\beta}_i\}). \tag{119}$$

We now consider the special class of operators that factorize as in equation (112). In this special case, it is easily seen that the corresponding Wigner and Weyl functions also factorize as

$$\begin{aligned} W(\Theta;\alpha,\beta) &= W(\Theta;\{\alpha_i\},\{\bar{\beta}_i\}) = \prod_{i=1}^{N} W_i(\Theta_i;\alpha_i,\bar{\beta}_i), \\ \tilde{W}(\Theta;\alpha,\beta) &= \tilde{W}(\Theta;\{\alpha_i\},\{\bar{\beta}_i\}) = \prod_{i=1}^{N} \tilde{W}_i(\Theta_i;\alpha_i,\bar{\beta}_i). \end{aligned} \tag{120}$$

For later use we consider the problem of finding the displacements in the phase-space $\mathcal{Z}(d)\times\mathcal{Z}(d)$ that lead to displacements in only one of the N factor phase-spaces $\mathcal{Z}(d_i)\times\mathcal{Z}(d_i)$. In other words, we want to find the special cases for which the product in the right-hand side of equation (114) contains only one non-trivial factor (the other factors are equal to **1**). We see easily that for given integers α_i, $\bar{\beta}_i$ in $\mathcal{Z}(d_i)$ the integers $\alpha = s_i\alpha_i$ and $\beta = r_i\bar{\beta}_i$ in $\mathcal{Z}(d)$ are mapped to

$$\alpha = s_i\alpha_i \quad\leftrightarrow\quad (0,\ldots,0,\alpha_i,0,\ldots,0), \tag{121}$$

$$\beta = r_i\bar{\beta}_i \quad\leftrightarrow\quad (0,\ldots,0,\bar{\beta}_i,0,\ldots,0), \tag{122}$$

according to the mappings of equations (104) and (105), correspondingly. Consequently, in this case

$$D(s_i\alpha_i, r_i\bar{\beta}_i) = D_i(\alpha_i,\bar{\beta}_i). \tag{123}$$

8.4. $SU(d)$ transformations

We consider $SU(d)$ transformations in the system descibed with the Hilbert space $\mathcal{H}(d)$. We have explained earlier that the displacement operators are generators of these transformations and that infinitesimal $SU(d)$ transformations are given by equation (98). Taking into account the factorization discussed in this section, and in particular equation (114), we can express the infinitesimal $SU(d)$ transformations as

$$g = \mathbf{1} + \sum_{\alpha,\beta} \epsilon\,(\alpha,\beta)\,D(\alpha,\beta) = \mathbf{1} + \sum_{\{\alpha_i\},\{\beta_i\}} \epsilon\left(\{\alpha_i\},\{\bar{\beta}_i\}\right)\prod_{i=1}^{N} D_i(\alpha,\bar{\beta}_i) \tag{124}$$

where $\epsilon(\alpha,\beta)$ are infinitesimal coefficients and $\epsilon(\{\alpha_i\},\{\bar{\beta}_i\})$ are the same coefficients relabelled.

Finite $SU(d)$ transformations are given by equations (99) and (100). Taking into account equations (114) and (119), we rewrite them as

$$U = \frac{1}{d}\sum_{\alpha,\beta}\tilde{W}(U;-\alpha,-\beta)D(\alpha,\beta) = \frac{1}{d}\sum_{\{\alpha_i\},\{\beta_i\}}\tilde{W}(U;\{-\alpha_i\},\{-\bar{\beta}_i\})\prod_{i=1}^{N} D_i(\alpha_i,\bar{\beta}_i). \tag{125}$$

A special case of $SU(d)$ transformations is the ones that factorize as in equation (112), i.e. $U = U_1\otimes\cdots\otimes U_N$. These transformations form the

$$F = SU(d_1)\times\cdots\times SU(d_N) \tag{126}$$

subgroup of $SU(d)$. This group has $\sum(d_i^2-1)$ generators, and we can choose them to be the displacement operators $D_i(\alpha_i,\bar{\beta}_i)$. In this special case the product of displacement operators appearing in equation (114) has only one non-trivial factor (the other factors are **1**). Taking into account equation (123), we see that transformations in the group F have as generators $D(\alpha,\beta)$, where $\alpha = s_i\alpha_i$, $\beta = r_i\bar{\beta}_i$ and α_i, $\bar{\beta}_i$ take all values in $\mathcal{Z}(d_i)$. Infinitesimal F transformations can be written as

$$g_F = \mathbf{1} + \sum_{\alpha_i,\bar{\beta}_i} \epsilon(\alpha_i, \bar{\beta}_i) D_i(\alpha, \bar{\beta}_i) = \mathbf{1} + \sum_{\alpha_i,\bar{\beta}_i} \epsilon(\alpha_i, \bar{\beta}_i) D(s_i\alpha_i, r_i\bar{\beta}_i) \tag{127}$$

For finite F transformations we take into account that for factorizable operators the corresponding Wigner and Weyl functions factorize (equation (120)). Therefore, equation (125) becomes

$$U = \frac{1}{d}\prod_{i=1}^{N}\left(\sum_{\alpha_i,\bar{\beta}_i} \tilde{W}(U_i; -\alpha_i, -\bar{\beta}_i) D_i(\alpha_i, \bar{\beta}_i)\right). \tag{128}$$

9. Transformations in composite finite quantum systems

We consider a composite finite quantum system composed of two parts described with the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The total Hilbert space of the system is $\mathcal{H}_1 \otimes \mathcal{H}_2$. In this section, we study local and entangling unitary transformations of this system, and in more detail local and entangling symplectic transformations.

For simplicity, we consider the case where both Hilbert spaces have the same dimension d. Our notation is the same as above, with an index i to indicate the appropriate Hilbert space. For example, $Z_1 = Z \otimes \mathbf{1}$, $Z_2 = \mathbf{1} \otimes Z$; also $|X_1; n\rangle$ and $|X_2; n\rangle$ are position states in $\mathcal{H}_1$ and $\mathcal{H}_2$, correspondingly, etc.

Z_i has the d eigenvalues ω^m, and there is a degeneracy with d eigenvectors corresponding to each eigenvalue. For example, $Z_1 = Z \otimes \mathbf{1}$ has the eigenvalues ω^m and corresponding eigenvectors $|X_1; m\rangle|s_2\rangle$, where $|s_2\rangle$ is any vector in the d-dimensional Hilbert space $\mathcal{H}_2$. We consider the common eigenvectors of Z_1 and Z_2 (which commute with each other). To each set of eigenvalues $\omega(m_1)$, $\omega(m_2)$ corresponds (up to a phase factor) one normalized eigenvector,

$$Z_i|X; m_1, m_2\rangle = \omega(m_i)|X; m_1, m_2\rangle, \qquad |X; m_1, m_2\rangle \equiv |X_1; m_1\rangle|X_2; m_2\rangle. \tag{129}$$

In a similar way, we consider the common eigenvectors of both X_i with $i = 1, 2$. To each set of eigenvalues $\omega(-m_1)$, $\omega(-m_2)$ corresponds (up to a phase factor) one normalized eigenvector

$$X_i|P; m_1, m_2\rangle = \omega(-m_i)|P; m_1, m_2\rangle, \qquad |X; m_1, m_2\rangle \equiv |X_1; m_1\rangle|X_2; m_2\rangle. \tag{130}$$

The position space of this system is $\mathcal{Z}(d) \times \mathcal{Z}(d)$ and the momentum space is $\mathcal{Z}(d) \times \mathcal{Z}(d)$. Therefore the phase-space is $[\mathcal{Z}(d)]^4$.

The displacement operators $D_1(\alpha_1, \beta_1) \equiv D(\alpha_1, \beta_1) \otimes \mathbf{1}$ perform displacements in phase-space in the first system only. Similarly $D_2(\alpha_2, \beta_2) \equiv \mathbf{1} \otimes D(\alpha_2, \beta_2)$ perform displacements in phase-space in the second system only. The operators $D_1(\alpha_1, \beta_1)D_2(\alpha_2, \beta_2) \equiv D(\alpha_1, \beta_1) \otimes D(\alpha_2, \beta_2)$ perform displacements in phase-space in both systems.

9.1. Local and entangling unitary transformations in bi-partite systems

We first study local unitary transformations on the composite system considered above. They form the group $SU(d) \times SU(d)$, which consists of transformations of the form $U_1 \otimes U_2$, where U_i denotes $SU(d)$ transformations in $\mathcal{H}_i$. They are local transformations in the sense that acting on a factorizable pure state, they give another factorizable pure state.

The $SU(d) \times SU(d)$ group has $2(d^2 - 1)$ generators, and as previously we choose them to be the displacement operators $D_i(\alpha_i, \beta_i)$ $(i = 1, 2)$. Infinitesimal transformations can be written as

$$g = \mathbf{1} + \sum_{\alpha_1,\beta_1} \epsilon_1(\alpha_1, \beta_1) D_1(\alpha_1, \beta_1) + \sum_{\alpha_2,\beta_2} \epsilon_2(\alpha_2, \beta_2) D_2(\alpha_2, \beta_2), \tag{131}$$

where $\epsilon_1(\alpha_1, \beta_1)$ and $\epsilon_2(\alpha_2, \beta_2)$ are infinitesimal coefficients.

For finite transformations, we use equations (99),(100) and the fact that the Weyl function of $U_1 \otimes U_2$ factorizes. We get

$$U_1 \otimes U_2 = \frac{1}{d^2} \sum_{\alpha_1,\beta_1,\alpha_2,\beta_2} \tilde{W}(U_1; -\alpha_1, -\beta_1)\tilde{W}(U_2; -\alpha_2, -\beta_2)[D_1(\alpha_1, \beta_1)D_2(\alpha_2, \beta_2)]. \tag{132}$$

We next consider more general unitary transformations that describe both local and entangling transformations. $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a d^2-dimensional Hilbert space, and the general unitary transformations are $SU(d^2)$. This group has $d^4 - 1$ generators which we take to be $D_1(\alpha_1, \beta_1)D_2(\alpha_2, \beta_2)$. From them the $2(d^2 - 1)$ generators contain only one non-trivial factor (the other factor is $\mathbf{1}$) and are associated as we explained earlier with local $SU(d) \times SU(d)$ transformations. The rest of the $(d^2 - 1)^2$ generators contain two non-trivial factors and describe entangling transformations. Infinitesimal $SU(d^2)$ transformations can be written as

$$\begin{aligned} g = \mathbf{1} &+ \left[\sum_{\alpha_1,\beta_1} \epsilon_1(\alpha_1, \beta_1) D_1(\alpha_1, \beta_1) + \sum_{\alpha_2,\beta_2} \epsilon_2(\alpha_2, \beta_2) D_2(\alpha_2, \beta_2) \right] \\ &+ \sum_{\alpha_1,\beta_1,\alpha_2,\beta_2} \epsilon_3(\alpha_1, \beta_1, \alpha_2, \beta_2) D_1(\alpha_1, \beta_1) D_2(\alpha_2, \beta_2), \end{aligned} \tag{133}$$

where $\epsilon_1(\alpha_1, \beta_1)$, $\epsilon_2(\alpha_2, \beta_2)$ and $\epsilon_3(\alpha_1, \beta_1, \alpha_2, \beta_2)$ are infinitesimal coefficients. Here the $2(d^2 - 1)$ generators appearing in the first two sums describe local transformations, and the rest of the $(d^2 - 1)^2$ generators appearing in the last sum describe entangling transformations.

We stress that $SU(d) \times SU(d)$ is not a normal subgroup of $SU(d^2)$ as can be seen from the fact that the commutators of the generators of $SU(d) \times SU(d)$ with the generators of $SU(d^2)$ are not necessarily generators of $SU(d) \times SU(d)$. Therefore the space of entangling transformations $SU(d^2)/SU(d) \times SU(d)$ is not a group (the left cosets are different from the right cosets).

Finite $SU(d^2)$ transformations can be written as

$$U = \frac{1}{d^2} \sum_{\alpha_1,\beta_1,\alpha_2,\beta_2} \tilde{W}(U; -\alpha_1, -\beta_1, -\alpha_2, -\beta_2)\, [D_1(\alpha_1, \beta_1)D_2(\alpha_2, \beta_2)]\,. \tag{134}$$

9.2. Local and entangling symplectic transformations in bi-partite systems

In this section, we study symplectic transformations in the phase-space of the composite quantum system. For the reasons that we explained earlier, we consider Galois quantum systems with $d = p^n$.

The $[GF(p^n)]^4$ phase-space of the composite quantum system is 'four-dimensional' (if such a term can be used for a finite set), and general symplectic transformations form the $Sp(4, GF(p^n))$ group. This describes both local and entangling symplectic transformations and contains $[Sp(2, GF(p^n))]^2$ of local symplectic transformations as a subgroup. In order to study it, we consider the transformations

$$X_i' = SX_iS^\dagger = (X_1^{\kappa_{1i}}Z_1^{\lambda_{1i}})(X_2^{\kappa_{2i}}Z_2^{\lambda_{2i}}),$$
$$Z_i' = SZ_iS^\dagger = (X_1^{\mu_{1i}}Z_1^{\nu_{1i}})(X_2^{\mu_{2i}}Z_2^{\nu_{2i}}), \tag{135}$$

where all exponents belong in $GF(p^n)$. S is a unitary operator that will be constructed explicitly below. We require that these transformations preserve equations (27) and also that X_1', Z_1' commute with X_2', Z_2'. In this case, X_i', Z_i' are new displacement operators in phase-space, that displace in different directions, in comparison with X_i, Z_i. This requirement leads to the constraints

$$(\kappa_{11}\lambda_{12} - \lambda_{11}\kappa_{12}) + (\kappa_{21}\lambda_{22} - \lambda_{21}\kappa_{22}) = 0,$$
$$(\mu_{11}\nu_{12} - \nu_{11}\mu_{12}) + (\mu_{21}\nu_{22} - \nu_{21}\mu_{22}) = 0, \tag{136}$$
$$(\kappa_{1i}\nu_{1k} - \lambda_{1i}\mu_{1k}) + (\kappa_{2i}\nu_{2k} - \lambda_{2i}\mu_{2k}) = \delta(i,k).$$

There are 16 integer parameters in these transformations and six constraints. Therefore there are ten independent integer parameters. In Galois quantum systems ($d = p^n$) we choose ten integer parameters and then solve the constraints (136) (because the inverses exist) to find the rest of the parameters.

We can show easily that the transformations (135) form a group that we have called $Sp(4, GF(p^n))$. In the special case

$$\kappa_{21} = \lambda_{21} = \mu_{21} = \nu_{21} = \kappa_{12} = \lambda_{12} = \mu_{12} = \nu_{12} = 0, \tag{137}$$

the above transformations form the $[Sp(2, GF(p^n))]^2$ subgroup of local symplectic transformations (independent symplectic transformations in each of the two subsystems). In this special case, X_1', Z_1' are displacement operators in the phase-space $[GF(p^n)]^2$ of the first subsystem (but they displace in different directions in comparison with X_1, Z_1). In the more general case of equation (135) the X_1', Z_1' displace outside the phase-space of the first subsystem (within the larger combined phase-space $[GF(p^n)]^4$ of the two subsystems).

9.3. Numerical calculation of the symplectic operator S

In this section we discuss how to evaluate the symplectic operator S numerically. We first consider the common eigenvectors of the commuting operators Z_1' and Z_2', which we denote as $|X'; m_1, m_2\rangle$:

$$Z_i'|X'; m_1, m_2\rangle = \omega(m_i)|X'; m_1, m_2\rangle, \qquad |X'; m_1, m_2\rangle = S|X; m_1, m_2\rangle. \tag{138}$$

To each set of eigenvalues m_1, m_2 corresponds (up to a phase factor) one normalized eigenvector which we calculate numerically.

In order to calculate the phases, we start from the lowest state, $|X'; 0, 0\rangle$ (whose phase we choose arbitrarily), and use numerically the equation

$$(X_1')^{m_1}(X_2')^{m_2}|X'; 0, 0\rangle = |X'; m_1, m_2\rangle. \tag{139}$$

$|X'; m_1, m_2\rangle$ calculated through this equation differ from the corresponding vectors calculated above as common eigenvectors of the matrices Z_1', Z_2', only by a phase factor. This is a test that the numerical work is correct and at the same time it provides the phases. We stress that the phases are very important for the self-consistency of the formalism.

We can now calculate the matrix elements of the operator S:

$$S(m_1, m_2|n_1, n_2) \equiv \langle X; m_1, m_2|S|X; n_1, n_2\rangle = \langle X; m_1, m_2|X'; n_1, n_2\rangle. \tag{140}$$

As an example, we consider five-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and the transformations

$$\begin{aligned} X_1' &= SX_1S^\dagger = (X_1^{-1}Z_1)(X_2^2), \\ Z_1' &= SZ_1S^\dagger = (X_1Z_1^2)(X_2Z_2^2), \\ X_2' &= SX_2S^\dagger = (X_1Z_1)(Z_2), \\ Z_2' &= SZ_2S^\dagger = (X_1)(X_2^{-2}Z_2^{-2}), \end{aligned} \tag{141}$$

which obey the constraints (136).

In order to find the common eigenvectors $|X'; m_1, m_2\rangle$ of Z_1', Z_2', we first consider Z_1' in the basis $|X_1; n_1\rangle|X_2; n_2\rangle$. This is the Kronecker product of the two 5×5 matrices $\delta(n_1, m_1 + 1)\omega(2m_1)$ and $\delta(n_2, m_2 + 1)\omega(2m_2)$ corresponding to the operators $X_1Z_1^2$ and $X_2Z_2^2$. We calculate numerically this Kronecker product (which is a 25×25 matrix) and its eigenvalues and eigenvectors. There are five different eigenvalues $\omega(m_1)$ where $m_1 \in \mathcal{Z}(5)$. To each of these eigenvalues correspond five eigenvectors, and we form 25×5 matrices that have as columns these eigenvectors and that we denote as $B(m_1)$. The $|X'; m_1, m_2\rangle$ is in the five-dimensional subspace spanned by the eigenvectors corresponding to the eigenvalue $\omega(m_1)$, and therefore it can be written as $B(m_1)A(m_1, m_2)$, where $A(m_1, m_2)$ is a 5×1 vector to be determined.

We next calculate numerically the Kronecker product of the two 5×5 matrices $\delta(n_1, m_1 + 1)$ and $\delta(n_2, m_2 - 2)\omega(-2m_2)$ corresponding to the operators X_1 and $X_2^{-2}Z_2^{-2}$ in Z_2'. Since $B(m_1)A(m_1, m_2)$ is an eigenvector of Z_2' with eigenvalue $\omega(m_2)$, this implies that $(Z_2' - \omega(m_2)\mathbf{1})B(m_1)A(m_1, m_2) = 0$. Consequently, $A(m_1, m_2)$ belongs in the null space of $(Z_2' - \omega(m_2)\mathbf{1})B(m_1)$, which is one-dimensional and which is readily available in most computer libraries (e.g. MATLAB). With this method we have calculated all the eigenvectors $|X'; m_1, m_2\rangle$ up to phase factors.

In order to calculate the phases, we start from the lowest state, $|X'; -2, -2\rangle$ (for which we choose arbitrarily a phase), and use numerically equation (139). We then calculate the matrix elements of the operator $S(n_1, n_2|m_1, m_2)$ defined in equation (140). In table 3 we give $S(n_1, n_2|1, 1)$.

9.4. Symplectic transformations in multi-partite systems

Above we have studied symplectic transformations $Sp(4, GF(p^n))$ in bi-partite quantum systems. Here we study more general $Sp(2N, GF(p^n))$ in N-partite quantum systems with $[GF(p^n) \times GF(p^n)]^N$ phase-space.

We consider the transformations

$$\begin{aligned} X_i' &= SX_iS^\dagger = (X_1^{\kappa_{1i}}Z_1^{\lambda_{1i}}) \cdots (X_N^{\kappa_{Ni}}Z_N^{\lambda_{Ni}}), \\ Z_i' &= SZ_iS^\dagger = (X_1^{\mu_{1i}}Z_1^{\nu_{1i}}) \cdots (X_N^{\mu_{Ni}}Z_N^{\nu_{Ni}}), \end{aligned} \tag{142}$$

Table 3. The coefficients $S(n_1, n_2|1, 1)$ for the transformations of equation (141). Here, $z_1 = -0.0494 - 0.1938\mathrm{i}$, $z_2 = z_1\omega^2$, $z_3 = z_1\omega$, $z_4 = z_1\omega^{-2}$ and $z_5 = z_1\omega^{-1}$.

	$n_2 = -2$	$n_2 = -1$	$n_2 = 0$	$n_2 = 1$	$n_2 = 2$
$n_1 = -2$	z_1	z_1	z_4	z_5	z_4
$n_1 = -1$	z_2	z_3	z_4	z_4	z_3
$n_1 = 0$	z_2	z_1	z_3	z_1	z_2
$n_1 = 1$	z_1	z_2	z_2	z_1	z_3
$n_1 = 2$	z_3	z_2	z_3	z_4	z_4

where all exponents belong in $GF(p^n)$. We require that these transformations preserve equations (27) and also that for $i \neq k$, X'_i, Z'_i commute with X'_k, Z'_k. In this case, X'_i, Z'_i can also be used as displacement operators in phase-space. In comparison with X_i, Z_i, they displace in different directions. This requirement leads to the constraints

$$\begin{aligned}
&\sum_{\ell=1}^{N}(\kappa_{\ell i}\lambda_{\ell k} - \lambda_{\ell i}\kappa_{\ell k}) = 0,\\
&\sum_{\ell=1}^{N}(\mu_{\ell i}\nu_{\ell k} - \nu_{\ell i}\mu_{\ell k}) = 0,\\
&\sum_{\ell=1}^{N}(\kappa_{\ell i}\nu_{\ell k} - \lambda_{\ell i}\mu_{\ell k}) = \delta(i,k).
\end{aligned} \tag{143}$$

There are $4N^2$ integer parameters in these transformations and $2N^2 - N$ constraints. Therefore there are $2N^2 + N$ independent integer parameters.

We note that in the special case

$$\kappa_{ik} = \lambda_{ik} = \mu_{ik} = \nu_{ik} = 0, \qquad i \neq k, \tag{144}$$

the above transformations form the $[Sp(2, GF(p^n))]^N$ subgroup of local symplectic transformations (independent symplectic transformations in each of the N subsystems).

The numerical work presented earlier for the calculation of the symplectic operator S in bi-partite quantum systems can also be generalized to multi-partite systems.

10. Angle and angular momentum states

Angular momentum is a well studied topic in the literature and there are several textbooks and review articles on it (e.g. [57–59]). The aim of this section and the next section is to make a connection between the general theory of finite quantum systems presented earlier and angular momentum.

We consider the usual angular momentum states, which we denote as $|J; jm\rangle$. The extra J to the usual notation is not a variable but simply indicates angular momentum states. We also consider the usual angular momentum operators, J_z, J_+, J_-, which form the $SU(2)$ algebra

$$[J_z, J_+] = J_+, \qquad [J_z, J_-] = -J_-, \qquad [J_+, J_-] = 2J_z. \tag{145}$$

The corresponding Casimir operator is

$$J^2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) = j(j+1)\mathbf{1}. \tag{146}$$

The angular momentum operators act on the angular momentum states as follows:

$$\begin{aligned}
&J_+|J; jm\rangle = [j(j+1) - m(m+1)]^{1/2}|J; jm+1\rangle,\\
&J_-|J; jm\rangle = [j(j+1) - m(m-1)]^{1/2}|J; jm-1\rangle,\\
&J_z|J; jm\rangle = m|J; jm\rangle,\\
&J^2|J; jm\rangle = j(j+1)|J; jm\rangle.
\end{aligned} \tag{147}$$

We next introduce a polar decomposition of the ‘Cartesian operators’, J_+ and J_-, in terms of the ‘radial operator’, J_r, and the ‘exponential of the phase operator’, which we denote as X

because we will show that it is similar to the operator X used earlier.

$$J_+ = J_r X, \qquad J_- = X^\dagger J_r,$$
$$J_r = (J_+J_-)^{1/2}, \qquad X = \sum_m |J; jm+1\rangle\langle J; jm|. \tag{148}$$

It is easily seen that X is a unitary operator and that

$$J_r|J; jm\rangle = [j(j+1) - m(m-1)]^{1/2}|J; jm\rangle,$$
$$J_r = [j(j+1)\mathbf{1} - J_z^2 + J_z]^{1/2}, \tag{149}$$
$$[J_r, J_z] = 0.$$

J_r and X do not commute, and consequently the ordering of these operators in the polar decomposition is important. An alternative polar decomposition is

$$J_+ = XJ'_r, \qquad J_- = J'_r X^\dagger, \qquad J'_r = (J_-J_+)^{1/2}, \tag{150}$$

where in this case

$$J'_r|J; jm\rangle = [j(j+1) - m(m+1)]^{1/2}|J; jm\rangle,$$
$$J'_r = [j(j+1)\mathbf{1} - J_z^2 - J_z]^{1/2}, \tag{151}$$
$$[J'_r, J_z] = 0.$$

In the following, we distinguise the Bose sector (integer j) from the Fermi sector (half-integer j). This is because many relations are based on equation (3), which is valid for integers. As pointed out in [14], a consequence of this is that some of the formulae are slightly different in the two cases of Bose and Fermi sectors (the Fermi sector requires a 'correction by $1/2$').

10.1. Bose sector

The Fourier operator of equation (2) is in this case

$$F = (2j+1)^{-1/2} \sum_{m,n} \omega(mn)|J; jm\rangle\langle J; jn|. \tag{152}$$

We introduce angle states by acting with the Fourier operator on the angular momentum states:

$$|\theta; jm\rangle = F|J; jm\rangle = (2j+1)^{-1/2} \sum_n \omega^{mn}|J; jn\rangle. \tag{153}$$

In the harmonic oscillator, we have momentum and position states related through a Fourier transform. Here we have angular momentum and angular position (angle) states related through a finite Fourier transform.

Acting with the Fourier operator on the angular momentum operators we get the angle operators

$$FJ_zF^\dagger = \theta_z, \qquad FJ_+F^\dagger = \theta_+, \qquad FJ_-F^\dagger = \theta_- \tag{154}$$

which form the $SU(2)$ algebra

$$[\theta_z, \theta_+] = \theta_+, \qquad [\theta_z, \theta_-] = -\theta_-, \qquad [\theta_+, \theta_-] = 2\theta_z. \tag{155}$$

The Casimir operator is

$$\theta^2 = \theta_z^2 + \tfrac{1}{2}(\theta_+\theta_- + \theta_-\theta_+) = j(j+1)\mathbf{1} = J^2. \tag{156}$$

The angle operators act on the angle states in a way analogous to the angular momentum operators acting on the angular momentum states:

$$\begin{aligned}
\theta_+|\theta;\, jm\rangle &= [j(j+1) - m(m+1)]^{1/2}|\theta;\, jm+1\rangle,\\
\theta_-|\theta;\, jm\rangle &= [j(j+1) - m(m-1)]^{1/2}|\theta;\, jm-1\rangle,\\
\theta_z|\theta;\, jm\rangle &= m|\theta;\, jm\rangle,\\
\theta^2|\theta;\, jm\rangle &= j(j+1)|\theta;\, jm\rangle.
\end{aligned} \tag{157}$$

We next consider a polar decomposition of θ_+ and θ_- in terms of the 'radial operator', θ_r, and the 'exponential of the phase operator', which we denote as Z:

$$\begin{aligned}
&\theta_+ = \theta_r Z, \qquad \theta_- = Z^\dagger\theta_r,\\
&\theta_r = (\theta_+\theta_-)^{1/2} = F J_r F^\dagger, \qquad Z = FXF^\dagger = \sum_m |\theta;\, jm+1\rangle\langle\theta;\, jm|.
\end{aligned} \tag{158}$$

We can show that the two exponentials of the phase operators X and Z obey equation (27) and form the Heisenberg–Weyl group in this context. They also obey relations analogous to equations (25) and (26), with the states $|X;\, m\rangle$ replaced by the angular momentum states, and the states $|P;\, m\rangle$ replaced by the angle states

$$Z^\alpha|\theta;\, jm\rangle = |\theta;\, jm+\alpha\rangle, \qquad Z^\alpha|J;\, jm\rangle = \omega(\alpha m)|J;\, jm\rangle, \tag{159}$$

$$X^\beta|\theta;\, jm\rangle = \omega(-m\beta)|\theta;\, jm\rangle, \qquad X^\beta|J;\, jm\rangle = |J;\, jm+\beta\rangle. \tag{160}$$

It is seen easily that

$$X = \exp\left(-\mathrm{i}\frac{2\pi}{2j+1}\theta_z\right), \qquad Z = \exp\left(\mathrm{i}\frac{2\pi}{2j+1}J_z\right). \tag{161}$$

We can introduce cosine and sine operators C_X, S_X and also $C_Z$and S_Z as in equations (34) and (36):

$$C_X = \cos\left(\frac{2\pi}{2j+1}\theta_z\right), \qquad S_X = \sin\left(\frac{2\pi}{2j+1}\theta_z\right), \tag{162}$$

$$C_Z = \cos\left(\frac{2\pi}{2j+1}J_z\right), \qquad S_Z = \sin\left(\frac{2\pi}{2j+1}J_z\right). \tag{163}$$

We next calculate the expectation values $\langle K^i\rangle \equiv \langle\theta;\, jn|K^i|\theta;\, jn\rangle$ and the corresponding variances $(\Delta K)^2 \equiv \langle K^2\rangle - \langle K\rangle^2$ of various operators K with respect to angle states:

$$\begin{aligned}
&\langle J_r\rangle = (2j+1)^{-1}\sum_m [j(j+1) - m(m-1)]^{1/2},\\
&\langle J_x\rangle = \langle J_r\rangle\cos\left(\frac{2\pi n}{2j+1}\right), \qquad \langle J_y\rangle = -\langle J_r\rangle\sin\left(\frac{2\pi n}{2j+1}\right),\\
&\langle C_X\rangle = \cos\left(\frac{2\pi n}{2j+1}\right), \qquad (\Delta C_X)^2 = 0,\\
&\langle S_X\rangle = \sin\left(\frac{2\pi n}{2j+1}\right), \qquad (\Delta S_X)^2 = 0,\\
&\langle C_Z\rangle = 0, \qquad (\Delta C_Z)^2 = \tfrac{1}{2},\\
&\langle S_Z\rangle = 0, \qquad (\Delta S_Z)^2 = \tfrac{1}{2}.
\end{aligned} \tag{164}$$

These quantities elucidate the physical meaning of the angle states.

10.2. Fermi sector

Some of the relations in the Fermi sector (half-integer j) are slightly different from the corresponding ones in the Bose sector (integer j). This is because many relations are based on equation (3), which is valid for integers. Consequently, the j of the Bose sector is replaced with $j + 1/2$ in the Fermi sector.

The Fourier operator in the Fermi sector is given by

$$F = (2j+1)^{-1/2} \sum_{m,n} \omega \left[\left(m + \frac{1}{2} \right) \left(n + \frac{1}{2} \right) \right] |J; jm\rangle\langle J; jn| \tag{165}$$

and the angle states are defined as

$$|\theta; jm\rangle = F|J; jm\rangle = (2j+1)^{-1/2} \sum_{n} \omega \left[\left(m + \frac{1}{2} \right) \left(n + \frac{1}{2} \right) \right] |J; jn\rangle. \tag{166}$$

Equations (154)–(158) are the same in both Bose and Fermi sectors. Equations (159) and (160) are replaced in the Fermi sector with

$$Z^{\alpha}|\theta; jm\rangle = |\theta; jm+\alpha\rangle, \qquad Z^{\alpha}|J; jm\rangle = \omega\left[\alpha\left(m+\tfrac{1}{2}\right)\right]|J; jm\rangle, \tag{167}$$

$$X^{\beta}|\theta; jm\rangle = \omega\left[-\beta\left(m+\tfrac{1}{2}\right)\right]|\theta; jm\rangle, \qquad X^{\beta}|J; jm\rangle = |J; jm+\beta\rangle. \tag{168}$$

Equation (161) is replaced with

$$X = \exp\left[-\mathrm{i}\frac{2\pi}{2j+1}\left(\theta_z + \frac{1}{2}\right)\right], \qquad Z = \exp\left[\mathrm{i}\frac{2\pi}{2j+1}\left(J_z + \frac{1}{2}\right)\right]. \tag{169}$$

Equations (167)–(169) are proved using equations (165) and (166). The analogues of equations (162)–(164) for the Fermi sector require '1/2 corrections' and are not presented here.

10.3. The Holstein–Primakoff $SU(2)$ formalism

In this section, $\mathcal{H}$ is the infinite dimensional Hilbert space associated with the harmonic oscillator. Let $a^{\dagger}$, a be the creation and annihilation operators, correspondingly, and $|N\rangle$ be the number eigenstates. The Holstein–Primakoff $SU(2)$ formalism [60] considers the $(2j+1)$-dimensional Hilbert space $\mathcal{H}_{\mathrm{tr}}$ spanned by the number states with $0 \leqslant N \leqslant 2j$. The index 'tr' in the notation indicates truncated Hilbert space. In $\mathcal{H}_{\mathrm{tr}}$ we consider the operators

$$\begin{aligned} &J_+ = [(2j+1) - a^{\dagger}a]^{1/2}a^{\dagger}, \qquad J_- = a[(2j+1) - a^{\dagger}a]^{1/2}, \\ &J_z = a^{\dagger}a - j, \qquad J^2 = j(j+1)\mathbf{1}. \end{aligned} \tag{170}$$

Here $\mathbf{1}$ is the unit operator in $\mathcal{H}_{\mathrm{tr}}$. It is easily seen that these operators obey the usual angular momentum commutation relations of equation (145). The corresponding angular momentum states are the number eigenstates

$$|J; jm\rangle = |N\rangle, \qquad N = j + m, \tag{171}$$

and we can easily check the validity of equation (147). Clearly, the whole formalism on finite quantum systems discussed earlier can be applied to $\mathcal{H}_{\mathrm{tr}}$.

The Holstein–Primakoff $SU(2)$ formalism expresses the angular momentum operators in terms of harmonic oscillator creation and annihilation operators. Other formalisms that do that are the Schwinger formalism, which will be studied later, and the Dyson formalism, which will not be discussed here.

11. $SU(2)$ coherent states and Q and P representations

Coherent states play an important role in various branches of physics: quantum optics, condensed matter, nuclear physics, particle physics (especially in connection with infrared divergences), etc. They are usually related to various groups in the sense that transformations from the group take a coherent state into another coherent state ('temporal stability'). Coherent states associated with the Heisenberg–Weyl group, the $SU(1, 1)$ group and the $SU(2)$ group have been studied extensively in the literature. There are already several textbooks and review articles on coherent states and their applications (e.g. [28–30, 33]). The aim of this section is to make a connection between the theory of finite quantum systems presented earlier and $SU(2)$ coherent states.

11.1. $SU(2)$ coherent states

A sphere is topologically equivalent to the extended complex plane ($C \cup \{\infty\}$). A stereographic projection provides a one-to-one mapping between the points on a sphere and the points on the extended complex plane. A point on a sphere described in spherical coordinates with the angles (α, β) $(0 \leqslant \alpha \leqslant \pi, 0 \leqslant \beta < 2\pi)$ is mapped into the point z in the extended complex plane, where

$$z = -\tan\left(\frac{\alpha}{2}\right) e^{-i\beta}. \tag{172}$$

The south pole is mapped to the point $z = 0$ and the north pole to ∞.

With this notation in mind we consider the unitary $SU(2)$ operators

$$U_1(\alpha, \beta, \gamma) = \exp\left[-\tfrac{1}{2}\alpha e^{-i\beta} J_+ + \tfrac{1}{2}\alpha e^{i\beta} J_-\right] \exp(i\gamma J_z). \tag{173}$$

For later use we give without proof [28, 57] the relation

$$\begin{aligned} &U_1(\alpha, \beta, \gamma) J_z [U_1(\alpha, \beta, \gamma)]^\dagger = J_\epsilon, \\ &J_\epsilon = \epsilon \cdot J, \qquad J = (J_x, J_y, J_z), \qquad \epsilon = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha), \end{aligned} \tag{174}$$

and also the relation

$$\exp\left[-\tfrac{1}{2}\alpha e^{-i\beta} J_+ + \tfrac{1}{2}\alpha e^{i\beta} J_-\right] = \exp(zJ_+)(1 + |z|^2)^{J_z} \exp(-z^* J_-), \tag{175}$$

where z is given in terms of (α, β) in the stereographic projection relation of equation (172).

$SU(2)$ coherent states are defined as

$$|J_{\mathrm{coh}}; \alpha, \beta\rangle = U_1(\alpha, \beta, 0)|J; j - j\rangle, \tag{176}$$

where the index 'coh' in the notation indicates coherent states. We have taken U_1 with $\gamma = 0$, but a non-zero value of γ gives the same coherent state with an extra trivial phase factor.

One of the most important properties of coherent states is the 'temporal stability' [61]. In the present context, this is the fact that the operators $U_1(\alpha, \beta, \gamma)$ acting on coherent states produce other coherent states (i.e. under these transformations the states remain in the same 'family'). This is indeed the case because the product of two U_1 operators is an operator of the same type (they form the $SU(2)$ group).

A second definition of $SU(2)$ coherent states that is equivalent to (176) is as eigenstates of J_ϵ:

$$J_\epsilon |J_{\mathrm{coh}}; \alpha, \beta\rangle = -j|J_{\mathrm{coh}}; \alpha, \beta\rangle. \tag{177}$$

The equivalence of these two definitions is easily proved with the use of equation (174).

A third definition of $SU(2)$ coherent states that is equivalent to both (176) and (177) is

$$|J_{\rm coh}; z\rangle = (1+|z|^2)^{-j} \sum_n d(j,n) z^{j+n} |J; jn\rangle, \qquad d(j,n) = \left[\frac{(2j)!}{(j+n)!(j-n)!}\right]^{1/2}, \tag{178}$$

where z belongs in the extended complex plane. The equivalence of (178) with (176) is proved with the use of equation (175), and it is easily seen that

$$|J_{\rm coh}; \alpha, \beta\rangle = |J_{\rm coh}; z\rangle, \tag{179}$$

where z is given in terms of α, β in the stereographic projection relation of equation (172).

The overlap of two coherent states is given by

$$\langle J_{\rm coh}; z_1 | J_{\rm coh}; z_2\rangle = \left[\frac{(1+z_1^* z_2)^2}{(1+|z_1|^2)(1+|z_1|^2)}\right]^j. \tag{180}$$

There is a connection between this result and the distance $\delta(z_1, z_2)$ between the points z_1 and z_2, in spherical geometry (extended complex plane) given by

$$\tan^2\left[\frac{1}{2}\delta(z_1, z_2)\right] = \left|\frac{z_1 - z_2}{1 + z_1^* z_2}\right|^2. \tag{181}$$

It is easily seen that

$$\langle J_{\rm coh}; z_1 | J_{\rm coh}; z_2\rangle = \left\{1 + \tan^2\left[\tfrac{1}{2}\delta(z_1, z_2)\right]\right\}^{-j}. \tag{182}$$

The resolution of the identity in terms of $SU(2)$ coherent states is

$$\frac{2j+1}{\pi}\int |J_{\rm coh}; z\rangle\langle J_{\rm coh}; z|\, {\rm d}\mu(z) = \mathbf{1}, \qquad {\rm d}\mu(z) = (1+|z|^2)^{-2}\, {\rm d}^2 z, \tag{183}$$

where the integration is over the extended complex plane. ${\rm d}\mu(z)$ is the spherical geometry metric, as expressed in the extended complex plane. Equation (183) is proved if we substitute equation (178) into equation (183) and perform the integration. The resolution of the identity (183) shows that the set of all $SU(2)$ coherent states is at least complete. In fact, it is highly overcomplete because we will prove below that small subsets of these coherent states are also overcomplete.

We note that if we act with any unitary operator V on both states and operators we get another set of coherent states. The states

$$|J'_{\rm coh}; z\rangle \equiv V|J_{\rm coh}; z\rangle \tag{184}$$

are $SU(2)$ coherent states with respect to the operators

$$\begin{aligned} &U'_1(\alpha, \beta, \gamma) = V U_1(\alpha, \beta, \gamma) V^\dagger = \exp\left[-\tfrac{1}{2}\alpha {\rm e}^{-{\rm i}\beta} J'_+ + \tfrac{1}{2}\alpha {\rm e}^{{\rm i}\beta} J'_-\right] \exp({\rm i}\gamma J'_z) \\ &J'_z = V J_z V^\dagger, \qquad J'_+ = V J_+ V^\dagger, \qquad J'_- = V J_- V^\dagger, \end{aligned} \tag{185}$$

but they are not coherent states with respect to the original operators of equation (173). However, it may be interesting to study the properties of the states (184) with respect to the operators (173).

An example of this general comment is to take $V = F$, i.e. to act with the Fourier operator on the above states and operators. We get the states

$$|\theta_{\rm coh}; z\rangle \equiv F|J_{\rm coh}; z\rangle = (1+|z|^2)^{-j} \sum_n d(j,n) z^{j+n} |\theta; jn\rangle, \tag{186}$$

which are $SU(2)$ coherent states with respect to the operators

$$U_2(\alpha,\beta,\gamma) = FU_1(\alpha,\beta,\gamma)F^\dagger = \exp\left[-\tfrac{1}{2}\alpha e^{-i\beta}\theta_+ + \tfrac{1}{2}\alpha e^{i\beta}\theta_-\right]\exp(i\gamma\theta_z) \tag{187}$$

but they are not coherent states with respect to the original operators of equation (173). It is seen easily that

$$\langle J_{\text{coh}};\zeta|\theta_{\text{coh}};z\rangle = (2j+1)^{-1/2}(1+|\zeta|^2)^{-j}(1+|z|^2)^{-j}\sum_{m,n} d(j,n)d(j,m)(\zeta^*)^{j+n}z^{j+m}\omega(nm) \tag{188}$$

for the Bose case (integer j).

11.2. Q and P representations

Q and P representations can be defined with respect to the J-coherent states or with respect to the θ-coherent states; and for this reason we use the indices J and θ, correspondingly.

The Q_J representation of an operator A is defined as

$$Q_J(A;z) = \langle J_{\text{coh}};z|A|J_{\text{coh}};z\rangle. \tag{189}$$

In a similar way, we can define $Q_\theta(A;z)$, and the relationship between the two can be found using equation (188).

The P_J representation of an operator A is defined as

$$A = \frac{2j+1}{\pi}\int P_J(A;z)|J_{\text{coh}};z\rangle\langle J_{\text{coh}};z|\,d\mu(z). \tag{190}$$

In a similar way we can define $P_\theta(A;z)$.

It is seen easily that

$$\text{Tr}A = \frac{2j+1}{\pi}\int Q_J(A;z)\,d\mu(z) = \frac{2j+1}{\pi}\int P_J(A;z)\,d\mu(z) \tag{191}$$

and also that

$$\text{Tr}(AB) = \frac{2j+1}{\pi}\int Q_J(A;z)P_J(B;z)\,d\mu(z). \tag{192}$$

Inserting equation (190) into equation (189) and using equation (180), we prove that

$$Q_J(A;z) = \frac{2j+1}{\pi}\int P_J(A;w)\left[\frac{|1+z^*w|^2}{(1+|z|^2)(1+|w|^2)}\right]^{2j} d\mu(w). \tag{193}$$

The Q and P functions are also known as covariant and contravariant symbols. There are many relations that connect the Q and P functions with the Wigner and Weyl functions and will not be reviewed here [50].

12. Analytic representations based on *SU*(2) coherent states

Intimately related to coherent states are the analytic representations that represent the various quantum states with analytic functions. In this way we can use the powerful theory of analytic functions to derive strong results in a quantum mechanical context. We can also perform easily the transformations of the corresponding group as conformal mappings. The Bargmann analytic representation in the complex plane [62] is intimately related to ordinary coherent states and Euclidean geometry. Analytic representation in the unit disc (or equivalently, half-plane) [63] is intimately related to $SU(1,1)$ coherent states and hyperbolic geometry. Both these representations are associated with infinite-dimensional Hilbert spaces and will not be discussed here. We will study an analytic representation in the extended complex plane that is intimately related to $SU(2)$ coherent states and spherical geometry.

Another analytic representation is the so-called Dirac contour representation. This has been studied in the context of ordinary coherent states and Euclidean geometry in [64]. Here, we will present the analogue of this representation in the context of the extended complex plane and $SU(2)$ coherent states.

12.1. Analytic representation in the extended complex plane: quantum states

We consider the $2j+1$-dimensional space spanned by the angular momentum states $|J;\, jm\rangle$. In this Hilbert space we consider an arbitary state

$$|f\rangle = \sum_n f_n |J;\, jn\rangle, \qquad \sum_n |f_n|^2 = 1. \tag{194}$$

We use the following notation:

$$\langle f| = \sum_n f_n^* \langle J;\, jn|, \qquad |f^*\rangle = \sum_n f_n^* |J;\, jn\rangle, \qquad \langle f^*| = \sum_n f_n \langle J;\, jn|. \tag{195}$$

In the analytic representation, the state $|f\rangle$ is represented by the polynomial

$$f(z) = (1+|z|^2)^j \langle J_{\rm coh};\, z^*|f\rangle = (1+|z|^2)^j \langle f^*|J_{\rm coh};\, z\rangle = \sum_n d(j,n) f_n z^{j+n}, \tag{196}$$

which in general is of order $2j$ (but in special cases it may be of lower order).

Examples of various states and their analytic representations are

$$\begin{aligned}
|J;\, jm\rangle &\rightarrow d(j,m) z^{j+m}, \\
|\theta;\, jm\rangle &\rightarrow (2j+1)^{-1/2} \sum_n d(j,n)\omega(nm) z^{j+n}, \\
|J_{\rm coh};\, \zeta\rangle &\rightarrow \frac{(1+z\zeta)^{2j}}{(1+|\zeta|^2)^j}, \\
|\theta_{\rm coh};\, \zeta\rangle &\rightarrow (2j+1)^{-1/2}(1+|\zeta|^2)^{-j} \sum_{n,k} \zeta^{j+k}\omega^{nk} z^{j+n}.
\end{aligned} \tag{197}$$

In this representation the scalar product of two states $|f\rangle$ and $|g\rangle$ represented by the functions $f(z)$ and $g(z)$, correspondingly, is given by

$$\langle g|f\rangle = \frac{2j+1}{\pi} \int [g(z)]^* f(z) (1+|z|^2)^{-2j}\, {\rm d}\mu(z), \tag{198}$$

where $d\mu(z)$ has been given in equation (183). This is easily proved with the use of the resolution of the identity (183).

The angular momentum operators are represented as

$$J_+ = -z^2\partial_z + 2jz, \qquad J_- = \partial_z, \qquad J_z = z\partial_z - j. \tag{199}$$

Indeed we can check easily that these operators acting on $d(j,m)z^{j+m}$ which represent the states $|J;\, jm\rangle$ give the expected results of equation (147). Other operators are represented with various kernels, as will be discussed in the next subsection.

We next consider a state $|f\rangle$ represented by the function $f(z)$. If w is a zero of the function $f(z)$ (i.e. $f(w) = 0$), then the state $|f\rangle$ is orthogonal to the state $|J_{\rm coh};\, w^*\rangle$, as can be seen from equation (196). A more general result is that if w is a zero of order k of the function $f(z)$

(i.e. $f(z) = (z - w)^k f_1(z)$), then the state $|f\rangle$ is orthogonal to all states

$$|J_{\rm coh}; w^*\rangle, \qquad J_+|J_{\rm coh}; w^*\rangle, \qquad \ldots, \qquad (J_+)^{k-1}|J_{\rm coh}; w^*\rangle. \tag{200}$$

Indeed $\partial_z^\ell f(z) = (1 + |z|^2)^j \langle J_{\rm coh}; z^*|(J_-)^\ell|f\rangle$ and w is a zero of $\partial_z^\ell f(z)$ for $0 \leqslant \ell \leqslant k-1$.

Using this we can prove that any set of more than $2j$ $SU(2)$ coherent states is at least complete. Indeed, if this is not a complete set, then there exists a state $|f\rangle$ that is orthogonal to all of them. Correspondingly, there exists a polynomial $f(z)$ with more than $2j$ zeros. But this is not possible because as we explained $f(z)$ is of order $2j$.

When a set of states is at least complete, in principle we can expand an arbitrary state in terms of these states. However, in practice it is not always easy to find the coefficients of such expansion. In this sense a resolution of the identity is much stronger, because not only does it show that the states involved form a set that is at least complete but it also provides the coefficients of the expansion. For example, the resolution of the identity of equation (183) shows that an arbitrary state $|f\rangle$ can be expanded in terms of $SU(2)$ coherent states as

$$|f\rangle = \frac{2j+1}{\pi} \int |J_{\rm coh}; z\rangle\langle J_{\rm coh}; z|f\rangle\, {\rm d}\mu(z). \tag{201}$$

This expansion involves all coherent states in the extended complex plane. Later, we will show how to expand an arbitrary state in terms of $SU(2)$ coherent states on a contour around the origin, in the extended complex plane.

12.2. Analytic representation in the extended complex plane: transformations

We have seen in equation (199) that the angular momentum operators are nicely represented by simple differential operators. An arbitrary operator

$$Q = \sum_{n,m} Q_{nm}|J; jn\rangle\langle J; jm| \tag{202}$$

cannot be easily expressed in terms of differential operators. It can be represented by a kernel $Q(z, \zeta)$ such that the action of Q on the state $|f\rangle$ represented by $f(z)$, is given by

$$Q|f\rangle \to \int Q(z, \zeta) f(\zeta)\, {\rm d}\mu(\zeta), \tag{203}$$

where ${\rm d}\mu(\zeta)$ has been given in equation (183). It is easily seen that the kernel should be given by

$$\begin{aligned} Q(z, \zeta) &= \frac{2j+1}{\pi} \left(\frac{1+|z|^2}{1+|\zeta|^2}\right)^j \langle J_{\rm coh}; z^*|Q|J_{\rm coh}; \zeta^*\rangle, \\ &= \frac{2j+1}{\pi}(1+|\zeta|^2)^{-2j} \sum_{n,m} [d(j,n) z^{j+n}] Q_{nm} [d(j,m)(\zeta^*)^{j+m}]. \end{aligned} \tag{204}$$

As a first example, we consider the unit operator for which

$$\mathbf{1} \to Q(z, \zeta) = \frac{2j+1}{\pi} \frac{(1+z\zeta^*)^{2j}}{(1+|\zeta|^2)^{2j}}. \tag{205}$$

This implies that for any state $|f\rangle$, $f(z)$ is related to $f(\zeta)$ through the relation

$$f(z) = \frac{2j+1}{\pi} \int \frac{(1+z\zeta^*)^{2j}}{(1+|\zeta|^2)^{2j}} f(\zeta)\, {\rm d}\mu(\zeta). \tag{206}$$

For this reason, the $Q(z, \zeta)$ of equation (205) is called a ‘reproducing kernel’.

As a second example, we consider the angular momentum operator J_z for which $Q_{nm} = n\delta_{nm}$. A straightforward calculation gives

$$J_z \to Q(z,\zeta) = -\frac{j(2j+1)}{\pi}\frac{(1+z\zeta^*)^{2j-1}(1-z\zeta^*)}{(1+|\zeta|^2)^{2j}}. \tag{207}$$

This is an alternative to the differential form of J_z given in equation (199).

One of the merits of the analytic representation in the extended complex plane is that $SU(2)$ transformations are easily performed using conformal mappings. Below we give the formulae without proof. We first consider the Möbius mapping

$$w = \frac{\kappa z - \lambda^*}{\lambda z + \kappa^*}, \qquad |\kappa|^2 + |\lambda|^2 = 1. \tag{208}$$

The action of the $SU(2)$ operator $U_1(\alpha, \beta, \gamma)$ of equation (173) on the arbitrary state $|f\rangle$ represented by the polynomial $f(z)$, transforms $f(z)$ into the polynomial

$$f(z) \to f\left(\frac{\kappa z - \lambda^*}{\lambda z + \kappa^*}\right)(\lambda z + \kappa^*)^{2j} = \sum_n d(j,n) f_n [\kappa z - \lambda^*]^{j+n}[\lambda z + \kappa^*]^{j-n}. \tag{209}$$

The relation between the complex coefficients κ and λ appearing in (208) and the variables α, β, γ in (173) is

$$\kappa = \cos\left(\frac{\alpha}{2}\right)\exp\left[\mathrm{i}\left(\frac{\beta+\gamma}{2}\right)\right], \qquad \lambda = \sin\left(\frac{\alpha}{2}\right)\exp\left[\mathrm{i}\left(\frac{\beta-\gamma+\pi}{2}\right)\right]. \tag{210}$$

As an example, we consider the coherent states $|J_{\mathrm{coh}};\zeta\rangle$, whose analytic representation has been given in (197). In order to find the state $U_1(\alpha,\beta,\gamma)|J_{\mathrm{coh}};\zeta\rangle$, we perform the Möbius mapping of equation (209) and get

$$f(z) = \mathrm{e}^{\mathrm{i}2j\psi}\frac{(1+z\zeta')^{2j}}{(1+|\zeta'|^2)^j}, \qquad \zeta' = \frac{\lambda + \kappa\zeta}{\kappa^* - \lambda^*\zeta}, \qquad \psi = \arg(\kappa^* - \lambda^*\zeta). \tag{211}$$

This function corresponds to the coherent state $\mathrm{e}^{\mathrm{i}2j\psi}|J_{\mathrm{coh}};\zeta'\rangle$. κ and λ have been given in terms of α, β, γ in (210).

12.3. The Dirac contour representation in the extended complex plane

In this representation, an arbitrary ket state $|f\rangle$ is represented by a function $f_{\mathrm{k}}(z)$ (which is identical to the one used in equation (196)) but the corresponding bra state $\langle f|$ is represented by a different function $f_{\mathrm{b}}(z)$:

$$|f\rangle = \sum_n f_n |J; jn\rangle \to f_{\mathrm{k}}(z) = \sum_n d(j,n) f_n z^{j+n}, \tag{212}$$

$$\langle f| = \sum_n f_n^* \langle J; jn| \to f_{\mathrm{b}}(z) = \sum_n \frac{f_n^*}{d(j,n) z^{j+n+1}}. \tag{213}$$

We use 'k' and 'b' in the notation to indicate ket and bra states, correspondingly. The function $f_{\mathrm{k}}(z)$ is a polynomial of z of order $2j$ and has singularity at ∞ (which is the north pole). The function $f_{\mathrm{b}}(z)$ is a polynomial of z^{-1} of order $2j+1$ and has singularity at 0 (which is the south pole).

As examples, we give the angular momentum states

$$\begin{aligned} |J; jm\rangle &\to f_{\mathrm{k}}(z) = d(j,m) z^{j+m}, \\ \langle J; jm| &\to f_{\mathrm{b}}(z) = \frac{1}{d(j,m) z^{j+m+1}} \end{aligned} \tag{214}$$

and the coherent states

$$
\begin{aligned}
|J_{\mathrm{coh}};\zeta\rangle &\rightarrow f_{\mathrm{k}}(z)=\frac{(1+z\zeta)^{2j}}{(1+|\zeta|^2)^j},\\
\langle J_{\mathrm{coh}};\zeta| &\rightarrow f_{\mathrm{b}}(z)=\frac{1}{(1+|\zeta|^2)^j}\frac{z^{2j+1}-(\zeta^*)^{2j+1}}{(z-\zeta^*)z^{2j+1}}.
\end{aligned}
\tag{215}
$$

We note that the $f_{\mathrm{b}}(z)$ corresponding to $\langle J_{\mathrm{coh}};\zeta|$ has a singularity at $z=0$ but has no singularity at $z=\zeta^*$.

In this analytic representation the scalar product is given by

$$\langle f|g\rangle=\oint_C\frac{\mathrm{d}z}{2\pi\mathrm{i}}f_{\mathrm{b}}(z)g_{\mathrm{k}}(z)=\sum f_n^*g_n, \tag{216}$$

where C is an anticlockwise contour around the origin, which is the singularity for $f_{\mathrm{b}}(z)$.

The following transformations take the Dirac bra representation to the Dirac ket representation and vice versa:

$$\oint_C\frac{\mathrm{d}w}{2\pi\mathrm{i}}f_{\mathrm{b}}(w)(1+z^*w)^{2j}=[f_{\mathrm{k}}(z)]^*, \tag{217}$$

$$\frac{2j+1}{z}\int_0^\infty\frac{\mathrm{d}t}{(1+t)^{2j+2}}\left[f_k\left(\frac{t}{z^*}\right)\right]^*=f_{\mathrm{b}}(z). \tag{218}$$

We can prove these equations if we substitute the $f_{\mathrm{k}}(z)$ and $f_{\mathrm{b}}(z)$ from equation (212) and perform the integration.

An arbitrary operator

$$Q=\sum_{n,m}Q_{nm}|J;jn\rangle\langle J;jm| \tag{219}$$

is represented by a kernel $Q(z_1,z_2)$ such that

$$Q|f\rangle\rightarrow\oint_C\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}Q(z,\zeta)f_{\mathrm{k}}(\zeta), \tag{220}$$

$$\langle f|Q\rightarrow\oint_C\frac{\mathrm{d}w}{2\pi\mathrm{i}}f_{\mathrm{b}}(w)Q(w,z), \tag{221}$$

where C is an anticlockwise contour around the origin. It is seen easily that the kernel $Q(z_1,z_2)$ is given by

$$Q(z_1,z_2)=\sum_{n,m}Q_{nm}\frac{d(j,n)z_1^{j+n}}{d(j,m)z_2^{j+m+1}}. \tag{222}$$

As a first example, we consider the unit operator for which

$$\mathbf{1}\rightarrow Q(z_1,z_2)=\frac{z_1^{2j+1}-z_2^{2j+1}}{(z_1-z_2)z_2^{2j+1}}. \tag{223}$$

This implies that for any state $|f\rangle$, $f_{\mathrm{k}}(z)$ is related to $f_{\mathrm{k}}(\zeta)$ through the relation

$$f_k(z)=\oint_C\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\frac{z^{2j+1}-\zeta^{2j+1}}{(z-\zeta)\zeta^{2j+1}}f_{\mathrm{k}}(\zeta) \tag{224}$$

and $f_{\mathrm{b}}(z)$ is related to $f_{\mathrm{b}}(w)$ through the relation

$$f_{\mathrm{b}}(z)=\oint_C\frac{\mathrm{d}w}{2\pi\mathrm{i}}f_{\mathrm{b}}(w)\frac{w^{2j+1}-z^{2j+1}}{(w-z)z^{2j+1}}. \tag{225}$$

C is an anticlockwise contour around the origin. We can check directly these relations if we insert into them equations (212) and (213) and perform the contour integrals.

As a second example, we consider the angular momentum operator J_z for which $Q_{nm} = n\delta_{nm}$. A straightforward calculation gives

$$J_z \to Q(z_1, z_2) = \frac{j(z_1^{2j+2} - z_2^{2j+2}) - (j+1)z_1 z_2(z_1^{2j} - z_2^{2j})}{(z_1^2 - z_2^2)z_2^{2j+1}}. \tag{226}$$

As explained in the previous subsection, in the extended complex plane, $SU(2)$ transformations are performed easily using conformal mappings. Action of the $SU(2)$ operator (173) on the arbitrary ket state $|f\rangle$ represented by the polynomial $f_k(z)$ is implemented with the Möbius mapping as described in equations (208) and (209):

$$U_1|f\rangle \to f_{\rm k}\left(\frac{\kappa z - \lambda^*}{\lambda z + \kappa^*}\right)(\lambda z + \kappa^*)^{2j} = \sum_n d(j,n) f_n[\kappa z - \lambda^*]^{j+n}[\lambda z + \kappa^*]^{j-n}. \tag{227}$$

With regard to bra states, it is not easy to find an analogous simple formula for the function corresponding to $\langle g|U_1$. But of course we can calculate the ket function for $U_1^\dagger|g\rangle$ and then calculate the corresponding bra function using equation (218) or expand the ket function as in equation (212), find the coefficients f_n and then calculate the bra function using equation (213).

12.4. Expansion of a state in terms of $SU(2)$ coherent states on a contour

As an application of the above formalism, we will show how we can expand an arbitrary state in terms of $SU(2)$ coherent states on a contour around the origin, in the extended complex plane. This problem has been considered in [65–67]. We introduce the 'complementary states':

$$\begin{aligned} |J_{\rm compl}; z\rangle &= [\mathcal{N}(|z|)]^{-1} \sum_n \frac{1}{d(j,n)(z^*)^{j+n+1}} |J; jn\rangle, \\ \langle J_{\rm compl}; z| &= [\mathcal{N}(|z|)]^{-1} \sum_n \frac{1}{d(j,n) z^{j+n+1}} \langle J; jn|, \end{aligned} \tag{228}$$

where $\mathcal{N}(|z|)$ is a normalization factor,

$$\mathcal{N}(|z|) = \left[\sum_n \frac{(j+n)!(j-n)!}{(2j)!} \frac{1}{|z|^{2(j+n+1)}}\right]^{1/2}. \tag{229}$$

They are not $SU(2)$ coherent states, but they play an auxiliary role. The representation of an arbitrary bra state $\langle f|$ given in equation (213) can be written in terms of these states as

$$f_{\rm b}(z) = \sum_n \frac{f_n^*}{d(j,n) z^{j+n+1}} = \mathcal{N}(|z|)\langle f|J_{\rm compl}; z^*\rangle = \mathcal{N}(|z|)\langle J_{\rm compl}; z|f^*\rangle. \tag{230}$$

We can easily prove that

$$\begin{aligned} &\oint_C \frac{{\rm d}z}{2\pi {\rm i}} (1+|z|^2)^j \mathcal{N}(|z|) |J_{\rm coh}; z\rangle\langle J_{\rm compl}; z| = \mathbf{1}, \\ &\oint_C \frac{{\rm d}z}{2\pi {\rm i}} (1+|z|^2)^j \mathcal{N}(|z|) |J_{\rm compl}; z^*\rangle\langle J_{\rm coh}; z^*| = \mathbf{1}, \end{aligned} \tag{231}$$

where C is an anticlockwise contour around the origin. Using this, we can prove that an arbitrary ket state $|f\rangle = \sum f_n|J; jn\rangle$ can be expanded in terms of $SU(2)$ coherent states on a contour around the origin as

$$
\begin{aligned}
|f\rangle &= \oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} (1+|z|^2)^j a(z)|J_{\mathrm{coh}}; z\rangle, \\
a(z) &= \mathcal{N}(|z|)\langle J_{\mathrm{compl}}; z|f\rangle = \sum_n \frac{f_n}{d(j,n)z^{j+n+1}}.
\end{aligned}
\tag{232}
$$

It is seen that the coefficients $a(z)$ are the Dirac representation for the bra state $\langle f^*|$. We can also prove that an arbitrary bra state $\langle g| = \sum g_n^* \langle J; jn|$ can be expanded in terms of $SU(2)$ coherent states on a contour around the origin as

$$
\begin{aligned}
\langle g| &= \oint_C \frac{\mathrm{d}z}{2\pi\mathrm{i}} (1+|z|^2)^j b(z)\langle J_{\mathrm{coh}}; z^*|, \\
b(z) &= \mathcal{N}(|z|)\langle g|J_{\mathrm{compl}}; z^*\rangle = \sum_n \frac{g_n^*}{d(j,n)z^{j+n+1}}.
\end{aligned}
\tag{233}
$$

It is seen that the coefficients $b(z)$ are the Dirac representation for the bra state $\langle g|$.

The overlap between a $SU(2)$ coherent state and a complementary state is

$$
\langle J_{\mathrm{compl}}; z|J_{\mathrm{coh}}; w\rangle = [\mathcal{N}(|z|)]^{-1}(1+|w|^2)^{-j}\frac{w^{2j+1}-z^{2j+1}}{(w-z)z^{2j+1}}. \tag{234}
$$

It is equal to zero when $w = z\omega^k$, where $k \neq 0$ and $\omega = \exp(\mathrm{i}2\pi/(2j+1))$. Other properties of the complementary states can be found in [67].

13. Systems described by a direct sum of finite Hilbert spaces

In this section, we consider systems with an infinite dimensional Hilbert space, which however is ‘naturally’ expressed as the direct sum of finite Hilbert spaces because a certain class of transformations leaves these finite Hilbert spaces invariant.

One example is functions on a sphere (in connection with problems expressed in spherical coordinates). With regard to rotations, it is useful to express the relevant infinite-dimensional Hilbert space as the direct sum of the $2j+1$-dimensional Hilbert spaces spanned by the spherical harmonics $Y_{jm}(\alpha, \beta)$ (with $-j \leqslant m \leqslant j$). This is because each of these $2j+1$-dimensional Hilbert spaces remains invariant under rotations.

In such problems we can apply the formalism of finite quantum systems as exemplified below for the case of spherical harmonics and the Schwinger $SU(2)$ formalism for two-mode systems.

13.1. Dual spherical harmonics

We consider the $(2j+1)$-dimensional Hilbert space $\mathcal{H}(2j+1)$ spanned by the angular momentum states $|J; jm\rangle$. Let $\mathcal{H}$ be the infinite-dimensional Hilbert space which is the direct sum of the Hilbert spaces $\mathcal{H}(2j+1)$, for all integer values of j. In $\mathcal{H}$ we consider the states

$$
|J; \alpha, \beta\rangle = \sum_{j,m} Y_{jm}^*(\alpha, \beta)|J; jm\rangle, \tag{235}
$$

where $0 \leqslant \alpha \leqslant \pi$ and $0 \leqslant \beta < 2\pi$ are angles on a sphere and $Y_{jm}(\alpha, \beta)$ are the usual spherical harmonics. The states $|J; \alpha, \beta\rangle$ form an orthonormal basis in $\mathcal{H}$:

$$
\begin{aligned}
&\int |J; \alpha, \beta\rangle\langle J; \alpha, \beta|\, \mathrm{d}\cos\alpha\, \mathrm{d}\beta = \mathbf{1}, \\
&\langle J; \alpha_1, \beta_1|J; \alpha_2, \beta_2\rangle = \delta(\cos\alpha_1 - \cos\alpha_2)\delta(\beta_1 - \beta_2).
\end{aligned}
\tag{236}
$$

We call F_{2j+1} the Fourier operator within the Hilbert space $\mathcal{H}(2j+1)$, and we introduce the Fourier operator in $\mathcal{H}$ as

$$F = \sum_j F_{2j+1}. \tag{237}$$

We also introduce the 'dual spherical harmonics' [14], which are related to the usual spherical harmonics through a finite Fourier transform:

$$X_{jn}(\alpha, \beta) = (2j+1)^{-1/2} \sum_m Y_{jm}(\alpha, \beta)\omega(nm). \tag{238}$$

Acting with F on the states $|J; \alpha, \beta\rangle$, we get the states

$$|\theta; \alpha, \beta\rangle = F|J; \alpha, \beta\rangle = \sum_{j,m} Y^*_{jm}(\alpha, \beta)|\theta; jm\rangle = \sum_{j,m} X^*_{jm}(\alpha, \beta)|J; j \; -m\rangle. \tag{239}$$

The states $|\theta; \alpha, \beta\rangle$ form an orthonormal basis in $\mathcal{H}$.

An arbitrary state $|s\rangle$ in $\mathcal{H}$ can be represented with the function $s_J(\alpha, \beta) = \langle J; \alpha, \beta|s\rangle$ (which we call J-representation) or with the function $s_\theta(\alpha, \beta) = \langle \theta; \alpha, \beta|s\rangle$ (which we call θ-representation). In order to find a transform between these two representations, we calculate the matrix elements of the Fourier operator

$$\begin{aligned} \mathcal{F}(\alpha, \beta; \gamma, \delta) \equiv \langle J; \alpha, \beta|F|J; \gamma, \delta\rangle &= \sum_{j,m} Y^*_{jm}(\alpha, \beta) X^*_{j,-m}(\gamma, \delta) \\ &= \sum_{j,m} X_{jm}(\alpha, \beta) Y^*_{jm}(\gamma, \delta). \end{aligned} \tag{240}$$

We see easily that

$$\begin{aligned} s_J(\alpha, \beta) &= \int \mathrm{d}\cos\gamma \, \mathrm{d}\delta \mathcal{F}(\alpha, \beta; \gamma, \delta) s_\theta(\gamma, \delta), \\ s_\theta(\alpha, \beta) &= \int \mathrm{d}\cos\gamma \, \mathrm{d}\delta [\mathcal{F}(\alpha, \beta; \gamma, \delta)]^* s_J(\gamma, \delta). \end{aligned} \tag{241}$$

13.2. The Schwinger $SU(2)$ formalism

We consider a two-mode harmonic oscillator with Hilbert space $\mathcal{H}_1 \times \mathcal{H}_2$ spanned by the number eigenstates $|N_1, N_2\rangle$. Let $a_1^\dagger, a_1$ and $a_2^\dagger, a_2$ be the creation and annihilation operators for the two modes.

In the Schwinger representation of $SU(2)$ the angular momentum operators are expressed as

$$J_+ = a_1^\dagger a_2, \qquad J_- = a_1 a_2^\dagger, \qquad J_z = \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2). \tag{242}$$

Indeed we can see easily that they obey the standard angular momentum commutation relations. If n_s is the sum of the number operators for the two modes, then

$$\begin{aligned} &n_s = a_1^\dagger a_1 + a_2^\dagger a_2, \qquad [n_s, J_+] = [n_s, J_-] = [n_s, J_z] = 0, \\ &J^2 = \frac{n_s}{2}\left(\frac{n_s}{2} + 1\right). \end{aligned} \tag{243}$$

A consequence of this is that the number eigenstates $|N_1, N_2\rangle$ are also the angular momentum states $|J; jm\rangle$:

$$|N_1, N_2\rangle = |J; jm\rangle, \qquad j = \tfrac{1}{2}(N_1 + N_2), \qquad m = \tfrac{1}{2}(N_1 - N_2). \tag{244}$$

Using the notation $\mathcal{H}(2j+1)$ for the $(2j+1)$-dimensional Hilbert space spanned by the angular momentum states $|J;\, jm\rangle$ with a fixed j, we see that

$$\mathcal{H}_1 \times \mathcal{H}_2 = \sum_j \mathcal{H}(2j+1), \qquad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots, \tag{245}$$

where the summation here indicates a direct sum of Hilbert spaces. We call π_{2j+1} the projection operator into the Hilbert space $\mathcal{H}(2j+1)$, which is written in terms of number eigenstates as

$$\pi_{2j+1} = \sum_{N=0}^{2j} |N, 2j-N\rangle\langle N, 2j-N|, \qquad \sum_j \pi_{2j+1} = \mathbf{1}. \tag{246}$$

All π_{2j+1} commute with the angular momentum operators of equation (242). Therefore, any operator that is a function of the angular momentum operators leaves the Hilbert spaces $\mathcal{H}(2j+1)$ invariant (i.e. acting on a state in $\mathcal{H}(2j+1)$ produces another state that belongs to the same Hilbert space).

As an application of this, we consider a two-mode system described by the Hamiltonian

$$\begin{aligned} H &= \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \lambda a_1^\dagger a_2 + \lambda^* a_1 a_2^\dagger = \Omega n_s + \omega J_z + \lambda J_+ + \lambda^* J_-, \\ \Omega &= \tfrac{1}{2}(\omega_1 + \omega_2), \quad \omega = \omega_1 - \omega_2. \end{aligned} \tag{247}$$

This Hamiltonian is used for the description of frequency converters in quantum optics [68]. We assume that at $t = 0$ the system is in a state $|s\rangle$, and we want to calculate the state of the system at a later time t. We first write the state $|s\rangle$ as a sum of its projections to the Hilbert spaces $\mathcal{H}(2j+1)$:

$$|s\rangle = \sum_j |s_{2j+1}\rangle, \qquad |s_{2j+1}\rangle = \pi_{2j+1}|s\rangle. \tag{248}$$

Taking into account that n_s commutes with the angular momentum operators (equation (243)) and also that $n_s|s_{2j+1}\rangle = 2j|s_{2j+1}\rangle$, we prove that

$$\begin{aligned} \exp(\mathrm{i}tH)|s\rangle &= \sum_j \exp(\mathrm{i}tH)|s_{2j+1}\rangle \\ &= \sum_j \exp(\mathrm{i}2jt\Omega)\exp[\mathrm{i}t(\omega J_z + \lambda J_+ + \lambda^* J_-)]|s_{2j+1}\rangle. \end{aligned} \tag{249}$$

We note that the state $\exp[\mathrm{i}t(\omega J_z + \lambda J_+ + \lambda^* J_-)]|s_{2j+1}\rangle$ is in the Hilbert space $\mathcal{H}(2j+1)$. We will not present the rest of the calculation, which is straightforward but lengthy.

14. Applications

The theory of finite quantum systems is a subject in its own right but it also has a variety of applications. In this section we give a guide to the literature on the applications.

14.1. Quantum optics

The practical implementation of some of the transformations that we discussed earlier, using beam splitters, has been discussed in [69]. Frequency converters [68] is another application using the Schwinger formalism, which we presented earlier. In this case, as we explained, the Hilbert space is infinite, but it is the direct sum of many finite Hilbert spaces that remain invariant under the action of the Hamiltonian.

The Pegg–Barnett formalism of phase states [70, 71] starts with a finite Hilbert space and uses extensively the theory of finite systems. Only at the end does it consider the limit $d \to \infty$.

The theory of quantum multi-pole radiation uses the theory of finite quantum systems and is reviewed in [72]. Coherent states in truncated (finite) Hilbert space [73] are also another application.

A different formalism for finite quantum systems that is based on different boundary conditions has been discussed in [74].

14.2. Qudits in quantum information processing

The theory of finite quantum systems is essential for the area of quantum information processing [75]. For example, the quantum Fourier transform [76] is intimately related to the Fourier transform studied earlier. Also, quantum computing with Wigner functions has been studied in [77].

Most of the work on quantum information processing has been with two-dimensional Hilbert spaces (qubits). More recently the use of d-dimensional Hilbert spaces (qudits) as a potentially more powerful tool has been studied [78]. In this context, the formalism on finite quantum systems discussed in this paper plays an important role. Experimental realization of qudits has been discussed in [79].

$SU(2)$ transformations and the Pauli matrices used in the context of qubits are replaced with $SU(d)$ transformations and the diplacement operators $D(\alpha, \beta)$ in the context of qudits. The displacement operators form the Heisenberg–Weyl group (also called the Pauli group by the quantum information community). Arbitrary transformations on qudits are expressed in terms of the displacement operators, as shown in equation (95).

The symplectic group of transformations (also called Clifford group by the quantum information community) preserves the structure of the Heisenberg–Weyl group given in equations (27). The symplectic group on one qudit and more importantly the symplectic group on multiqudits (direct products of many qudits) are important in quantum coding.

Quantum coding introduces redundancy in order to protect qubits from errors. The simplest coding schemes are the three-qubit repetition codes (reviewed in [75]). However it is easily seen that they protect qubits from a very limited class of errors. For example, the three-qubit bit flip code cannot protect against phase errors; and the three-qubit phase flip code cannot protect against bit flip errors. For protection against larger classes of errors, more qubits are required. Arbitrary errors at known positions (erasures) require at least four qubits [80]. More general errors require at least five qubits [81]. Other coding schemes that provide protection against any one-qubit error are Shor's nine qubit code [82] and the seven qubit code [83]. Generalization of coding and quantum computation with qubits into coding and quantum computation with qudits is currently in progress.

14.3. Other applications

There are many applications of the theory of finite systems in various problems within the general area of Mathematical Physics. Some examples are string theory [84], quantum maps [85], hydrodynamics [86], etc.

Another application is the magnetic translation group in condensed matter. Two-dimensional electron systems in a uniform magnetic field and in a toroidal topology and the relevant magnetic translation group have been studied in [87–89]. Related also are applications to the quantum Hall effect [90] and to the Hofstadter butterfly [91].

Some of the mathematical formalism relevant to the theory of finite systems has been presented in [38].

15. Discussion

In this paper we have reviewed the work on finite quantum systems. This is quantum mechanics, in a position space that is a finite lattice with periodicity. A lot of the results are general. However some results are for a particular class of finite systems (e.g. those with an odd dimension or those with a dimension that is the power of a prime), and in our discussion we have indicated this very clearly.

In this context we have introduced position and momentum states and the Fourier transform that relates them. Consequently, an arbitrary state can be expressed in the position representation or in the momentum representation. The uncertainty principle states that these two distributions cannot be very narrow simultaneously, and it is quantified with entropic quantities in equation (22). We have also introduced the functions Δ_m of equation (18), which are the analogues of the delta functions in the harmonic oscillator and which are useful in practical calculations.

The phase-space is the toroidal lattice $\mathcal{Z}(d) \times \mathcal{Z}(d)$. We have studied displacements in this phase-space and the corresponding Heisenberg–Weyl group. We have also considered symplectic transformations in this phase-space. We have explained that when the dimension d is the power of a prime number ($d = p^n$), the phase-space has a geometrical structure (it is a finite geometry). In this case, symplectic transformations are well defined and they form the $Sp(2, GF(p^n))$group. We have constructed explicitly the symplectic operator $S(\kappa, \lambda, \mu)$ both analytically and numerically, and we gave several examples (for the simple case where the dimension is the first power of a prime) that elucidate the nature of these transformations.

An important tool in phase-space methods is the displaced parity operators and the displacement operators. They are related to each other through a two-dimensional Fourier transform. We have studied their marginal properties and their Radon transforms.

Intimately related to the displaced parity operators and the displacement operators are the Wigner and Weyl functions, correspondingly. The Wigner function is a pseudoprobability distribution of the quantum mechanical particle in phase-space. The Weyl function is a generalized correlation function. We have explained that they are related to each other through a two-dimensional Fourier transform, and we have studied their marginal properties and their Radon transforms.

General transformations in the Hilbert space have also been studied. They can be written as a sum of displacement operators with the Weyl functions as coefficients and also as a sum of displaced parity operators with the Wigner functions as coefficients.

A factorization of the Hilbert space in terms of smaller ones can be useful because calculations in large Hilbert spaces can be tedious. We have discussed such a factorization based on the Chinese remainder theorem. This method has originally been used by Good in the context of fast Fourier transforms, and here we used it in the context of finite quantum systems. We have shown that all unitary transformations and more generally the whole quantum mechanical formalism in the large Hilbert space reduce to calculations in the smaller spaces, which should be performed and combined appropriately to produce the results in the large Hilbert space.

Composite systems and their entanglement is a subject of enormous interest. We have discussed one aspect of this problem, which is transformations in composite finite quantum systems. We have made the distinction between local $SU(d) \times SU(d)$ unitary transformations and more general $SU(d^2)$ unitary transformations that can entangle the two subsystems. We have also discussed in more detail local and entangling symplectic transformations and calculated numerically the symplectic operators.

An important example of finite systems is systems with angular momentum j. A Fourier transform on the usual angular momentum states gives the angle states. In this context, we have made the distinction between the Bose sector (with odd dimension) and the Fermi sector (with even dimension) because the formulae are slightly different in these two cases. We have also discussed $SU(2)$ coherent states and in particular the connection of $SU(2)$ coherent states with the theory of finite quantum systems. Related to coherent states are the analytic representations. In connection with $SU(2)$ coherent states we have studied the analytic representation in the extended complex plane and the Dirac contour representation in the extended complex plane.

Some systems have an infinite-dimensional Hilbert space, which is naturally expressed as the direct sum of finite Hilbert spaces, because a certain class of transformations leaves these finite Hilbert spaces invariant. In such problems we can apply the formalism of finite quantum systems. We have discussed two such examples: functions on a sphere and spherical harmonics, and the Schwinger $SU(2)$ formalism for two-mode systems.

Applications of the theory of finite quantum systems include quantum optics, quantum computing, two-dimensional electron systems in magnetic fields and the magnetic translation group, the quantum Hall effect, hydrodynamics, mathematical physics, applied mathematics, etc. We have not discussed these in detail, but we have given a brief guide to the relevant literature. We hope that this paper will motivate researchers to apply the theory of finite systems into their own field.

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