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POWERFUL RELATIVISTIC ELECTRON BEAMS IN A PLASMA AND IN A VACUUM (THEORY)

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ABSTRACT. The possibility of using intense relativistic electron beams (REBs) for heating plasmas in open systems is discussed. Within this context the following three sets of problems are discussed:

1. REB transport in a vacuum with a strong magnetic field; beam equilibrium, stability and critical currents in a vacuum.
2. Beam transport in a plasma; charge and current neutralization of the beam; reverse-current heating of the plasma, and macroscopic REB instabilities in the plasma.
3. The theory of collective relaxation of REBs in a plasma, including quasi-linear and non-linear relaxation models; the role of plasma non-uniformity, and macroscopic effects during REB relaxation.

1. INTRODUCTION

The aim of this survey is to examine the interaction between high power relativistic electron beams (REBs) and a plasma within the context of the problem of plasma heating. The possibilities of using REBs in this connection are rather varied.

First, there is the traditional method, for non-relativistic beams, of heating a plasma in straight systems: injection of the beam into the plasma from the end of the device along the magnetic lines of force. This was in fact the way that the first experiments on heating a plasma by means of REBs [1-9], which brought such very encouraging results, were set up.

Second, there is the method of heating a plasma in toroidal systems by a beam circulating in a torus round a large circumference. Experimental data on the interaction between the injected beam and the plasma in these circumstances are unavailable, but it is natural to assume that the interaction is qualitatively the same as in straight systems. The most difficult problem here appears to be the injection of powerful REBs through a strong magnetic field. The first results relating to this problem were published by Meixel et al. [10] and Rudakov et al. [11].

And third, there is the heating method proposed by Winterberg and Zavojskij, in which small particles of a D-T mixture are heated by collective absorption of a focussed electron beam [12, 13]. At present, no experimental results are available for this scheme.

In this survey we shall focus our attention on the first of the above possibilities (i.e. heating the plasma in open systems), since it is along this line that most of the results have been obtained both experimentally and in theoretical work.

In the case of open systems the range of plasma densities of interest for controlled nuclear fusion runs from $n \sim 10^{14}$ cm^{-3} (normal open traps) to $n \sim 10^{18}$-10^{19} cm^{-3} (θ pinches with a liner [14] and multimirror traps [15]). The energy required to heat the plasma up to $T \sim 10^4$ eV in the devices described ranges from tens of kJ to tens of MJ. At present there are in existence REB sources with an energy of $\sim$1 MJ [16], and we can hope to see sources with energies of 10 MJ in the near future. Hence the energy requirements do not seem to be an insuperable problem. Furthermore, they may become much more moderate if we bear in mind the use of REBs as a means of preheating a plasma, let us say up to $T \sim 10^3$ eV, with the final heating being effected by another method (e.g. adiabatic compression).

In all the above open systems the absence of contact between the plasma and the ends of the device is essential for longitudinal thermal insulation, or, in other words, between the electron beam source and the plasma there must be a vacuum space (or a space filled with plasma of such low density that the thermal flux through it can be disregarded). Hence the bulk of the plasma can be heated only in a case in which the transport of the beam through this space can proceed without great losses of energy.

We discuss the problems of beam transport in Sections 2 and 3, assuming from the outset that in the system considered there is a strong longitudinal magnetic field. It predetermines to a considerable extent the motion of the beam particles and makes the problem of beam equilibrium much easier to solve. More specifically, we can thereby eliminate the problem of beam reflection by the curvature of the electron trajectories in the magnetic self field. Even in this relatively simple case, however, the problem of transport through a vacuum still retains some rather interesting, though seemingly not very well known, physical effects, hence we have devoted a certain amount of space to it (Section 2).

Beam transport in a case in which the space between the source and the bulk of the plasma is filled by a second plasma (rarefied as compared with the principal plasma, but not necessarily so compared with the beam) is discussed in Section 3. There we deal with the problems of charge and current neutralization of the beam and consider macroscopic beam instabilities in the plasma (by macroscopic instabilities we mean those whose scale is not less than the transverse dimension of the beam). Instabilities of this kind are extremely dangerous since they may lead to displacement of the beam as a whole and to its being spilled onto the walls of the vacuum chamber.

The next problem (and apparently the most important one so far as beam heating is concerned) is to find a mechanism by which the beam energy can be effectively transferred to the bulk of the plasma. This problem arises because the collisional deceleration length required for relativistic electrons in a plasma with $n \lesssim 10^{19}$ cm^{-3} is too great ($\gtrsim$1 km) to provide for effective dissipation of the beam energy in a device of reasonable length. Hence all prospects here are based on the collective effects occurring during the development of small-scale beam instability, which can in principle reduce the beam relaxation length in the plasma by a considerable amount. The problems involved are discussed in Section 4. Relaxation is considered on the basis of weak turbulence theory, the applicability conditions of which enable us to examine a number of rather interesting practical effects. The most difficult problem is to find the turbulent oscillation spectrum, which is determined by the balance between the linear excitation of the oscillations by the beam and the non-linear spreading of the oscillations over the spectrum. If this problem is solved, it is not difficult to derive the electron beam deceleration rate and, accordingly, the relaxation length. In Section 4 we discuss several relaxation models corresponding to various physical conditions.

Such is the general arrangement of our survey. The areas of plasma physics covered are still a long way from being able to give specific recommendations on constructing devices. There is a great deal more to be explained and clarified (especially as regards REB relaxation in a plasma). Hence we seek only to give the reader general guidance concerning the results obtained so far and to draw attention to those problems that still remain to be solved.

2. BEAM TRANSPORT IN A VACUUM

2.1. Transport in a strong longitudinal magnetic field; critical currents

A typical set-up for experiments on the transport of a REB in a vacuum is shown in Fig. 1. The beam emitted by the cathode (1) and accelerated in the diode space passes through the anode (2) (which may be a thin metal foil or grid) into the drift space (3), after which it reaches the collector (4) connected to the anode by the return current conductor (5). Henceforth we shall take this conductor to be cylindrical in all cases.

The duration of the beam τ in the transport experiments is usually long compared with the time of flight of the electrons through the drift space (L/c). Hence we shall mainly discuss steady-state transport. Non-steady-state effects with $c\tau \gg L$ reduce mainly to some degree of deceleration of the particles at the leading edge of the current and to acceleration at the trailing edge (due to the induced e.m.f.).

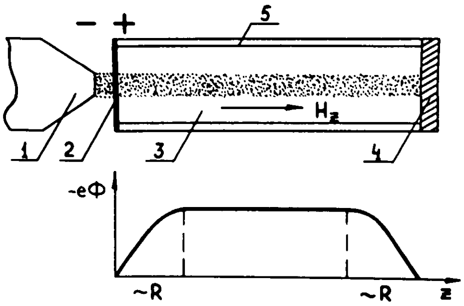

FIG. 1. Scheme of an experiment for REB transport studies in a vacuum: (1) cathode; (2) anode; (3) drift space; (4) collector; (5) return current conductor. The distribution of the potential φ along the axis of the device is shown at the bottom.

To ensure vacuum transport of the beam over long distances, use is made of a strong longitudinal magnetic field, which suppresses transverse broadening of the beam. In this section we shall assume that the magnetic field is infinitely large so that the beam electrons are tied to the lines of force and can only move in a longitudinal direction. The conditions under which this approximation is valid will be formulated below (see Eq. (2.19)). The presence of a strong magnetic field hinders the transverse broadening of the beam as a result of the space charge, but there is still a restriction from above on the amount of current passed through by the system for a given particle energy. This limitation is associated with the effect of the Z component of the electric field close to the point at which the beam enters the drift space, but when calculating the critical current it proves possible (see Smith and Hartman [17]) to get round the very difficult problem of ascertaining the field structure in this region, and just to calculate the field in a region some distance away from the ends[1].

We shall start with a simple case in which the cross-section of the beam is shaped like a thin ring (the thickness of the ring a is taken to be small compared with both the beam radius r_b and the distance between the beam boundary and the chamber wall).

In the steady state a long way from the ends of the system (at a distance several times greater than the radius of the return current conductor R) the distribution of the electrostatic potential ϕ is almost the same along the entire length (it depends on the radius alone). In the case of an annular beam

$$\phi(r) = -2\frac{|I_b|}{v}\begin{cases}\ln\dfrac{R}{r}, & r_b < r < R\\[2ex] \ln\dfrac{R}{r_b}, & r \le r_b\end{cases} \qquad (2.1)$$

[1] Here it is assumed that the length of the drift tube is much greater than its radius. The critical current problem for a shorter space ($L \ll R$) is considered by Godyak et al. [18].

Here I_b is the total current, and v is the electron velocity (the beam is taken to be monoenergetic). On the chamber wall the potential is taken equal to zero. The velocity of the particles is associated with $\phi_b \equiv \phi(r_b)$ by the law of the conservation of energy. Using v_0 to represent the electron velocity at the input to the drift space, we get

$$v = c\left\{1 - \left(\gamma_0 + \frac{e\phi_b}{mc^2}\right)^{-2}\right\}^{\frac{1}{2}} \tag{2.2}$$

where

$$\gamma_0 = \left(1 - \frac{v_0^2}{c^2}\right)^{-\frac{1}{2}}$$

Eqs (2.1) and (2.2) give the following equation for ϕ_b:

$$F(\phi_b) \equiv -\frac{e\phi_b}{mc^2}\left[1 - \left(\gamma_0 + \frac{e\phi_b}{mc^2}\right)^{-2}\right]^{\frac{1}{2}} = \frac{2e|I_b|}{mc^3} \ln\frac{R}{r_b} \tag{2.3}$$

The function $F(\phi_b)$ on the left-hand side of this equation is plotted in Fig.2. It has a maximum at the point $e\phi_b/mc^2 = -\gamma_0 + \gamma_0^{1/3}$. The maximum is equal to $(\gamma_0^{2/3} - 1)^{3/2}$. It can be seen from the plot that for small currents

$$|I_b| < I_{cr} \equiv \frac{mc^3}{2e\ln\frac{R}{r_b}} (\gamma_0^{2/3} - 1)^{3/2} \tag{2.4}$$

Eq. (2.3) has two solutions. The first corresponds to a small deceleration potential, i.e. to particles of high velocity and low density inside the drift space. The second solution corresponds to a high deceleration potential, which implies a lower steady-state velocity and higher density.

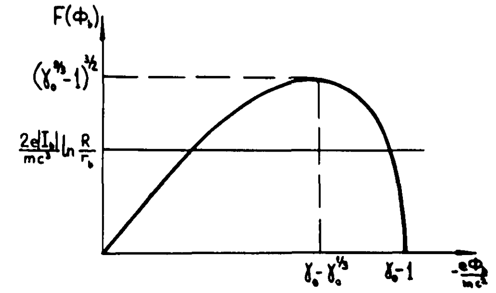

FIG.2. Graph showing the function $F(\phi_b)$.

It can easily be verified that the total energy in the system (the kinetic energy of the particles plus the energy of the field) is less in the first case than in the second. It may therefore be assumed that the second state is unstable and that the system must inevitably change from the second state to the first, although there is still no rigorous proof in support of this statement.

We may note that in the case of finite conductivity of the chamber walls, even the first state becomes unstable. This is the so-called "wall" instability, well known in the physics of accelerators (see e.g. Lopukhin and Vedenov [19], Kolomenskij [20] and Neil and Sessler [21]). The reason for this is that oscillations moving at a velocity close to the flow velocity (but smaller than it) possess negative energy in the laboratory frame of reference. If there is dissipation (in this case Joule dissipation in the walls), these oscillations become unstable. For the growth rate of this instability with $\gamma \gg 1$, $I_b/I_{cr} \ll 1$ the following evaluation is valid (see Appx I):

$$\mathrm{Im}\,\omega \simeq \frac{c}{2R}\sqrt{\frac{c}{8\pi\sigma R}}\sqrt{\frac{\gamma I_b}{I_{cr}}}\frac{1}{\ln\frac{R}{r_b}} \tag{2.5}$$

Here σ is the conductivity of the wall material. It should be noted that since this instability is of the convective type, it can occur only in fairly long systems. A stabilizing effect is exerted on it by the spread of the beam electrons over the longitudinal velocities. Under typical experimental conditions with high-current pulsed beams (see e.g. Abrashitov et al. [3], and Kapetanakos and Hammer [7]), the dissipative instability does not develop because of the limitation of the system in a longitudinal direction and the short duration of the beam.

At $I_b > I_{cr}$ there is no solution for Eq.(2.3), which indicates that it is impossible to have the steady-state flow of a current higher than critical through the drift space. As can be seen from (2.4), at $r_b \to R$ the critical current for an annular beam, when calculated formally, becomes infinite. In actual fact, at $r_b \to R$ we have to take into account the finite width of the beam a. We then get the following result (see Bogdankevich and Rukhadze [22]):

$$I_{cr} = \frac{mc^3}{e}(\gamma_0^{2/3} - 1)^{3/2}\frac{R}{a} \tag{2.6}$$

In deriving (2.4) we took the beam to be annular, but it is easy to show that the same equation determines the critical current for an arbitrary axisymmetric beam, provided that its radius is fairly small ($r_b \ll R$). Or, to be more exact, at $r_b \ll R$, Eq.(2.4) gives us the first term of the expansion of I_{cr} with respect to the parameter $(\ln(R/r_b))^{-1}$. The dependence of the critical current on the beam configuration shows up in the next order with respect to $(\ln(R/r_b))^{-1}$:

$$I_{cr} = \frac{mc^3}{2e\ln\frac{R}{r_b}}(\gamma_0^{2/3} - 1)^{3/2} \times \left[1 - \frac{1}{\ln\frac{R}{r_b}}\int_0^{r_b} r\,dr \int_r^{r_b}\frac{dr'}{r'}\int_0^{r''} r''\,f(r'')\,dr''\right] \tag{2.7}$$

Here $f(r) \equiv 2\pi j_b(r)/I_b$, where $j_b(r)$ is the current density and I_b is the total beam current.

The second term on the right-hand side of (2.7) is calculated on the assumption that $\ln R/r_b \gg \gamma_0^{2/3} - 1$. In the opposite limiting case ($\gamma_0^{2/3} - 1 \gg \ln R/r_b$) we can calculate the critical current by taking the beam electrons to be ultra-relativistic and not making the additional assumption that $R \gg r_b$. The result takes the following form:

$$I_{cr} = \frac{mc^3\gamma_0}{2e}\left[\ln\frac{R}{r_b} + \int_0^{r_b}\frac{dr}{r}\int_0^r r'f(r')\,dr'\right]^{-1} \qquad (2.8)$$

This equation is given by Bogdankevich and Rukhadze [22] for the case of a beam with a uniform current distribution over the cross-section.

Quantitative verification of the rather obvious results described above was provided in experiments of Mkheidze et al. [23] and Nation and Read [24]. Reasonable agreement with the theoretical values was found.

It is conceivable that by injecting the beam in the form of separate clumps, or by modulating the input current with respect to time in some other way, we might bring about an increase in the mean current $\bar{I}_b$ above I_{cr}. It seems obvious, however, that in order of magnitude the relationship $\bar{I}_b \lesssim I_{cr}$ remains valid for any method of modulating the input current. Indeed, at $\bar{I}_b \sim I_{cr}$ the space potential is equal in order of magnitude to the energy of the injected electrons, while at $\bar{I}_b \gg I_{cr}$ the potential energy would considerably exceed the kinetic energy which would certainly lead to most of the current being reflected.

We should point out an interesting effect relating to the steady-state injection of the beam. It can easily be seen that for all the cases considered above (Eqs (2.4), (2.6)-(2.8)) at $I_b = I_{cr}$ the space potential is somewhat less than the initial kinetic energy of the electrons. For example, in the case of (2.4),

$$\frac{e\phi_b}{mc^2} + \gamma_0 - 1 = \gamma_0^{1/3} - 1$$

i.e. at $\gamma_0 = 2$-3, the residual kinetic energy amounts to about 20% of its initial value. At first sight it might seem unclear why we do not get a supercritical current under steady-state conditions if each particle has a finite reserve of energy in the critical state. This can be understood by considering small deviations of the system from the steady state. We shall do this using the example of an annular beam.

Away from the ends of the waveguide, where in the steady state none of the quantities is dependent on the longitudinal co-ordinate z, propagation of small axisymmetric perturbations is described by the following dispersion equation (see Appx I):

$$\frac{I_0(\kappa R)}{I_0(\kappa r_b)} = \frac{2e|I_b|}{mv^3\gamma^3}\,\frac{\kappa^2}{(k_z - \frac{\omega}{v})^2} \times \left[I_0(\kappa R)K_0(\kappa r_b) - I_0(\kappa r_b)K_0(\kappa R)\right] \qquad (2.9)$$

Here I_0 and K_0 are modified zero-order Bessel functions, and $\kappa^2 = k_z^2 - (\omega^2/c^2)$. For long waves ($|\kappa R| \ll 1$) Eq. (2.9) can be substantially simplified, i.e.

$$1 = \frac{k_z^2 - \frac{\omega^2}{c^2}}{(k_z - \frac{\omega}{v})^2}\,\frac{2e|I_b|}{mv^3\gamma^3}\ln\frac{R}{r_b} \qquad (2.10)$$

As can be seen from this equation, long-wave oscillations possess a linear dispersion law ($\omega = \text{const} \times k_z$), there being two types of perturbations – the fast and the slow. A simple examination shows that for small current values both types of perturbation propagate in the direction of motion of the electrons[2]. As the current nears the critical value the velocity of the slow perturbations tends to zero[3]. Let us now assume that at the initial moment of time the system was in a steady state with current I_b, and that the current at the waveguide input then increased by a small value ΔI_b. Under subcritical conditions the slow perturbations associated with this change in current will drift together with the stream, and the system will change to a new steady state with current $I_b + \Delta I_b$. In the critical conditions, however, the slow potential perturbations will build up until they stop the flux. Further investigation of this situation is of course impossible within the framework of linearized equations.

For the time being, no categorical answer can be given to the question of what happens when a current greater than critical is injected into the waveguide. In such a case we would have the following possibilities:

(a) A steady-state case in which a non-monotonic distribution of the potential along the length[4] is established in the system (see Fig. 3), and where the excess current is reflected from the hump of the potential. If the amount by which I_b exceeds I_{cr} is not very great the turning point of the particles should be at a distance of the order of several radii from the point where the beam enters the waveguide (there is no other scale in the problem).

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[2] As pointed out above for small (sub-critical) currents there can be two steady states with different particle velocities v. Our statement refers to the solution with the higher velocity. In the case of the lower-velocity solution the slow perturbations propagate opposite to the electron stream.

[3] It should be stressed that in Eq. (2.10) the value of v corresponding to the electron energy behind the decelerating layer is used.

[4] Generally speaking, the problem of the monotonicity of the solution also exists for sub-critical currents. Proof of the monotonicity exists for only a few special cases.

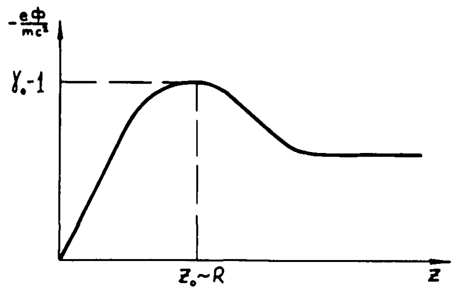

FIG. 3. Distribution of the potential along the device upon injection of a supercritical current.

(b) Non-steady flow of the supercritical current. The point here is that the steady-state solution (should there be one) may prove unstable on account of two-stream motion in the region $z < z_0$ (Fig. 3). Study of the instability is now a highly complex procedure, since even the steady-state distribution of the potential close to the point of beam input into the waveguide is not yet known. An indication of the fact that the pattern is really not steady-state can be gained from experiments [25] in which microwave oscillations were observed when supercritical currents were injected into a vacuum. The experimental data, however, do not enable us to say for certain that the generation of microwaves is associated with the supercritical nature of the injected current rather than with some other factor, e.g. the presence of residual plasma, a slipping instability connected with the finite magnitude of the magnetic field (see below), or something else.

2.2. Effects associated with the finite magnitude of the magnetic field

When going from the case $H_z \to \infty$ to finite magnetic fields it is essential, first and foremost, to consider the problem of radial beam equilibrium. We shall illustrate the nature of the problems involved by using the example of the ultrarelativistic beam $\gamma \gg 1$, assuming, in addition, that the field H_z is not too small, so that the electric and magnetic self fields of the beam can be regarded as a perturbation (see Hammer and Rostoker [26]). We shall make one further assumption: that the characteristic Larmor radius of the beam electrons r_H is small compared with the beam radius, i.e.

$$r_H \sim \frac{mc^2}{eH_z}\gamma\Delta\theta \ll r_b \qquad (2.11)$$

where $\Delta\theta$ is the angular spread of the particle velocities which may occur, e.g., through scattering of the beam electrons in the anode foil[5]. In the case of an annular beam we have to replace r_b by a on the right-hand side of this inequality.

Inequality (2.11) can be represented as

$$\frac{H_z^2}{8\pi} \gg \gamma n_b mc^2\Delta\theta^2 \frac{\gamma mc^3}{eI_b} \qquad (2.12)$$

It follows that when the condition $r_H \ll r_b$ is satisfied, the magnetic pressure $H_z^2/8\pi$ automatically becomes much greater than the transverse gas-kinetic pressure of the beam electrons $\mathcal{P}_\perp \sim \gamma n_b mc^2\Delta\theta^2$ (to see that this is so we may note that $I_b < I_{cr} \lesssim \gamma mc^3/e$). Hence we shall first consider the equilibrium of the beam without any angular spread.

In this case the projections of the particle trajectories onto the plane perpendicular to the z axis are circles[6], and the beam equilibrium condition can be written as

$$\frac{v_\varphi p_\varphi}{r} - eE_r + \frac{e}{c}v_z H_\varphi - \frac{e}{c}v_\varphi H_z = 0 \qquad (2.13)$$

(here we are using a cylindrical frame of reference r, φ, z). The azimuthal magnetic field H_φ and the radial electric field E_r are, respectively,

$$H_\varphi = \frac{4\pi}{rc}\int_0^r r' j_z(r')\,dr' \qquad (2.14)$$

$$E_r = \frac{4\pi}{r}\int_0^r \frac{r' j_z(r')\,dr'}{v_z(r')}\,dr' \qquad (2.15)$$

The function $j_z(r')$ is determined by the beam source, and we shall take it as given. To obtain a closed system of equations we shall also write the function

$$p(r) \equiv mc\sqrt{v_z^2+v_\varphi^2}\left(1-\frac{v_z^2+v_\varphi^2}{c^2}\right)^{-\frac{1}{2}} \qquad (2.16)$$

(it should be noted that $p(r) \neq$ const, even if the energy $\mathcal{E}_0$ of the electrons injected into the device is constant over the cross-section; see Section 2.1). For the given functions $j_z(r)$ and $p(r)$ Eqs (2.13)-(2.16) form a closed system from which the functions $v_\varphi(r)$, $v_z(r)$, $E_r(r)$, $H_\varphi(r)$ can be determined (as stated at the beginning of this section, we take the field H_z to be uniform; the condition under which this approximation can be applied will be shown below).

It is easy to see that if we can satisfy the fairly soft inequality:

$$H_z \gtrsim \frac{mc^2}{er_b} \qquad (2.17)$$

(or, in "practical" units, $H_z(\mathrm{Oe}) \gtrsim 1.5\times 10^3/r_b$ (cm)) the inequalities $v_\varphi/c < 1/\gamma$ and $p_\varphi/r < eH_z/c$

[5] The reason for the angular spread could also be non-collinearity of the electrical and magnetic fields in the diode space. This effect decreases, of course, as the longitudinal magnetic field increases.

[6] If there is spread, the trajectories become cycloidal because of Larmor rotation; for condition (2.11) the amplitude of the cycloid is small compared with the beam radius.

are automatically satisfied and considerably facilitate the solution of the problem of equilibrium. The first of these enables v_z to be expressed in terms of p(r), i.e.

$$v_z \simeq c\left[1 - \frac{1}{2\gamma^2(r)}\right], \quad \gamma(r) \equiv \left[\frac{p^2(r)}{m^2c^4} + 1\right]^{\frac{1}{2}}$$

while the second one means that we can disregard the first term on the left-hand side of Eq. (2.13), as compared with the last one. As a result we find that

$$E_r = H_\varphi + \frac{2\pi}{rc}\int_0^r \frac{r' j_z(r')}{\gamma^2(r')}\,dr'$$

$$v_\varphi = -\frac{2\pi}{rH_z}\left[\frac{1}{\gamma^2(r)}\int_0^r r' j_z(r')\,dr' + \int_0^r \frac{r' j_z(r')\,dr'}{\gamma^2(r')}\right] \qquad (2.18)$$

It is easy to see that, given condition (2.17), the distortion of the z component of the magnetic field by the azimuthal current j_φ is small: $\Delta H_z/H_z \ll 1$. Nevertheless, we shall give an equation for the variation in the magnetic flux:

$$\Delta\psi = 2\pi\int_0^{r_b} r\Delta H_z(r)\,dr$$

since $\Delta\Psi$ can easily be measured (see, e.g., Abrashitov et al. [4]) and gives an idea of the beam parameters[7]:

$$\Delta\psi = -\frac{4\pi^2 I_b}{c^2 H_z}\int_0^{r_b}\frac{r j_b(r)}{\gamma^2(r)}\,dr \quad , \quad \gamma \gg 1$$

If we make the fairly realistic assumption $\gamma(r) \simeq$ const, this formula can be simplified to

$$\Delta\psi = -\frac{2\pi I_b^2}{\gamma^2 c^2 H_z}$$

(the distribution of the current over the cross-section may be arbitrary).

The angular spread of the beam particles results in two new effects. First of all, a transverse gas-kinetic pressure $\mathscr{P}_\perp \sim n_b mc^2\gamma\Delta\theta^2$ is created, and second, there is variation in the electrostatic repulsion force. The variations become important if the angular spread exceeds γ^{-1}. In this case the difference $c-\overline{v}_z$ (where $\overline{v}_z$ is the mean beam electron velocity) becomes approximately equal to $c\Delta\theta^2/2$ (and not to c/γ^2, as was the case for $\Delta\theta = 0$).

[7] The expression given for $\Delta\psi$ relates to a case in which there is a longitudinal slot in the return current conductor through which the magnetic flux can be forced out.

For these reasons the conditions for compensation of the electric and magnetic forces are changed, and the resulting radial force:

$$e\left(E_r - \frac{\overline{v}_z}{c}H_\varphi\right)$$

is equal in order of magnitude to $eE_r\Delta\theta^2$ (instead of eE_r/γ^2). At the same time, one can verify that at $\Delta\theta > \gamma^{-1}$, $I_b < I_{cr}$, one can disregard the self fields of the beam in the equilibrium equations as compared with the gas-kinetic pressure $\mathscr{P}_\perp$, so that the variation in the magnetic flux is determined by

$$\Delta\psi \simeq -\frac{4\pi}{H_z}\int_0^{r_b} r\mathscr{P}_\perp dr$$

In conclusion we should stress that all the results given are valid when only two conditions (2.11) and (2.17) are satisfied. These can be combined as

$$H_z \gtrsim \frac{mc^2}{er_b}\max\{1;\ \gamma\Delta\theta\} \qquad (2.19)$$

It need only be remembered that we are dealing with beam equilibrium a long way beyond the transition layer at the entrance to the drift space, so that we can consider the inequality $I_b < I_{cr}$ automatically satisfied. In the transition layer itself, when supercritical currents are injected, the conditions under which the perturbation theory can be applied to the problem of radial equilibrium are more restrictive (since there is both a direct and a return current flow during injection of supercritical currents in the transition layer, and their pressures are added).

Let us now go on to the problem of beam stability in a finite magnetic field. As can be seen from Eq. (2.18), the angular velocity v_φ/r of the electron drift rotation depends, generally speaking, on the radius (the rotation is differential). Differential rotation may be the reason for the so-called diocotron instability [27]. We shall study it for the example of a beam with an angular spread that is negligibly small, while the longitudinal velocities are the same for all particles.

Let us pass to the frame of reference associated with the beam and consider, within this reference frame, the low-frequency ($\omega \ll \omega_{He}$) potential oscillations with $k_z = 0$. Bearing in mind that the z components of the electric field and current are equal to zero in the wave, and that transverse motion of the electrons is of the drift type under condition (2.19), we get the following equation for the electric potential of the wave $\delta\phi$:

$$r\frac{\partial}{\partial r}r\frac{\partial\delta\phi}{\partial r} + \delta\phi\left[-\ell^2 + \frac{2\ell r}{1 - \dfrac{\ell v_\varphi'}{\omega r}}\frac{\partial}{\partial r}\frac{\Omega}{\omega}\right] = 0 \quad (2.20)$$

Here $v_\varphi'(r)$ is the unperturbed value of the electron drift velocity;

$$\Omega(r) \equiv \frac{2\pi ec}{H_z'}n_b'(r)$$

is the so-called diocotron frequency, and ℓ is the number of the azimuthal harmonic ($\ell \neq 0$). The prime on n_b and v_φ indicates that they are taken in the beam frame of reference and not in the laboratory:

$$n_b' = \frac{n_b}{\gamma}, \quad v_\varphi' = \frac{4\pi e c}{H_z}\int_0^r r' n_b(r')\, dr'$$

We should point out that although, when deriving (2.20), we assume $k_z = 0$, it is also applicable to finite values of k_z, provided the latter is not too large, i.e.

$$k_z \ll \frac{1}{r_b}\left(\frac{n_b' m c^2}{H_z^2}\right)^{\frac{1}{2}} \tag{2.21}$$

In the simplest case, in which the radial distribution of the density $n_b'(r)$ takes the form shown in Fig.4, Eq.(2.20), taking into account boundary conditions at $r \to 0$ and $r = R$, gives the following dispersion relation (for details see Levy [27]):

$$\frac{\omega^2}{\Omega_0^2} - \frac{\omega}{\Omega_0}\left(\frac{r_2^{2\ell} - r_1^{2\ell}}{R^{2\ell}} + \ell\,\frac{r_2^2 - r_1^2}{r_2^2}\right)$$
$$-\left(\frac{R^{2\ell} - r_2^{2\ell}}{R^{2\ell}}\,\frac{r_2^{2\ell} - r_1^{2\ell}}{r_2^{2\ell}} - \ell\,\frac{R^{2\ell} - r_1^{2\ell}}{R^{2\ell}}\,\frac{r_2^2 - r_1^2}{r_2^2}\right) = 0 \tag{2.22}$$

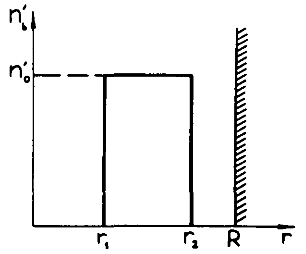

FIG.4. Density profile used to solve Eq.(2.20).

Here $\Omega_0 \equiv (2\pi e c/H_z) n_0'$ and the other notations are shown in Fig.4. It can be seen from (2.22) that the stability criterion:

$$\left(\frac{r_2^{2\ell} - r_1^{2\ell}}{R^{2\ell}} + \ell\,\frac{r_2^2 - r_1^2}{r_2^2}\right)^2 + 4\left(\frac{R^{2\ell} - r_2^{2\ell}}{R^{2\ell}}\,\frac{r_2^{2\ell} - r_1^{2\ell}}{r_2^{2\ell}}\right.$$
$$\left. - \ell\,\frac{R^{2\ell} - r_1^{2\ell}}{R^{2\ell}}\,\frac{r_2^2 - r_1^2}{r_2^2}\right) > 0 \tag{2.23}$$

contains only the geometric characteristics of the problem and the harmonic number.

We shall show another simple sufficient stability criterion for Eq.(2.20), valid for the arbitrary density distribution $n_b'(r)$. It is arrived at by the standard method (see, e.g. Mikhajlovskij [28] and Timofeev [29]; Eq.(2.20) is multiplied by $\delta\phi^*/r$ and integrated over the interval $0 < r < R$. This leads to the following equality:

$$0 = \mathrm{Im}\int_0^R \frac{|\delta\phi|^2 \frac{d\Omega}{dr}}{\omega - \frac{\ell v_\varphi'}{r}}\, dr \equiv -\left[\int_0^R \frac{|\delta\phi|^2 \frac{d\Omega}{dr}}{\left|\omega - \frac{\ell v_\varphi'}{r}\right|^2}\, dr\right]\cdot \mathrm{Im}\,\omega$$

It can be seen from this that, if $\Omega(r)$ is a monotonic function, all the solutions of Eq.(2.20) are stable. This is true, in particular, for the state with a uniform density distribution along the radius. It is interesting to see that this state is stable with respect to all classes of perturbation as a whole, including electromagnetic ones (see Wong et al. [30]).

In a situation where the amount by which the instability threshold is exceeded is not too small, the characteristic growth rate in the reference frame connected with the beam is equal in order of magnitude to Ω_0. In this frame of reference the instability is absolute. In the laboratory frame it is of the convective type and can therefore only be observed in fairly long devices. Actually, the condition for instability development in the beam system takes the form $\tau' \mathrm{Im}\,\omega \gtrsim 10$, where $\tau' \sim L/\gamma c$ is the time of flight of the relativistic electrons through the drift tube, as calculated within their own frame of reference. Taking into account that $\mathrm{Im}\,\omega \sim \Omega_0$, we arrive at the next instability development condition, namely $L \gtrsim 10\,\gamma H_z/2\pi e n_b' \sim 2\gamma^2 H_z/e n_b$. We should also point out that the diocotron instability should become stabilized if there is fairly good neutralization of the beam charge and current, since in this case the drift rotation of the electrons is considerably reduced.

The part played by the diocotron instability in specific experiments was discussed by Carmel and Nation [31] and by Kapetanakos et al. [32]. We shall deal first with the former study, whose authors consider the instability observed to be a possible reason for the destruction of the beam in their experiments. The instability growth rate and its dependence on the magnetic field intensity agree satisfactorily with the theoretical estimates. An interesting feature of the experiments is that the instability leads to the space inside the annular beam becoming filled. This seems natural when it is considered that the beam relaxes into the stable state, and that the state with a uniform density distribution over the cross-section is, as already pointed out, a stable one. The drop in current along the device, as measured by Faraday cups, may be due to the fact that the current injected into the system is close to critical. Since the critical current is somewhat smaller for a solid beam than for an annular one, the "smearing" of the beam over the cross-section may have been accompanied by reflection of some of the electrons.

In the study of Kapetanakos et al. [32] the steady-state pattern observed for the division of the beam into azimuthal segments is explained in terms of oscillations with zero frequency (in the laboratory frame). It remains unclear, however, why these oscillations build up (in space) more rapidly than the others. Furthermore, when deriving the dispersion relation, the authors disregard the magnetic field perturbations in the laboratory frame of reference. Consideration of the effect of these perturbations on movement of the particles gives a correction of the order of unity and therefore seems essential in making a quantitative interpretation of the experimental results.

Yet another source of instability when a beam propagates through a vacuum is anisotropy of the electron distribution function in the frame of reference moving at the mean velocity of the beam [33, 34]. Occurrence of this anisotropy may be due partly to scattering of the beam in the anode foil, since if the foil is made of a material with $Z \gg 1$, the relative electron spread over the energies is small compared with the angular spread $\Delta\theta$. Hence we can take it that the beam electrons are monoenergetic after passing through the foil. Under these conditions the following estimate is valid for the spread in momentum of the beam particles over the longitudinal and transverse directions:

$$\Delta p_z' \sim \gamma mc \frac{\Delta\theta^2}{\Delta\theta + \gamma^{-1}} \quad , \quad \Delta p_\perp' \sim \gamma mc \Delta\theta$$

It can be seen from this that at $\Delta\theta \lesssim \gamma^{-1}$ the inequality $\Delta p_\perp' > \Delta p_z'$ is satisfied, and there may be cyclotron build-up of the eigenmodes of the waveguide due to the normal Doppler effect. Here the longitudinal wave vector for the excited wave is determined from the resonance condition, which in a laboratory frame of reference takes the form:

$$\omega(k_z) - k_z v_z = eH_z/\gamma mc$$

where $\omega(k_z)$ is the waveguide mode frequency. It appears that an anisotropic instability has been observed experimentally [33, 34].

3. BEAM TRANSPORT IN A PLASMA

3.1. Charge and current neutralization of the beam

In this section we consider a situation in which the drift space ((3) in Fig. 1) is filled with a plasma. We shall assume that the plasma density n is much greater than the beam density n_b. When the first portions of the beam are injected, the drift space becomes negatively charged and some of the plasma electrons $\Delta n \simeq n_b$ are forced onto the anode foil and collector ((2) and (4) in Fig. 1, respectively) in such a way that the drift space becomes electrically neutral[8]. Because of the induced e.m.f. associated with the variation in beam current with respect to time, a current is created in the plasma directed opposite to the beam current. If the conductivity of the plasma is fairly high, there may be almost entire compensation of the beam current, so that the magnetic self-field vanishes[9].

The charge neutralization does away with the restrictions on the limiting current which are due to electrostatic reflection (Section 2.1), and the current limitation then becomes due to various instabilities produced by the motion of the beam with respect to the plasma. The relevant problems are dealt with in Section 3.2. In this section, however, we shall leave aside the problem of stability for the moment and deal only with the purely "laminar" effects associated with the charge and current neutralization.

These effects have been studied theoretically in numerous articles [26, 37-41]. In all these publications the plasma is considered infinite[10]. The effect of the electric and magnetic fields created by the beam on the motion of the beam particles was disregarded[11]. The main difference between these studies is that they deal with one group or other of the specific effects influencing the plasma current (absence or presence of electron/ion collisions, direction and magnitude of magnetic field, etc.).

The most thoroughly studied case is one in which the beam current is distributed uniformly over the beam cross-section and is a step function of time. The corresponding calculations are extremely unwieldy and we shall not give them here; the reader can find them in the original papers. Of particular interest is the article by Lee and Sudan [38] in which they trace the transition from infinitely weak to infinitely strong longitudinal magnetic fields in the beam current neutralization problem.

For the experimental set-up which we are considering (Fig. 1) the formulation of the problem as described above and based on the approximation of an infinite plasma is of interest provided that the duration of the electron beam τ is fairly small, i.e. $\tau \ll L/c$ (L is the length of the drift space). However, experimentally we more often find the opposite limiting case, i.e. $\tau \gg L/c$ (under typical conditions $\tau \sim 100$ ns; $L = 2\text{-}3$ m). In such a case the problem of neutralization becomes much more trivial, especially as regards analytic calculations.

The specific calculations depend on the type of Ohm's law. We shall perform them for a case in which the device has a strong longitudinal magnetic

[8] In principle it is sufficient to have a plasma with $n = n_b$ for charge neutrality. After ejection of the plasma electrons only ions with $n_i = n_b$ are left in the drift tube. But as the beam density generally varies rapidly with time (the injection pulse length is ~100 ns), the exact neutrality condition $n_i = n_b$ cannot be satisfied during the whole of the pulse. Hence from the very outset we consider only the case $n \gg n_b$.

[9] Current neutralization has been observed in experiments (see Abrashitov et al. [4], Miller and Kuswa [6], Korn et al. [9], Roberts and Bennett [35]). See also Agafonov's review of experimental work on beam transport [36].

[10] The exception is the study of Rosinskij et al. [40] in which the presence of a cylindrical conducting sheath was considered, although the length of the plasma was still assumed to be infinite.

[11] The latter is valid either for fairly complete charge and current neutralization, or else for the injection of currents that are considerably smaller than the critical current.

field that suppresses the transverse conductivity, i.e. we shall assume that

$$j_r = j_\varphi = 0, \ j_z = \hat{\sigma} E_z$$

where $\hat{\sigma}$ is the longitudinal conductivity operator:

$$\hat{\sigma} = \frac{\omega_p^2}{4\pi}\left(\nu + \frac{\partial}{\partial t}\right)^{-1}$$

ν is the frequency of electron/ion collisions (which, like ω_p, we shall take to be independent of space and time, for the sake of brevity). In this case r and z components of the electric field and the φ component of the magnetic field are produced (see Appx I), so that the problem can be described by means of the following equations:

$$\frac{\partial E}{\partial z} - \frac{\partial E_r}{\partial r} = -\frac{1}{c}\frac{\partial H_\varphi}{\partial t}$$

$$\frac{\partial H_\varphi}{\partial z} = -\frac{1}{c}\frac{\partial E_r}{\partial t} \qquad (3.1)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rH_\varphi\right) = \frac{1}{c}\frac{\partial E_z}{\partial t} + \frac{4\pi}{c}\left(\hat{\sigma}E_z + j_b\right)$$

where $j_b(r, z, t)$ is the beam current density. Assuming that the beam particles move at a constant velocity v and assuming the function $j_{b_0}(r,t) \equiv j_b(r, 0, t)$ to be known, we can find the current at an arbitrary point in the plasma:

$$j_b = j_{b_0}\left(r, t - \frac{z}{v}\right)$$

The fact that the boundaries of the drift space are conductors is taken into account by the boundary conditions:

$$E_z = 0 \quad \text{when} \ r = R$$
$$E_r = 0 \quad \text{when} \ z = 0, L \qquad (3.2)$$

The formal solution of Eq. (3.1) presents no difficulty, but it is very cumbersome. We shall therefore only consider the limiting case $\tau \gg L/c$. Here, to make things clearer we shall assume that the beam current $j_{b_0}(r,t)$ is a smooth function of time (of the type shown in Fig. 5), and that the shape of the current can be described by a single parameter τ;[12] the beam current is assumed to be likewise a smooth function of the radius.

In the limit $c\tau \gg L$ the return current induced by the beam in the plasma may be taken as independent of z, i.e.

$$\left|j_b(z = L) - j_b(z = 0)\right| \simeq \frac{L}{c}\left|\frac{\partial j_b}{\partial t}\right| \ll j_b$$

[12] It should be pointed out that if for the total duration $\tau \gg L/c$ the beam has a steep leading edge (the current build-up time < L/c), our results are applicable to the diminishing part of the current.

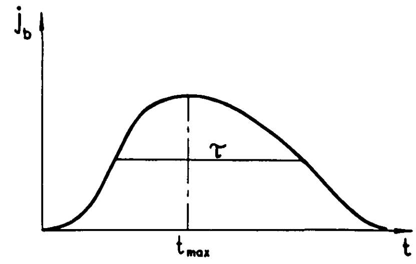

FIG. 5. Beam current density as a function of time (for fixed r).

The same applies, of course, to the charge density $\rho_b \equiv j_b/v$. However, the most important simplification is that at $c\tau \gg L$ we can limit ourselves to a quasisteady-state approximation (see Landau and Lifshits [42]). In this case there is the natural splitting of the electric field into two parts: a solenoidal part independent of z, $\vec{E}_s = (0;0;E_s(r,t))$, and a part derived from a scalar potential which is dependent on z, $E_p = -(\partial\phi(r,z,t)/\partial r; 0; \partial\phi(r,z,t)/\partial z)$, where ϕ is the electrostatic potential.

The solenoidal part of the field is determined from the equations:

$$\frac{\partial E_s}{\partial r} = \frac{1}{c}\frac{\partial H_\varphi}{\partial t} \qquad (3.3)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rH_\varphi\right) = \frac{4\pi}{c}\left(\hat{\sigma}E_s + j_b\right) \qquad (3.4)$$

while the potential ϕ is calculated from

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2} = -4\pi(\rho + \rho_b)$$
$$\frac{\partial\rho}{\partial t} + \operatorname{div}\vec{j} = 0 \qquad (3.5)$$

A more formal derivation of Eqs (3.3) - (3.5) is given in Appx II. Bearing in mind that $j_r = j_\varphi = 0$ and $j_z = \hat{\sigma}(E_s - (\partial\phi/\partial z))$ instead of the latter equation we get

$$\frac{\partial\rho}{\partial t} - \hat{\sigma}\frac{\partial^2\phi}{\partial z^2} = 0 \qquad (3.6)$$

Let us recall that the functions j_b and ρ_b are considered here to be independent of the co-ordinate z.

Thus, in the quasisteady-state approximation, the problems of current compensation (Eqs (3.3), (3.4)) and charge neutralization (Eqs (3.5), (3.6)) are split up[13].

We shall not give the full solution of Eqs (3.3) - (3.6) in this paper but merely explain the conditions under which this neutralization occurs.

[13] We should point out that inasmuch as the solenoidal part of the electric field has only a z component, the effect of current compensation does not depend on the value of transverse conductivity; more particularly, the set of equations (3.3) and (3.4) does not vary when the conductivity is isotropic. In this respect the quasisteady-state considered by us differs considerably from the current compensation problem in a boundless plasma ($L \to \infty$); in the latter case the compensating effect depends to a great extent on transverse conductivity [38].

More particularly, in order to determine the conditions for satisfactory charge neutralization we shall, assuming the conductivity to be fairly high, obtain a solution of the equations (3.5) - (3.6) in the form of an expansion in inverse powers of σ.[14] It is obvious that $\rho^{(0)} = -\rho_b(r,t)$, $\phi^{(0)} = 0$. Substituting $\rho^{(0)}$ in (3.6) and taking into account the symmetry of the problem with respect to the plane $z = L/2$, we get $\phi^{(1)}$, i.e.

$$\phi^{(1)} = -\frac{2\pi}{\omega_p^2}\left[\left(z - \frac{L}{2}\right)^2 - \frac{L^2}{4}\right] \left(\nu + \frac{\partial}{\partial t}\right)\frac{\partial \rho_b(r,t)}{\partial t}$$

Finally, substituting $\phi^{(1)}$ in (3.5), we obtain the correction for $\rho^{(0)}$ (which is actually equal to the unneutralized charge density), i.e.

$$\rho^{(1)} = \frac{1}{2\omega_p^2}\left[\left(z - \frac{L}{2}\right)^2 - \frac{L^2}{4}\right] \times \left(\nu + \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \rho_b(r,t)}{\partial r}\right]$$

(here we have taken into consideration the fact that $L \gg R$, and have disregarded the second term on the left-hand side of (3.5). The condition for charge neutrality $|\rho^{(1)}| \ll |\rho^{(0)}|$ can thus be approximately written:

$$\sigma \gg \frac{1}{4\pi\tau}\frac{L^2}{r_b^2} \tag{3.7}$$

Similarly, we can find the following expression for the current from Eqs (3.3) and (3.4):

$$j = -j_b(r,t) - \frac{c^2}{\omega_p^2}\int_0^t dt' \left(\nu + \frac{\partial}{\partial t'}\right)\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial j_b(r,t')}{\partial r}$$

It can be seen from this that the current compensation condition takes the form:

$$\sigma \gg \frac{\tau}{4\pi}\frac{c^2}{r_b^2} \tag{3.8}$$

This condition is considerably more restrictive than (3.7) (let us recall that $c\tau \gg L$). It can also be rewritten in the form:

$$\delta \ll r_b \tag{3.8$'$}$$

where $\delta \sim (c^2\tau/4\pi\sigma)^{\frac{1}{2}}$ is the skin depth.

If the conductivity is high, and condition (3.8) is satisfied with a large margin, the plasma contains neither an electric nor a magnetic field induced by the beam. In this case the beam equilibrium conditions become considerably less restrictive, since the external magnetic field has to balance only the gas-kinetic pressure associated with the angular spread.

[14] In these estimates we take σ to be of the order of magnitude $\omega_p^2/4\pi(\nu + 1/\tau)$.

But since (3.8) is much more stringent than (3.7), one often encounters situations in which there is charge neutrality but no current compensation, the radial electric field of the beam becoming much smaller than in the case of a vacuum, while the azimuthal magnetic field remains the same as before. As a result the radial repulsion of the beam is replaced by compression. The beam equilibrium in this case is studied in the same way as in Section 2.2, but since there is no cancellation of the electric and magnetic forces, the conditions of applicability of the perturbation theory to the equilibrium problem become more stringent (see Hammer and Rostoker [26]). It is quite clear, however, that by creating a fairly strong longitudinal magnetic field we can pass as large a current as we like through the plasma (under charge neutralization conditions)[15]. Beam equilibrium in the case of partial or complete charge neutrality is studied in Refs [43 - 51] (for the results obtained in this field up to 1970 see the review article by Benford and Book [52]).

Interesting effects occur when the conductivity of the plasma contains non-axisymmetric inhomogeneities. In this case the return current is, generally speaking, displaced with respect to the beam; hence there can be displacement of the beam as a whole in the return-current magnetic field. To illustrate a situation which may occur, let us consider the limiting case shown in Fig. 6(a), where the conductivity of the plasma is high within a cylindrical area, the axis of which is displaced through a distance Δ from the beam axis, and is small outside this area. Clearly, the return current will only flow in the high-conductivity area, so that the lines of force of the magnetic field created by it represent circles with their centre at point A (Fig. 6(a)), displaced with respect to the beam axis. When acted on by the longitudinal magnetic field and the return-current field, the beam moves in a spiral that winds onto the high-conductivity area. Clearly, as a result the beam

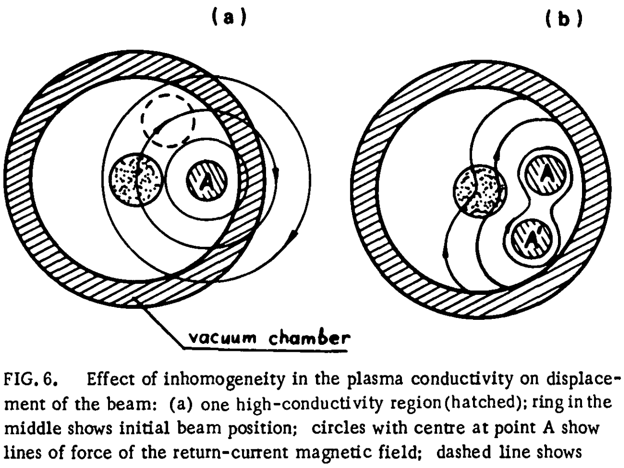

FIG. 6. Effect of inhomogeneity in the plasma conductivity on displacement of the beam: (a) one high-conductivity region (hatched); ring in the middle shows initial beam position; circles with centre at point A show lines of force of the return-current magnetic field; dashed line shows the beam position at some distance from the drift space entrance; (b) two high-conductivity regions.

[15] We are talking only about limitations associated with beam equilibrium and are not dealing with stability.

may shift through a distance 2Δ with respect to its initial position and, if Δ is sufficiently high, may strike the wall of the drift tube. The pitch of the spiral is $2\pi(H_z/H_I)\Delta$, where H_I is the return-current field on the beam axis. Obviously, the effects described above are important only when the length of the device is comparable with the pitch of the spiral or exceeds it (otherwise the beam cannot shift with respect to the azimuth). If the conductivity of the plasma is distributed in a more complex manner than in Fig. 6(a) (e.g. if there are two areas of satisfactory conductivity displaced with respect to one another (Fig. 6(b)) then wandering of the beam may assume an extremely complex and irregular character. It is possible that the irregular blurring of the beam during transport through a plasma, observed in some experiments, e.g. those described by Abrashitov et al. [3], can be explained by this mechanism.

3.2. Plasma heating by the return current

Joule dissipation of the return current relates more to the problem of heating a plasma than to that of beam transport. However, since it is a matter related directly to current compensation, we shall deal with it at this point (and not in Section 4 on plasma heating).

Return-current Joule dissipation has been discussed by Altyntsev et al. [1], Rudakov [53] and Lovelace and Sudan [54], the most detailed calculations having been made by the last-mentioned authors. In actual fact, however, they considered a beam in a plasma unbounded in the longitudinal direction. We shall present calculations for the case $L \ll c\tau$, which at present seems the most realistic one from an experimental point of view.

Assuming that the duration of the beam τ is greater than the time taken by the plasma electrons to be scattered by the ions (or by the fluctuations in the electric field if we are talking of the anomalous resistance of the plasma), we shall apply the following Ohm's law:

$$j_z = \sigma E_z, \quad \sigma = \sigma(r,t) = \frac{\omega_p^2(r,t)}{4\pi\,\nu(r,t)}$$

(as already pointed out, in a quasisteady-state the transverse conductivity is not part of the problem). Here the return-current dissipation is described by Eqs (3.3) and (3.4) in which we need only to replace $\hat{\sigma}$ by $\sigma(r,t)$.[16]

The energy dissipated by the return current per unit length of the plasma column is

$$Q = 2\pi \int_{-\infty}^{+\infty} dt \int_0^R \frac{r\, j_z^2(r,t)\, dr}{\sigma(r,t)}$$

Making use of Eqs (3.3) and (3.4) it is easy to see that

$$Q = \frac{2}{c^2} \int_{-\infty}^{+\infty} dt \int_0^R \frac{dr}{r}\, [I^*(r,t) + I_b^*(r,t)]\, \frac{\partial I^*(r,t)}{\partial t} \qquad (3.9)$$

in which we introduce the notation:

$$\left.\begin{matrix} I^*(r,t) \\ I_b^*(r,t) \end{matrix}\right\} = 2\pi \int_0^r r' \begin{Bmatrix} j(r',t) \\ j_b(r',t) \end{Bmatrix} dr'$$

The total currents for the beam and plasma (i.e. $I_b^*(R,t)$, $I^*(R,t)$) will be designated simply by $I_b(t)$ and $I(t)$.

In the case where the current is switched on in a "step", i.e.

$$j_b(r,t) = \begin{cases} 0, & t < 0 \\ j_b(r), & t > 0 \end{cases}$$

one can reduce Eq. (3.9) to the form:

$$Q = \frac{1}{c^2} \int_0^R \frac{dr}{r}\, I_b^{*2}(r) \equiv W_M$$

in which W_M is used to designate the magnetic energy per unit of beam length in the vacuum. We should stress that this result does not depend on the specific of the function $\sigma(r,t)$.

The energy W_M is released in the plasma over a time of the order of the field penetration [17]: $\tau_s \sim 4\pi r_p^2 \sigma/c^2$. A contribution is in fact made to the integral (3.9) at those values of t for which the quantity $I^*(r,t) + I_b^*(r,t)$ becomes essentially non-zero, and it is at the field penetration times that this happens. At the upper limit, $t > \tau_s$, the integral converges through attenuation of the plasma current.

If the beam has a sharp trailing edge and its duration exceeds τ_s, then when it switches off an additional energy W_M is released in the plasma, so that the total energy dissipation will be $2W_M$.

A simple estimate of Q is possible in the more general case as well: $I_b^*(r,t) = f(r)g(t)$. Here we may assume, without limiting the generality, that $f, g > 0$. Furthermore we assume that the function $g(t)$ has a single maximum, i.e. $g(t)$ takes the form shown in Fig. 5. The following inequality is then valid (see Appx III):

$$Q \le 2\eta \max W_M(t)$$

where η is a numerical factor of the order of unity and dependent on the beam current distribution over

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[16] In this section we do not make the assumption that σ is independent of space and time.

[17] We consider that $r_b \sim r_p \sim R$; but if $r_b \ll r_p$, then the dissipation time is of the order of the penetration time, calculated for the beam radius. It is interesting to observe that if at $r_b \ll r_p$ the duration of the beam satisfies the conditions $4\pi r_b^2 \sigma/c^2 \ll \tau \ll 4\pi r_p^2 \sigma/c^2$, then throughout the beam injection time the current measured by the Rogowski coil, external with respect to the plasma, is exponentially small, so that a rough estimate of the heating $Q \sim \int IU\,dt$ (where $U = -\mathscr{L}\,d(I + I_b)/dt$ is the induced e.m.f.) will produce a result that is much too low.

the cross-section (the exact expression for η is given in Appx III). It should be stressed that when deriving this inequality we do not make any assumptions with regard to the function $\sigma(r,t)$.

Taking into account the fact that the total energy imparted to the beam by the accelerator is equal in order of magnitude to $I_bU\tau$, where U is the voltage on the accelerator diode, it is easy to see that if $\tau \lesssim \tau_s$ the efficiency of the heating can be calculated in the following way:

$$\text{efficiency} \sim \left(\frac{LI_b^2}{c^2}\frac{\tau}{\tau_s}\right)(I_bU\tau)^{-1} \sim \frac{I_b}{I_{cr}}\frac{L}{c\tau_s} \qquad (3.10)$$

When $\tau > \tau_s$ the efficiency decreases in proportion to τ^{-1}. It can be seen that the efficiency may be of the order of unity only when the beam current is much greater than the critical vacuum current, namely at $I_b \sim I_{cr}(c\tau_s/L)$. In this case the e.m.f. induced in the plasma is exactly of the order of the kinetic energy of the beam electrons. The passage of a current $I_b > I_{cr}(c\tau_s/L)$ is impossible, since in such a case the induced e.m.f. would be greater than the electron energy.

Unfortunately, it appears to be that under conditions where the conductivity is determined by Coulomb collisions, the penetration time is so long that the optimal beam current $I_b \sim I_{cr}(c\tau_s/L)$ is too high. For example, in the cases of interest for thermonuclear applications τ_s amounts to 10^{-2} – 1 second, and for reasonable values of L ($L \lesssim 10^4$ cm) the optimal current is greater than $3\times10^4\ I_{cr}$.

Return-current heating may become acceptable if there is anomalous resistance in the plasma, i.e. if the collision frequency is much higher than the Coulomb frequency [53, 54]. The condition required for anomalous resistance is that the directed velocity of the plasma electrons u should exceed the threshold value of αv_{te}, where v_{te} is the thermal velocity of the plasma electrons and α is a numerical factor dependent on the ratio between the electron and ion temperatures and on the shape of the electron distribution function. It ranges from $(m/M)^{\frac{1}{2}}$ (for a plasma with $T_e \gg T_i$ and with an electron distribution function that is isotropic in a frame of reference moving at velocity u) to unity (for plasma with $T_e \sim T_i$).

Taking into account the fact that under conditions of zero net current $u \sim cn_b/n$, we get the following criterion for the existence of anomalous resistance:

$$\alpha v_{te} \lesssim cn_b/n$$

and, accordingly, the following expression for the limiting temperature up to which the plasma may be heated through anomalous resistance:

$$\frac{T_e}{mc^2} \lesssim \frac{1}{2\alpha^2}\left(\frac{n_b}{n}\right)^2$$

Heating up to $T_e \sim 10^4$ eV is possible at $n_b/n \gtrsim 0.2\alpha$. Even for the lowest possible value of α ($\alpha \sim 2\times10^{-2}$ for a deuterium plasma) this condition, as applied to plasmas with $n \gtrsim 10^{14}$ - 10^{15} cm^{-3}, necessitates the use of extremely dense beams. It should also be pointed out that during the heating of a plasma the electron distribution function may elongate along the electric field direction [55, 56]. In this case, even at $T_e \gg T_i$ (by these symbols we denote the mean particle energy in a frame of reference moving with the mean velocity of particles of the given kind), the parameter α may become much larger than $(m/M)^{\frac{1}{2}}$, right up to $\alpha \sim 1$ (see the theoretical studies of Vekshtejn et al. [55, 56] and the results of computer calculations of Biskamp and Chodura [57]), while an increase in α leads in turn to the condition $n_b/n > 0.2\alpha$ requiring beam densities that are too high.

The possibility that the distribution function is deformed was not taken into account by Guillory and Benford [58], whose investigation of the excitation of ion-acoustic oscillations by the return current is based on the assumption of electron and ion distribution functions of the Lorentz type. For this reason the conclusions given by these authors cannot be regarded as quite reliable. A similar objection can be raised against the study of Lovelace and Sudan [54], whose use of Sagdeev's equation [59] for the effective collision frequency does not seem to be fully justified since, strictly speaking, it relates only to the case of a current perpendicular to the magnetic field in which there is a cyclotron mixing effect, and the electron distribution function cannot become elongated along the electric field[18].

The situation at $\alpha \gg (m/M)^{\frac{1}{2}}$ can be slightly improved by creating non-uniform plasma density distribution along the device. In this case the heating conditions in the reduced density regions are made easier and the energy released in them through electron thermal conductivity is uniformly distributed along the device. On the whole, however, the prospects for heating plasma with $n \gtrsim 10^{15}$ cm^{-3} by return current are not too good at present. Nevertheless there is no doubt that it would be possible to use this heating method in the region of relatively low plasma densities ($n \sim 10^{14}$ cm^{-3}).

3.3. Macroscopic instabilities in connection with beam transport in a plasma

As was demonstrated in Section 3.1, when a beam is transported in a plasma it is comparatively easy to neutralize the space charge. This eliminates restrictions on the value of the current due to electrostatic reflection of the beam. If, moreover, there is current neutrality, the beam equilibrium conditions, even at $I_b > I_{cr}$, do not impose any unduly hard limitations on the value of the longitudinal magnetic field. The corresponding limitations, as can be seen from what follows, are due to the stability requirement.

[18] Sagdeev's formula can be used only if there is no external longitudinal magnetic field (then magnetic field lines are perpendicular to the current).

Unstable oscillations in a plasma/beam system can be conveniently subdivided into two groups: large-scale, or macroscopic, and small-scale oscillations. The first group includes those with a characteristic longitudinal scale of the order of (or larger than) the beam radius. The second subdivision contains the perturbations of shorter wavelength. To some extent this is an arbitrary grouping in that, depending on the relationship between the plasma and beam parameters, instabilities of the same physical nature may sometimes come within the first group, and at other times the second. Nevertheless we shall use this classification, since both the macro- and micro-instabilities differ in their outward manifestations. The former give rise to considerable distortion of the equilibrium beam configuration (sausage-type deformations, kink instabilities, and so on), while the latter are mainly responsible for heating and diffusion of the particles.

In this section we examine the large-scale instabilities, since it is they that constitute the most obvious hazard in beam transport (small-scale instabilities are studied in detail in Section 4 in connection with the plasma heating problem).

Being concerned with pulsed beams, we shall focus our attention here on purely electron oscillations. As far as oscillations involving ions are concerned (disregarding the ion-acoustic instability of the return current, which was discussed in detail in Section 3.2), they have considerably lower characteristic frequencies and growth rates, and under actual conditions often fail to grow during the beam injection time.

Let us first consider collisionless instabilities, i.e. those with frequencies and growth rates large compared with the effective collision frequency ν_{eff}, which takes into account both the Coulomb collisions between electrons and ions, and electron scattering by microfluctuations. An example of an instability of this kind is the build-up of helicon waves (see Shafranov [60] with a frequency below cyclotron frequency in a dense plasma ($\omega_p \gg \omega_H$). These waves are essentially whistlers. We shall demonstrate below that they are unstable when the density of the beam kinetic energy is higher than the density of the external magnetic field energy.

For the sake of clarity let us solve the problem of the stability of a beam moving along an external magnetic field H_z in a metal waveguide of radius $R \gg c/\omega_p$ filled with plasma of density $n \gg n_b$. The equilibrium distribution of the beam density n_b and electron velocity v_z over the waveguide cross-section will be considered to be uniform. The current and space charge of the beam in a state of equilibrium are assumed to be completely neutralized.

Taking the charge and current neutralization into account, we arrive at the following expressions for the transverse components of the current perturbation in the wave:

$$\delta j_r = -i\frac{\omega}{4\pi}\frac{k_z^2 c^2}{\omega^2}\beta_b E_r - \frac{\omega_p^2}{4\pi\omega_H}E_\varphi \qquad (3.11)$$

$$\delta j_\varphi = \frac{\omega_p^2}{4\pi\omega_H}E_r - i\frac{\omega}{4\pi}\frac{k_z^2 c^2}{\omega^2}\beta_b E_\varphi \qquad (3.12)$$

Here $\beta_b \equiv 4\pi n_b m v_z^2 \gamma / H_z^2$.

Instead of Ohm's law for the longitudinal current component we shall apply the condition $E_z = 0$, which means that the longitudinal conductivity in the system is substantially higher than the transverse conductivity. When a whistler is propagated through a plasma without a beam, the condition, as is well known, makes for a limitation from above on the value of the wave vector ($kc \ll \omega_p$). It can be demonstrated that when a beam is present there is a further limitation:

$$\beta_b \ll \frac{\omega_p}{\omega_H}\frac{v_z}{c}\gamma \qquad (3.13)$$

which we shall take to be satisfied.

Let us substitute the currents (3.11) and (3.12) in Maxwell's equations and consider axisymmetric perturbations. As a result we get the following equation for the φ component of the wave electric field:

$$\left(1 - \beta_b\right)\left(1 - \beta_b - \frac{1}{k_z^2}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r\right)E_\varphi = \Omega^2 E_\varphi \qquad (3.14)$$

where $\Omega^2 \equiv \omega^2\omega_p^4/\omega_H^2 k_z^4 c^4$.

The solution of this equation, which behaves properly at the origin, takes the form:

$$E_\varphi = aJ_1\left[k_z r\left(\beta_b - 1 - \frac{\Omega^2}{\beta_b - 1}\right)^{\frac{1}{2}}\right] \qquad (3.15)$$

(where a is an arbitrary constant and J_1 is a Bessel function).

The boundary condition on the wall of the waveguide ($E_\varphi(R) = 0$) gives us the following dispersion relation:

$$\omega^2 = \omega_H^2\frac{k_z^4 c^4}{\omega_p^4}\left(\beta_b - 1\right)\left(\beta_b - 1 - \frac{\mu_\ell^2}{k_z^2 R^2}\right) \qquad (3.16)$$

$\mu_\ell(\ell = 1, 2, \ldots)$ are the roots of function J_1.

At $\beta_b = 0$ (without a beam) this equation gives us the dispersion relation for whistlers in waveguide. As β_b increases, the dispersion of the wave varies slightly but, right up to $\beta_b = 1$, all the oscillations remain stable. If β_b exceeds unity, however, there is an aperiodic instability at $\mu_\ell^2/k_z^2R^2 > \beta_b - 1$. The maximum growth rate for k_z is attained at $k_z^2 = \mu_\ell^2/2R^2(\beta_b - 1)$, i.e.

$$\max \operatorname{Im}\omega = \omega_H \frac{\mu_\ell^2 c^2}{2R^2\omega_p^2} \qquad (3.17)$$

As the number of the harmonic increases, so does the growth rate, in proportion to μ_ℓ^2, but at high values of μ_ℓ ($\mu_\ell \sim R\omega_p/c$) the conditions under which Eq. (3.16) is applicable cease to be valid, since when deriving the dispersion relation we can no longer disregard the longitudinal electric field E_z. The value k_z is restricted from above by the condition $k_z c \ll \omega_H/\gamma$ (at $k_z c \gtrsim \omega_H/\gamma$ we have to take into account corrections of the order of $k_z c\gamma/\omega_H$ in the expressions for the transverse current (3.11) and (3.12)).

Generally speaking, a completely neutralized beam is unstable even in a strong magnetic field ($\omega_H \gg \omega_p$). The characteristic frequency of the unstable perturbations is in this case $\sim\omega_p$. Although this instability can apparently not lead to the beam being diverted onto the chamber walls (there is a strong longitudinal magnetic field), it may bring about a considerable spread over the longitudinal energies, which is undesirable in many respects (see Section 4). We shall therefore consider in greater detail the conditions under which this instability arises.

If $\omega_H > \omega_p$, we can disregard the transverse displacement of the beam particles and plasma, and apply the following Ohm's law:

$$\delta j_\perp = 0, \quad \delta j_z = \frac{i\omega_p^2}{4\pi\omega}\left(1 + \frac{\omega^2}{(\omega - k_z v)^2}\,\frac{n_b}{n\gamma^3}\right)E_z \qquad (3.18)$$

(see Eq. (A. I. 1)). For this Ohm's law the presence of the beam and plasma inside the drift tube affects only the TM mode (the TE mode is uncoupled and described by the same equations as in the case of a vacuum — see Appx I).

We shall assume that the beam and the plasma are uniform and completely fill the drift tube. For the sake of brevity we shall consider only axisymmetric perturbations (as being the most unstable). In these perturbations the z component of the electric field satisfies the following equation (derived in similar fashion to (A. 1.3)):

$$\frac{1}{r}\frac{\partial}{\partial r}\, r\,\frac{\partial E_z}{\partial r} + \left(\frac{\omega^2}{c^2} - k_z^2\right)\left[1 - \frac{\omega_p^2}{\omega^2} - \frac{n_b}{\gamma^3 n}\,\frac{\omega_p^2}{(\omega - k_z v)^2}\right]E_z = 0$$

Taking into account the fact that the solution of this equation should be well behaved at zero and vanish at $r = R$, we can easily get the following dispersion equation:

$$1 = \frac{k_{\perp\ell}^2}{\frac{\omega^2}{c^2} - k_z^2} + \frac{\omega_p^2}{\omega^2} + \frac{n_b}{\gamma^3 n}\,\frac{\omega_p^2}{(\omega - k_z v)^2} \qquad (3.19)$$

where $k_{\perp\ell} = \mu_\ell/R$, and μ_ℓ is the ℓ-th root of $J_0(\mu)$. Stability analysis of this dispersion equation is carried out by the standard method (see e.g. Mikhajlovskij [61]). It can easily be seen that for beams that are not unduly dense, $n_b \ll n/\gamma$, the stability condition takes the form:

$$k_z^2 + \gamma^2 k_{\perp\ell}^2 > \frac{\omega_p^2}{c^2} \qquad (3.20)$$

This gives the following result for the maximum permissible tube radius for which the beam is still stable:

$$R_{max} = \frac{\gamma\mu_1 c}{\omega_p}$$

This criterion stems from the requirements of stabilization for the most unstable oscillations with $k_z = 0$.[19]

A more detailed study of the instability under consideration is contained in a paper by Berk [62] in which he describes, more particularly, the effects associated with the finiteness of a longitudinal magnetic field. Some findings relating to the same problem are also to be found in an article by Grishin and Kolomenskij [63].

The instabilities occurring during beam transport through a plasma in a magnetic field satisfying the inequality

$$H_z^2 \gg 8\pi n_b \gamma m c^2 \qquad (3.21)$$

are described in detail in a survey by Bogdankevich and Rukhadze [22]. However, some degree of caution should be exercised with regard to their results, since the authors have proceeded on the assumption that, given (3.21), the unstable oscillations can be considered electrostatic, whereas in actual fact for oscillations with $\omega/k_z \simeq c$ condition (3.21) (and even $H_z \to \infty$) is, generally speaking, insufficient for the electrostatic approximation. Hence many of the relationships given by them are not quantitatively valid, and at $\gamma \gg 1$ they are not even qualitatively acceptable (they give a wrong dependence on γ). More particularly, at the limit $H_z \to \infty$, instead of the dispersion relation (3.19) another dispersion relation is implied in the survey [22], namely

$$1 = -\frac{k_{\perp\ell}^2}{k_z^2} + \frac{\omega_p^2}{\omega^2} + \frac{n_b}{\gamma^3 n}\,\frac{\omega_p^2}{(\omega - k_z v)^2}$$

which at $|(\omega/k_z) - c| \ll c$ is very different from (3.19).

Let us now consider the instabilities that are caused by incomplete compensation of the beam current. Since the charge neutralization condition in the plasma is more easily satisfied than the current compensation condition, we shall assume that there is no space charge in the equilibrium state and that the net current is equal to the beam current. One would naturally expect, in such a case, the development of a kink instability of the kind studied by Kruskal and Shafranov. The following approximation holds for its growth rate:

$$\mathrm{Im}\,\omega \sim \frac{2I_b}{c r_b^2 \sqrt{4\pi n M}} \qquad (3.22)$$

(we take the radius of the plasma column to be equal in order of magnitude to the beam radius r_b). The condition under which this instability arises takes the form:

$$\frac{I_b}{c r_b H_z} \gtrsim \frac{\pi r_b}{L} \qquad (3.23)$$

[19] In a system which is limited along the z direction, such oscillations are obviously impossible. The problem of the stability of beams limited longitudinally is discussed in the survey of Bogdankevich and Rukhadze [22].

where L is the length of the system and H_z is the external magnetic field. The specific features of excitation of this instability by a relativistic electron beam are considered by Lee [64].

The growth rate of the Kruskal-Shafranov instability is relatively small, since the magnetic field is frozen into the plasma, and ions are drawn into the motion. Should the freezing condition be disrupted on account of the finite nature of the conductivity (e.g. due to a return-current instability), a build-up of purely electron kink instability oscillations becomes theoretically possible. Here the instability development time is determined by the time taken for diffusion of the magnetic field perturbations into the plasma. Instabilities of this type have been studied by Benford [65] on the assumption of a uniform current distribution over the beam cross-section. The beam boundary was assumed to be sharp. When formulated more realistically (for a smoothly decreasing current distribution), the solution of the problem is made more complicated by the fact that the stabilizing effect of shear must be taken into account.

In conclusion we should point out that the macroscopic beam instabilities in the plasma can of course also occur in a case where the longitudinal magnetic field is negligibly small or absent entirely. Such instabilities were treated theoretically by Rosenbluth [66], Weinberg [67, 68], Ivanov and Rudakov [69]. The hose and the sausage instabilities were both discussed. The experimental observation of the hose instability was reported by Moses, Bauer and Winter [70].

4. COLLECTIVE EFFECTS IN THE RELAXATION OF REBs IN A PLASMA

4.1. General formulation of the beam relaxation problem

As was pointed out in Section 1, binary collisions alone cannot ensure sufficiently fast dissipation of the energy of a relativistic beam to heat a plasma to thermonuclear temperatures. Hence the principal part in the problem of the beam heating of a plasma should be played by collective effects. Collective relaxation of the beam is connected with the development of a beam instability in the plasma, followed by scattering of the beam by the resulting electric field fluctuations. The relaxation length is determined by the fluctuation spectrum[20].

The beam instability development time is normally extremely small compared to both the beam injection time and the characteristic time for variation of the plasma parameters. Thus our main interest lies in solving the problem of the steady-state injection of the beam into the plasma and in finding the steady-state oscillation spectrum.

[20] By fluctuation spectrum we mean the distribution of the energy of the fluctuations over the wave vectors, which is characterized by the spectral function $W(\vec{k})$.

The next step should be to describe the self-consistent variation of the plasma parameters (temperature, density, etc.) due to the beam heating.

A steady-state oscillation spectrum in the beam relaxation problem is established because of the fact that the generation of waves is balanced, first, by their removal at the group velocity from the generation region, and, second, by various non-linear processes. Depending on which of the effects is stronger, we are dealing with either the quasi-linear or non-linear approximation.

The quasi-linear approximation has been applied to the problem under consideration by various authors [53, 71, 72]. It produces such high values for the oscillation energy density that the conditions of applicability of a quasi-linear approximation are very restrictive and in most cases the non-linear interaction of the waves has to be considered.

In many situations of practical interest this can be done within the framework of weak turbulence theory. This is based on the relative smallness of the oscillation energy as compared with the thermal energy of the plasma, and includes averaging over the random phases of the interacting waves (see the various surveys [73-75]). Theories going beyond the approximation of weak turbulence do not, in our opinion, have a sufficiently reliable formal basis at present. On the other hand, we can already formulate a number of problems that the weak turbulence theory would seem unable to answer. As a general observation it should be pointed out that the non-linear theory of relaxation is only just emerging and that many of the results now available require further elucidation.

The problem of beam relaxation is characterized by a large number of physical parameters (n_b/n; γ; T_e/mc^2; T_e/T_i; ω_H/ω_p; ν_{ei}/ω_p; $c/\omega_p R$) and, correspondingly, a wide variety of limiting cases. A complete discussion would be impossible within the confines of this survey, hence we shall deal in detail only with the most broadly studied case of beam relaxation in an isothermal plasma without a magnetic field, after which we shall discuss some possible modifications of the theory.

4.2. Excitation of Langmuir oscillations by a REB

If there is no magnetic field in a plasma or if it is small enough that $\omega_H \ll \omega_p$, then, as it follows from the linear theory, the beam excites predominantly Langmuir oscillations [76]. The mechanism involved is Cherenkov excitation. As is known, the condition for Cherenkov interaction between a particle with velocity $\vec{v}$ and an oscillation with a wave vector $\vec{k}$ and frequency $\omega(\vec{k})$ is

$$\omega - \vec{k}\vec{v} = 0 \tag{4.1}$$

Estimation of the instability growth rate depends strongly on the distribution of the energy and angular spread of the beam. In the case of a monochromatic beam ($\Delta\vec{v} = 0$) the instability is hydrodynamic in nature and the growth rate when maximized along the longitudinal component of the wave vector is given by the following equation (see Fajnberg et al. [71] and Bludman et al. [77]):

$$\mathrm{Im}\,\omega \sim \omega_p \left(\frac{n_b}{n\gamma}\right)^{\frac{1}{3}} \left(\frac{k_z^2}{k^2\gamma^2} + \frac{k_\perp^2}{k^2}\right)^{\frac{1}{3}} \tag{4.2}$$

Here $k_z = \omega_p/v$. This result is valid for

$$|\vec{k}\Delta\vec{v}| < |\mathrm{Im}\,\omega(\vec{k})| \tag{4.3}$$

If we have the opposite inequality, then the instability goes over to the kinetic stage and the approximation of the growth rate changes: if the velocity spread Δv_z of the beam electrons along the longitudinal direction is not too large we can have a situation in which for some values of $k_\perp$ the instability is kinetic, while for others it is hydrodynamic. Finally, if Δv_z and $\Delta v_\perp$ are fairly large (see below for the exact criterion), the instability is kinetic for all values of $k_\perp$. We shall examine only this situation, bearing in mind that the production of monochromatic beams involves great experimental difficulties. A detailed study of the dependence $\mathrm{Im}\,\omega(\vec{k})$ for the intermediate case is given by Rudakov [53]. A description of the hydrodynamic and the intermediate stages of relaxation is given by Rudakov [53] and Fajnberg et al. [71].

Let us formulate the condition under which the kinetic approximation is applicable in the case of a relativistic beam ($\gamma \gg 1$) with an angular spread $\Delta\theta \lesssim 1$ and an energy spread $\Delta\mathscr{E} \lesssim \mathscr{E}$. In this case we may assume that $v \simeq c$ and $\Delta v_\perp \simeq c\Delta\theta$. For determining Δv_z we use the equation:

$$\Delta v_z \simeq c\,\frac{\Delta\mathscr{E}}{\gamma^2\mathscr{E}} + c\Delta\theta^2 \tag{4.4}$$

It can be seen from this that at $\Delta\theta > \gamma^{-1}(\Delta\mathscr{E}/\mathscr{E})^{\frac{1}{2}}$ the longitudinal spread is mainly determined by the angular divergence of the beam. For a beam of 2 - 3 MeV energy, at $\Delta\theta \gtrsim 10° - 15°$ the even stronger inequality:

$$\Delta\theta \gtrsim \gamma^{-1} \tag{4.5}$$

is satisfied. Since the chances of producing powerful beams with a smaller angular spread are at present only slight, we will take the condition (4.5) to be satisfied and assume $\Delta v_z \sim c\Delta\theta^2$. The condition for the applicability of the kinetic approximation can then be written as

$$k_z c\Delta\theta^2 + k_\perp c\Delta\theta \gtrsim \omega_p \left(\frac{n_b}{n\gamma}\right)^{\frac{1}{3}} \left(\frac{k_z^2}{k^2\gamma^2} + \frac{k_\perp^2}{k^2}\right)^{\frac{1}{3}} \tag{4.6}$$

where $k_z = \omega_p/c$. It can be readily seen that at

$$\Delta\theta \gtrsim \max\left\{\left(\frac{n_b}{n\gamma}\right)^{1/4}; \left(\frac{n_b}{n\gamma^3}\right)^{1/6}\right\} \tag{4.7}$$

the inequality (4.6) is satisfied for all values of $k_\perp$.

It is clear from the resonance condition (4.1) that at $\Delta\theta \gtrsim \frac{1}{\gamma}(\Delta\mathscr{E}/\mathscr{E})^{\frac{1}{2}}$ the beam can interact only with those oscillations whose wave vectors satisfy the following relationship:

$$\left|k_z - \frac{\omega_p}{c}\right| \lesssim \frac{\omega_p}{c}\Delta\theta^2 + k_\perp\Delta\theta \tag{4.8}$$

The region determined by this inequality in the wave-vector space is shown in Fig. 7(a). From here on we shall call this region the resonance region.

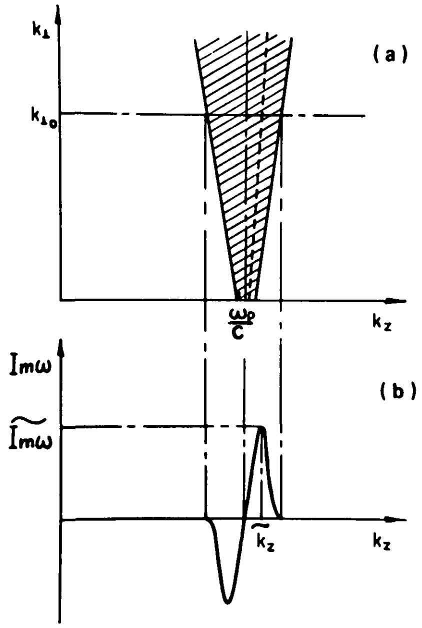

FIG. 7. Resonance region and growth rate for beam instability: (a) resonance region as determined by Eq. (4. 8) (hatched); region boundaries subtend an angle $\Delta\theta$ with the vertical; the growth rate is positive on the right-hand side of the resonance region and negative on the left; dashed line shows maximum growth rate; (b) growth rate as a function of longitudinal wave-vector component for a fixed $k_\perp$ (denoted by $k_{\perp 0}$ in the upper part of the graph).

The calculations of Brejzman and Mirnov [78] demonstrate that the expression for the growth rate at the kinetic limit under the assumptions made above takes the form:

$$\mathrm{Im}\,\omega = \pi\omega_p \frac{n_b}{n}\left(\frac{\omega_p}{kc}\right)^3 \int_{\theta_1}^{\theta_2} \frac{d\theta}{[(\cos\theta_1 - \cos\theta)(\cos\theta - \cos\theta_2)]^{1/2}} \times \left[-2g\sin\theta + \left(\cos\theta - \frac{kc}{\omega_p}\cos\theta'\right)\frac{\partial g}{\partial\theta}\right]$$

$$g = mc\int_0^\infty pf(p,\theta)dp \tag{4.9}$$

$$\cos\theta_{1,2} = \frac{\omega_p}{kc}\left(\cos\theta' \pm \sin\theta'\sqrt{\frac{k^2c^2}{\omega_p^2} - 1}\right)$$

Use is made in these calculations of the spherical co-ordinates p, θ, φ in the momentum space, and k, θ', φ' in the wave-vector space. The angles θ and θ' are measured from the direction of beam injection. The symbol f denotes the momentum distribution function for the beam electrons normalized by the condition:

$$2\pi \int_0^\infty p^2\,dp \int_0^\pi f \sin\theta\, d\theta = 1$$

An analysis of Eq. (4.9) shows that the growth rate is positive on the right-hand side of the resonance region and negative on the left-hand side (Fig. 7(a)). For a fixed value of $k_\perp$ the dependence of $\mathrm{Im}\,\omega$ on k_z takes the form shown in Fig. 7(b). The position of the peak growth rate $\tilde{k}_z$ depends on $k_\perp$: $\tilde{k}_z = \tilde{k}_z(k_\perp)$.

The peak points for the growth rate (with respect to k_z) form the broken line in Fig. 7(b). The dependence of the growth rate on $k_\perp$ on this line takes the form shown in Fig. 8. Beyond the narrow region of small $k_\perp$ ($k_\perp \sim \Delta\theta\omega_p/c$) the dependence of $\widetilde{\mathrm{Im}\,\omega}$ on $k_\perp$ takes the form:

$$\widetilde{\mathrm{Im}\,\omega} \sim \omega_p \frac{n_b}{n\gamma} \frac{1}{\Delta\theta^2} \frac{\omega_p^2}{\omega_p^2 + k_\perp^2 c^2} \tag{4.10}$$

In calculating the growth rate we have disregarded collisions between plasma electrons and ions. To take them into account we have to include the term $-\nu/2$ on the right-hand side of Eq. (4.9). In a dense plasma, collisions may lead to suppression of the instability.

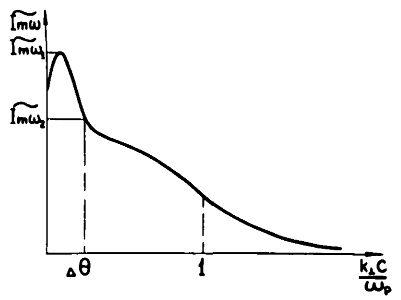

FIG. 8. Growth rate for beam instability maximized for k_z as a function of the transverse component of the wave vector (the value for growth rate on the dashed line shown in Fig. 7(a)). The ratio of $\widetilde{\mathrm{Im}\,\omega_1}$ to $\widetilde{\mathrm{Im}\,\omega_2}$ is of the order of 1.5 - 2 and depends on particulars of the beam distribution function.

4.3. Effect of the inhomogeneity of the plasma on the relaxation process

When an oscillation propagates through a non-uniform medium, its wave vector varies according to the equation given by Landau and Lifshits [79]:

$$\frac{d\vec{k}}{dt} = -\frac{\partial\omega}{\partial\vec{r}} \tag{4.11}$$

where $\omega = \omega(\vec{k}, \vec{r})$ is the solution of the dispersion relation. Let us consider first the part played by the longitudinal inhomogeneity of the plasma (i.e. let us assume that ω depends only on z). It then follows from Eq. (4.11) that only the longitudinal component of the wave vector is time-dependent:

$$\frac{dk_z}{dt} \sim \frac{\omega_p}{L_\parallel} \tag{4.12}$$

where $L_\parallel$ is the characteristic length scale of the longitudinal inhomogeneity[21]. We should further point out that interaction with the relativistic beam, and consequently build-up, is possible only in the case of those oscillations for which k_z satisfies the inequality (4.8). But since k_z varies with time (see Eq. (4.12)), a given oscillation will interact with the beam only over the small interval of time:

$$\Delta t \sim \frac{L_\parallel}{c}\left(\Delta\theta^2 + \frac{k_\perp c}{\omega_p}\Delta\theta\right)$$

Relaxation of the beam will only occur in a case where the oscillation is able to build up substantially from the thermal level over the time Δt, i.e. only on condition $\widetilde{\mathrm{Im}\,\omega}\,\Delta t \gtrsim \Lambda$ where Λ is a numerical factor of the order of 10 and takes into account the smallness of the initial oscillation energy (for greater detail see Brejzman and Ryutov [80]). On the basis of (4.10), this condition can be written as

$$\frac{1 + \dfrac{k_\perp c}{\omega_p \Delta\theta}}{1 + \left(\dfrac{k_\perp c}{\omega_p}\right)^2} \gtrsim \mu_\parallel \Lambda \tag{4.13}$$

where the parameter $\mu_\parallel$ is determined by the equation:

$$\mu_\parallel = \frac{c}{\omega_p L_\parallel} \frac{n}{n_b} \gamma$$

It is clear from (4.13) that inhomogeneity influences oscillations with different $k_\perp$ in different ways. For values of $k_\perp$ at which the criterion (4.13) is satisfied, the part played by the inhomogeneity is small, since the corresponding oscillations have time for considerable amplification over the period during which they interact with the beam. Conversely, if it is not satisfied for certain $k_\perp$, the oscillations are not in fact excited at all.

It can easily be seen that at $\mu_\parallel\Lambda > 1$ and $\Delta\theta > 1/\mu_\parallel\Lambda$ the inequality (4.13) is not satisfied for any value of $k_\perp$. Hence, if the initial angular spread of the beam is large enough, $\Delta\theta_0 \gtrsim 1/\mu_\parallel\Lambda$, then there is no relaxation at all. But if $\Delta\theta_0 \lesssim 1/\mu_\parallel\Lambda$, relaxation proceeds; though as soon as the angular spread reaches a small value, $\Delta\theta \sim 1/\mu_\parallel\Lambda$, it is terminated [80]. Finally, if

$$\mu_\parallel \Lambda \lesssim 1 \tag{4.14}$$

[21] By the inhomogeneity length scale we mean $|\partial \ln n/\partial z|^{-1}$.

inhomogeneity has no effect to speak of on the relaxation.

Similar considerations indicate that transverse inhomogeneity does not affect the relaxation when [81]

$$\mu_\perp \Lambda \lesssim 1 \qquad (4.15)$$

where the parameter $\mu_\perp$ is determined in the same way as $\mu_\parallel$.

If the relaxation is non-linear, as was pointed out by Rudakov [53], we have to modify conditions (4.14) and (4.15) slightly by replacing the factor Λ by unity.

Since we are in fact dealing with non-linear conditions in the problem of beam heating of a plasma (see below), let us consider which limitations on the inhomogeneity scale the conditions $\mu_\parallel$, $\mu_\perp < 1$ lead to. Assuming $n \sim 10^{15}$ cm^{-3}, $n_b \sim 10^{12}$ cm^{-3}, $\gamma = 5$, we get $L_\parallel, L_\perp \gtrsim 100$ cm. In terms of longitudinal inhomogeneity this inequality may be satisfied without much difficulty; in terms of transverse inhomogeneity, however, it means that for a plasma of radius ~ 10 cm the variation in density along it should not exceed 10%.

By regulating the inhomogeneity we should be able, theoretically, to switch the beam instability "on" and "off". More particularly, by purposely making the plasma highly inhomogeneous we can suppress this and certain other instabilities during beam transport.

4.4. Quasi-linear approximation

The quasi-linear approximation has a very limited range of applicability in the problem of the relaxation of a high-current relativistic beam. Nevertheless, special consideration of it is fully justified as it represents a necessary first step in formulating the general theory of relaxation.

As already mentioned, in the quasi-linear approximation it is assumed that the generation of oscillations by a beam instability is compensated by their drift at the group velocity deep into the plasma. Let us single out a layer in the plasma (Fig. 9), in which the beam imparts to the oscillations, let us say, 50% of its initial energy, and let us agree that we will call the thickness of this layer ℓ the relaxation length[22].

The electron beam enters this layer from the left, while on the right both the beam and the oscillations flow out of it. To find the density of the oscillation energy on the right of the relaxation region, we should note that here the oscillation energy flux $v_g U_\ell$ (where U_ℓ is the energy density of the Langmuir oscillations, while v_g is their

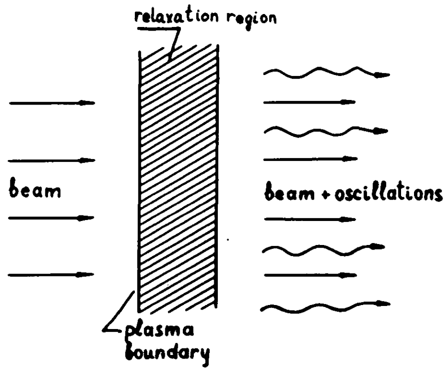

FIG. 9. Geometry for the determination of relaxation lengths.

characteristic group velocity) becomes, by definition, comparable with the initial energy flux of the beam $\gamma n_b mc^3$. Since the group velocity for Langmuir oscillations excited by a REB is equal in order of magnitude to v_{te}^2/c, from the given condition we find that

$$U_\ell \sim \gamma n_b mc^2 \, \frac{mc^2}{T} \qquad (4.16)$$

We then proceed to calculate the relaxation length. It should be estimated on the assumption that waves propagating deep into the plasma have enough time to build up over the length:

$$\ell \sim \Lambda \, \frac{v_g}{\operatorname{Im}\omega} \qquad (4.17)$$

where $\operatorname{Im}\omega$ is the instability growth rate. An important feature of a relativistic beam is the presence of a sharp growth rate peak (of width $\Delta\theta$) for the oscillations propagating more or less along the beam (Fig. 8). Consequently interactions between the waves and the beam only result in a spread of the longitudinal component of the momentum. In this case, even when the electron energy is reduced by an order of the initial energy, the angle between the electron momentum and the beam axis remains small ($\sim 2\Delta\theta_0$, where $\Delta\theta_0$ is the initial angular spread of the beam which is assumed to satisfy the inequality (4.5)). In other words, in the plasma the beam releases energy of the order of its initial energy without substantially increasing the angular spread[23]. We must therefore substitute into (4.17) the value of $\operatorname{Im}\omega$ calculated using the initial angular spread, which gives us the following expression:

$$\ell \sim \Lambda \, \frac{c}{\omega_p} \, \frac{n}{n_b} \, \frac{T}{mc^2} \, \gamma \Delta\theta_0^2 \qquad (4.18)$$

This result was obtained by Rudakov [53] and Brejzman and Ryutov [72][24].

[22] The finite size of the plasma in a transverse direction within the quasi-linear approximation is insignificant so long as the relaxation length is small compared with the transverse dimension of the plasma r_p. But if $r_p < \ell$, then when formulating the theory we have to recognize that oscillations with fairly large $k_\perp$ leave the plasma without having had time to build up. Thus, at $r_p \ll \ell$ the spectrum of the oscillations excited by the beam should be almost one-dimensional.

[23] We should point out that in the case of a non-relativistic beam the release of energy is accompanied by an increase in angular spread up to $\Delta\theta \sim 1$.

[24] Unidimensionality of relativistic beam relaxation was first pointed out by Fajnberg et al. [71].

Numerical calculations based on (4.18) give very small values for the relaxation length. For example, at $n = 10^{15}\ \mathrm{cm}^{-3}$, $n_b = 10^{12}\ \mathrm{cm}^{-3}$, $T = 10^3$ eV, $\Lambda = 10$ and $\gamma = 5$, the relaxation length is equal to 2 cm, even at $\Delta\theta_0 \sim 1$.

Waves excited within the relaxation region are absorbed upon further propagation into the plasma because of various dissipative processes. In particular, there is a highly effective dissipation mechanism associated with the inhomogeneity of the plasma density. This occurs if the Langmuir oscillations reach a lower density area, where their phase velocity may drop to a value of the order of v_{te}; they are then Landau damped, i.e. absorbed by the plasma electrons.

It is worth noting that although the quasi-linear relaxation of a relativistic beam is one-dimensional, it is accompanied by the appearance of a considerable number of accelerated electrons [72][25]. Indeed, the condition for interaction between the ultrarelativistic particles and the oscillation moving along the z axis takes the form:

$$\omega_p - ck\,\frac{p_z}{p} = 0$$

i.e. interaction with this oscillation is possible in the case of all the electrons lying on the cone subtending the angle $\theta = \arccos(\omega_p/ck)$ with the p_z axis in momentum space. But if the oscillations excited in the plasma have wave vectors ranging from a certain k_+ to a certain k_-, there may be interaction with them by all the electrons lying on the cones with angles θ ranging from $\theta = \theta_- = \arccos(\omega_p/ck_-)$ to $\theta = \theta_+ = \arccos(\omega_p/ck_+)$ (Fig. 10).

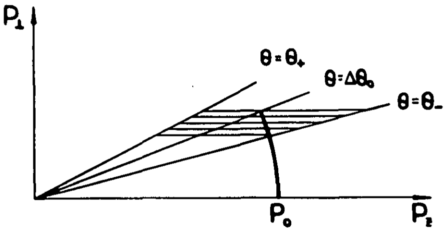

FIG. 10. Relaxation of an ultrarelativistic electron beam for one-dimensional oscillation spectrum. At the beginning of relaxation the electrons lie on the arc of a circle with radius $p = p_0$ in momentum space.

Since the quasi-linear interaction between electrons and oscillations moving along the z axis results in diffusion of the electrons along the lines $p_\perp = \mathrm{const}$, then, as can be seen from Fig. 10, the beam relaxation is inevitably accompanied by the appearance of accelerated electrons.

Formally the quasi-linear problem of steady-state injection of a beam into a homogeneous plasma can be reduced to solving the following set of equations [72]:

$$\frac{3kv_{te}^2}{\omega_p}\cos\theta'\,\frac{\partial W}{\partial z} = 2W\,\mathrm{Im}\,\omega \tag{4.19}$$

$$c\cos\theta\,\frac{\partial f}{\partial z} = \frac{1}{p^2}\frac{\partial}{\partial p}\,p^2\left(\mathscr{D}_{pp}\frac{\partial f}{\partial p} + \frac{\mathscr{D}_{p\theta}}{p}\frac{\partial f}{\partial\theta}\right) + \frac{1}{p\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left(\mathscr{D}_{p\theta}\frac{\partial f}{\partial p} + \frac{\mathscr{D}_{\theta\theta}}{p}\frac{\partial f}{\partial\theta}\right) \tag{4.20}$$

Here use is made of the spherical co-ordinate systems p, θ, φ and k, θ' and φ'.

The growth rate of the instability $\mathrm{Im}\,\omega$ is expressed in terms of the beam electron distribution function f in Eq. (4.9), while the diffusion tensor components are expressed in terms of the spectral density of the oscillation energy $W(k, \theta', z)$:

$$\left.\begin{matrix}\mathscr{D}_{pp}\\ \mathscr{D}_{p\theta}\\ \mathscr{D}_{\theta\theta}\end{matrix}\right\} = 2\pi\,\frac{m\omega_p^4}{nc^3}\int_{\omega_p/c}^{\infty}\frac{dk}{k} \times \int_{\theta_1'}^{\theta_2'}\frac{\sin\theta'\,W(k,\theta')\,d\theta'}{\sqrt{(\cos\theta_1' - \cos\theta')(\cos\theta' - \cos\theta_2')}}\left\{\begin{matrix}1\\ \zeta\\ \zeta^2\end{matrix}\right\} \tag{4.21}$$

$$\zeta = \left(\cos\theta - \frac{kc}{\omega_p}\sin\theta'\right)\Big/\sin\theta$$

Here

$$\cos\theta_{1,2} = \frac{\omega_p}{kc}\left(\cos\theta \pm \sin\theta\sqrt{\frac{k^2c^2}{\omega_p^2} - 1}\right)$$

Analysis of the set of equations (4.19) and (4.20) given in Ref. [72] affords the basis for a quantitative assessment of the relaxation process, i.e. for finding the oscillation energy density and the variation in θ_- and θ_+ (see Fig. 10). A graph showing the dependence of θ_- and θ_+ on the z co-ordinate at the initial stage of relaxation is given in Fig. 11. The calculations were made for a case

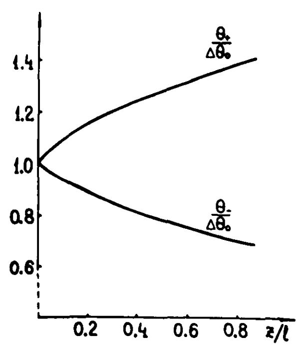

FIG. 11. θ_+ and θ_- (see Fig. 10) as a function of the longitudinal co-ordinate z in relaxation of an ultrarelativistic electron beam in a plasma. Numerical integration is performed for a case in which the initial distribution function is constant along the arc $p = p_0$ in Fig. 10.

[25] This is all the more surprising in that accelerated electrons do not appear during the one-dimensional relaxation of a non-relativistic beam (see Ivanov and Rudakov [82]).

in which the beam distribution function at the input to the plasma takes the following form:

$$f\big|_{z=0} = \frac{n_b\delta(p-p_0)}{2\pi p_0^2}\begin{cases}\dfrac{2}{\Delta\theta_0^2}, & \theta < \Delta\theta_0 \\ 0, & \theta > \Delta\theta_0\end{cases}$$

4.5. Classification of non-linear processes

When the Langmuir oscillation level is not too high (which is actually the case under the conditions in which the weak turbulence theory is applicable) we need merely take into account the non-linear processes in the lower (second) order of the perturbation theory. It is well known [74, 75, 83, 84] that in an isothermal plasma ($T_e = T_i = T$) without a magnetic field the Langmuir oscillations may take part in the following second-order processes (Fig. 12):

(1) Scattering by plasma electrons and ions (Fig. 12(a));
(2) Scattering by plasma electrons and ions with transformation into electromagnetic waves (Fig. 12(b)) and the reverse process (Fig. 12(c));
(3) Merging of two Langmuir oscillations into an electromagnetic wave (Fig. 12(d)) and the reverse process (Fig. 12(e));
(4) Merging of a Langmuir and an electromagnetic wave into an electromagnetic wave (Fig. 12(f)) and the reverse process (Fig. 12(g)).

Let us now describe each of these processes, beginning with the first.

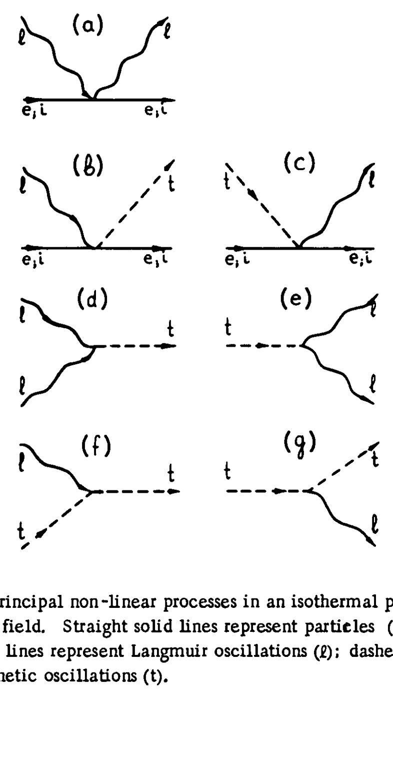

FIG. 12. Principal non-linear processes in an isothermal plasma without a magnetic field. Straight solid lines represent particles (electrons or ions); wavy lines represent Langmuir oscillations (ℓ); dashed lines show electromagnetic oscillations (t).

If a wave with a wave vector $\vec{k}$ and frequency $\omega(\vec{k})$ is scattered by a particle possessing a velocity $\vec{v}$, the wave vector $\vec{k}'$ and the frequency $\omega(\vec{k}')$ of the secondary wave satisfy the relationship:

$$\omega(\vec{k}) - \omega(\vec{k}') = (\vec{k} - \vec{k}')\vec{v} \tag{4.22}$$

which is a trivial consequence of the laws of the conservation of energy and momentum ($\hbar\omega$ is the wave energy and $\hbar\vec{k}$ is its momentum). If the scattering is effected by particles with a Maxwellian distribution, on the average the waves give up their energy to the particles [84]. Since the number of quanta during each elementary scattering event is conserved, the result is that the frequency of the secondary wave becomes less than that of the primary one.

For the process described in Fig. 12(a), Eq. (4.22) takes the form:

$$\frac{3}{2}\frac{v_{te}^2(k^2 - k'^2)}{\omega_p} = (\vec{k} - \vec{k}')\vec{v} \tag{4.23}$$

In scattering by electrons the absolute value of the vector $\vec{v}$ on the right-hand side is equal in order of magnitude to v_{te},[26] and the right-hand side, roughly approximated as $kv_{te} \sim \omega_p v_{te}/c$, proves to be formally much greater than the left-hand side, which can be estimated as $k^2v_{te}^2/\omega_p \sim \omega_p(v_{te}/c)^2$. But in actual fact this means that the scattering of wave $\vec{k}$ into wave $\vec{k}'$ can be effected only by particles whose velocity is almost perpendicular to the vector $\vec{k} - \vec{k}'$ (in this way the equality of the left- and right-hand sides of (4.23) is maintained). But since a plasma with an isotropic distribution function contains electrons with an arbitrary velocity vector orientation, condition (4.23) does not impose any limitation in practice on the permissible values of $\vec{k}'$.

Calculations (see Tsytovich [83]), show that the probability of scattering is maximal at $k' \sim (\frac{1}{2} - \frac{1}{3})k$, or in other words, in electron scattering the spectral transfer is towards oscillations with a wave vector smaller than the original one by a factor of 2 or 3. The characteristic scattering time is determined as

$$\tau_e^{\ell\ell} \sim \omega_p^{-1}\frac{nT}{U_\ell}\left(\frac{\omega_p}{\bar{k}v_{te}}\right)^3 \tag{4.24}$$

where $\bar{k}$ is the characteristic wave vector of the oscillations. Here we have introduced notations that will be used in what follows: the superscript on τ means that this quantity relates to scattering of the Langmuir oscillations into Langmuir oscillations, and the subscript indicates that the scattering is due to electrons.

In scattering by ions, the absolute value of the vector $\vec{v}$ on the right-hand side of (4.23) is equal in order of magnitude to v_{ti}, and the estimate for the right-hand side is

$$kv_{ti} \sim \omega_p(T/Mc^2)^{\frac{1}{2}}$$

[26] We are not considering scattering by beam electrons, since there are very few of them, $n_b \ll n$.

From this it can be seen that if

$$\frac{T}{mc^2} \gg \frac{1}{10}\frac{m}{M} \qquad (4.25)$$

the left-hand side of (4.23) is formally much larger than the right-hand side (in contrast with the previous instance) and that there is a considerable limitation on the reduction in the wave vector modulus in a single scattering event:

$$\Delta k \sim \tfrac{1}{3}\frac{\omega_p}{c}\left(\frac{m^2c^2}{MT}\right)^{\frac{1}{2}} \ll \bar{k} \qquad (4.26)$$

For a hydrogen plasma inequality (4.25) is satisfied at $T \gg 25$ eV, and for a deuterium plasma it is satisfied at $T \gg 10$ eV.

In this way, in each elementary scattering act due to ions the wave vector modulus is only reduced by a small value ($\Delta k \ll \bar{k}$), throughout the temperature region of practical interest. In this case by convention the spectral transfer is said to be differential. No substantial limitations are placed on the angle between $\vec{k}$ and $\vec{k}'$ by condition (4.23) (in particular, scattering through large angles is permissible (see Fig. 13)).

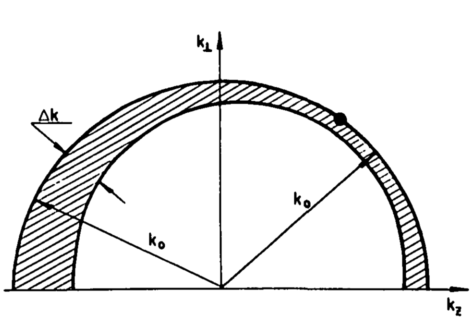

FIG. 13. Wave-vector region into which ℓ-ℓ scattering by ions is permitted by the conservation laws in a plasma with no magnetic field. Initial oscillation is indicated by a dot.

The reduction in the wave vector by its own magnitude takes place within $\bar{k}/\Delta k$ steps. The calculations show that if the oscillation spectrum has a characteristic width of the order of $\bar{k}$ from the outset, the time taken by the process can be estimated in the following way:

$$\tau_i^{\ell\ell} \sim \frac{1}{\omega_p}\frac{nT}{U_\ell}\left(\frac{\bar{k}}{\Delta k}\right)^2 \qquad (4.27)$$

It is important for what follows to note that at temperatures that are not too high ($T \lesssim mc^2(m/10\,M)^{2/5}$, i.e. $T \lesssim 10$ keV in the case of a deuterium plasma) for oscillations excited by the beam ($\bar{k} \sim \omega_p/c$), ion scattering is much more effective than electron scattering. A qualitative explanation for this effect is given e.g. by Kaplan and Tsytovich (Ref. [84], p. 67).

Let us now consider the scattering of Langmuir oscillations by plasma particles with transformation into transverse (electromagnetic) waves (which we shall represent by the subscript t) described by the dispersion equation:

$$\omega_t = \sqrt{\omega_p^2 + k_t^2c^2} \qquad (4.28)$$

The principal contribution to this process is likewise made by the plasma ions.

Taking into account the fact that the variation in frequency in ion scattering is very small, it can be said that the frequency of the t wave is close to ω_p. This means that $k_t \ll \omega_p/c$ and the dispersion equation for the t waves can be simplified:

$$\omega_t \simeq \omega_p\left(1 + \tfrac{1}{2}\frac{k_t^2c^2}{\omega_p^2}\right)$$

Accordingly, equality (4.22) takes the form:

$$\frac{3}{2}k^2v_{te}^2 - \frac{1}{2}k_t^2c^2 = \omega_p(\vec{k}\vec{v}_i) \qquad (4.29)$$

(on the right-hand side we have disregarded k_t since $k_t \ll k \sim \omega_p/c$). If condition (4.25) is satisfied, as a first approximation we can disregard the right-hand side, after which we determine from (4.29) the wave vector of the t wave, $k_t = \sqrt{3}\,kv_{te}/c$. The characteristic time for ℓ-t scattering by ions is calculated (see Tsytovich [83]) as $\omega_p^{-1}(nT/U_\ell)(\bar{k}/\Delta k)^2$, where Δk is determined by (4.26), i.e. $\tau_i^{\ell t}$ is equal in order of magnitude to the ℓ-ℓ scattering time. But the ℓ-t scattering process is highly sensitive to the inhomogeneity of the plasma. Indeed, in accordance with Eq. (4.11), the wave vector for the electromagnetic wave varies by a magnitude of order k_t over time:

$$\Delta t \sim r_p\frac{k_t}{\omega_p}$$

where it is assumed that the transverse scale of inhomogeneity coincides with the plasma radius. If the scale is small enough, then t oscillation rapidly leaves the region of permitted k_t values (4.29) and is consequently not excited. On the basis of the evaluation given above for $\tau_i^{\ell t}$, the condition for suppression of the ℓ-t scattering can be written as

$$r_p < 5\,\frac{c}{\omega_p}\frac{nT}{U_\ell}\frac{M}{m}\frac{v_{te}}{c} \qquad (4.30)$$

This inequality is easily satisfied even at $U_\ell/nT \sim 1$, and we shall therefore assume in what follows that there is no ℓ-t scattering[27]. However, it can be demonstrated, as for example by Kaplan and Tsytovich [84], that even if ℓ-t scattering is permitted, as far as the Langmuir oscillation spectrum is concerned the results are qualitatively the same as for ℓ-ℓ scattering.

[27] Since $k_t \ll c/\omega_p$ the effect of the inhomogeneity on ℓ-ℓ scattering is less important.

Let us go on to the process shown in Fig. 12(d). This process results in electromagnetic radiation with a frequency close to $2\omega_p$ (since the frequency of each Langmuir wave is close to ω_p). The wave vector for the t waves as determined from the dispersion equation (4.28) is $\sqrt{3}\omega_p/c$. The generation rate is determined from the relationship:

$$\frac{\partial U_t^{(2)}}{\partial t} \sim \omega_p U_\ell \frac{U_\ell}{nmc^2} \qquad (4.31)$$

where the superscript 2 above U_t means that the radiation is generated at the second harmonic of the plasma frequency. If the level of t oscillations is high enough, the decay process $t \to \ell + \ell$ may occur, and to take it into account we have to add the term $-\omega_p U_t^{(2)} U_\ell / nmc^2$ to the right-hand side of (4.31) [83]. Furthermore, we have to consider the possibility that the electromagnetic radiation escapes at the group velocity (which in the case of oscillations at twice the plasma frequency is equal to $\sqrt{3}/2c$) and also the absorption of the radiation due to pair collisions. As a result, instead of (4.31) we get the following schematic equation for U_t^2:

$$\frac{\partial U_t^{(2)}}{\partial t} \sim \omega_p U_\ell \frac{U_\ell}{nmc^2} - U_t^{(2)}\left(\omega_p \frac{U_\ell}{nmc^2} + \frac{c}{r_p} + \nu_{ei}\right) \qquad (4.32)$$

where ν_{ei} is the frequency of electron/ion collisions. Since $\nu_{ei} \ll c/r_p$ is normally the case in a hot plasma, the collisions in (4.32) can be disregarded.

If the time taken for the escape of the radiation from the plasma is small compared with the decay time for $t \to \ell + \ell$, i.e. if the following condition is satisfied:

$$\frac{c}{r_p} > \omega_p \frac{U_\ell}{nmc^2} \qquad (4.33)$$

then the radiation leaves the plasma freely. In the opposite case, the radiation remains trapped and its energy density is determined from the balance between the forward and reverse processes, which gives us $U_t^{(2)} \sim U_\ell$.

When the radiation is not trapped, then, as can be seen from what follows, it may remove a great deal of energy from the plasma, and we therefore have to seek ways of trapping it. For this reason there is a certain advantage to be gained by creating density distributions for which the density increases by more than a factor of 4 on all sides of the relaxation region (Fig. 14). The radiation is then reflected from the higher density regions and becomes trapped. Formally, this means that we can drop the term $U_t^{(2)} c/r_p$ in (4.32). What happens to the radiation then depends on the relationship between ν_{ei} and $\omega_p(U_\ell/nmc^2)$. If the former is greater than the latter[28], the radiation is effectively absorbed by the plasma and is kept at a low level. But if the former is smaller than the latter, then $U_t^{(2)}$ becomes equal to U_ℓ. At this level of electromagnetic energy density the processes $t + \ell \rightleftarrows t$ (Fig. 12 (f) and (g)) are initiated. These lead to the occurrence of radiation at a frequency of $\sim 3\omega_p$, and it will no longer be confined in the plasma at a reasonable level of inhomogeneity. It may therefore be advisable to surround the plasma with a reflecting sheath. But to ensure that the radiation is absorbed by the plasma and not by the walls, the latter should have a very small absorption factor (less than $\nu_{ei} r_p/c$).

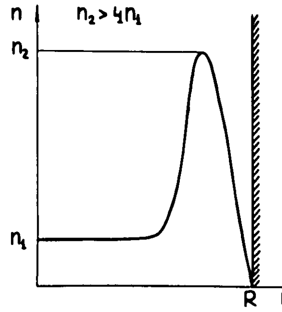

FIG. 14. Radial distribution of plasma density for which there is no radiation at twice the plasma frequency. The beam passes through the region with density n_1. Hatched area shows vacuum chamber wall.

4.6. Non-linear relaxation

The part played by non-linear processes in beam relaxation will be considered for the simplest case of an isothermal plasma ($T_e = T_i = T$) without a magnetic field. This problem was studied by Altyntsev et al. [1] and in greater detail by Brejzman, Ryutov and Chebotaev [86], which is the one we shall describe here.

If we substitute the oscillation energy density calculated from Eq. (4.16) on the basis of a quasi-linear approximation in expression (4.27) for the spectral transfer time due to the principal non-linear process (ℓ-ℓ scattering by ions), the time will in virtually every case be small in comparison with the inverse growth rate of the beam instability. This means that in actual fact a steady-state spectrum is created not through drift of the oscillations but through their transfer from the resonance to the non-resonance region of the spectrum. As a result, the energy density for resonance waves is substantially reduced, while the relaxation length increases with respect to quasi-linear relaxation length. The relaxation length may become considerably greater than the plasma radius r_p. Nevertheless, if the condition

$$r_p > \Lambda v_g / \mathrm{Im}\,\omega$$

is satisfied (and, as mentioned in Section 4.4, it is very soft), we can disregard the escape of the oscillations both in the longitudinal and transverse directions, and apply the following equation for the spectral function $W(\vec{k}, z, t)$:

$$\frac{\partial W}{\partial t} = 2(\mathrm{Im}\,\omega + \Gamma)W \qquad (4.34)$$

[28] It should be noted that the electromagnetic waves may be absorbed more effectively close to the reflection points than within the plasma. This is the result of spatial dispersion effects (see Golant and Piliya [85]).

The term ΓW describes deformation of the Langmuir oscillation spectrum through ion scattering. Γ is a linear integral functional of W:

$$\Gamma = \int A(\vec{k},\vec{k}')\, W(\vec{k}')\, d\vec{k}'$$

The expression for the kernel $A(\vec{k},\vec{k}')$ can be found, e.g. in the surveys of Galeev et al. [75] and Tsytovich [83]. In order of magnitude Γ^{-1} coincides with $\tau_i^{\ell\ell}$ derived in Section 4.5.

As was demonstrated in Section 4.5, only oscillations with wave-vectors close in the modulus (see Eq. (4.26)) can interact with each other during ion scattering. It follows from this fact, and also from the structure of the beam instability growth rate (which is positive in the region shown in Fig. 7), that for non-linear relaxation the spectrum of oscillations interacting with the beam should be essentially tridimensional, i.e. in the resonance region there should be oscillations with all values of $k_\perp$.[29] The tridimensionality of the resonance oscillation spectrum is one of the distinctive features of the relaxation mechanism under discussion, i.e. the electric field of the oblique waves ($k_\perp \sim k_z$) directed at an angle with respect to the beam axis brings about a considerable increase in angular spread.

Let us consider the initial relaxation stage where the beam momentum spread is still small ($|\Delta\vec{p}| \lesssim p_0$). In this case it follows from (4.20) by definition of $\mathscr{D}_{\theta\theta}$ that

$$c\frac{d}{dz}\Delta\theta^2 \simeq \frac{1}{p_0^2}\mathscr{D}_{\theta\theta} \tag{4.35}$$

Taking into account, furthermore, the fact that for the broad-angle oscillation spectrum ($k_\perp \sim k_z$) all three diffusion factors are of the same order of magnitude, it can be stated that the increase in angular spread is accompanied by spreading of the beam electron energies in the direction of lower values, and that at $\Delta\theta \gg \Delta\theta_0$

$$\frac{\Delta\mathscr{E}}{\mathscr{E}} \sim \Delta\theta \tag{4.36}$$

To find the relaxation length, i.e. the length at which $\Delta\mathscr{E}/\mathscr{E}$ and correspondingly $\Delta\theta$ become of the order of unity, we have to determine the oscillation spectrum $W(\vec{k})$ and then estimate the diffusion factor $\mathscr{D}_{\theta\theta}$. In the steady state the spectrum is determined from the condition:

$$\begin{aligned} &\mathrm{Im}\,\omega(\vec{k}) + \Gamma(\vec{k}) = 0, \qquad && W(\vec{k}) \neq 0 \\ &\mathrm{Im}\,\omega(\vec{k}) + \Gamma(\vec{k}) \leq 0, \qquad && W(\vec{k}) = 0 \end{aligned} \tag{4.37}$$

The corresponding problem reduces to the solution of a first-order integral equation, for which we know that, except for certain degenerate cases, there are no regular solutions. It is therefore not clear whether there are any steady-state spectra at all in the problem we are considering. However, on physical grounds it is obvious that even if the system lacks a steady-state solution, there should be a quasi-steady state in which the spectral function $W(\vec{k},t)$ fluctuates around a mean level $\langle W(\vec{k},t)\rangle$, where the brackets show time averaging. Inasmuch as the conservation laws permit scattering over large angles (see Fig. 15), it is natural to assume in this case that the oscillation spectrum is more or less isotropic (the anisotropy is of the order of unity). We then get from Eq. (4.21) the following approximation for $\mathscr{D}_{\theta\theta}$:

$$\mathscr{D}_{\theta\theta} \sim \frac{m\omega_p}{n} U_\ell \tag{4.38}$$

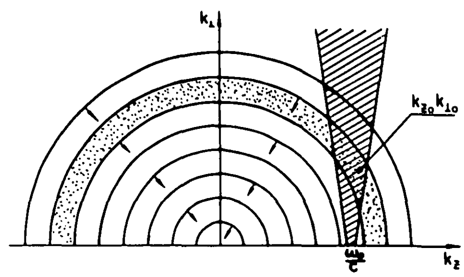

FIG. 15. Theory of non-linear relaxation of an ultrarelativistic beam in a plasma with no magnetic field. The concentric circles are lines along which there is isotropization of the oscillation spectrum. Arrows indicate direction of spectral transfer. The resonance region is hatched. The region permitted by conservation laws for scattering of waves k_{z0}, $k_{\perp 0}$ is illustrated.

The energy density of the oscillations U_ℓ is estimated from the condition $\mathrm{Im}\,\omega \sim \Gamma$ in the region $k \sim \omega_p/c$. On the basis of Eqs (4.10) and (4.27) we get

$$U_\ell \sim 10 n_b T \frac{M}{m}\frac{T}{mc^2}\frac{1}{\gamma_0 \Delta\theta^2} \tag{4.39}$$

and, consequently, that

$$c\frac{d}{dz}\Delta\theta^2 = 10\omega_p \frac{n_b}{n}\frac{M}{m}\left(\frac{T}{mc^2}\right)^2 \frac{1}{\gamma_0^3 \Delta\theta^2}$$

i.e.

$$\Delta\theta = \left(\Delta\theta_0^4 + \frac{z}{\ell}\right)^{1/4} \tag{4.40}$$

Here we have introduced the notation[30]

$$\ell = \frac{\gamma_0^3}{20}\frac{c}{\omega_p}\frac{n}{n_b}\frac{m}{M}\left(\frac{mc^2}{T}\right)^2 \tag{4.41}$$

The quantity ℓ is obviously the beam relaxation length.

[29] In general, we might have a situation in which there is only a spectrum of non-resonance oscillations suppressing noise in the resonance region, but analysis shows that such states are not actually possible.

[30] We should point out that there is a misprint in the paper of Altyntsev et al. [1]; the factor $k/\Delta k$ has been omitted in the estimate of the spectral transfer time and in the formula for ℓ.

Let us now find the energy q released by the beam per unit plasma volume per unit time:

$$q = \frac{d}{dz} n_b c \Delta \mathscr{E} = \frac{n_b c \mathscr{E}}{4\ell}\left(\Delta\theta_0^4 + \frac{z}{\ell}\right)^{-3/4} \qquad (4.42)$$

One can see that q has a sharp peak at small z ($z \sim \ell\Delta\theta_0^4$), after which it rapidly diminishes. At a distance of the order of $\ell\Delta\theta_0^4$ the beam loses energy of the order of $\mathscr{E}\Delta\theta_0$ (calculated per beam particle). The bulk of the energy $\sim \mathscr{E}$, however, is lost by the beam in a considerably greater distance ($\sim \ell$).

The results given above were obtained on the assumption of non-steady state and, in fact, the isotropy, of the turbulence spectrum. It should be emphasized, however, that the problem cannot be considered definitively solved. In particular, it can be demonstrated [87] that in the so-called diffusion approximation based on a simplified expression for Γ that makes allowance for the smallness of $\Delta k/k$, there is a genuine steady-state solution for (4.34). Let us look at this in greater detail.

In the diffusion approximation, Eq. (4.34) takes the following form:

$$\frac{\partial}{\partial t} W(k,x) = W(k,x)\left[2\,\mathrm{Im}\,\omega + \frac{\partial}{\partial k} k^2 \int_{-1}^{+1} T(x,y)\, W(k,y)\, dy\right] \qquad (4.43)$$

Here we use the notations:

$$x = \cos\theta, \quad y = \cos\theta'$$

$$T(x,y) = \frac{\pi^2}{9}\frac{m}{M}\frac{\omega_p}{nT}\frac{m\omega_p^2}{T} \times [1 - x^2 - y^2 + 3x^2y^2 - 3xy + 3xy^3 + 3x^3y - 5x^3y^3] \qquad (4.44)$$

The kernel of the integral T(x,y) is degenerate and constitutes a polynomial of the third order with respect to x. Generally speaking, the angular dependence of the growth rate of the instability $\mathrm{Im}\,\omega$ does not take this form. From this it follows directly that in the genuine steady state the function W may differ from zero for the given k only in the discrete set of values x ($x = x_i(k)$). The lines $x_i(k)$ on which the spectral distribution is concentrated will be termed rays. In accordance with what has been said, the function W should take the following form:

$$W(k,x) = \sum_{i=1}^{\mathscr{N}} W_i(k)\,\delta(x - x_i(k)) \qquad (4.45)$$

The shape of the rays, $x_i(k)$, the number of them, $\mathscr{N}$, and the distribution of intensities, $W_i(k)$, now have to be determined from the conditions (4.37), which give us a set of $2\mathscr{N}$ normal differential equations for $x_i(k)$ and $W_i(k)$. In a case in which the angular spread of the beam is small ($\Delta\theta \ll 1$), the ray located in the resonance region does in fact coincide with the line of the peak growth rate $\mathrm{Im}\,\omega(k,x)$ with respect to x, there being no other rays in the region of large k ($k \gg \omega_p/c$) [87]. In this region $W \sim k^{-3}$. For smaller k the function W(k) becomes more complex, but the ray-like nature of the spectrum is of course conserved. Since in this case a larger portion of oscillation energy is concentrated in the resonance region than in the case of the isotropic spectrum ($\sim \frac{1}{2}$ instead of $\sim \Delta\theta$), the estimate of the diffusion factor $\mathscr{D}_{\theta\theta}$ changes as compared with (4.38) and takes the form:

$$\mathscr{D}_{\theta\theta} \sim \frac{m\omega_p}{n\Delta\theta} U_\ell \qquad (4.46)$$

This results only in a slight change in the dependence of the angular and the energy spread of the beam on the longitudinal co-ordinate:

$$\Delta\theta \sim \frac{\Delta\mathscr{E}}{\mathscr{E}} \sim \left(\frac{z}{\ell}\right)^{1/5} \qquad (4.47)$$

However, the relaxation length ℓ is the same as before (see Eq. (4.41)).

In both the relaxation schemes formulated above, the energy lost by the beam is transferred into the long-wavelength area of the spectrum by ion scattering. We shall point out below a few of the mechanisms limiting the long-wavelength oscillation level. Which one is the principal mechanism has to be decided taking into account the concrete experimental conditions. We should stress, however, that if the removal of energy from the long-wavelength region of the spectrum is efficient enough, the beam relaxation does not depend on the mechanism by which the long-wavelength oscillations are damped. The words "efficient enough" mean that the characteristic damping time for long-wavelength Langmuir oscillations should not exceed the time taken for them to be transferred from the region $k \sim \omega_p/c$ into $k \ll \omega_p/c$ by ion scattering (which is equal to $\tau_i^{\ell\ell}$; see Eq. (4.27)).

The most effective mechanism for the damping of long-wavelength Langmuir oscillations ($k < \omega_p/c$) is their transformation into electromagnetic waves by interaction with oscillations from the region $k > \omega_p/c$ (see Brejzman et al. [86] and Brejzman and Ryutov [88]). The process in question is shown in Fig. 12(d). Using the estimate in Section 4.5, we can easily establish that this process may hamper the build-up of oscillations in the long-wavelength region if

$$T \gtrsim mc^2 \sqrt{\frac{m}{10M}}$$

i.e. at $T \gtrsim 2$ keV (for a deuterium plasma).

If the electromagnetic radiation is not trapped it will remove from the plasma a large amount of the energy lost by the beam, which is of course undesirable when the beam is being used as a means of heating the plasma. Some of the arrangements which can be used to counteract this effect are described in Section 4.5.

At temperatures of less than 2 keV, the removal of energy from the long-wavelength region is not efficient enough, and therefore the spectral transfer due to ℓ-ℓ scattering by ions causes a build-up of these oscillations in the region $k \lesssim \Delta k$, where concentration of energy becomes very high [86] (see Fig. 16). Limitation of the energy density of these oscillations may be due to pair collisions. But if the collision frequency is small, it is obviously necessary, in calculating the wave absorption rate, to make use of effects which reach beyond the approximation of weak turbulence (see below).

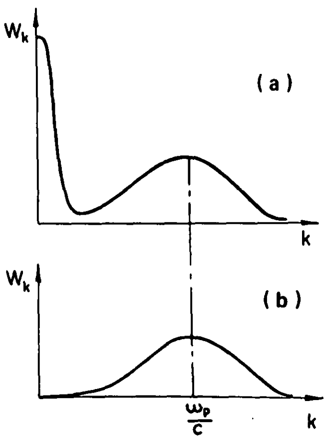

FIG. 16. Shape of the Langmuir oscillation spectrum: (a) without damping in the region of small k; (b) with damping. In (a) a high-energy density peak forms in the region of small k.

A relaxation model differing substantially from the one discussed above was formulated by Rudakov [53]. He suggested that the chief non-linear process is scattering of the oscillations by plasma electrons and in fact totally disregarded scattering by ions.

The chain of events leading to the establishment of the steady state in this model can be roughly represented in the following way. Oscillations are generated by the beam in the resonance region (see Fig. 7) and because of scattering by electrons are continuously transferred to the $k < \omega_p/c$ region, where, as was assumed by Rudakov [53], they are absorbed as a result of binary collisions. The energy density of the oscillations in the $k < \omega_p/c$ region is estimated by Rudakov assuming that the time $\tau_e^{\ell\ell}$ calculated from their energy density U'_ℓ should be exactly equal to the reciprocal of the maximum growth rate of the beam instability. This means the non-linear processes suppress the instability at all points, except at the maximum growth rate point, i.e. only oscillations moving almost strictly along the beam are to be found in the resonance region. In this case, as pointed out in Section 4.4, the inequalities $\mathscr{D}_{pp} \gg \mathscr{D}_{p\theta}, \mathscr{D}_{\theta\theta}$ are satisfied, and relaxation is practically one-dimensional (since the beam energy losses are accompanied by only a small increase in angular spread). This is the most characteristic feature of Rudakov's model.

According to this model the relaxation length is calculated in the following way. From the condition $\tau_e^{\ell\ell} \sim (\max \operatorname{Im}\omega)^{-1}$ we estimate the energy density U'_ℓ of the non-resonance oscillations $(k < \omega_p/c)$:

$$\omega_p \frac{U'_\ell}{nT}\left(\frac{T}{mc^2}\right)^{3/2} \sim \max \operatorname{Im}\omega$$

Knowing U'_ℓ, we can find the energy q dissipated by the beam in a unit of plasma volume per unit time: $q \sim \nu_{ei} U'_\ell$ (where ν_{ei} is the frequency of electron/ion collisions). Further, by dividing the beam energy flux by q we can estimate the relaxation length:

$$\ell \sim \gamma_0^2 \Delta\theta_0^2 \frac{v_{te}}{\nu_{ei}} \tag{4.48}$$

It can be seen that the relaxation length is a strong function of the initial angular spread of the beam $\Delta\theta_0$, this being a consequence of the one-dimensionality of the relaxation referred to above.

However, in the most important temperature region from the practical point of view (2 keV < T < 10 keV), Rudakov's solution is apparently not realizable. Indeed, if we make his assumption [53] that the non-resonance oscillations fill the entire range of wave vectors fairly uniformly from $k = 0$ to $k = \omega_p/c$ and estimate the time taken for spectral transfer of these oscillations as a result of ion scattering, we can easily see that this period of time ($T \lesssim 10$ keV; see Section 4.5) is considerably less than the inverse growth rate of the instability. Consequently, ion scattering causes "bunching" of the oscillation spectrum near the point $k = 0$, after which, as we know from Tsytovich [83], the non-linear growth rate due to electron scattering is sharply diminished. Thus we arrive at a situation in which the instability in the resonance region remains large, and a step-by-step transfer of oscillations due to the ion scattering considered above sets in.

It should be pointed out, however, that models of the type discussed by Rudakov are possibly valid in cases where the system produces a spectrum with a high energy density at $k \to 0$ (i.e. at $T \lesssim 2$ keV). But even in this case the relaxation length differs from (4.48), since the smallness of the wave vectors forming the bulk of the spectrum alters the expression for the rate of scattering of the Langmuir oscillations by electrons.

We shall now deal with the limits of applicability of the relaxation length (4.41). When the beam density n_b and consequently the Langmuir oscillation energy density increases, we encounter effects that do not fit into the picture drawn above. These effects consist for the most part in a modification of the dispersion properties of the plasma under the influence of turbulence. The most significant modification takes place in the acoustic branch of the oscillations, which at a fairly high level of turbulence becomes unstable. This fact is related to the "negativity" of the Langmuir oscillation pressure. It can be shown [89] that when the plasma density is perturbed by a value δn the pressure of the Langmuir oscillations varies by

$$\delta p \simeq -\frac{\delta n}{n}\frac{U_\ell}{\bar{k}^2 r_D^2} \simeq -\frac{\delta n}{n}\left(\frac{T}{mc^2}\right)U_\ell$$

with $\delta p < 0$ at $\delta n > 0$. It is clear from this that at

$$U_\ell > nT(\bar{k}r_D)^2 \sim nT\left(\frac{T}{mc^2}\right) \qquad (4.49)$$

low-frequency density fluctuations can occur spontaneously in the plasma. Bearing in mind that U_ℓ can be expressed, using Eq.(4.39), in terms of the beam and plasma parameters, we shall reformulate the inequality set out above in the form of a restriction on these parameters:

$$\frac{n_b}{n} > \frac{1}{10}\frac{m}{M}\gamma_0 \Delta\theta^2 \qquad (4.50)$$

The instability growth rate occurring under condition (4.49) can be expressed as follows:

$$\mathrm{Im}\,\omega \sim \kappa\left(\frac{U_\ell}{Mn\bar{k}^2 r_D^2}\right)^{1/2} \sim \frac{\kappa}{\bar{k}}\omega_p\left(\frac{T}{Mc^2}\right)^{1/2} \qquad (4.51)$$

where κ is the wave vector of the acoustic perturbation. The fastest growing perturbations are those with the highest possible value of κ, which is equal to $\bar{k}$ (at $\kappa > \bar{k}$ we cannot speak of a Langmuir plasmon gas). Under typical experimental conditions the growth rate (4.51) is very high compared with the inverse beam injection time.

The development of the instability in the plasma causes density inhomogeneities which begin to increase and finally reach the point where the generation rate for the Langmuir oscillations is substantially slowed down by effects similar to those described in Section 4.3.

Reliable estimates of the relaxation length in the limit (4.50) are not available. If it is assumed that the density of the Langmuir oscillation energy is "frozen" at the level of (4.49), which corresponds to the excitation threshold for an acoustic instability, we get for the relaxation length an expression which does not depend on the beam density:

$$\ell \sim \frac{c}{\omega_p}\left(\frac{\mathscr{E}}{T}\right)^2 \qquad (4.52)$$

Since the wave energy density in the limit (4.50) still appears to increase with n_b (though more slowly than for small n_b), Eq. (4.52) should be understood more as an upper estimate of the relaxation length for high-density beams.

When going from (4.49) to (4.50), we applied (4.39) for the energy density of the Langmuir oscillations. Of course, this is valid only if there is no peak in the spectrum in the region of small k. Otherwise the source of the instability may be the spectrum peak itself, and acoustic fluctuations may occur at much smaller values of U_ℓ, as compared with (4.49). The initial stage of this instability was studied by Zakharov [90].

There have been attempts of late to make both analytical [91] and numerical [92-94] studies of the effects under conditions where the acoustic instability described above begins to manifest itself. The general idea behind these studies is to determine the deformation of the Langmuir oscillation spectrum due to interaction with plasma inhomogeneities[31] (see Section 4.3). As a result of the interaction there is, first, a decrease in the effective growth rate of the beam instability (see Ryutov [95]) and, second, at a fairly high level of inhomogeneity there is a possibility of the absorption of the oscillations by plasma electrons. Unfortunately, all the studies listed here deal with a highly idealized case of one-dimensional turbulence.

4.7. Effect of a magnetic field on relaxation

We have not yet considered effects related to the presence of a magnetic field in the plasma. In actual fact, however, even a weak magnetic field ($\omega_H \ll \omega_p$) may substantially affect relaxation of the beam. And indeed, the conclusions reached in Sections 4.5 and 4.6 on the structure of the Langmuir oscillation spectrum and transfer to the region with small wave vectors were based on an analysis of the conservation law (4.23), which is largely dependent on the dispersion properties of the Langmuir oscillations. Taking the magnetic field into account, we get an additional term to the Langmuir oscillation frequency, equal to

$$\frac{\omega_p}{2}\frac{k_\perp^2}{k^2}\frac{\omega_H^2}{\omega_p^2}\left(1-\frac{\omega_p^2}{k^2c^2}\right)$$

From (4.23) it is clear that the part played by the magnetic field is considerable when this additional term exceeds $(3/2)(kr_D)^2\,\omega_p$. Assuming $k \sim \omega_p/c$ and $k_\perp \sim k$ (this situation occurs in the case of the spectrum considered above), we get a condition for which we have to take the magnetic field into account: $\omega_H^2 > \omega_p^2(v_{te}/c)^2$. This condition can be rewritten as $\beta < 1$. Since in most experiments it is just this inequality that is satisfied, it would be very helpful to improve the theory by taking the magnetic field into account.

Progress along these lines is attended by certain difficulties of a formal nature, namely, the fact that the equations for the Langmuir oscillation spectrum are more complicated, and the

[31] Similar effects were discussed earlier in connection with non-relativistic beams [95-97] in the approximation of the given inhomogeneity.

relaxation process involves new types of oscillations, such as helicons, cyclotron oscillations, and so on. However, if we assume that the relaxation is associated with the excitation only of Langmuir oscillations with an energy density restricted through ion scattering, then we can already make some qualitative statements on the role of the magnetic field.

We now apply the conservation law (4.23) modified so as to take into account the additional (magnetic field) term for the oscillation frequency:

$$\frac{3}{2}\frac{v_{te}^2(k^2-k'^2)}{\omega_p} + \frac{\omega_p}{2}\frac{\omega_H^2}{\omega_p^2}\left[\frac{k_\perp^2}{k^2}\left(1-\frac{\omega_p^2}{k^2c^2}\right) - \frac{k_\perp'^2}{k'^2}\left(1-\frac{\omega_p^2}{k'^2c^2}\right)\right] = (\vec{k}-\vec{k}')\,\vec{v}_i \qquad (4.53)$$

It can be seen from this that for $\beta \ll 1$ an oscillation with wave vector $k \sim \omega_p/c$ may be scattered into the region of $\vec{k}$ space depicted in Fig.17, i.e. the scattering in the magnetic field leads, first, to "blurring" of the spectrum over the wave vector magnitude and, second, to slow ("differential") angular transfer (the angular dimension of the region into which transfer is allowed is of the order of $\beta(k\Delta kc^2/\omega_p^2)$ In this case we can write down the following expression for $\tau_i^{\ell\ell}$:

$$\tau_i^{\ell\ell} \sim \frac{1}{\omega_p}\frac{nT}{U_\ell}\frac{M}{m}\frac{T}{mc^2\beta} \qquad (4.54)$$

where the additional factor β^{-1} is associated with a reduction in the volume of phase space into which the transfer is allowed. Comparing Eqs

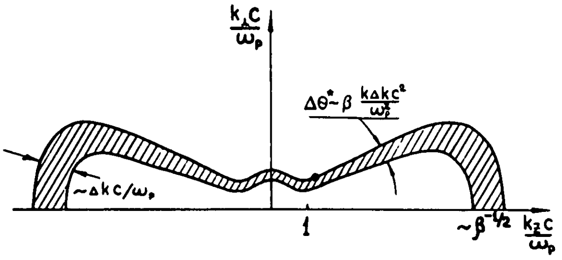

FIG. 17. Wave-vector region into which ℓ-ℓ scattering by ions in a magnetic field with $\beta < 1$ is permitted by the conservation laws. Initial oscillation is indicated by a dot.

(4.54) and (4.27), we see that at $\beta < 1$ the quantity U_ℓ, as estimated from the condition $\mathrm{Im}\,\omega \sim (\tau_i^{\ell\ell})^{-1}$, increases by a factor of β^{-1} compared with the zero-field case, which results in a reduced relaxation length. To take this effect into account it was proposed by Altyntsev et al. [1] that the relaxation length estimated on the basis of (4.41) should be multiplied by β. It should be noted, however, that this conclusion still requires careful checking.

In a plasma with $\beta < 1$ the scattering of the Langmuir oscillations has yet another important feature: as the magnetic field increases, so does the portion of energy which the oscillations impart to the plasma ions (this is proportional to the magnetic term added to the oscillation frequency). We should recall that in a plasma without a magnetic field the portion of energy transferred to the ions in the scattering process does not exceed $3/2\ (T/mc^2)$.

Even though for $\beta > 1$ the magnetic field only slightly affects the dispersion of the Langmuir oscillations interacting with the beam, and the non-linear processes in which they participate, it may still substantially alter the relaxation pattern. The reason for this is that the beam excites oscillations which do not exist at $H = 0$. More particularly, as shown by Brejzman [98] and by Brejzman and Feizov [99], relaxation may involve the excitation of whistlers.

It follows from linear theory that whistlers are excited by the beam less effective than Langmuir oscillations; but under conditions where the energy density of the Langmuir oscillations is limited by non-linear processes, it may well be the whistlers that are responsible for the beam relaxation. A formal solution of the beam-whistler interaction problem is derived by Brejzman and Feizov [99]. In the present paper we shall estimate the part played by this effect on the basis of qualitative considerations.

In the presence of a magnetic field H_z the condition for interaction between the beam electrons and the wave is

$$\omega(\vec{k}) - k_z v_z - n\omega_H' = 0 \qquad (4.55)$$

where $\omega_H' \equiv \omega_H/\gamma$ is the relativistic electron cyclotron frequency.

Equation (4.55) follows from application of the energy and momentum conservation relations during a radiation (or absorption) event of a wave by a particle. Here $\hbar\omega$ is the energy of the radiated wave, while $\hbar n\omega_H'$ is the change in the "transverse" energy of the particle during radiation.

As can be seen from the dispersion equation for the whistlers, i.e.

$$\omega(\vec{k}) = \omega_H k|k_z|c^2/\omega_p^2 \qquad (4.56)$$

in a weak magnetic field ($\omega_H \ll \omega_p$) the phase velocity of these waves is small compared with the speed of light. Hence the Cherenkov resonance ($n = 0$) with beam electrons is impossible for whistlers. But if $n \neq 0$, then we can disregard $\omega(\vec{k})$ in the resonance condition (4.55). In other words, the change in the "transverse" energy of the particle is much greater than the energy losses in radiation of the wave. This means that the action of the whistlers on the beam results in almost elastic scattering of the particles.

We now derive the change in angular spread of the particles $\Delta\theta$ for steady-state injection of the beam into the plasma. To do this we shall estimate the distance from the plasma boundary at which the energy density of oscillation excited by the beam with a spread of $\Delta\theta$ attains a level considerably exceeding the thermal level:

$$z \sim \Lambda\, v_{gz}/\mathrm{Im}\,\omega$$

Here Λ is a factor of the order of 10, v_{gz} is the z component of the wave group velocity, and $\mathrm{Im}\,\omega$ is the instability growth rate. For $\mathrm{Im}\,\omega$ we have the expression (see Brejzman and Feizov [99]):

$$\mathrm{Im}\,\omega \sim \omega_H \frac{n_b}{n} \frac{kc}{\omega_H} \frac{1}{\Delta\theta^2} \qquad (4.57)$$

This gives us

$$z \sim \Lambda \frac{c}{\omega_p} \frac{n}{n_b} \frac{\omega_H}{\omega_p} \Delta\theta^2$$

If the non-linear interaction of the waves with each other is negligibly small, this relationship will also give the z dependence of the angular spread of the beam. Thus, in a quasi-linear approximation[32]:

$$\Delta\theta(z) \sim (z/\ell)^{1/2} \qquad (4.58)$$

where the quantity:

$$\ell_w \equiv \Lambda \frac{c}{\omega_p} \frac{\omega_H}{\omega_p} \frac{n}{n_b} \qquad (4.59)$$

is the beam relaxation length under conditions where it is the excitation of the whistlers that determines the relaxation.

The whistler energy density U_w at a distance z from the plasma boundary can be found from the law of conservation of momentum flux. There are no oscillations at the input to the plasma, but the electron momentum flux is equal to $n_b vp$, where v and p are the velocity and momentum of the electron, respectively. Taking into account the fact that the relaxation reduces essentially to an increase in the angular spread of the beam, we get

$$n_b vp = n_b vp(1 - \Delta\theta^2(z)) + v_g k_z \frac{U_w}{\omega(\vec{k})}$$

It follows from this that

$$U_w(z) \sim n_b vp \Delta\theta^2(z) \qquad (4.60)$$

At a distance ℓ_w from the plasma boundary the oscillation energy density is comparable with the beam energy density. The wave energy flux, which is equal to $v_{gz} U_w$, is still substantially smaller than the beam energy flux, i.e. the relative loss of energy by the electrons $\Delta\mathscr{E}/\mathscr{E}$ is small:

$$\frac{\Delta\mathscr{E}}{\mathscr{E}} \sim \left(\frac{\omega_H}{\omega_p}\right)^2 \frac{1}{\gamma_0} \ll 1$$

(we have assumed that at $\Delta\theta \sim 1$ the beam excites oscillations with a wave vector $k \sim \omega_H'/c$).

As can be seen from (4.59), the relaxation length diminishes as the magnetic field decreases. The limitation on the field magnitude from below is due to the fact that as $H_z \to 0$ the phase velocity of the whistlers becomes the order of v_{ti}, and we have to take their absorbtion by plasma ions into account. As a result we get the following condition for the applicability of (4.59):

$$\omega_H > \omega_p(\gamma_0^2 T/Mc^2)^{1/4}$$

By comparing (4.59) and (4.41) we get the condition at which the excitation of the whistlers substantially influences the relaxation process:

$$\frac{\omega_H}{\omega_p} < \frac{\gamma_0^3}{20\Lambda} \frac{m}{M} \left(\frac{mc^2}{T}\right)^2$$

We should point out that these results all relate to the case of a beam infinite in the transverse direction, i.e. it is understood that the inequality $r_b \gg \ell_w$ is satisfied in which r_b is the beam radius. In the opposite case ($r_b \ll \ell_w$) the waves propagating in a radial direction leave the region of interaction with the beam without having had time to be amplified. But the limitation $r_b \gg \ell_w$ may prove unimportant if, on account of the radial non-uniformity of the plasma density and magnetic field, the plasma column constitutes a waveguide. In the case of whistlers this state of affairs is quite feasible. The qualitative picture of the relaxation in this case remains the same as in the case of an infinite beam.

Obviously, the whistlers may be excited by the beam not only in the case $\beta > 1$ which we have discussed here, but also when $\beta < 1$. It is not yet clear, however, whether the whistler mechanism of relaxation can be the predominant one when $\beta \ll 1$.

To answer this question we need to study other relaxation mechanisms more thoroughly than has been done so far (primarily relaxation of the beam through interaction with Langmuir waves).

So far we have been discussing effects occurring in the region of weak magnetic fields, i.e. $\omega_H < \omega_p$. Naturally, when moving on to strong magnetic fields ($\omega_H \gg \omega_p$), we find that the relaxation process is completely different. More particularly, there is a considerable change in the dispersion relation, the expressions for the growth rates, and so on. Formulation of a theory for this region of parameters is in some measure facilitated by the fact that in a strong longitudinal field the motion of the beam electrons and plasma can be considered one-dimensional.

An analytical solution of the problem has not been found. We do, of course, have numerical calculations (see Toepfer and Poukey [92,100], Thode and Sudan [93], Sudan [94]), but they are based on the assumption of the one-dimensionality of both the particle motion and the oscillation spectrum. Such models are undoubtedly of use,

[32] As shown by Brejzman and Feizov [99], non-linear processes for whistlers are unimportant in the case under consideration.

though it must be borne in mind that in a real situation, even when $\omega_H \to \infty$, the oscillation spectrum for non-linear relaxation can obviously not be considered one-dimensional[33]. Indeed, as can be seen from the dispersion relation for electron oscillations in an infinitely strong longitudinal magnetic field, namely

$$\omega^4 - \omega^2(\omega_p^2 + k^2c^2) + \omega_p^2 k_z^2 c^2 = 0$$

their spectrum is of the decay type and the laws of conservation permit decay of a wave spreading strictly in the direction of the field into two other waves (with finite values of $k_\perp$). As the estimates show, the decay instability may develop within times comparable to the inverse growth rate of the beam instability, and hence it must definitely be taken into account in the problem under consideration.

4.8. Macroscopic effects in the relaxation of REBs in a plasma

In studying the relaxation of an electron beam we assumed that the parameters of the plasma (density profile and temperature) were fixed. If the beam is used to heat the plasma we can limit ourselves to this approximation only for the rather small interval of time during which the plasma parameters cannot be altered to any great extent by the beam. To describe the whole heating process we need to solve the self-consistent problem of beam relaxation and motion of the beam-heated plasma.

The most important effect in this connection is the occurrence of inhomogeneities in the plasma density. If the density gradient exceeds the critical value

$$|\nabla n| > |\nabla n|_{crit} \equiv \frac{n_b \omega_p}{\gamma c}$$

(see Section 4.3), the relaxation is terminated. This effect is most pronounced for the condition:

$$|\nabla n|_{crit} < \frac{n}{\ell} \qquad (4.61)$$

where ℓ is the relaxation length. The condition (4.61) is a completely realistic one.

As an illustration of the problems that arise, let us consider the beam heating of an initially uniform plasma filling the half-space $z > 0$, and assume furthermore that condition (4.61) is satisfied. Immediately after the beam is switched on, its energy is released in a layer of thickness $z \sim \ell$. As the plasma in the layer gradually heats up, it expands at an ever greater rate and becomes more and more inhomogeneous. When the characteristic density gradient reaches the critical value for suppression of the instability, the energy begins to be released in the next layer of thickness ℓ, where the plasma, in its turn, becomes inhomogeneous, and so on. Of course, in actual fact the relaxation region shifts into the plasma continuously and not in spurts (this can be called a relaxation wave).

[33] With the exception of the initial stage of beam relaxation (Matsiborko et al. [101]).

Depending on the ratio of the mean free path of the plasma electrons to ℓ, we use one or the other set of gas-dynamic equations for solving the problem. For the sake of clarity we shall consider a situation in which the mean free path is small and thermal conductivity can be disregarded. We can then say that over a small interval of time (small as compared with $\ell(T/M)^{-\frac{1}{2}}$) the plasma density in the region in which the beam energy is released diminishes by $\Delta n \sim (\Delta t^2 T/M\ell^2)n$, where T is the temperature to which the plasma is heated by the beam over the time Δt: $T \sim \mathscr{E}(n_b/n)(c\Delta t/\ell)$. The longitudinal density gradient attains a critical value over the time $\Delta t_0 \sim \ell(\epsilon M/T)^{\frac{1}{2}}$, where the small ratio $|\nabla n|_{crit}/n\ell$ is denoted as ϵ.

After this time interval the relaxation region shifts in the direction $z > 0$ by a distance of the order of ℓ so that the velocity of the relaxation wave may be expressed as

$$v \sim \frac{\ell}{\Delta t_0} \sim \sqrt{\frac{T}{M\epsilon}} \gg \sqrt{\frac{T}{M}} \qquad (4.62)$$

where T is now taken to mean the temperature which the plasma possesses behind the relaxation wave front:

$$T \sim \frac{n_b}{n}\left(\frac{c\Delta t_0}{\ell}\right)\mathscr{E}$$

It can be seen from (4.62) that the relaxation wave is propagated at a velocity greatly exceeding that of sound in a beam-heated plasma.

The results of more accurate computer calculations described by Brejzman et al. [86] are shown in Fig. 18, which gives density profiles at successive moments of time. The time is measured in units of Δt_0, and the distance in units of ℓ.

5. CONCLUSIONS

The experiments conducted so far [1-9] demonstrate that the collective dissipation of REB energy in a plasma is a fact. For example, in the experiments described by Abrashitov et al. [3] and by Kapetanakos and Hammer [7] it was established that the beam transfers to the plasma as much as 20% - 25% of its initial energy under conditions in which the binary collisions were known to be insignificant. The heating of the plasma may have been due either to direct dissipation of the beam energy (beam instability) or to Joule dissipation of the return current, due to anomalous resistance. In some of the experiments it proved possible to create conditions under which only one of these mechanisms was active and each of them could be observed separately. For example, in the study of

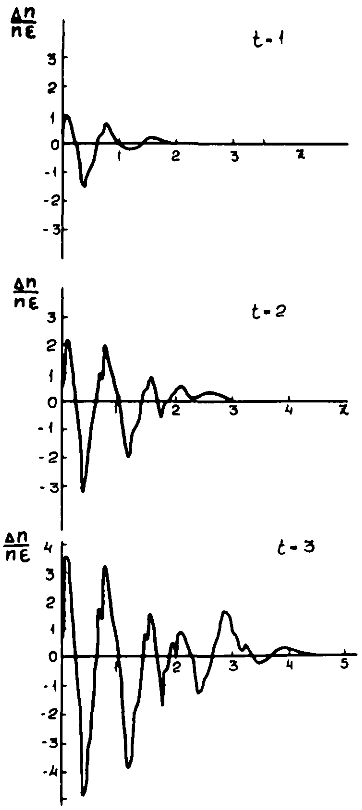

FIG. 18. Propagation of relaxation wave (result of numerical integration). Plasma density perturbation profiles are shown for three successive moments of time.

Korn et al. [9] the heating can be ascribed with a high degree of certainty to the return current. In the investigations of Altyntsev et al. [1] (at a plasma density $n < 10^{12}$ cm^{-3}) and those of Kojdan et al. [2] (at $n > 10^{14}$ cm^{-3}) it can be attributed to a beam instability.

Literal application of the results contained in Section 4 to explain experiments of the type reported in Refs [1-9] is impossible because the plasma was in some cases markedly non-isothermal ($T_e \gg T_i$), and in a plasma of that kind, as we know, other types of oscillations (as compared with those considered in Section 4) occur and, correspondingly, other non-linear processes. Their role is illustrated most strikingly in cases in which a current flows through the plasma and creates ion-acoustic turbulence with a characteristic spatial scale of the order of a Debye length (for example, this is the situation when there is anomalous resistance). Under such conditions the ℓ-s scattering of Langmuir oscillations by electrons and the two-quantum absorption of ℓ and s plasmons by electrons (the symbol s relates to sound waves) proceed very effectively. Outwardly these processes appear to be collisional damping of the Langmuir oscillations with a decrement of the order of $\omega_p(U_s/nT_e)$. In the case of fairly high values of U_s the beam instability may be completely suppressed.

Direct comparison of the results in Section 4 with data reported in Refs [1-9] is impossible also because the density ratio n_b/n used in those studies is not small enough. At the limit of applicability the theoretical results can be used to estimate the relaxation length only for the smallest available experimental values of n_b/n ($n_b/n \sim 10^{-3} - 5\times10^{-4}$). They do not contradict the experimental data in this region.

If one has in mind the use of REBs in controlled nuclear fusion research, one should apparently consider using plasmas with a higher density than is usual in Refs [1-9] (e.g. $n \gtrsim 10^{16}$ cm^{-3}). In this case the region of beam and plasma parameters of interest in experiments and the limits over which the theory is applicable overlap to a large extent. As an illustration we shall give a numerical example: $n = 10^{17}$ cm^{-3}; $T = 5$ keV; $n_b = 10^{12}$ cm^{-3}; $\gamma = 2$; $8\pi nT/H^2 \sim 1$. For these parameters we satisfy the condition for the applicability of weak turbulence theory (see (4.50)), hence the relaxation length may be calculated from (4.41). This calculation gives $\ell \sim 400$ cm (for a hydrogen plasma), which is quite acceptable. Generally speaking, at $T \sim 3\times10^3 - 10^4$ eV the relaxation lengths are small even for very moderate beam parameters.

However, since at lower plasma temperatures the relaxation length is considerably increased, the initial heating stage should be the result not of a beam instability but some other mechanism, for example Joule dissipation of the return current.

Since the heating of a plasma with density $n > 10^{16}$ cm^{-3} appears to require the use of beams with highly supercritical currents, for purposes of transport we shall need neutralization of the beam charge. Current neutralization does not seem to be essential, at least in a fairly strong magnetic field. The main difficulty involved in transport under these conditions is the beam instability. However, the most hazardous large-scale instabilities are suppressed by a strong magnetic field, and to counteract small-scale instabilities we can make the plasma highly inhomogeneous by artificial means.

On the whole, the prospects for using REBs for heating plasma can be considered encouraging; in principle it appears possible to solve the two main problems, namely, transport of the beam and rapid energy deposition in the bulk of the plasma.

APPENDIX I

In an infinitely strong longitudinal magnetic field we can disregard the displacement of the particles across the lines of force. Accordingly, only the z component of the current:

$$\delta j_z = -en_b\,\delta v - ev\,\delta n_b$$

will be perturbed in the wave. On the basis of the continuity equation:

$$-i\omega\,\delta n_b + ik_z v\,\delta n_b + ik_z n_b\,\delta v = 0$$

and the equation of motion:

$$-i\omega\,\delta v + ik_z v\,\delta v = -\frac{eE_z}{m\gamma^3}$$

we get an Ohm's law in the form:

$$\delta j_z = \frac{ie^2 n_b \omega E_z}{m\gamma^3(\omega - k_z v)^2} \qquad (A.1.1)$$

For this Ohm's law the Maxwell equations split into equations for the TM wave, in which the φ component of the magnetic field and the r and z components of the electric field are perturbed, and into equations for the TE wave, in which the r and z components of the magnetic field and the φ component of the electric field are perturbed (we are dealing here with axisymmetric perturbations). The presence of the beam has no effect whatsoever on the TE wave, and we shall therefore consider only the TM wave:

$$ik_z E_r - \frac{\partial E_z}{\partial r} = \frac{i\omega}{c} H_\varphi$$

$$k_z H_\varphi = \frac{\omega}{c} E_r \qquad (A.1.2)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\varphi) = -\frac{i\omega}{c} E_z + \frac{4\pi}{c}\delta j_z$$

The set of equations (A.1.1), (A.1.2) can be reduced to a single equation for E_z:

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial E_z}{\partial r} - \kappa^2\left[1 - \frac{\omega_b^2(r)}{\gamma^3(\omega - k_z v)^2}\right]E_z = 0 \qquad (A.1.3)$$

where

$$\kappa^2 \equiv k_z^2 - \frac{\omega^2}{c^2}$$

and

$$\omega_b^2(r) \equiv 4\pi e^2 n_b(r)/m$$

For an infinitely thin annular beam

$$n_b(r) = \frac{|I_b|}{2\pi e r_b v}\,\delta(r - r_b)$$

In this case by integrating Eq.(A.1.3) with respect to r from $r_b - 0$ to $r_b + 0$ we can easily see that the following relationships hold:

$$E_z\big|_{r_b+0} = E_z\big|_{r_b-0}$$

$$E_z'\big|_{r_b+0} - E_z'\big|_{r_b-0} + \frac{\kappa^2}{(\omega - k_z v)^2}\frac{2e|I_b|}{mvr_b\gamma^3}E_z\big|_{r_b-0} = 0 \qquad (A.1.4)$$

In the region $r < r_b$

$$E_z = AI_0(\kappa r) \qquad (A.1.5)$$

(the solution $K_0(\kappa r)$ is discarded from the boundary condition for the field at the point $r = 0$). In the region $r > r_b$

$$E_z = BI_0(\kappa r) + CK_0(\kappa r) \qquad (A.1.6)$$

Assuming that the conductivity σ of the wall is such that the skin depth is small compared with the tube radius, we can apply the Leontovich boundary condition at the point $r = R$ (see Landau and Lifshits [42] p.355):

$$E_z = (i-1)\sqrt{\frac{\omega}{8\pi\sigma}}\,H_\varphi = -\frac{\omega c}{\omega^2 - k_z^2 c^2}(i+1)\sqrt{\frac{\omega}{8\pi\sigma}}\,\frac{\partial E_z}{\partial r} \qquad (A.1.7)$$

Substituting the solution (A.1.5) and (A.1.6) in the boundary conditions (A.1.4) and (A.1.7), we get a system of equations for the constants A, B and C, and from their condition of solvability arrive at the dispersion relation:

$$\frac{I_0(\kappa R) + \epsilon I_0'(\kappa R)}{I_0(\kappa r_b)} = \xi\,\frac{k_z^2 - \frac{\omega^2}{c^2}}{\left(k_z - \frac{\omega}{v}\right)^2}\Big\{[I_0(\kappa R)K_0(\kappa r_b) - I_0(\kappa r_b)K_0(\kappa R)] + \epsilon[K_0(\kappa r_b)I_0'(\kappa R) - I_0(\kappa r_b)K_0'(\kappa R)]\Big\}$$

$$\epsilon = -(i+1)\frac{\omega}{\kappa c}\sqrt{\frac{\omega}{8\pi\sigma}}, \qquad \xi = \frac{2e|I_b|}{mv^3\gamma^3}$$

At the limit $\sigma \to \infty$ it changes to (2.9), and at $\kappa R \ll 1$ to the following equation:

$$1 = \xi\,\frac{k_z^2 - \frac{\omega^2}{c^2}}{\left(k_z - \frac{\omega}{v}\right)^2}\left(\ln\frac{R}{r_b} + \frac{\epsilon}{\kappa R}\right) \qquad (A.1.8)$$

If we assume here that $\sigma = \infty$ (i.e. $\epsilon = 0$), then we arrive at the dispersion relation (2.10).

To study the resistive wall instability one only needs to consider the limit $\kappa R < 1$, since at $\kappa R \gg 1$ the oscillation field diminishes exponentially between the beam boundary and the wall (we assume that $R - r_b \sim R$), and the part played by dissipation in the wall becomes negligibly small. We shall therefore make use of an approximate dispersion relation (A.1.8). Assuming the parameter ε to be small, we can easily derive an equation for the growth rate using the perturbation theory:

$$\mathrm{Im}\,\omega = \xi\frac{v^2}{2Rc}\,\frac{\omega}{k_z v - \omega\left(1 + \frac{v^2}{c^2}\xi\ln\frac{R}{r_b}\right)}\sqrt{\frac{\omega}{8\pi\sigma}}$$

where ω is determined from the "unperturbed" dispersion relation (2.10). The latter has two solutions, a fast wave and a slow wave:

$$\omega = \frac{k_z v}{1 + \frac{v^2}{c^2}\xi\ln\frac{R}{r_b}} \times\left(1 \pm \sqrt{\left(1 - \frac{v^2}{c^2}\right)\xi\ln\frac{R}{r_b} + \frac{v^2}{c^2}\xi^2\ln^2\frac{R}{r_b}}\right)$$

As is readily verifiable, it is the slow wave solution that is unstable. Direct verification shows that the slow wave possesses negative energy (in the laboratory frame of reference).

We shall not give an unwieldy general expression for the growth rate, but merely one for the special case $\gamma \gg 1$, $I_b/I_{cr} \ll 1$. Here $\xi \ln R/r_b = I_b/(\gamma^2 I_{cr})$ and

$$\mathrm{Im}\,\omega = \frac{c}{2R \ln \frac{R}{r_b}} \left(\frac{ck_z}{8\pi\sigma}\right)^{\frac{1}{2}} \left(\frac{I_b}{I_{cr}}\right)^{\frac{1}{2}}$$

The maximum growth rate is reached at the limit of applicability of the theory, where $\kappa R \sim 1$. It can easily be seen that this corresponds to $k_z R \sim \gamma$, i.e.

$$\mathrm{Im}\,\omega \simeq \frac{c}{2R \ln \frac{R}{r_b}} \left(\frac{c}{8\pi\,\sigma R}\right)^{\frac{1}{2}} \left(\frac{\gamma I_b}{I_{cr}}\right)^{\frac{1}{2}}$$

APPENDIX II

Let us represent the functions E_r, E_z, H_φ and j_b in the form of the following Fourier series for z:

$$E_r = \sum_{\ell=1}^{\infty} E_r^{(\ell)} \sin \frac{\ell\pi}{L} z$$

$$\left.\begin{matrix} E_z \\ H_\varphi \\ j_b \end{matrix}\right\} = \sum_{\ell=0}^{\infty} \left\{\begin{matrix} E_z^{(\ell)} \\ H_\varphi^{(\ell)} \\ j_b^{(\ell)} \end{matrix}\right\} \cos \frac{\ell\pi}{L} z$$

We get the following expressions for the Fourier coefficients:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\ell^2\pi^2}{L^2}\right) E_r^{(\ell)} = \frac{\ell\pi}{L}\frac{\partial E_z^{(\ell)}}{\partial r}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\ell^2\pi^2}{L^2}\right) H_\varphi^{(\ell)} = \frac{1}{c}\frac{\partial^2 E_z^{(\ell)}}{\partial r\,\partial t} \qquad (A.2.1)$$

$$\frac{1}{r}\frac{\partial}{\partial r} r H_\varphi^{(\ell)} = \frac{1}{c}\frac{\partial}{\partial t} E_z^{(\ell)} + \frac{4\pi}{c}(\hat{\sigma} E_z^{(\ell)} + j_b^{(\ell)})$$

In the quasi-steady state $\frac{\partial}{\partial t} \sim \frac{1}{\tau} \ll \frac{c}{L}$. For this reason there is a major difference between the equations with $\ell = 0$ and $\ell \neq 0$. At $\ell = 0$ we get

$$E_r^{(0)} = 0$$

$$\frac{\partial E_z^{(0)}}{\partial r} = \frac{1}{c}\frac{\partial H_\varphi^{(0)}}{\partial t}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r H_\varphi^{(0)} = \frac{4\pi}{c}(\hat{\sigma} E_z^{(0)} + j_b^{(0)})$$

(the displacement current can be disregarded since we know that $1/\tau \ll c/r$). Thus, for $E_z^{(0)}$ and $H_\varphi^{(0)}$, (3.3) and (3.4) are the very equations we get, i.e. E_s should be identified with $E_z^{(0)}$.

At $\ell \neq 0$ we can disregard the terms $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ in Eq. (A.2.1), as compared with $\ell^2\pi^2/L^2$. As a result we find that

$$\frac{\ell\pi}{L} E_r^{(\ell)} = \frac{\partial E_z^{(\ell)}}{\partial r}, \qquad \ell \neq 0$$

Hence, by introducing the notations $E_{pz} \equiv E_z - E_z^{(0)}$, $E_{pr} \equiv E_r$, we can write $\partial E_{pz}/\partial r = \partial E_{pr}/\partial z$. Consequently, there exists a function which is such that the relationships

$$E_{pz} = -\frac{\partial\phi}{\partial z}; \qquad E_{pr} = -\frac{\partial\phi}{\partial r}$$

are satisfied. Further, using the exact equation $\mathrm{div}\,\vec{E} = 4\pi(\rho + \rho_b)$, we arrive at (3.5).

APPENDIX III

From Eqs (3.3) and (3.4), in which $\hat{\sigma}$ is replaced by $\sigma(r,t)$, we can easily arrive at the following equation for the function $I^*(r,t)$:

$$\frac{\partial I^*}{\partial t} = \frac{c^2}{4\pi} r \frac{\partial}{\partial r}\frac{1}{r\sigma}\frac{\partial I^*}{\partial r} - \frac{\partial I^*_b}{\partial t} \qquad (A.3.1)$$

The initial and boundary conditions for (A.3.1) take the form

$$I^*\big|_{t\to-\infty} = 0 \qquad (A.3.2)$$

$$\left.\frac{1}{\sigma}\frac{\partial I^*}{\partial r}\right|_{r=R} = 0$$

$$I^*\big|_{r\to 0} = 0(r^2) \qquad (A.3.3)$$

Let us use $i(r,t',t)$ to denote the solution of (A.3.1) for the special form of the function $g(t)$, i.e. $g(t) = \theta(t - t')$, where $\theta = 0$ for negative values of the argument and $\theta = 1$ for positive values. Clearly, $i(r, t', t)$ is the solution of the equation:

$$\frac{\partial i}{\partial t} = \frac{c^2}{4\pi} r \frac{\partial}{\partial r}\frac{1}{r\sigma}\frac{\partial i}{\partial r} \qquad (A.3.4)$$

for the boundary conditions (A.3.3) and the initial condition

$$i\big|_{t=t'} = -f(r)$$

In accordance with the principle of the maximum for an equation of the parabolic type (see e.g. Smirnov [102]) at $t > t'$:

$$-\max f(r) < i(r, t', t) < 0 \qquad (A.3.5)$$

The solution of (A.3.1) for the arbitrary form of the function g(t) is expressed in terms of i(r, t', t) as

$$I^*(r,t) = \int_{-\infty}^{t} \frac{dg}{dt'}\, i(r,t',t)\, dt'$$

Bearing in mind that g(t) has one maximum, and that i(r, t', t) satisfies inequalities (A.3.5), it can be stated that

$$|I(r,t)| \le \max_r f(r) \max_t g(t) = \max I_b(t)$$

We then apply Eq. (3.9), from which it follows that

$$Q \le \frac{2}{c^2} \int_{-\infty}^{+\infty} dt \int_0^R \frac{dr}{r} |I^*(r,t)| \left| \frac{\partial I_b^*(r,t)}{\partial t} \right|$$

$$\le \frac{2}{c^2} \max I_b(t) \int_0^R \frac{dr}{r} \int_{-\infty}^{+\infty} \left| \frac{\partial I_b^*(r,t)}{\partial t} \right| dt$$

$$= \frac{4}{c^2} \max I_b(t) \int_0^R \frac{dr}{r} \max_t I_b^*(r,t)$$

$$= 2\eta \max W_M$$

where

$$\eta = 2\left[\int_0^R \frac{dr}{r} f^2(r)\right]^{-1} \left[f_{max} \int_0^R \frac{dr}{r} f(r)\right]$$

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NOTE ADDED IN PROOF

Some papers connected with the subject of this review have been published during the last few months.

The beam propagation in a vacuum was considered in papers by Grishin [103] and Friedman [104]. In Ref.[103], the possibility is pointed out of increasing the vacuum critical current by means of division of the waveguide into many narrow channels with conducting walls (honeycomb structure). In Ref.[104] the influence of local broadening of the waveguide diameter on the beam parameters is investigated experimentally.

More detailed theoretical treatment of some problems of current neutralization is presented in works by Chu and Rostoker [105], Rosinskij et al. [106] and Küppers et al. [107]. In the experimental work by Wachtel and Safran [108] a surprising phenomenon is discovered: when the beam is injected into the neutral gas which is ionized by the beam itself, then at some conditions the maximum value of the net current is twice as large as the beam maximum current.

Lee and Lampe [109] have simulated numerically the macroscopic beam-plasma instability in a weak magnetic field.

The paper by Chu and Rostoker [110] considers the possibility of plasma heating by the return current excited in a plasma by the rotating beam. It is well known that rotating beams can be obtained by injecting a non-rotating beam into the cusped magnetic field. The efficiency of this method was studied experimentally by Kapetanakos [111], who has shown that it is possible to transmit into a rotating beam up to 85% of the initial beam energy. More detailed discussion of the possibilities of regulating beam parameters by means of the cusped magnetic field is presented in the work by Levin et al. [112].

Kingsep et al.[113] and Degtyaryev et al. [114] have continued the one-dimensional computer simulation of non-linear phenomena at a high level of Langmuir turbulence. Rudakov [115] has noted that the possibility exists that in a low-density plasma in a strong turbulence regime all the beam energy will be transmitted only to a small group of high-energy electrons, while the bulk of plasma electrons and ions will remain cold.

Experiments on beam-plasma interaction in open-ended systems were continued [116,117].

To conclude, let us mention the appearance of first experiments on the heating of pellets by focussed E-beams [118].

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