State space control of frequency standards

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Erratum

Erratum: State space control of frequency standards (2016 *Metrologia* 53 R60)

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During the publication process a misprint appeared in the line following equation (19). That sentence should read:

The LQG control gain is

$$u = -\widehat{\mathbf{G}}\widehat{\mathbf{X}}, \tag{19}$$

where $\widehat{\mathbf{G}} = \left(\mathbf{B}^T\widehat{\mathbf{D}}\mathbf{B} + \mathrm{W}_R\right)^{-1}\mathbf{B}^T\widehat{\mathbf{D}}\boldsymbol{\Phi}$, and $\widehat{\mathbf{D}}$ is a solution to the steady state Ricatti equation

$$\widehat{\mathbf{D}} = \boldsymbol{\Phi}^T\widehat{\mathbf{D}}\boldsymbol{\Phi} + \mathbf{W}_Q - \boldsymbol{\Phi}^T\widehat{\mathbf{D}}\mathbf{B}\left(\mathbf{B}^T\widehat{\mathbf{D}}\mathbf{B} + \mathrm{W}_R\right)^{-1}\mathbf{B}^T\widehat{\mathbf{D}}\boldsymbol{\Phi}.$$

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Review

State space control of frequency standards

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Abstract
Discrete time control systems are widely used in time and frequency applications. Control systems can be designed to reduce phase and frequency offsets to a given reference either more or less aggressively depending on overall system needs. There are many methods available to aid in designing control systems, including techniques based on state space system models. Several state space control design methods will be shown. The control concepts covered in this paper are pole placement, minimum control effort, and linear quadratic Gaussian. The effects of steering limits on system performance will also be discussed.

Keywords: control, state space, frequency standards

(Some figures may appear in colour only in the online journal)

Introduction

Control systems align some output or parameter to a given reference. Synchronizing two signals aligns them in time and two signals are said to be syntonized if they have zero frequency offset. Steering goals can also include offsets to compensate for known biases, signal delays, and calibrations.

The control of oscillators and signals is a common practice in time and frequency systems [1–4]. Physical realizations of timescales are created by steering signals derived from frequency standards toward a reference paper clock (see figure 1). Timing laboratories around the world steer local timescales in the long-term toward coordinated universal time (UTC) [5, 6]. The frequency standards themselves have many internal control systems that lock voltage controlled oscillators to atomic based signals, stabilize temperatures, and compensate for drift. Environmentally regulated rooms that house precise timing equipment also utilize sophisticated control system designs. This paper will concentrate on several discrete time control system design techniques. Designs in this paper will use a two-state system model. It is straightforward to then apply these techniques to systems with three or more states [3, 4]. The terms steer and control will be used interchangeably throughout the paper.

Pole placement

Using pole placement methods, a control can be designed based on the desired transient response of a system [7, 8]. The individual frequency steers for a proportional control are calculated by multiplying the control gain, $\mathbf{G}$, by the current time and frequency offset values. The general form of a two-state control system for timing applications with a time interval of τ between steers is

$$\mathbf{X}_{t+\tau} = \mathbf{\Phi}\mathbf{X}_t + \mathbf{B}u_t, \tag{1}$$

where $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \text{time offset} \\ \text{frequency offset} \end{bmatrix}$, the state transition matrix $\mathbf{\Phi} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \tau \\ 1 \end{bmatrix}$, assumes frequency steps are used to implement the control, and the proportional control amount $u = -\mathbf{G}\mathbf{X} = -\begin{bmatrix} g_x & g_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Placing these values into (1) gives

$$\begin{aligned}\begin{bmatrix}x\\y\end{bmatrix}_{t+\tau} &= \boldsymbol{\Phi}\begin{bmatrix}x\\y\end{bmatrix}_t - \mathbf{BG}\begin{bmatrix}x\\y\end{bmatrix}_t\\ &= (\boldsymbol{\Phi}-\mathbf{BG})\begin{bmatrix}x\\y\end{bmatrix}_t\\ &= \begin{bmatrix}1 & \tau\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}_t - \begin{bmatrix}\tau\\1\end{bmatrix}[g_x \;\; g_y]\begin{bmatrix}x\\y\end{bmatrix}_t\\ &= \begin{bmatrix}1-g_x\tau & (1-g_y)\tau\\ -g_x & 1-g_y\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}_t\\ &= \mathbf{A}\begin{bmatrix}x\\y\end{bmatrix}_t.\end{aligned} \tag{2}$$

The eigenvalues of **A** correspond to the system poles, and are related to how the control system responds to offsets. The control systems can be classified as over-damped, under-damped, and critically damped. The under-damped system has two complex conjugate poles, an over-damped system has two real and unequal poles, and a critically damped system has real and equal poles.

The poles are found by solving for the eigenvalues of **A**. The eigenvalue equation

$$\lambda^2 + (g_y + g_x\tau - 2)\lambda + 1 - g_y = 0 \tag{3}$$

has solutions

$$\lambda_d = \frac{-(g_y + g_x\tau - 2) + (-1)^d\sqrt{(g_y + g_x\tau - 2)^2 - 4(1-g_y)}}{2}, \quad \text{for } d = 1, 2. \tag{4}$$

For example, in critical damping the poles need to be real and equal, therefore

$$\lambda_d = \lambda = \frac{-(g_y + g_x\tau - 2)}{2} \tag{5}$$

and the relationship between the gain components is

$$(g_y + g_x\tau - 2)^2 - 4(1-g_y) = 0. \tag{6}$$

The positive solution, using (5) and (6), noting that the absolute value of the poles need to be less than or equal to one to ensure stability, is

$$\lambda = 1 - \sqrt{g_x\tau} = \sqrt{1-g_y}. \tag{7}$$

Figure 2 gives an example of critically damped control system responses to a frequency step.

Minimum control effort

The goal for this design is to minimize the control effort, or so-called control energy,

$$\frac{1}{2}\sum_{k=0}^{N-1} u^2(k), \tag{8}$$

necessary to drive the initial time offset, $x(0)$, and the frequency offset, $y(0)$, to zero in N steps [7].

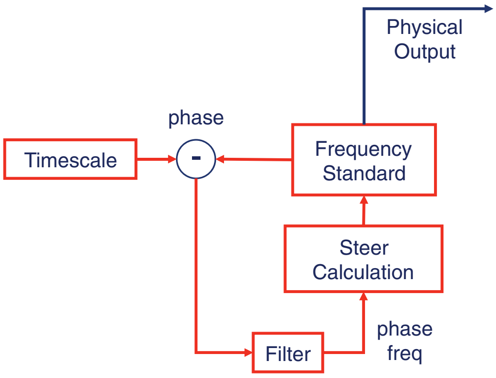

Figure 1. Block diagram of a control system that creates a physical realization to timescale.

The recursive property of the state space model (1) can be used to find the value of the states at any point for $k > 0$.

$$\begin{aligned}\mathbf{X}(1) &= \boldsymbol{\Phi}\mathbf{X}(0) + \mathbf{B}u(0)\\ \mathbf{X}(2) &= \boldsymbol{\Phi}\mathbf{X}(1) + \mathbf{B}u(1)\\ &= \boldsymbol{\Phi}(\boldsymbol{\Phi}\mathbf{X}(0) + \mathbf{B}u(0)) + \mathbf{B}u(1)\\ &= \boldsymbol{\Phi}^2\mathbf{X}(0) + \boldsymbol{\Phi}\mathbf{B}u(0) + \mathbf{B}u(1)\\ &\;\;\vdots\\ \mathbf{X}(k) &= \boldsymbol{\Phi}^k\mathbf{X}(0) + \boldsymbol{\Phi}^{k-1}\mathbf{B}u(0) + \boldsymbol{\Phi}^{k-2}\mathbf{B}u(1) + \cdots + \mathbf{B}u(k-1).\end{aligned} \tag{9}$$

Now set $\mathbf{X}(N) = 0$ in (9) and solve for $\mathbf{X}(N)$,

$$\begin{aligned}\mathbf{X}(0) &= -\boldsymbol{\Phi}^{-1}\mathbf{B}u(0) - \boldsymbol{\Phi}^{-2}\mathbf{B}u(1) - \cdots - \boldsymbol{\Phi}^{-N}\mathbf{B}u(N-1)\\ &= -[\boldsymbol{\Phi}^{-1}\mathbf{B} \,\vdots\, \boldsymbol{\Phi}^{-2}\mathbf{B} \,\vdots \cdots \vdots\, \boldsymbol{\Phi}^{-N}\mathbf{B}]\begin{bmatrix}u(0)\\u(1)\\ \vdots\\ u(N-1)\end{bmatrix}\\ &= -\begin{bmatrix}0 & -\tau & \cdots & -(N-1)\tau\\ 1 & 1 & & 1\end{bmatrix}\begin{bmatrix}u(0)\\u(1)\\ \vdots\\ u(N-1)\end{bmatrix}\end{aligned} \tag{10}$$

$$= -\mathbf{SU}. \tag{11}$$

The minimal control effort solution for **U** is found by applying the right pseudo-inverse [7] to solve for **U** in (11),

$$\mathbf{U} = -\mathbf{S}^{\dagger}(\mathbf{SS}^{\dagger})^{-1}\mathbf{X}(0), \tag{12}$$

where † denotes the conjugate transpose.

Solving (12) gives

$$\mathbf{U} = -\frac{6}{N(N+1)}\begin{bmatrix}\frac{1}{\tau} & \frac{2N-1}{3}\\ \frac{1}{\tau}\left(1 - \frac{2}{N-1}\right) & \frac{2N-1}{3} - 1\\ \vdots & \\ -\frac{1}{\tau} & \frac{2N-1}{3} - (N-1)\end{bmatrix}\mathbf{X}(0). \tag{13}$$

The individual steers are

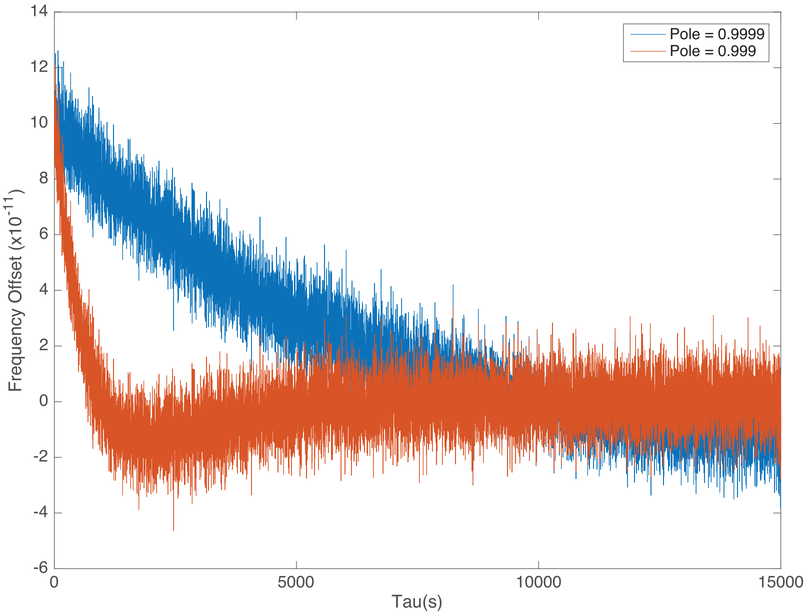

Figure 2. Example of the response to a frequency offset for two critically damped controls with poles at 0.9999 and 0.999.

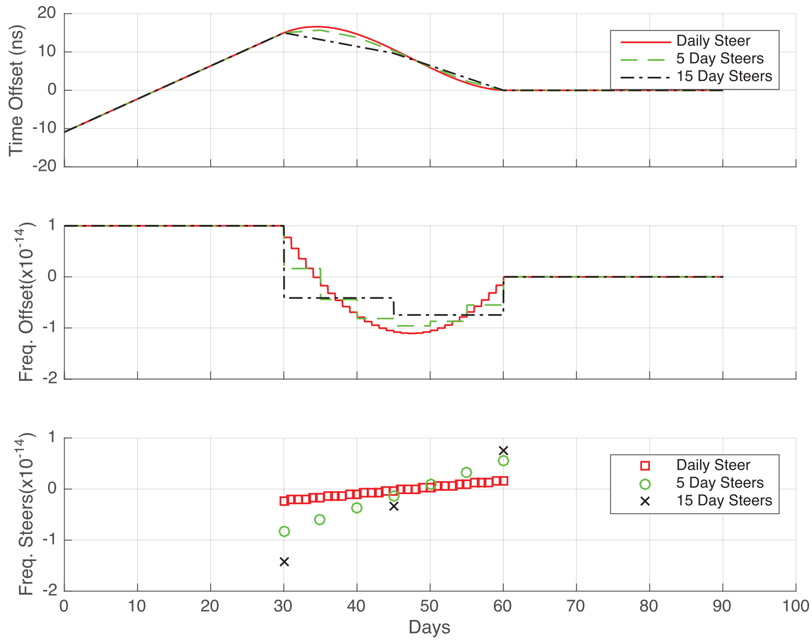

Figure 3. Removal of a time offset of 15 ns and frequency offset of 1×10^{-14} over one month using a minimal control design with steering intervals of one, five, and fifteen days.

$$u(k) = -\frac{6}{N(N+1)}\left[\left(1 - \frac{2k}{N-1}\right)\tau^{-1}x(0) + \left(\frac{2N-1}{3} - k\right)y(0)\right]$$
$$\text{for} \quad k = 0 \dots N-1. \tag{14}$$

Figure 3 gives an example of a minimal control effort design removing time and frequency offsets with different steering intervals.

Linear quadratic Gaussian stochastic control

Linear quadratic Gaussian (LQG) is an optimal control design technique that incorporates Kalman filtering [9] to estimate time and frequency errors that are used as the input into a steer calculation [1–4, 7–11]. It can be shown that the optimal control and optimal estimation problems can be designed independently. This property is known as the separation principle.

Kalman filters estimate the time and frequency offsets from noisy measurements of signals that have inherent stochastic properties. A noise vector $\boldsymbol{w}(k)$, corresponding to the stochastic properties of the state elements, is added to the model in (1) giving a state equation

$$\mathbf{X}(k+1) = \mathbf{\Phi X}(k) + \boldsymbol{w}(k). \tag{15}$$

The noisy measurement is related to the state by

$$z(k) = \mathbf{HX}(k) + v(k), \tag{16}$$

where $z(k)$ is the time difference measurement,

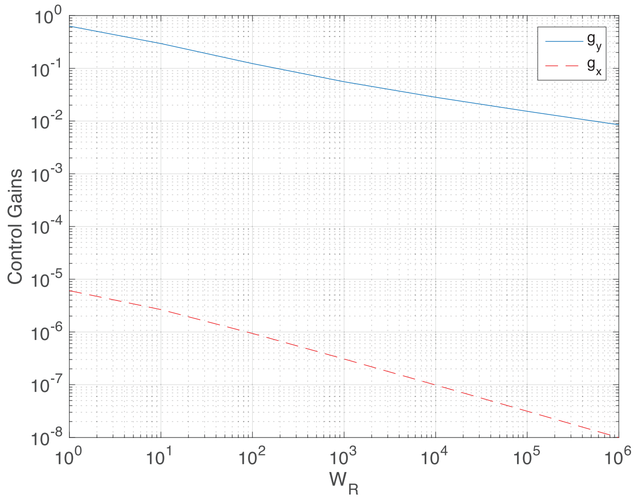

Figure 4. The plot shows how the two-state LQG calculated control gains vary with the control weight penalty W_R while the state penalties are held constant at $\mathbf{W}_Q = \begin{bmatrix} 10^{-10} & 0 \\ 0 & 1 \end{bmatrix}$. The control gains are reduced, as expected, as the penalty for the control amount increases.

$$\mathbf{H} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and $v(k)$ is the measurement noise.

The Kalman filter state estimates are calculated using

$$\widehat{\mathbf{X}}(k+1) = \boldsymbol{\Phi}\widehat{\mathbf{X}}(k) + \mathbf{B}u(k) + \mathbf{K}_g(z(k+1) - \mathbf{H}(\boldsymbol{\Phi}\widehat{\mathbf{X}}(k) + \mathbf{B}u(k))), \quad (17)$$

where $\mathbf{K}_g$ is the Kalman gain and the hat $(\widehat{\ })$ over the parameter denotes an estimated value.

In the LQG design a discrete frequency steer is calculated by minimizing the quadratic cost function,

$$J = \sum [\mathbf{X}_k^T \mathbf{W}_Q \mathbf{X}_k + u_k^T W_R u_k]. \quad (18)$$

$\mathbf{W}_Q$ and W_R are matrices that are chosen by the designer in order to set relative penalties assessed to the state offsets and control effort as they vary. In general, if the magnitude of W_R is increased compared to $\mathbf{W}_Q$, the penalty is increased for the system attempting to drive the state vector toward zero rapidly (see figure 4). If the magnitude of W_R decreases the system faces a smaller penalty for large control effort and the system is driven toward zero more aggressively.

The LQG control gain is

$$u = -\widehat{\mathbf{G}}\mathbf{X}, \quad (19)$$

where $\widehat{\mathbf{G}} = (\mathbf{B}^T\widehat{\mathbf{G}}\,\mathbf{B} + W_R)^{-1}\mathbf{B}^T\widehat{\mathbf{G}}$, and $\widehat{\mathbf{D}}$ is a solution to the steady state Ricatti equation

$$\widehat{\mathbf{D}} = \boldsymbol{\Phi}^T\widehat{\mathbf{D}}\,\boldsymbol{\Phi} + \mathbf{W}_Q - \boldsymbol{\Phi}^T\widehat{\mathbf{D}}\mathbf{B}(\mathbf{B}^T\widehat{\mathbf{D}}\,\mathbf{B} + W_R)^{-1}\mathbf{B}^T\widehat{\mathbf{D}}\boldsymbol{\Phi}$$

This gives a statistically optimal control $u(k)$ for the given cost function with the designer specified parameters $\mathbf{W}_Q$ and W_R.

Control considerations

Limiting the amount of control allowed can aid in the robustness of a control system [12, 13]. The limits can take a physical form, for example, limiting the magnitude of any given steer sent to a frequency synthesizer. Limits can also be utilized to raise alarms when control parameters exceed some defined levels. Trends can also be monitored. An issue with a system could show if many steers in the same direction are detected. Caution should be used to keep control limits as broad as practical. Over limiting a system can excessively clamp stochastic signals or possibly invalidate the assumptions of a linear system. Limits should also be broad enough to handle changes in the reference standard or external factors like environmental disturbances. Simulating control designs is very valuable in ensuring system designs operate as intended. Be sure to create good system models. Check how the system responds to steps, ramps, and outliers in the input.

Common control applications in precise timing can include maintaining a given time offset, frequency offset, or could be utilized to improve the frequency stability of a given device. Caution should be applied when aggressively steering so as to not overly disturb the frequency stability by causing large frequency shifts. Being overly cautious about minimizing control effort can lead to long-term system wander.

Conclusion

There are many techniques to choose from when designing control systems for precise timing applications. The choice of design technique is dependent upon the system along with the desired control response and system performance. This paper has outlined several discrete time control design methods.

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