

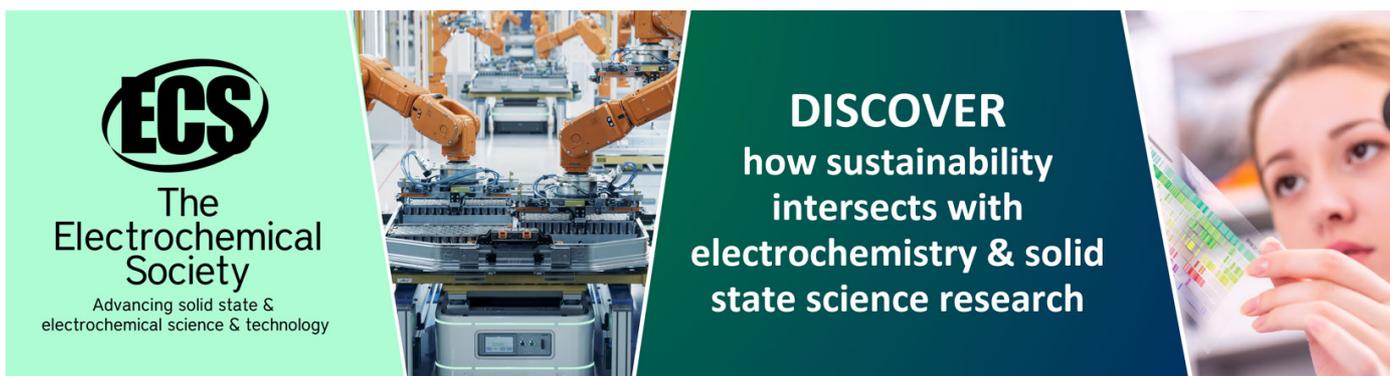
The Bohm criterion and sheath formation

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REVIEW ARTICLE

The Bohm criterion and sheath formation

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Abstract. In the limit of a small Debye length ($\lambda_D \rightarrow 0$) the analysis of the plasma boundary layer leads to a two-scale problem of a collision free sheath and of a quasi-neutral presheath. Bohm's criterion expresses a necessary condition for the formation of a stationary sheath in front of a negative absorbing wall. The basic features of the plasma–sheath transition and their relation to the Bohm criterion are discussed and illustrated from a simple cold-ion fluid model. A rigorous kinetic analysis of the vicinity of the sheath edge allows one to generalize Bohm's criterion accounting not only for arbitrary ion and electron distributions, but also for general boundary conditions at the wall. It is shown that the generalized sheath condition is (apart from special exceptions) marginally fulfilled and related to a sheath edge field singularity. Due to this singularity smooth matching of the presheath and sheath solutions requires an additional transition layer. Previous investigations concerning particular problems of the plasma–sheath transition are reviewed in the light of the general relations.

1. Introduction

The problem of sheath formation at the plasma boundary is of importance for nearly all applications where a plasma is confined to a finite volume. It is one of the oldest problems in plasma physics and yet is still not fully understood. Because of its particular importance in plasma technology and fusion research it remains of undiminished or even growing interest.

In its simplest form the interaction of a plasma with a (more or less) absorbing wall can be characterized as follows: due to the high mobility of the electrons the wall potential will adjust itself to be negative with respect to the surrounding plasma (this holds not only for floating walls but is usually even true for the anode of a gas discharge!). The repulsion of electrons results in the formation of a positive space-charge region ('sheath') shielding the neutral plasma from the negative wall. The typical extension of the sheath is given by the electron Debye length λ_D . Usually the Debye length is small compared with all other characteristic lengths L of the plasma (e.g. ion mean free path) and the sheath is planar and collision free. This usual case, however, runs into a fundamental complication: the substantial distortion of the ion distribution due to wall losses renders shielding impossible unless the 'Bohm criterion' is fulfilled. This condition for sheath formation demands that the ions enter the sheath region

with a high velocity, which cannot be generated by thermal ion motion. (A corresponding condition for an electron sheath in front of a positive wall is easy to fulfil but without practical importance). Consequently, the ions must be accelerated by an electric field penetrating a 'presheath' region. The boundary layer is thus split up into the separate model zones of a collision free sheath (scale λ_D) and of a quasi-neutral presheath (scale L). The complete interchange of physical mechanisms dominating these regions results in a formal singularity at the sheath edge, merging sheath and presheath in the asymptotic limit $\lambda_D/L \rightarrow 0$. The singularity, as well as the strong condition imposed on the presheath acceleration, has always attracted considerable interest in arguments questioning the Bohm criterion.

Basic features of the plasma–sheath transition have been revealed in the early works of Langmuir (1929)—especially in the famous kinetic analysis of the low-pressure column due to Tonks and Langmuir (1929). In these investigations the essence of the Bohm criterion was already used in an implicit form. The explicit formulation and clear interpretation of the sheath condition is due to Bohm (1949). Harrison and Thompson (1959) were able to solve the Tonks–Langmuir problem analytically and to find a kinetic formulation of Bohm's criterion valid under rather general conditions.

The first investigations on the plasma–sheath transition were concerned with collisionless plasmas. Boyd

(1951) introduced the Bohm criterion by an artificial cut-off into the diffusion controlled theory of the collision dominated plasma. Persson (1962) was apparently the first to recognize the universal role of ion inertia in the boundary region later called the 'presheath'. The first self-consistent kinetic analysis of a collisional presheath was given by Riemann (1981).

The term 'presheath' is due to Hu and Ziering (1966) and was originally addressed to the Knudsen layer of a collision dominated plasma. With a different meaning it was used by Franklin (1976) to designate a transition layer required for a smooth matching of plasma and sheath. Gradually the application of the name to the inertia-dominated boundary region—regardless of the plasma collisionality—prevailed. In the case of a bounded collisionless plasma the term 'presheath' (here first used by Emmert *et al* (1980)) refers to the whole plasma. This coincidence may have increased some continuing confusion in nomenclature pointed out by Franklin (1989).

Boyd (1951) investigated a quasi-neutral 'extra sheath', Ecker and McClure (1962, 1965) considered zones of mobility and inertia limited motion. Chodura (1986) distinguished a plasma presheath, a quasi-neutral magnetic sheath and an electrostatic sheath. Main (1987) analysed a presheath governed by Poisson's equation, and Zawaideh *et al* (1990) related the term 'sheath' to the use of Poisson's equation in the numerical analysis. The confusion in the model regions is easily transferred to the region limit, the *sheath edge*, which according to Emmert *et al* (1980) is 'a rather ill defined point'. It is exactly to this point that the Bohm criterion is related—no wonder that this uncertainty was a continuous source of criticism in the strange sheath condition: Hall (1961) pointed to the neglect of a transition region in Bohm's derivation; Ecker and McClure (1962, 1965) questioned the criterion by taking into account finite sheath edge fields; Godyak and co-workers (1982, 1989) related the Bohm criterion to contradictory boundary conditions of the sheath and presheath; Bakhst *et al* (1969) argued that in collisional systems the Bohm criterion could never be satisfied, and that collisions in the sheath had to be considered in any case to obtain a reasonable solution at all; Zawaideh *et al* (1990) claimed to have obtained new reasonable solutions violating the Bohm criterion by accounting for collisions in the sheath.

Other investigators advanced arguments confirming the validity of Bohm's criterion in the equality form (Allen and Thonemann 1954). This more distinct statement was used as a boundary condition both for the sheath and for the presheath. The singularity related to the plasma-sheath transition was considered a reasonable consequence of the scale transition $\lambda_D/L \rightarrow 0$ confirming the marginal validity of the sheath condition (Stangeby and Allen 1970, Riemann 1980). This interpretation again became questionable when Emmert *et al* (1980) presented the analytic model of a plasma sheath transition *without* singularity. Bissel and Johnson (1987) analysed a slightly different model and

obtained again the sheath edge singularity using the equality form of Bohm's criterion as a boundary condition. Was this procedure still conclusive? Scheuer and Emmert (1988b) reinvestigated the same problem *without* imposing the criterion and *verified* the results of Bissel and Johnson closing with the 'interesting open question, *why* the Bohm criterion is satisfied in the equality form'.

In this review I shall try to answer the question 'why'—and 'why not' under slightly modified conditions. I shall further attempt to set aside the confusion arising from the definition of the model zones, to discuss the objections to Bohm's criterion and to clarify its accurate meaning. A kinetic analysis of the sheath edge region will allow definite conclusions to be drawn on the plasma-sheath transition and to generalize the sheath condition. The review is concerned only with theoretical investigations of the asymptotic case $\lambda_D/L \rightarrow 0$ governed by the Bohm criterion. It is further restricted to the formation of stationary wall sheaths. For the discussion of related problems in the theory of double layers see reviews by Allen (1985), Raadu and Rasmussen (1988) and Raadu (1989).

To illustrate the basic concept and the main objections, the entire topic of the Bohm criterion and sheath formation will be discussed in section 2 in a simplified form. Starting from a cold-ion fluid model we shall rederive the original Bohm criterion (section 2.1), introduce the two-scale formalism to describe sheath and presheath (section 2.2) and investigate the presheath mechanism (section 2.3). Section 2.4 deals with the plasma-sheath transition for small but finite λ_D/L and section 2.5 examines arguments questioning Bohm's criterion. Finally section 2.6 points to problems not covered in the scope of this review.

A rigorous kinetic theory of the plasma-sheath transition resulting in general sheath conditions will be developed in section 3. This analysis is not restricted to absorbing walls, but accounts for ion reflection and emission. Furthermore, it yields new results on the type of the sheath edge singularity. Section 4 discusses the sheath criterion in more detail and presents special forms adapted to specified conditions. The problem of matching the plasma and sheath solutions, and the definition of appropriate model zones, are considered in section 5. Section 6 reviews special problems and their relation to the points discussed in the preceding sections. The most important results and statements on the Bohm criterion are finally summarized in section 7.

2. The mechanism of sheath formation: elementary theory

In this section the basic properties of the plasma-sheath transition will be discussed and fundamental concepts will be introduced. To this end a simplified model exhibiting the basic mechanisms but avoiding mathematical exertion is studied. More rigorous

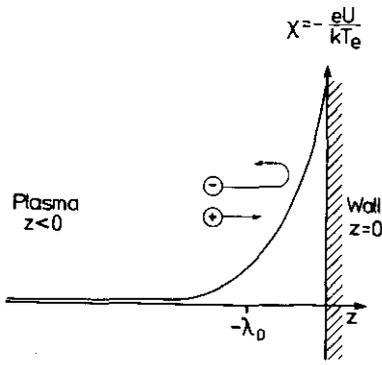


Figure 1. Shielding positive sheath in front of a negative wall (schematically).

models of the boundary layer will be considered in following sections. In particular, the assumptions made here are monoenergetic cold ions, Boltzmann distributed electrons, and completely absorbing walls. In addition our general presupposition $\varepsilon = \lambda_D/L \rightarrow 0$ is recalled, where L is any relevant characteristic length of the plasma boundary.

2.1. The space-charge region (sheath)

The case considered is a neutral non-magnetized plasma with one kind of singly-charged ions in contact with an absorbing wall. Due to the high mobility of the electrons the wall is usually negatively charged, so that most of the electrons are repelled (see figure 1). The thermal electrons are therefore hardly disturbed by wall losses and can be assumed to be in Boltzmann equilibrium. The corresponding decrease of the electron density is presumed to form a positive space-charge shielding the potential distortion in a typical distance of some electron Debye lengths λ_D in front of the wall. Since λ_D is the smallest characteristic length, the space-charge region can be regarded as a thin, collision-free planar 'sheath'.

Considering the problem more carefully, the presumed shielding becomes somewhat questionable, because the ions are strongly distorted by wall losses. To investigate this quantitatively the following dimensionless quantities are introduced

$$\begin{aligned} y &= \frac{m_i v_z^2}{2kT_e} & \chi &= -\frac{eU}{kT_e} \\ n_{e,i} &= \frac{N_{e,i}}{N_0} & \xi &= \frac{z}{\lambda_D} \end{aligned} \quad (1)$$

normalizing the kinetic ($m_i v_z^2/2$) and potential (eU) energy of the singly charged ions with the electron thermal energy (kT_e), the electron and ion densities ($N_{e,i}$) with the charged particle density (N_0) of the plasma (more precisely of the 'sheath edge', see below)

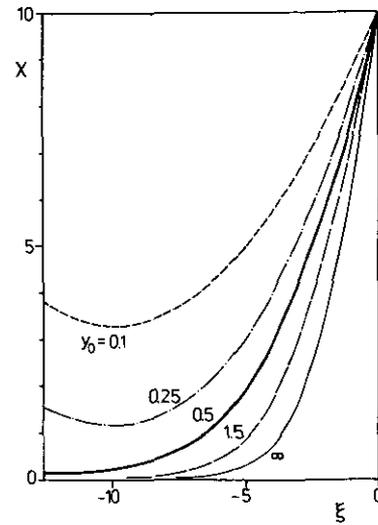


Figure 2. Sheath potential variation according to equation (9) for various ion energies y_0 .

and the space coordinate z (see figure 1) with the electron Debye length which is given by

$$\lambda_D = \left(\frac{\varepsilon_0 k T_e}{N_0 e^2} \right)^{1/2} \quad (2)$$

The sheath is represented by the equations

$$n_i y^{1/2} = y_0^{1/2} \quad \text{ion continuity} \quad (3)$$

$$y = y_0 + \chi \quad \text{ion energy conservation} \quad (4)$$

$$n_e = \exp -\chi \quad \text{electron Boltzmann factor} \quad (5)$$

$$d^2 \chi / d\xi^2 = n_i - n_e \quad \text{Poisson's equation.} \quad (6)$$

The zero potential and the subscript 0 refer to the 'sheath edge' where the sheath merges into the neutral (more or less undisturbed) plasma region. (The sheath edge will be defined more rigorously in the following.)

With the ion density (see equations (3) and (4))

$$n_i = (1 + \chi/y_0)^{-1/2} \quad (7)$$

Poisson's equation (equation (6)) can be integrated after multiplication with $d\chi/d\xi$. Using the boundary condition

$$\chi, \chi' \rightarrow 0 \quad \text{for } \xi \rightarrow -\infty \quad (8)$$

of a potential distortion fading away at the sheath edge, we obtain

$$\left(\frac{d\chi}{d\xi} \right)^2 = 4y_0[(1 + \chi/y_0)^{1/2} - 1] + 2(e^{-\chi} - 1). \quad (9)$$

The second integration must be performed numerically. The corresponding boundary condition (e.g. the wall potential $\chi(0)$), is not essential, because it results only in a parallel shift of the potential curves due to the spatial homogeneity of equation (9). In figure 2 we show solutions with $\chi(0) = 10$ for various values of the ion energy y_0 . Obviously, only for $y_0 \geq 0.5$ the boundary condition (8) is really met—for smaller ion energies

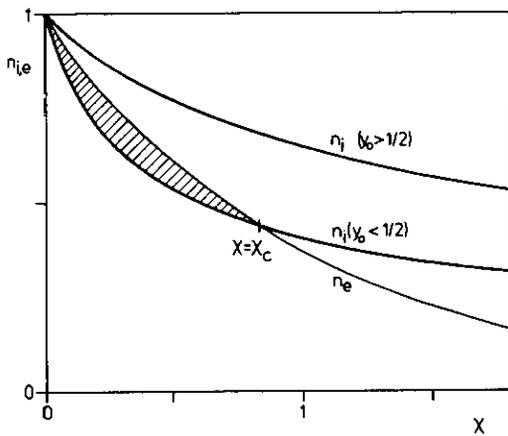


Figure 3. Electron and ion density variation in the sheath for $y_0 > \frac{1}{2}$ and $y_0 < \frac{1}{2}$.

no shielding is possible. This can be seen analytically from the expansion

$$\left(\frac{d\chi}{d\xi}\right)^2 = \left(1 - \frac{1}{2y_0}\right)\chi^2 + O(\chi^3) \quad (10)$$

of equation (9) for $\chi \rightarrow 0$, resulting in a contradiction if $2y_0 < 1$.

The necessary condition

$$y_0 \geq \frac{1}{2}$$

or

$$v_{0z}^2 \geq v_B^2 = \frac{kT_e}{m_i} \quad (11)$$

for the formation of a shielding sheath is known as the ‘Bohm criterion’. It was derived and formulated by Bohm (1949) (under the somewhat unfortunate notion ‘criterion for a stable sheath’ (Hall 1961)). Similar statements have already been given by Langmuir (1929, p 980).

To understand the physical reason, it is illustrative (Chen 1974, chapter 8.2) to compare the variation of the electron and ion densities (see figure 3 and equations (5) and (7)). Both n_e and n_i decrease with increasing χ , n_e according to the Boltzmann factor, and n_i due to the ion acceleration at constant current density. Only for sufficiently fast ions exceeding the ‘Bohm velocity’, $v > v_B$, the ion density n_i falls more slowly than n_e near the sheath edge ($\chi \rightarrow 0$), so that a positive space-charge shielding the negative wall distortion from the neutral plasma can be built up. To stress this aspect the Bohm criterion can be written in the form

$$\rho_0 = \left(\frac{dn_i}{d\chi} - \frac{dn_e}{d\chi}\right)\bigg|_{\chi=0} \geq 0 \quad (12)$$

(cf equations (5) and (7)), which is very suitable for generalization.

We conclude this interpretation with the linearization

$$\frac{d^2\chi}{d\xi^2} = \rho_0\chi \quad (\chi \rightarrow +0) \quad (13)$$

of Poisson’s equation at the sheath edge. For $\rho_0 > 0$ (fulfilled Bohm criterion) the presumed exponentially damped distortions corresponding to Debye shielding are obtained. In contrast, $\rho_0 < 0$ (violated Bohm criterion) yields only oscillatory solutions contradicting the boundary condition (8). These oscillations are the cause of the widely used but misleading interpretation of Bohm’s criterion as a criterion for a sheath with monotonic potential variation. In general, the Bohm criterion enables statements to be made about the *local* sheath formation near the sheath edge but *not* on the *global* sheath structure.

2.2. Sheath and presheath: two-scale theory

The description of the screening by the sheath is not satisfactory and cannot be complete: it ends at a (as yet hardly defined) ‘sheath edge’, which cannot be identified with the undisturbed plasma. According to the Bohm criterion the ions enter the sheath region with a velocity $v_{0z} \geq v_B = (kT_e/m_i)^{1/2}$. Usually we have $T_e \gg T_i$ and this implies the need for an electric field in the plasma region preceding the sheath in order to accelerate the ions to this high velocity. I follow the majority of authors and call this plasma region, where the accelerating field overcomes the ion inertia, the ‘presheath’—a notion introduced by Hu and Ziering (1966). With respect to the controversial nomenclature mentioned in the introduction the universal role of ion inertia for the presheath mechanism which was first recognized by Persson (1962) should be emphasized.

Depending on the particular physical situation, the presheath may be a part of the boundary layer (e.g. the Knudsen layer of the collision-dominated plasma) or the entire plasma (e.g. the collision free, low-pressure column). In any case, an extension L , large compared with the Debye length λ_D , is assumed:

$$L \gg \lambda_D = \epsilon L. \quad (14)$$

The sheath analysis would otherwise include the presheath, and the Bohm criterion would have to be fulfilled at the entrance of the presheath. Remembering that the Bohm criterion originates from ion continuity and energy conservation (equations (3) and (4)) (at least) one of these equations must be violated on the presheath scale. Therefore, L will usually be determined by the ion mean free path, by the ionization length or by the geometry of the system.

To consider the problem on the presheath scale it is natural to use the corresponding space coordinate

$$x = z/L = \epsilon\xi \quad (15)$$

and to write Poisson’s equation in the form

$$\epsilon^2 \frac{d^2\chi}{dx^2} = n_i(\chi, x) - n_e(\chi). \quad (16)$$

A characteristic feature of this differential equation is

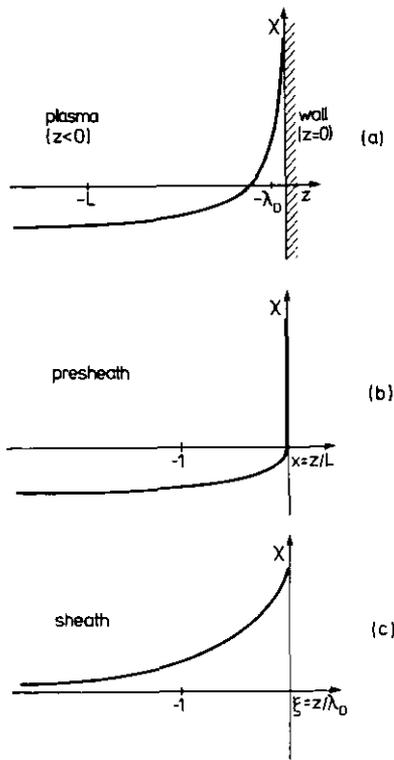


Figure 4. Schematic potential variation in front of a negative wall. (a) $\varepsilon = \lambda_D/L$ small but finite; (b) $\varepsilon \rightarrow 0$, presheath scale $x = z/L$; (c) $\varepsilon \rightarrow 0$, sheath scale $\xi = z/\lambda_D$.

that the coefficient of the highest derivative vanishes in the asymptotic limit $\varepsilon \rightarrow 0$, thus reducing the order of the problem and the number of possible boundary conditions. This implies that no uniformly valid solution, which is regular in the smallness parameter ε , is possible. Problems of this type belong to the class of 'singular perturbation theory' (Van Dyke 1964). They are solved on separate scales x and ξ and the solutions are 'matched' (see section 5).

From equation (16) and from the asymptotic limit $\varepsilon \rightarrow 0$ it is clear that the presheath is quasi-neutral. (It is important to bear in mind that quasi-neutrality is not based on a small curvature of the potential variation on the x scale, but on a small scale factor ε .) On the other hand, since the extension $x = \varepsilon\xi$ of the sheath ($\xi = 0(1)$) on the presheath scale tends to zero, the sheath is planar and collisionless.

The formation of the model zones 'sheath' and 'presheath' is sketched in figure 4. For small but finite $\varepsilon = \lambda_D/L$, the potential shows a steep variation ($d/dz \sim 1/\lambda_D$) in the sheath and a weak variation ($d/dz \sim 1/L$) in the presheath region (figure 4(a)). In the limit $\varepsilon = \lambda_D/L \rightarrow 0$ we must distinguish between the sheath and presheath scales. On the presheath scale the sheath is compressed into an infinitely thin layer and the scale transition is indicated by a formal field singularity (figure 4(b)). If the potential variation within the sheath is to be resolved one must consider the sheath scale, where the presheath is infinitely remote (figure 4(c)). In this concept the sheath edge is defined by the

presheath field singularity (exceptions will be discussed in sections 2.5, 3.3 and 6.3). Note that the position of the sheath edge is given by a special value of the potential (we use this potential value as reference point, $\chi = 0$) rather than by a special value of the space coordinate. We can only say that the sheath edge is located at $x = 0$, or at $\xi = -\infty$.

2.3. The presheath and its breakdown

Before discussing physical processes in the presheath the consequences of its quasi-neutrality are formulated. Equating the ion density

$$n_i = j_i y^{-1/2} \quad j_i = \left(\frac{m_i}{2kT_e} \right)^{1/2} J_i / N_0 \quad (17)$$

(where J_i is the ion current density) with the electron density (equation (5)) and differentiating logarithmically yields

$$\frac{1}{2y} \frac{dy}{dx} - \frac{d\chi}{dx} = \frac{1}{j_i} \frac{dj_i}{dx} \quad (18)$$

As long as the Bohm criterion is not yet fulfilled, i.e. for $y < \frac{1}{2}$, we therefore have

$$\frac{dy}{dx} - \frac{d\chi}{dx} < \frac{1}{j_i} \frac{dj_i}{dx} \quad (19)$$

By comparison with the ion energy law (see equation (4)) it can be concluded that a presheath, where slow ions are accelerated to the Bohm velocity v_B , is possible only if

(i) $dj_i/dx > 0$, i.e. the ion current density increases approaching the wall and/or

(ii) $dy/dx < d\chi/dx$, i.e. the ions suffer a retarding force (e.g. friction) in the presheath.

This requirement can be fulfilled in particular by the following isolated presheath mechanisms.

(a) *Geometric presheath* with current concentration $dj_i/dx > 0$ (e.g. spherical probe), $L =$ curvature radius.

(b) *Collisional presheath* with ion friction, $dy/dx < d\chi/dx$, $L =$ ion mean free path.

(c) *Ionizing presheath* with current increase $dj_i/dx > 0$ and mean ion retardation, $dy/dx < d\chi/dx$, $L =$ ionization length.

A further presheath mechanism, which is not contained in this analysis, should at least be mentioned here (see section 2.6).

(d) The *magnetic presheath*, where kinetic energy y perpendicular to the wall is converted into parallel kinetic energy, $L =$ ion gyroradius.

To study the presheath dynamics quantitatively the magnetic fields are again disregarded and a friction force

$$-m_i v_z \left(\frac{v_z}{\lambda(v_z)} + \frac{S_i}{N_i} \right) \quad (20)$$

is accounted for where $\lambda(v_z)$ is the ion momentum-

transfer mean free path and S_i the ionization rate. Returning to dimensionless quantities we use a dimensionless collision cross section q and ionization rate σ defined by

$$q(y) = \frac{L}{\lambda(v_z)} \tag{21}$$

$$\sigma(x) = L \left(\frac{m_i}{2kT_e} \right)^{1/2} \frac{S_i}{N_0}.$$

Further we restrict ourselves to simple geometries where the current continuity can be formulated one-dimensionally via the variation of an area element $A(x)$:

$$\text{div } j_i = \frac{1}{A} \frac{d}{dx} (A j_i) = \sigma. \tag{22}$$

($A(x) = (1 \mp x)^\beta$ with $\beta = 0$ for plane, $\beta = 1$ for cylindrical, and $\beta = 2$ for spherical geometry.) We then obtain from equation (18)

$$\frac{1}{2y} \frac{dy}{dx} - \frac{d\chi}{dx} = \frac{\sigma}{j_i} - \frac{A'}{A} \tag{23}$$

and the momentum balance reads

$$\frac{dy}{dx} - \frac{d\chi}{dx} = -2y[q + (\sigma/j_i)]. \tag{24}$$

Equations (23) and (24) represent a system of differential equations for $\chi(x)$ and $y(x)$. Solving for the derivatives we find a singularity (vanishing determinant) at $y = \frac{1}{2}$, i.e. at the Bohm velocity. Of course, this singularity is presumed to indicate the scale transition (Persson 1962).

From hydrodynamics we know the singularity of 'breaking the sound barrier', and in the hydrodynamic theory of the multicomponent plasma there occur removable singularities at the sound velocities of each particle component (Franklin 1976, chapter 4.6, Valentini 1988). In the simple model described here the electrons are not described dynamically and the ions are cold, consequently there is no singularity corresponding to the electron or ion sound velocity. The strong coupling via the quasi-neutrality, however, generates a new one-component fluid whose dynamical behaviour is determined by the ion inertia and by the electron pressure. The corresponding 'sound' velocity v_s of ion acoustic waves (Chen 1974, ch 4.6) is identical to the critical velocity v_B . It is therefore concluded that the plasma-sheath interface is defined on the presheath scale by the singularity which occurs when the (ion acoustic) 'sound barrier' is broken and that the Bohm criterion is automatically fulfilled with the equality sign (Stangeby and Allen 1970, Andrews and Stangeby 1970).

For some special cases equations (23) and (24) can be solved analytically. In particular the following special solutions are obtained.

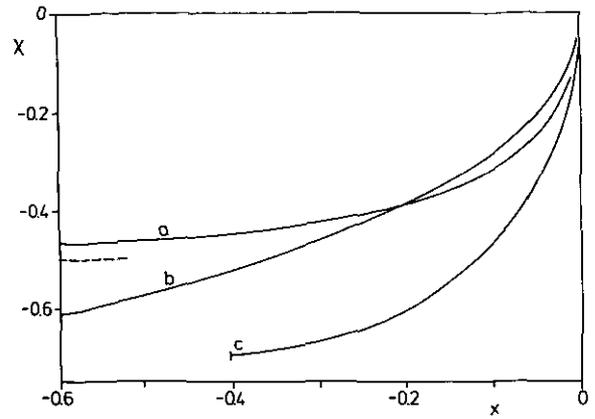


Figure 5. Potential variation of elementary presheath models. (a) Geometrical presheath, equation (26); broken line, undisturbed plasma potential. (b) Collisional presheath, equation (28), (c) Ionizing presheath, equation (30).

(a) Geometric presheath (spherical probe (Allen *et al* 1957)) with

$$A(x) = (1 - x)^2 \quad q = 0 \quad \sigma = 0 \tag{25}$$

$$x = 1 - e^{x/2} (1 + 2\chi)^{-1/4}. \tag{26}$$

(b) Collisional presheath with

$$A = 1 \quad q = 1 \quad \sigma = 0 \tag{27}$$

$$x = \frac{1}{2}(1 - e^{-2\chi} - 2\chi). \tag{28}$$

(c) Ionizing presheath (plane symmetric 'column' with ionization proportional to n_e (Kino and Shaw 1966)) with

$$A = 1 \quad q = 0 \quad \sigma = e^{-x}/2 \tag{29}$$

$$x = \sqrt{2}[\arctan X - \frac{1}{2}X + \frac{1}{2} - (\pi/4)] \tag{30}$$

$$X = (2e^x - 1)^{1/2}.$$

These solutions are plotted in figure 5. They show a quite different behaviour in the plasma region: the geometric presheath (a) relaxes to the undisturbed (field free) plasma; the collisional presheath (b) tends to a logarithmic potential shape, indicating that the ion transport requires a residual plasma field; and the ionizing presheath (c) ends with zero field at a finite point representing the midplane of a symmetric plasma. Despite these differences all the solutions run quite similarly with $y = \frac{1}{2}$ into the singularity at the sheath edge. The growing field inhomogeneity approaching this singularity indicates the formation of space charge and the breakdown of the quasi-neutral approximation.

2.4. The plasma-sheath transition

To extend this interpretation of space-charge formation at the ion acoustic singularity of the presheath let us

return to the general problem and use Poisson's equation

$$n_i = n_e + \varepsilon^2 \Delta \chi \quad (31)$$

rather than quasi-neutrality. In place of equations (18) and (23), we then obtain

$$\begin{aligned} \frac{1}{2y} \frac{dy}{dx} - \frac{d\chi}{dx} &= \frac{1}{j_i} \frac{dj_i}{dx} - \varepsilon^2 \frac{n_i}{n_e} \frac{d\Delta\chi}{dx} \\ &= \frac{\sigma}{j_i} - \frac{A'}{A} - \varepsilon^2 \frac{n_i}{n_e} \frac{d\Delta\chi}{dx}. \end{aligned} \quad (32)$$

Eliminating dy/dx from (24) and (32) yields

$$\left(1 - \frac{1}{2y}\right) \chi' + \left(q + 2\frac{\sigma}{j_i} - \frac{A'}{A}\right) = \varepsilon^2 \frac{n_i}{n_e} \frac{d\Delta\chi}{dx}. \quad (33)$$

The right-hand side (RHS) of this equation represents the space-charge contribution vanishing in the limit $\varepsilon \rightarrow 0$. In the first term on the LHS the difference between the density decrease due to the Boltzmann factor (χ') and that due to ion acceleration ($\chi'/2y$) can be recognized. For $y < \frac{1}{2}$ this term is negative. It is compensated for by the positive second term on the LHS, which clearly exhibits the joint effects of the previously discussed presheath mechanisms (collisional friction q , ionization σ and geometry $-A'$).

With increasing $y < \frac{1}{2}$ (thus decreasing the bracket) an increasing electric field is needed to avoid an excess of the second term. For $y \geq \frac{1}{2}$ this is not possible: both expressions on the LHS become positive and therefore the space-charge contribution must be finite. Formally, this contribution is of the order ε^2 ; to make it significant $\Delta\chi$ must grow to the order ε^{-2} . This explains the steepening of the potential variation according to the scale transition when breaking the sound barrier $y = \frac{1}{2}$.

Of course, for finite $\varepsilon = \lambda_D/L$, this steepening does not occur abruptly at the sound barrier, but begins smoothly in a 'transition region', which is influenced by the presheath mechanism and by the growing space-charge as well. This transition region can be described mathematically on an 'intermediate scale' (Lam 1965, 1967, Su 1967, Franklin and Ockendon 1970) $x = 0(\delta)$ with $\varepsilon \ll \delta \ll 1$. From the quasi-neutral approximation we have $\chi = 0$ ($\delta^{1/2}$) and we can estimate the order of magnitude $\varepsilon^2 \delta^{1/2}/\delta^3$ of the RHS of (33). Comparing with typical terms $O(1)$ of the LHS we see that the transition region is characterized (in this hydrodynamic approximation, see section 5) by the scaling

$$x = \varepsilon^{4/5} \zeta \quad \chi = \varepsilon^{2/5} w \quad (\zeta, w = O(1)). \quad (34)$$

I wish to demonstrate this transition by a numerical solution of the spherical probe problem (Allen *et al* 1957, see section 2.3) accounting for space charge and geometry. With $\chi = y - \frac{1}{2}$, $A(x) = (1-x)^2$ and $Aj_i = 1/\sqrt{2}$ we obtain from (5), (17) and (31)

$$y'' - \frac{2}{1-x} y' = \frac{1}{\varepsilon^2} \left(\frac{1}{\sqrt{2y(1-x)^2}} - e^{1/2-y} \right). \quad (35)$$

Numerical solutions for various finite $\varepsilon = \lambda_D/L$ are presented in figure 6(a). They show that with decreasing

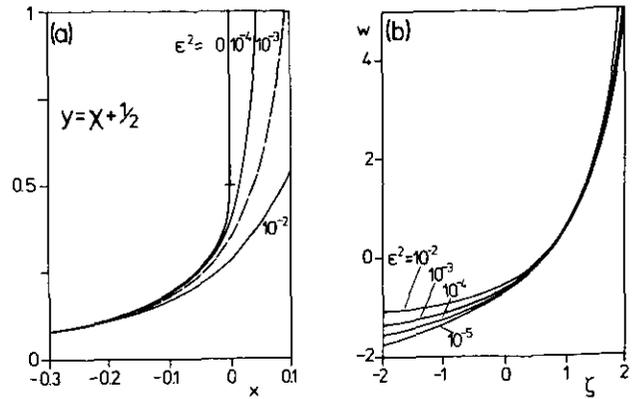


Figure 6. Numerical solutions of the spherical probe problem (equation (35)) for small $\varepsilon = \lambda_D/L$, $L = R$. (a) Presheath scale $x = z/L$; (b) intermediate scale according to equation (34).

ε , the quasi-neutral solution ($\varepsilon = 0$) is approached in the presheath region, which becomes universal for small ε . Near the sheath edge the curves diverge and the quasi-neutral solution is no longer a good approximation. Plotting (figure 6(b)) the same results in the transformed coordinates w and ζ (see equation (34)) the curves now come close together near the sheath edge and diverge in the presheath region: the solution becomes universal on the intermediate scale, which is the appropriate scale to describe the transition region.

With increasing sheath potential, the solutions for different ε again begin to diverge slightly. This indicates that the sheath scale now becomes the appropriate scale: all curves with small ε run (apart from a trivial parallel shift) into the curve $y_0 = 0.5$ of figure 2. This sheath solution is not only universal for different small ε , but also for all different presheath mechanisms. This universal sheath region is addressed by the Bohm criterion: it can only be formed with supersonic ions.

2.5. Is the Bohm criterion necessary for space-charge formation?

No: because the Bohm criterion refers exclusively to the sheath edge in the limit $\lambda/L \rightarrow 0$ (other assumptions can be removed by adapted generalizations, see sections 3 and 4). In particular, Bohm's criterion does not have to be satisfied if the (local, see section 5) Debye length is not small compared with the ion mean free path (Ingold 1972, Metzger *et al* 1989). This quite trivial statement is nevertheless important, since the incorrect application of the Bohm criterion to inadequate model zones and/or parameter regimes is a frequent source of misinterpretation and confusion.

However, because of the singular character of the asymptotic solution, and because in real situations λ_D/L may be small, but never exactly zero, there is a need—beyond the above clarification—to examine the validity and implications of the Bohm criterion very

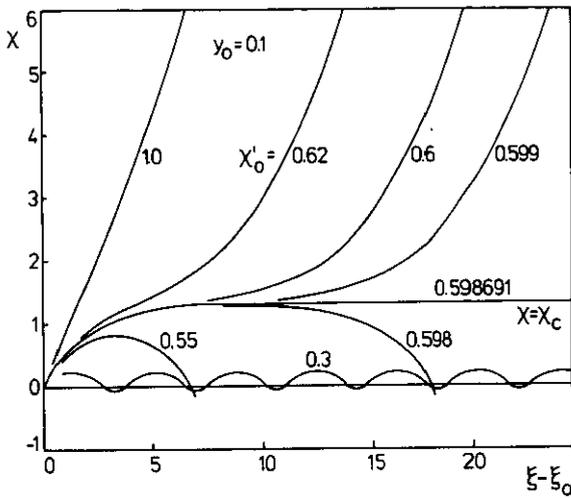


Figure 7. Formal 'sheath solutions' violating Bohm's criterion for various initial fields χ'_0 .

carefully. With this in mind let us recall Bohm's derivation (section 2.1) and discuss the main arguments against its validity.

Firstly, it must be observed that the criterion depends decisively on the boundary condition described by equation (8) and that this boundary condition representing the scale transition holds only in an approximate sense for small but finite λ_D/L (Hall 1961). The position ξ_0 (x_0) of the sheath edge will really be finite and χ , χ' and χ'' cannot vanish simultaneously (Chen 1965). It is, therefore, advisable to investigate whether a finite field (and/or space charge) at the sheath edge can yield physical solutions violating the Bohm criterion (Ecker and McClure 1962, 1965, Chekmarev 1972).

In figure 7 are plotted solutions of Poisson's equation (see equations (5), (6) and (7)) violating the Bohm criterion ($y_0 = 0.1$) for various initial fields χ'_0 . (For different conclusions from the same figure see Ecker (1973).) The figure exhibits two classes of solutions. For small χ'_0 we recognize the oscillatory solutions derived from the linearized equation (13). For initial fields exceeding a critical value ($\chi'_0 > 0.598691$) the limiting potential $\chi = \chi_c$ (cf figure 3) can be overcome and we obtain monotonic potential curves. (It is interesting to note that very similar solution structures for a somewhat different problem were discussed by Langmuir (1929, p 977).) These solutions cannot be considered physical because they show a strong potential variation (on the scale of the Debye length) at the 'sheath edge', contradicting the scale argument $L \gg \lambda_D$. The unphysical nature of these solutions is reflected by the subsequent mathematical breakdown shortly after passing the 'sheath edge', $\xi < \xi_0$: the solutions run into a region ($\chi < -y_0$), where the square root of equation (7) becomes imaginary.

The next criticism of the Bohm criterion concerns the influence of a transition region between the sheath

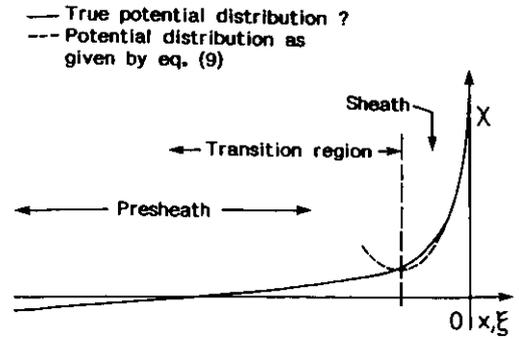


Figure 8. Presumed schematic potential variation, if Bohm's criterion is violated (after Hall 1961).

and the quasi-neutral presheath. Can the 'unphysical' solutions ($y_0 < \frac{1}{2}$) in figure 2 become 'physical', if we account for the 'presheath processes' on the sheath scale (see Hall 1961 and figure 8)? Within the parameter range $\lambda_D/L \ll 1$ the answer is again 'no': to become physical, the solutions would have to change completely their shape on the sheath scale, where the influence of the presheath processes is far too weak.

The corresponding question 'from the other end' of a small sheath field should be addressed: can the linearized equation (13) be improved to yield physical (i.e. growing or decaying) solutions for $y_0 < \frac{1}{2}$, if we account for an—arbitrarily small—influence of collisions (Zawaideh *et al* 1990), ionization or geometry? The answer is now in principle 'yes', but this 'improved' equation then refers to a point of the presheath region (note that the location of the zero potential is arbitrary), and it is not surprising that the presheath mechanism enables a growing potential accelerating subsonic ($y < \frac{1}{2}$) ions.

To illustrate this let us reconsider the numerical solution of (35) (cf figure 6) for a small but finite value $\epsilon^2 = 10^{-3}$ (figure 9(a)). No region is exactly collision free, no region is exactly neutralized, and we have a continuously growing field throughout the subsonic and sonic ranges. Let us tentatively introduce a 'sheath edge' in the subsonic range, i.e. let us choose an arbitrary point x_0 , $y_0 < \frac{1}{2}$ (to be specific, say $y_0 = 0.075$, $x_0 \approx -0.3$) on the quasi-neutral solution and let us start at this point to solve Poisson's equation (35) with a vanishing initial field ($y'_0 = \chi'_0 = 0$). The result is presented in figure 9(b); it shows a 'sheath solution' oscillating closely around the previous solution in figure 9(a). The oscillations can be seen more distinctly if we consider the derivatives (figure 9(c) and (d)). It can be seen that this has produced a finite space-charge with alternating sign, an oscillating positive definite electric field and a monotonically increasing potential. Does this show that a reasonable sheath solution violating the Bohm criterion (see Zawaideh *et al* 1990) has been found? No, it just shows that the oscillations on the scale of the Debye length derived from (13) make the presheath solution ($y < \frac{1}{2}$) numerically (and physically) stable. The error introduced by the incorrect boundary condition $y = y_0$, $y' = 0$ at $x = x_0$ (or a small physical

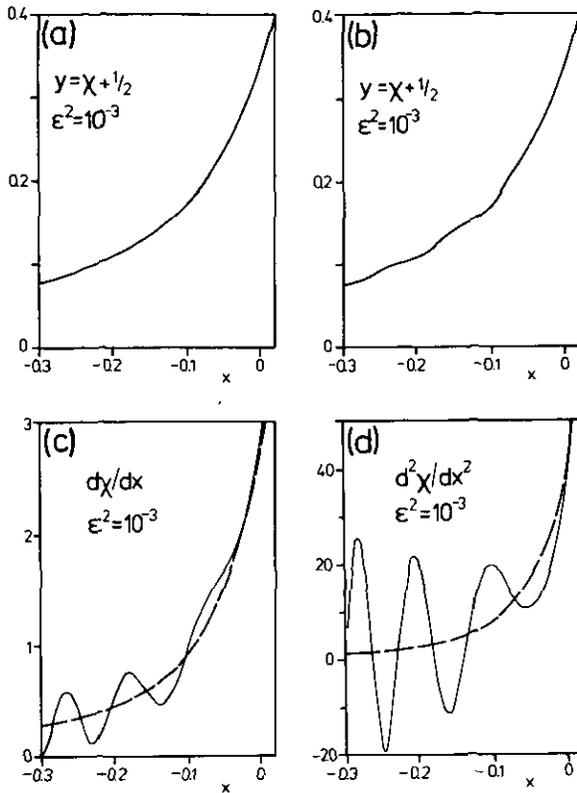


Figure 9. Numerical solution of Poisson's equation (35) for $\epsilon^2 = 10^{-3}$ (cf figure 6). (a) Correct probe solution; (b)–(d) oscillatory solutions resulting from the artificial boundary condition $y' = 0$ at $x = -0.3$ (see text).

distortion) results in stable oscillations round the undisturbed (presheath) solution. (This is in contrast to the sheath region, $y > \frac{1}{2}$.) Moreover, this discussion shows that we are not allowed to *choose* an arbitrary sheath edge x_0 : the sheath edge is uniquely defined by the singularity of the asymptotic solution.

Is this singularity caused by the contradictory assumptions of subsonic flow ($y < \frac{1}{2}$) in the presheath and supersonic flow ($y > \frac{1}{2}$) in the sheath region? And is this presumed contradiction reflected in contradictory boundary conditions of a zero sheath-edge field on the sheath scale and an infinite field on the presheath scale (Godyak 1982, Godyak and Sternberg 1990)?

We have seen in section 2.4 that there is no discrepancy between subsonic and supersonic flow velocities but that passing the 'sound barrier' $y = \frac{1}{2}$, changes a critical sign in equation (33), thus forcing strong space-charge formation. Due to the scaling $\lambda_D/L \rightarrow 0$ this results in the formal sheath-edge singularity. There is no contradiction between sheath-edge fields $d\chi/dx \rightarrow \infty$ and $d\chi/d\xi \rightarrow 0$ for $\epsilon \rightarrow 0$, $\chi \rightarrow 0$. Correctly interpreted these conditions read

$$E_0 \gg \frac{kT_e}{eL} \quad E_0 \ll \frac{kT_e}{e\lambda_D} \quad (36)$$

where E_0 is the typical field in the transition region at the sheath edge. To obtain a smooth transition avoiding singularities Godyak and Sternberg made the

suggestion to 'glue' the presheath and sheath models together with the common boundary condition

$$E_1 = kT_e/e\lambda_D. \quad (37)$$

In fact, this boundary condition postulates a stronger singularity in $\epsilon = \lambda_D/L$ than that described by equations (36). Equation (37) gives the characteristic field of the sheath region where the quasi-neutral approximation has broken down. Therefore the described 'gluing' can be considered as only a rough, qualitative 'patching'. A consistent 'matching' (see section 5) has to be performed on the intermediate scale (34) of the transition region. From the scaling of this region we read an order of magnitude

$$E_0 = \frac{kT_e}{e\lambda_D^{2/5}L^{3/5}} \quad (38)$$

of the sheath edge field confirming the presheath and sheath field limits of relations (36). (The scaling is changed slightly in a more rigorous kinetic treatment accounting for slow ions at the sheath edge, see section 5.)

Summarizing this discussion we conclude that, in the limit $\lambda_D/L \rightarrow 0$ of a small Debye length, the Bohm criterion is indeed a necessary condition for the formation of a stationary sheath. Furthermore, an understanding is obtained of why the sheath edge may be usually characterized (e.g. Allen and Stangeby 1970, Boozer 1975, Franklin 1976) by a field singularity terminating the quasi-neutral approximation or, equivalently, by the 'marginal' (i.e. with equality sign) validity of Bohm's criterion.

This usual relation, however, is not necessarily true. In widening geometries $A' > 0$ the second bracket on the LHS of equation (33) may become zero at some x_0 within the plasma. This will certainly happen, for instance, in a collision free ($q = 0$) cylindrical ($A' = 1$) column, if the experimental arrangement (e.g. laser beam) restricts ionization ($\sigma > 0$) to a limited region near the axis.

For $\epsilon \rightarrow 0$ this results in a zero ($y = \frac{1}{2}$) of the first bracket in equation (33) at $x = x_0$. For $x > x_0$ both brackets on the LHS change their sign so that a further quasi-neutral acceleration of the supersonic flow ($y > \frac{1}{2}$) is possible. Analogous situations are present in the supersonic acceleration of the solar wind (Hundhausen 1974) and in the 'cathodic plasma jet' of an arc discharge (Wieckert 1987.)

Approaching the wall, the Bohm criterion is oversatisfied and there is now no singularity and no breakdown of the quasi-neutral approximation. An exponentially decaying sheath in front of the wall is only formed if required by the boundary conditions. This situation is completely analogous to Debye screening.

2.6. Related problems

This section is concluded with supplementary topics not systematically dealt with in this review. The first

concerns the sheath formation in a magnetized plasma ($L = \text{ion gyroradius } \rho_i \gg \lambda_D$) where the presheath mechanism is provided by the Lorentz force. Unfortunately there are only few investigations on this topic, and these give no coherent picture.

Daybelge and Bein (1981) considered the sheath formation in a magnetic field exactly parallel to the wall without elementary processes. Obviously, this results in an artificial static model with no transport at all and in particular with no presheath ion acceleration.

To provide wall transport Chodura (1982) accounts for field lines intersecting the wall at a small angle Ψ . Beyond the Bohm criterion ($v_z \geq v_s$) at the sheath edge he postulates a second condition of supersonic flow along the field lines ($v_{\parallel} \geq v_s$ or $v_z \geq v_s \sin \Psi$) at the 'entrance' of the magnetic presheath. Consequently, he assumes an additional 'plasma presheath' to fulfil the second condition (unfortunately the model zones are differently named in Chodura's articles 1986 and 1988, see table 1). Chodura derives his additional condition from a dispersion relation including electrostatic effects and refuses oscillatory solutions. In the presheath region, however, there is no boundary condition of decaying electrostatic fields (cf equation (8)); on the contrary the oscillatory electrostatic modes on the presheath scale guarantee the stability of the quasi-neutral solution against charge imbalances (see section 2.5 and figure 9). Therefore the necessity of the condition $v_{\parallel} \geq v_s$ appears questionable.

Another way to provide ion transport to the wall was chosen by Behnel (1985a, b). He assumed an exactly parallel ($\Psi = 0$) magnetic field and accounted for collisions ($\lambda \gg L = \rho_i$). Behnel solved the ion Boltzmann equation in a given potential but did not obtain the self-consistent solution. He claimed that the presheath acceleration necessary to satisfy the Bohm criterion results in high potential drops ($\Delta\chi \sim \ln \lambda/\rho_i$) and high velocities in the $\mathbf{E} \times \mathbf{B}$ direction. This could possibly be the reason for a Kelvin-Helmholtz instability resulting in a highly turbulent state, a result obtained from particle simulations by Theilhaber and Birdsall (1989).

The second topic refers to the transient problem of sheath and presheath formation. Prewett and Allen (1973) investigated the evolution of a presheath in front of a cylindrical and spherical probe immersed in a collisionless cold ion plasma. The transient sheath edge is again distinguished by a field singularity, Bohm's criterion is fulfilled in a 'dynamic' form, where the ions attain a velocity v_B with respect to the moving sheath edge. The presheath disturbance propagates in the form of a rarefaction wave at ion sound velocity into the undisturbed plasma and approaches the stationary solution.

In the corresponding plane problem no stationary solution exists (see equation (19) and its discussion) and no *a priori* smallness parameter $\varepsilon = \lambda_D/L$. Nevertheless, the transient solution (Allen and Andrews 1970, Cipolla and Silevitch 1981) tends to a self-similar

structure with a disturbance propagating at sound velocity and accelerating the ions to fulfil Bohm's criterion in front of the sheath edge, which in this case exhibits no singularity.

Solved transient presheath models are compared and discussed in more detail by Braithwaite and Wickens (1983).

3. Refined theory of the plasma-sheath transition

Using a simplified fluid model, in section 2 the basic mechanism of sheath formation was discussed in great detail. The persuasive power of the conclusions suffers from the simplifications drawn, particularly from the assumption of cold, monoenergetic ions. In the fluid description, warm ions produce an additional pressure contribution to the momentum balance. This implies that the critical velocity v_B causing the sheath edge singularity is replaced by the sound velocity

$$v_s = \left(\frac{k(T_e + \gamma T_i)}{m_i} \right)^{1/2} \quad (39)$$

(see e.g. Persson 1962, Self and Ewald 1966, Zawaideh *et al* 1986). Here T_i is the ion temperature and $\gamma = 1$ for isothermal flow, $\gamma = \frac{5}{3}$ for adiabatic flow with isotropic pressure and $\gamma = 3$ for one-dimensional adiabatic flow (Chen 1974, p 84). The different values of γ reflect the uncertainty arising from the cut-off of the hydrodynamic hierarchy.

A more rigorous way to generalize Bohm's criterion is to account for the full ion velocity distribution. The easiest way to do this is to rewrite equations (7) and (12) for several ion groups k with different velocities v_{zk} . This results in

$$\sum_k \frac{c_k}{y_k} \leq 2$$

or

$$\sum_k \frac{c_k}{v_{zk}^2} \leq \frac{m_i}{kT_e} \quad (40)$$

where c_k designates the relative density contribution of group k . Proceeding to a continuous distribution we obtain the famous kinetic formulation of Bohm's criterion

$$\langle 1/v_z^2 \rangle \leq m_i/kT_e \quad (41)$$

due to Harrison and Thompson (1959). The derivation, however, is mathematically unsound as pointed out by Hall (1962). The analysis of section 2 assumes $\chi < y_k$ and strictly speaking does not permit the transition from equation (40) to (41). In view of the essential contribution of slow velocities in (41) this is a serious objection which forces a more careful investigation.

In the following we give a rigorous analysis of the plasma-sheath transition based on the kinetic equation for the velocity distribution $F_i(z, v)$ of positive ions.

This gives the opportunity to relax the model assumptions and to investigate the sheath edge singularity simultaneously (Riemann 1989a, b). (For simplicity reference is made for the most part to one positive-ion species; more species can be accounted for trivially by summing the corresponding charge density contributions.) An arbitrary (but given) density distribution of electrons and possibly of negative ions is considered. For the sake of completeness the boundary condition of a completely absorbing wall is dropped and positive ions entering the plasma from the wall (e.g. ion reflection or contact ionization of fast neutrals) are accounted for. Apart from these generalizations the basic concept described in section 2 is followed.

3.1. Kinetic analysis of the ions

As a starting point, Boltzmann's equation in the form

$$v_z \frac{\partial F_i}{\partial z} - \frac{e}{m_i} \frac{dU}{dz} \frac{\partial F_i}{\partial v_z} = C(z, v; F_i) + R(z, v; F_i) = S(z, v) \quad (42)$$

is used, where z again designates the space coordinate perpendicular to the wall. (We assume here a one-dimensional, but not necessarily plane, geometry, z may be a suitable curvilinear coordinate.) C is the collision term (including ionization and recombination) and R represents possible additional phase space convection terms arising, for example, from the system geometry or from a magnetic field. At present there is no need to specify C and R explicitly, because we are interested in general structures and not in special solutions. We only assume that they do *not* involve dU/dz or $\partial F_i/\partial z$ (which may become singular at the sheath edge) and combine them in a generalized 'source function' $S(z, v)$.

We return to the designations of section 2 (see equations (1), (15) and (17)) and introduce the dimensionless half distributions

$$f_i^\pm(\chi, y) = \frac{1}{N_0} \left(\frac{kT_e}{2m_i} \right)^{1/2} \int F_i(z, v_\perp, v_z \geq 0) d^2v_\perp \quad (43)$$

normalized by

$$\begin{aligned} n_i(\chi) &= \int_0^\infty y^{-1/2} (f_i^+ + f_i^-) dy \\ j_i &= \int_0^\infty (f_i^+ - f_i^-) dy. \end{aligned} \quad (44)$$

Due to the scale transition χ is used in place of x or ξ as an independent variable assuming a monotonic potential variation. The energy variable y forces one to distinguish between positive and negative v_z by separate functions f_i^+ and f_i^- . In an analogous way the source functions are defined as

$$s^\pm(\chi, y) = \frac{L}{N_0} \int S(z, v_\perp, v_z \geq 0) d^2v_\perp \quad (45)$$

(which in general depend on the full three-dimensional

distribution F_i and cannot be explicitly expressed in terms of f_i^+ and f_i^-) and Boltzmann's equation (equation (42)) is written in the form

$$\frac{\partial f_i^\pm}{\partial \chi} + \frac{\partial f_i^\pm}{\partial y} = \pm k'(\chi) y^{-1/2} s^\pm(\chi, y) \quad (46)$$

where $k(\chi)$ designates the inverse function of the self-consistent potential variation $\chi(x)$. The boundary condition at the wall ($\chi = \chi_w$) is represented in the form

$$f_i^-(\chi_w, y) = \alpha f_i^+(\chi_w, y) + g(y) \quad (47)$$

where α is the coefficient for specular reflection and g accounts for diffuse reflection and contact ionization, for example. The artificial splitting of the distribution function in velocity space ($v_z \leq 0$) requires the additional boundary condition

$$f_i^+(\chi, 0) = f_i^-(\chi, 0) \quad (48)$$

connecting f_i^+ and f_i^- at the turning points of the ion orbits. (If the source function S contains a singular contribution $\sim \nu \delta(v)$ as assumed in some simplified models (see e.g. Harrison and Thompson 1959, Riemann 1981), equation (48) does not hold, and the analysis then has to be changed slightly. The essential conclusions of this section, however, remain valid.)

The self-consistent potential finally obeys Poisson's equation

$$\left(\sum_i \right) n_i(\chi) - n_-(\chi) = \frac{d^2\chi}{d\xi^2} = \epsilon^2 \frac{d^2\chi}{dx^2} \quad (49)$$

where $n_-(\chi)$ —the total negative charge density including electrons and negative ions—is assumed to be a known function and the summation symbol in brackets refers to the way of accounting for more positive ion species. In the asymptotic limit $\epsilon = \lambda_D/L \rightarrow 0$ equations (46) and (49) result in the familiar two-scale problem: on the presheath scale $x = 0(1)$, equation (49) can be replaced by quasi-neutrality and on the sheath scale $x = k(\chi) = 0(\epsilon)$, equation (46) turns into the collision-free Boltzmann equation

$$\frac{\partial f_i^\pm}{\partial \chi} + \frac{\partial f_i^\pm}{\partial y} = 0 \quad (50)$$

in plane geometry.

3.2. Derivation of general sheath conditions

Equation (50) shows that the sheath distributions f_i^\pm are functions of the energy $y - \chi$ only. Complementing the boundary conditions (47) and (48) by a specification

$$f_i^+(0, y) = f_0(y) \quad (51)$$

of the distribution of the ions entering the sheath region from the sheath edge, we obtain the contributions

$$\begin{aligned} f_{i1}^+ &= f_0(y - \chi) \Theta(y - \chi) \\ f_{i2}^- &= \alpha f_i^+(\chi, y) \\ f_{i3}^- &= g(y + \chi_w - \chi) \\ f_{i4}^+ &= f_i^-(\chi, y) \Theta(\chi - y) \end{aligned} \quad (52)$$

corresponding to the types of orbits sketched in figure

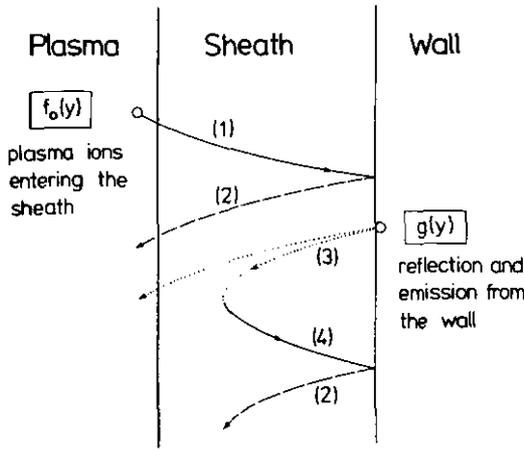


Figure 10. Different contributions to the ion distribution function in the sheath (schematic orbits): (1), ions coming from the sheath edge; (2), specular ion reflection; (3), diffuse reflection and emission; (4), contributions due to turning points in the sheath.

10. $\Theta(x)$ designates the Heaviside step function. Summing the contributions and solving for f_i^+ and f_i^- yields

$$f_i^+(\chi, y) = \Theta(y - \chi)f_0(y - \chi) + \Theta(\chi - y) \times \frac{g(y + \chi_w - \chi)}{1 - \alpha}$$

$$f_i^-(\chi, y) = \Theta(y - \chi) \times [\alpha f_0(y - \chi) + g(y + \chi_w - \chi)] + \Theta(\chi - y) \frac{g(y + \chi_w - \chi)}{1 - \alpha} \quad (53)$$

Calculating the ion density from (44) we find after some transformations

$$n_i = \frac{1 + \alpha}{1 - \alpha} \int_0^\infty \frac{\bar{f}(y)}{(y + \chi)^{1/2}} dy + \frac{2}{1 - \alpha} \int_0^\infty \frac{g(y - \chi + \chi_w)}{y^{1/2}} dy \quad (54)$$

with

$$\bar{f}(y) = (1 - \alpha)f_0(y) - g(y + \chi_w) = f_i^+(0, y) - f_i^-(0, y) \quad (55)$$

Following on from section 2.1 the derivation of a sheath condition by an expansion of the ion density for small χ is required. The natural way to do this seems to be to follow what is usually done when the kinetic form of Bohm's criterion is derived (Harrison and Thompson 1959, Boyd and Thompson 1959, Bertotti and Cavaliere 1965), i.e. to differentiate equation (54) or its less general analogue and to apply condition (12) or—in the case of equality—to compare the higher derivatives. This way, however, was criticized with good reason by Hall in 1962 (see discussion after relation (41)): due to the singularity of the first integral (physically caused by slow ions) $n_i(\chi)$ is not analytic in origin and cannot be represented by its Taylor series. In place of that it has to be expanded in a power

series in $\chi^{1/2}$. This expansion is performed in appendix 1 and results in

$$n_i = \sum_\nu a_\nu \chi^{\nu/2}$$

$$a_{2n} = \frac{(-1)^n}{n!} \int_0^\infty y^{-1/2} \frac{d^n f}{dy^n} dy \quad (56)$$

$$a_{2n+1} = \frac{1 + \alpha (-1)^{n+1} \pi^{1/2}}{1 - \alpha [n + (1/2)]!} \frac{d^n \bar{f}}{dy^n} \Big|_{y=0}$$

with

$$f(y) = (1 + \alpha)f_0(y) + g(y + \chi_w) = f_i^+(0, y) + f_i^-(0, y) \quad (57)$$

Simultaneously using

$$n_- = \sum_\nu b_\nu \chi^\nu \quad b_\nu = \frac{1}{\nu!} n_-^{(\nu)}(0) \quad (58)$$

Poisson's equation (49) may be written as

$$\frac{d^2 \chi}{d\xi^2} = \sum_\nu c_\nu \chi^{\nu/2}$$

$$c_\nu = \begin{cases} a_\nu - b_{\nu/2} & \text{for } \nu = 2\mu \\ a_\nu & \text{else} \end{cases} \quad (59)$$

Multiplying by $d\chi/d\xi$ and using again the boundary condition (equation (8)) of a decaying sheath (scale transition!) we can integrate (59):

$$\frac{d\chi}{d\xi} = 2 \left(\sum_\nu \frac{c_\nu}{\nu + 2} \chi^{(\nu+2)/2} \right)^{1/2} \quad (60)$$

If c_m is the first non-vanishing coefficient of the series, we conclude $c_m > 0$. Neglecting higher order terms we can integrate once more and obtain

$$\text{constant} - 2 \sqrt{\frac{c_m}{m+2}} \xi = \begin{cases} -\ln \chi & (m = 2) \\ \left(\frac{4}{2-m} \right) \chi^{(2-m)/4} & (m \neq 2). \end{cases} \quad (61)$$

Comparing again with the boundary condition (8) we conclude further $m \geq 2$. In particular this implies

$$a_0 = b_0 \quad a_1 = 0 \quad a_2 \geq b_1 \quad (62)$$

The first condition of (62) requires the quasi-neutrality of the sheath edge. It is fulfilled trivially by the normalization $a_0 = b_0 = n_{i,-}(0) = 1$. The second condition $a_1 = 0$ implies

$$\bar{f}(0) = f^+(0, 0) - f^-(0, 0) = 0 \quad (63)$$

and is always fulfilled due to the boundary condition (48). The third condition may be written (cf equations (56) and (58))

$$\int_0^\infty y^{-1/2} f'(y) dy \leq - \frac{dn_-}{d\chi} \Big|_{\chi=0} \quad (64)$$

and represents in principle the general sheath condition

we intended to derive. Returning to the boundary distributions f_0 and g (see equation (57)) f may be expressed in the form

$$f(y) = \frac{1 + \alpha}{1 - \alpha} [(1 - \alpha)f_0(y) - g(\chi_w + y)] + \frac{2}{1 - \alpha} g(\chi_w + y). \quad (65)$$

In the square brackets we recognize $\bar{f}(y)$ (see equation (55)). From equation (63) we see that the corresponding contribution in (64) may be integrated by parts. Therefore the general sheath condition may be written explicitly as

$$\frac{1 + \alpha}{1 - \alpha} \int_0^\infty \frac{(1 - \alpha)f_0(y) - g(\chi_w + y)}{2y^{3/2}} dy + \frac{2}{1 - \alpha} \int_0^\infty \frac{g'(\chi_w + y)}{y^{1/2}} dy \leq - \left. \frac{dn_-}{d\chi} \right|_{\chi=0}. \quad (66)$$

I shall discuss and specify this condition in section 4. Here our attention is drawn to another aspect: from the discussion of the Bohm criterion in section 2 one must be aware of the fact that the sheath condition (66) may hold with the equality sign. In this case we have an additional sheath condition $a_3 \geq 0$ or (cf equations (56) and (55))

$$\bar{f}'(0) = \frac{d}{dy} [(1 - \alpha)f_0(y) - g(y + \chi_w)]|_{y=0} \geq 0. \quad (67)$$

The question of whether the presheath distribution function will fulfil these sheath conditions at the sheath edge will now be addressed.

3.3. Presheath kinetics and sheath edge

To obtain information on the general sheath conditions resulting from the density variation in the sheath the density variation in the presheath is investigated. Remembering equation (44) we multiply Boltzmann's equation (46) by $y^{-1/2}$, add the contributions (+, -) and integrate:

$$\frac{\partial n_i}{\partial \chi} + \int_0^\infty \frac{\partial}{\partial y} [f_i^+(\chi, y) + f_i^-(\chi, y)] \frac{dy}{y^{1/2}} = k'(\chi) \int_0^\infty \frac{s^+(\chi, y) - s^-(\chi, y)}{y} dy. \quad (68)$$

Engaging quasi-neutrality $n_i(\chi) = n_-(\chi)$, considering the sheath edge $\chi = 0$ and using the abbreviation of equation (57) we obtain

$$\int_0^\infty y^{-1/2} f'(y) dy \approx - \left. \frac{dn_-}{d\chi} \right|_{\chi=0} + k'(0) \int_0^\infty \frac{s^+(0, y) - s^-(0, y)}{y} dy. \quad (69)$$

Comparing equations (69) and (64) we come to the

following important conclusion (Riemann 1980, 1989a, b). The general sheath condition (64) is automatically fulfilled marginally (i.e. with the equality sign), if the sheath edge exhibits the usual field singularity $k'(0) = 0$. The same is true for symmetric source functions $s^+ = s^-$ independently of $k'(0)$. Two questions must now be answered.

- (i) Can we conclude safely that the sheath edge is really characterized by the usual field singularity?
- (ii) If we really have $k'(0) = 0$ or $s^+ = s^-$, will then the *additional* sheath condition (67) also be fulfilled?

In the fluid mechanical model of section 2 the sheath-edge field singularity was related to a critical ion density variation at the 'sound barrier'. On the level of a kinetic description we do not recognize a sound barrier and expect essential contributions to the density variation from slow ions. Therefore a relation between the distribution of slow ions at the sheath edge and the potential shape by a formal integration of Boltzmann's equation is required. To do this the appropriate pre-sheath boundary condition at the sheath edge must be formulated. This is obtained from the physical boundary conditions (47) at the wall, because the sheath is collisionless (see equation (50)):

$$f_i^-(0, y) = \alpha f_i^+(0, y) + g(\chi_w + y). \quad (70)$$

(Notice, incidentally, that equation (70) together with quasi-neutrality is the definition of the sheath edge in the kinetic analysis.)

The formal integration of Boltzmann's equation is presented in appendix 2. At the sheath edge especially the ion distribution function depends only on the sum of the source half distributions s^+ , s^-

$$s = s^+ + s^-. \quad (71)$$

To reach specific conclusions let us assume that s can be represented for small y in the form

$$s(\chi, y) = s_0(\chi)y^a \quad y \rightarrow 0. \quad (72)$$

(In general, we expect that ions with zero velocity are influenced by the source, i.e. $s(\chi, 0) \neq 0$ and consequently $a = 0$.) Modelling the potential shape at the sheath edge further by the ansatz

$$k'(\chi) = k_0(-\chi)^b \quad \chi \rightarrow 0 \quad (73)$$

(where the presumed sheath-edge field singularity corresponds to $b > 0$) we obtain from appendix 2

$$\begin{aligned} \bar{f}(y) &\rightarrow C y^{a+b+(1/2)} \quad y \rightarrow 0 \\ f(y) &= \frac{1 + \alpha}{1 - \alpha} \bar{f}(y) + \frac{2}{1 - \alpha} g(\chi_w + y) \\ C &= k_0 s_0(0) \int_0^1 t^b (1 - t)^{a-1/2} dt. \end{aligned} \quad (74)$$

From the finite density slope $dn_-/d\chi$ and from the regularity and continuity of the source we can conclude

that the integral on the LHS of equation (69) must exist in any case. Inserting (74) we find

$$a + b > 0 \tag{75}$$

and conclude that in all ‘normal’ situations $a = 0$ ($s(\chi, 0) \neq 0$) the sheath edge necessarily shows the usual field singularity ($b > 0$). Only in the somewhat academic case of a source function (collision term) vanishing exactly at zero velocity, a finite sheath edge field is possible. This agrees with corresponding conclusions for completely absorbing walls (Bissel 1986, 1987, Riemann 1989a).

From the requirement of regular distributions f, \bar{f} (otherwise the expansion (56) would be meaningless) we can further conclude that $a + b + \frac{1}{2}$ is an integer—normally 1. This allows one to specify the field singularity in more detail: we expect $b = \frac{1}{2}$ for finite regular sources ($a = 0$) in contrast to singular sources (Harrison and Thompson 1959, Riemann 1981) and fluid models, where $b = 1$ holds (without proof here).

Since—apart from very special exceptions—the sheath condition (64) holds with the equality sign, the additional condition (67) must be fulfilled. In the case of absorbing walls ($\alpha = 0, g = 0, f(y) = \bar{f}(y) = f_0(y) \geq 0$) this condition is met trivially by the requirement of a non-negative distribution function. Generally we find that the validity of the sheath condition (67) depends on $\text{sign}(\dot{C}) = \text{sign}(k_0 s_0)$: presheath solutions with growing potential in front of the sheath edge ($k_0 > 0$) require a source providing an effective production of slow ions ($s_0 > 0$). (Note that this corresponds to some extent to the presheath mechanisms discussed in section 2.3.)

To illustrate the conclusions of this section explicitly let us refer to a simplified model of a plane, weakly ionized plasma in front of an absorbing wall and assume that the ion kinetics can be represented one-dimensionally by a BGK or charge exchange collision term and/or ionization (Emmert *et al* 1980, Riemann 1981, Bissel and Johnson 1987, Biehler *et al* 1988, Scheuer and Emmert 1988a, b, Koch and Hitchon 1989, see section 6). In this case the source function has the form

$$s^\pm(\chi, y) = \nu(y)[f_N(\chi, y) - f_i^\pm(\chi, y)] + \sigma(\chi)f_N(\chi, y) \tag{76}$$

where $\nu \geq 0$ represents the collision frequency, $\sigma \geq 0$ the ionization rate and f_N the symmetric neutral velocity distribution. With $\alpha = 0$ and $g = 0$ (cf equation (55)) this results in

$$\int_0^\infty \frac{s^+(0, y) - s^-(0, y)}{y} dy = - \int_0^\infty \frac{\nu(y)f_0(y)}{y} dy < 0 \quad \nu > 0. \tag{77}$$

For $\nu(0) \neq 0$ or $\sigma f_N(0) \neq 0$ we have a field singularity $k'(0) = 0$ ($s^+(0, 0) + s^-(0, 0) > 0 \Rightarrow a = 0, b > 0$) and

(69) indicates the marginal validity of the sheath condition (64). The additional condition (67) is satisfied.

Only for special models with $\nu(0) = 0$ and $\sigma f_N(0) = 0$ can the sheath edge field be finite and positive, $k'(0) > 0$ and we see from (69) and (77) that the sheath condition (64) will then be oversatisfied if collisions are present ($\nu > 0$).

4. The Bohm criterion

Let us recall that Bohm’s criterion is addressed to the asymptotic limit $\lambda_D/L \rightarrow 0$ and expresses a necessary condition for the decay of strong fields on the scale λ_D . It refers locally to the sheath edge and is definitely not a global condition for a monotonically varying sheath potential. In this sense equation (66) represents the most general sheath condition and should include all correct ‘generalizations’ of Bohm’s original condition. (The unspecific name ‘generalized Bohm criterion’ is avoided here because it is used too diversely.)

4.1. Kinetic and hydrodynamic formulations

To discuss various specifications and aspects of the sheath condition let us temporarily return to the assumption of completely absorbing walls $\alpha = 0, g = 0$. Introducing an effective temperature

$$\Theta = \left(- \frac{dn_-}{d\chi} \Big|_{x=0} \right)^{-1} \tag{78}$$

of the negative charge carriers we obtain from (66) the kinetic form of the Bohm criterion

$$\langle y^{-1} \rangle_0 = \int_0^\infty \frac{f_0(y)}{y} \frac{dy}{y^{1/2}} \leq \frac{2}{\Theta} \tag{79}$$

$$\langle v_z^{-2} \rangle_0 \leq \frac{m_i}{\Theta k T_e}$$

due to Harrison and Thompson (1959) and Boyd and Thompson (1959). (As stated above we want to avoid the term ‘generalized Bohm criterion’ frequently used for the kinetic form. The name ‘kinetic’ should not be confused with a ‘dynamic’ form used in the analysis of moving sheaths (see section 2.6).) Note that (79) is easily extended to account for more species i of positive ions:

$$\sum_i \frac{N_i e_i}{m_i} \langle v_z^{-2} \rangle_{i0} \leq \frac{1}{\Theta k T_e} \sum_i N_i e_i. \tag{80}$$

For simplicity reference is again made to one positive ion species.

To compare the condition with simplified versions referring to the mean ion velocity or kinetic energy we apply Schwarz’s inequality and find

$$\langle v_z^2 \rangle_0 \geq \langle v_z \rangle_0^2 \geq \langle v_z^{-2} \rangle_0^{-1} \geq \Theta \frac{k T_e}{m_i} \tag{81}$$

i.e. the simplified versions are fulfilled *a fortiori* if the

kinetic criterion holds. To obtain a consistent hydrodynamic formulation we assume $v_z = u + \xi$, $|\xi| \ll u$, $\langle \xi \rangle = 0$ and calculate

$$\begin{aligned} \langle v_z^{-2} \rangle_0 &= u^{-2} (1 - 2\xi/u + 3\xi^2/u^2 - \dots) \\ &\approx [u^2 - 3\langle \xi^2 \rangle]^{-1}. \end{aligned}$$

With $u = \langle v_z \rangle$ and $\langle \xi^2 \rangle = kT_i/m_i$ this results in the hydrodynamic Bohm criterion

$$m_i \langle v_z \rangle_0^2 \geq \Theta k(T_e + 3T_i) \quad (82)$$

showing that a consistent fluid approach must become one-dimensionally adiabatic at the sheath edge ($\gamma = 3$ in equation (39), see Zawaideh *et al* (1986)). This expresses the fact that there is no interchange of the parallel and perpendicular ion velocities on the Debye length scale.

4.2. Special ion distributions

The hydrodynamic approach ignores the delicate contribution of slow ions to the mean value in (79). Obviously $f_0(0)$ (and $f^-(y)$) must be zero and a shifted Maxwellian ion distribution cannot fulfil the sheath condition. The same is true for a Maxwellian half distribution (Tonks and Langmuir 1929). To overcome this difficulty some investigations (e.g. Chekmarev and co-workers 1972, 1983, 1984, Main and Lam 1987) use shifted Maxwellian half distributions with a gap extending from zero to a minimum energy. The results of such models are inconveniently involved and rather questionable due to the artificial gap neglecting slow ions.

In principle all specifications based on prescribed distributions f_0 must be refused: the Bohm criterion refers to the sheath-edge distribution f_0 which is uniquely determined from the presheath kinetics. However, if it is not known—and this will be the most frequent case—it appears reasonable and reliable to represent characteristic features by simple models. In this sense it is useful for special applications to represent different ion groups schematically by a distribution (Stangeby 1984a)

$$\begin{aligned} F_i(v_z) &= \sum_{\nu} \alpha_{\nu} F_{i\nu} \\ F_{i\nu} &= \frac{1}{2c_{\nu}} \begin{cases} 1 & u_{\nu} - c_{\nu} \leq v_z \leq u_{\nu} + c_{\nu} \\ 0 & \text{else} \end{cases} \quad (83) \end{aligned}$$

resulting in the special form

$$\sum_{\nu} \frac{\alpha_{\nu}}{u_{\nu}^2 - c_{\nu}^2} \leq \frac{m_i}{\Theta k T_e} \sum_{\nu} \alpha_{\nu} \quad (84)$$

of the Bohm criterion.

4.3. Electrons and negative ions

In a more accurate sense the Bohm criterion may be specified by distinguishing different components of negative particles. Two practically important examples are now given.

(a) The sheath in front of a 'hot cathode' may be modelled by Boltzmann distributed plasma electrons and a cold beam of emitted electrons (Hobbs and Wesson 1967, Prewett and Allen 1976):

$$n_-(\chi) = n_{ep} + n_{ec} = \frac{e^{-\chi} + \kappa(\chi_c - \chi)^{-1/2}}{1 + \kappa\chi_c^{-1/2}}. \quad (85)$$

(An analysis accounting for the temperature of the emitted electrons has been given by Schcherbinin 1973.)

This results in the effective temperature (see equations (78), (79))

$$\Theta = \frac{1 + \kappa\chi_c^{-1/2}}{1 - (\kappa\chi_c^{-3/2})/2} > 1 \quad (86)$$

intensifying the condition imposed on f_0 . It should be noticed that in spite of the validity of Bohm's criterion a double layer with negative space-charge in front of the cathode is formed (Langmuir 1929, Prewett and Allen 1976).

(b) An admixture of negative ions (or electrons with different temperature) may be represented in the form (Boyd and Thompson 1959, Takamura 1989)

$$n_-(\chi) = \frac{e^{-\chi} + \kappa e^{-\gamma\chi}}{1 + \kappa} \quad (87)$$

resulting in

$$\Theta = \frac{1 + \kappa}{1 + \gamma\kappa} \quad (88)$$

intensifying or relaxing Bohm's condition depending on the temperature $1/\gamma$ of the admixture. Observe again that due to the hotter component the space charge may become negative within the sheath. It should be noted further that the relative concentration κ refers to the sheath edge and depends on the presheath potential drop. The resulting self-consistency problem is discussed by Braithwaite and Allen (1988) for the special case of a spherical probe collecting cold ions.

4.4. The marginal validity of Bohm's criterion

As we have seen, the sheath condition is—except for a few artificial models—fulfilled by the equality sign. There are essentially three interpretations illustrating and/or establishing this equality.

(a) *Continuous density variation.* Bohm's criterion may be written in the form

$$\left. \frac{dn_+}{d\chi} \right|_{\chi=0+} \geq \left. \frac{dn_-}{d\chi} \right|_{\chi=0+} \quad n_+ = \sum_i n_i \quad (89)$$

and the equality sign appears to be a natural consequence of the quasi-neutral presheath solution $n_+(\chi) = n_-(\chi)$, $\chi \leq 0$ (Allen and Thonemann 1954, Chen 1965, Bissel 1987).

(b) *The breakdown of the quasi-neutral solution.* As discussed in section 2.3 the field singularity $k' \rightarrow 0$ due

to the sound barrier forcing the onset of space-charge formation. (The sheath edge as a Mach surface has been dealt with by Stangeby and Allen (1970) and Andrews and Stangeby (1970)). $k' \rightarrow 0$ is also related to the equality sign in the kinetic criterion (Cavaliere *et al* 1965, Riemann 1977, 1980).

(c) *Information loss at the absorbing wall.* Ion-acoustic waves cannot be propagated backwards from the sheath edge, because the flow there becomes supersonic (Bertotti and Cavaliere 1965, Cavaliere *et al* 1968, Franklin 1976 ch 9). The relation of Bohm's criterion to the sound velocity remains true also if the changed conditions ($\Theta \neq 1$) due to electron emission (Prewett and Allen 1976) or negative ions (Braithwaite and Allen 1988) are considered. The kinetic Bohm criterion corresponds to a root $\omega/k = 0$ of the ion-acoustic dispersion relation (Allen 1976, Raadu and co-workers 1988, 1989).

All three arguments interpret the equality sign from presheath considerations. 'Approaching the problem from the sheath side evidently always results in this type of ambiguity' (\geq sign, Stangeby 1986). The reason is clear: the sheath condition with the \geq sign is a necessary condition and must hold in any case. The equality sign, however, has—albeit somewhat artificial—exceptions: the statement is more distinct but less safe. Consequently the convincing arguments cannot be wholly conclusive.

Concerning the first argument (a) the limit $0+$ must be observed, which cannot be derived from quasi-neutrality. (The same argument applied to all higher derivatives would result in $n_+ = n_-$ everywhere.) With respect to the second argument (b) we know from the discussion in section 2.5 that the quasi-neutral solution does not in any case break down at sound velocity and that space-charge formation can—without singularity—also be due to the boundary conditions. The third argument (c) finally remains valid, even if Bohm's criterion is oversatisfied (Raadu and Rasmussen 1988 Appendix A).

After this critical discussion we should bear in mind, however, that the equality sign in Bohm's criterion represents the normal case. Carefully applied, the marginal condition provides a suitable means to formulate boundary conditions both for the presheath (Bissel and Johnson 1987, see section 6.3) and for the sheath region. Especially in the form of equation (89), it has been systematically used by Allen and co-workers to provide boundary conditions in the investigation of double layers (Andrews and Allen 1971, Prewett and Allen 1976, Allen 1985). It should be observed, however, that the validity of the equality form for double layers is not generally accepted, (see e.g. Raadu 1989 section 3).

4.5. Ion reflection and emission

In section 2.5 some arguments were resolved against the validity and necessity of the Bohm criterion by a

systematic application of the two-scale concept. One argument cannot be resolved in this way: the Bohm criterion is based on the assumption of completely absorbing walls and may be essentially affected by ions coming from the wall. Such ions can produce a contribution $n_w(\chi)$ to the ion density, growing with χ and relaxing—or even cancelling?—Bohm's condition of the presheath acceleration. On the other hand, we have in many applications little emission and small reflection coefficients. Moreover, ions emitted from sufficiently negative walls are trapped near the surface and cannot come to the sheath edge.

The pioneering works concerning the influence of wall processes on the sheath conditions are the papers by Hu and Ziering 1966 (specular and diffuse reflection) and Hassan 1968 (thermal accommodation and contact ionization). Both (rather involved) analyses represent the plasma ions by shifted half Maxwellian distributions. Attempting to remove the requirement of the artificial low-energy gap in the ion distribution, Hu and Ziering determined a 'permissible range of the ion reflection coefficients'. This argument appears strange, as indeed does the use of truncated Maxwellian sheath edge distributions.

There is an abundance of detailed information on the involved plasma wall interaction processes (see e.g. Post and Behrisch 1986). Such complex problems will not be considered in detail, since the interest of this review lies in the overall representation of ions coming from the wall to the sheath edge. This is done, more or less schematically, by the coefficient α and the function g in equation (47) and in the resulting sheath condition (66). (Using this notation we put aside the problem of coupling f_0 and g in the plasma and at the wall surface.)

To reach a comprehensible interpretation of the influence of α and g on the sheath condition let us introduce the ion distribution

$$f_w^+(\chi, y) = f_w^-(\chi, y) = \frac{g(\chi_w - \chi + y)}{1 - \alpha} \quad (90)$$

resulting from a given g , if the plasma was *not* present, and the corresponding density

$$\begin{aligned} n_w(\chi) &= \int_0^\infty \frac{f_w^+ + f_w^-}{y^{1/2}} dy \\ &= \frac{2}{1 - \alpha} \int \frac{g(\chi_w - \chi + y)}{y^{1/2}} dy. \end{aligned} \quad (91)$$

The influence of the plasma is then represented by an additional ion distribution

$$f_p(y) = f_p^+(y) + f_p^-(y)$$

with

$$f_p^-(y) = \alpha f_p^+(y) \quad (92)$$

and

$$f_p^+(y) = f_i^+(0, y) - f_w^+(0, y) = f_0(y) - \frac{g(\chi_w + y)}{1 - \alpha}$$

normalized by

$$\int_0^\infty y^{-1/2} f_p(y) dy = 1 - n_w(0). \quad (93)$$

Inserting equations (91)–(93) into (66) we obtain the sheath condition in the form

$$\begin{aligned} \int_0^\infty \frac{f_p(y)}{y^{3/2}} dy &= [1 - n_w(0)] \langle y^{-1} \rangle_p \\ &\leq 2 \left(\frac{dn_w}{d\chi} - \frac{dn_-}{d\chi} \right)_{\chi=0} \end{aligned} \quad (94)$$

and conclude the following.

(i) Ion emission (diffuse reflection, contact ionization) relaxes the sheath condition firstly by a density contribution n_w increasing with χ , $dn_w/d\chi > 0$, and secondly by a reduced share $1 - n_w(0)$ of plasma ions with decreasing density.

(ii) Specular reflection has only an indirect influence via the enhancement of n_w . In the case of no emission specular reflection does not affect the Bohm criterion because positive and negative velocities result in the same dynamic density variation.

To demonstrate this explicitly let us consider the example of a Maxwellian emission

$$g(y) = \kappa \sqrt{\frac{\beta}{\pi}} e^{-\beta y} \quad (95)$$

$$n_w(\chi) = \frac{2\kappa}{1-\alpha} e^{\beta(\chi-\chi_w)}.$$

Using again the abbreviation (78) we obtain

$$\begin{aligned} \langle y^{-1} \rangle_p &\leq \frac{2}{\Theta} \frac{1 + \beta\Theta q}{1 - q} \\ \langle v_z^{-2} \rangle_p &\leq \frac{m_i}{\Theta k T_c} \frac{1 + \beta\Theta q}{1 - q} \end{aligned} \quad (96)$$

with

$$q = n_w(0) = \frac{2\kappa}{1-\alpha} \exp(-\beta\chi_w)$$

and distinctly see the relaxation of the sheath condition with growing q . For $q \rightarrow 1$ finally (an academic case!) the RHS of (96) tends to infinity and there remains no sheath condition at all: the charge balance is now completely governed by thermal ions and we have Debye screening. However, as long, as this limiting case is not attained, there remains a (relaxed) sheath condition, and this remaining condition will, apart from the exceptions discussed, be fulfilled with the equality sign and will be related to a presheath field singularity. This singularity, however, can no longer be interpreted as ‘breaking the sound barrier’.

5. The plasma boundary layer: matching and model zones

The plasma sheath transition has been considered in the limit $\varepsilon \rightarrow 0$ on separate presheath and sheath scales

x , $\xi = x/\varepsilon$ yielding separate solutions $Y_p^0(x)$ and $Y_s^0(\xi)$ for any physical quantity Y . To account for finite ε the two-scale analysis may in principle be extended to higher order approximations $Y_p^\nu(x)$ and $Y_s^\nu(\xi)$ correct to the (possibly fractional) order ν in ε . Y_s^ν and Y_p^ν refer still to separate scales and must be related by a ‘matching principle’ assuming that there is an overlapping region where the approximations can be adapted to each other. A mathematical formulation of the matching principle proposed by Van Dyke (1964) can be used to determine free constants and allow one to construct uniformly (on both scales) valid ‘matched asymptotic expressions’. For the details of the method see, e.g., Van Dyke (1964) and Nayfeh (1973, 1981). I do not want here to apply the technique explicitly, but want to discuss a fundamental difficulty arising from it: the presheath and sheath solutions have no overlapping region.

If c_m is the first non-vanishing coefficient in equation (59), we obtain from (61) the limiting variation

$$\chi_s^0 \sim |\xi|^{-4/(m-2)} \quad \xi \rightarrow -\infty, m > 2 \quad (97)$$

of the sheath solution. Integrating (73) we find correspondingly

$$\chi_p^0 \sim |x|^{1/(1+b)} \quad x \rightarrow 0 \quad (98)$$

for the plasma solution. Apart from the common limiting sheath edge value $\chi_p^0(0) = \chi_s^0(-\infty) = 0$ there is obviously no overlap region to match the solutions smoothly (cf the discussion of the ‘contradictory’ presheath and sheath limits of the sheath edge field in section 2.5). So far this problem has been avoided by expressing all quantities as functions of χ rather than x or ξ .

According to a concept described by Kaplun (1967) the difficulty is overcome by introducing an ‘intermediate scale’ suitable for describing a transition layer accounting for (weak) space charges as well as for (weak) presheath processes. To find the appropriate scale transformation

$$x = \delta \zeta \quad \chi = \omega w \quad \varepsilon \ll \delta, \omega \ll 1 \quad (99)$$

we compare the space charge

$$\rho_p = \varepsilon^2 \frac{d^2 \chi_p^0}{dx^2} \sim \varepsilon^2 |x|^{-(1+2b)/(1+b)} \quad b \neq 0 \quad (100)$$

produced by the curvature of the quasi-neutral solution (cf equation (98)) with the space charge (cf (59))

$$\rho_s \sim \chi^{m/2} \quad (101)$$

of the sheath solution. Equating the orders of magnitude and assuming $x = 0(\delta)$ and $\chi = 0(\omega)$ (i.e. $\zeta, w = 0(1)$) we get from equations (98), (100) and (101)

$$\delta = \varepsilon^{(4+4b)/(m+2+4b)} \quad \omega = \varepsilon^{4/(m+2+4b)}. \quad (102)$$

The same result is obtained by equating the orders of magnitude of the potentials (97) and (98). From the

discussion of section 3.3 we expect $b = \frac{1}{2}$ and $b = 1$ respectively. Since the field singularity causes the marginal validity of the sheath condition we have $c_2 = 0$ and expect $m = 3$ in a kinetic and $m = 4$ (due to the existing expansion in χ rather than in $\chi^{1/2}$) in a hydrodynamic approach. We summarize these values as follows.

	m	b	δ	ω	
Kinetic approach with regular source	3	$\frac{1}{2}$	$\varepsilon^{6/7}$	$\varepsilon^{4/7}$	(103)
Kinetic approach with singular (cold) source	3	1	$\varepsilon^{8/9}$	$\varepsilon^{4/9}$	
Hydrodynamic approach	4	1	$\varepsilon^{4/5}$	$\varepsilon^{2/5}$	

(In the case of an oversatisfied sheath condition ($m = 2$) there is no field singularity and no transition layer; the sheath field decays exponentially.) It is remarkable that the scaling depends on the kind of approach. This is due to a different modelling of the production of slow ions. On the other hand it reflects the fact that there is no dominant process 'stabilizing' the solution in an overlapping region. This is in sharp contrast to the collision dominated 'continuum case' $L \gg \lambda_D \gg \lambda$ (Su and Lam 1963, Cohen 1963, Blank 1968), where plasma and sheath can be matched without an additional transition layer, because the electric field is everywhere balanced by ion friction (resistivity).

In the scaling $\delta = \varepsilon^{4/5}$ of the hydrodynamic approach we recognize the transition region discussed in section 2.4 (see equation (34)). The smooth transition of figure 6(b) indicates that the two 'ends' of an intermediate solution indeed provide the overlaps required for matching with the plasma and the sheath solution. This matching (with hydrodynamic scaling) was performed by Lam (1965, 1967) and by Su (1967) in the calculation of spherical probe characteristics. Franklin and Ockendon (1970) performed a systematic higher order matching for the Tonks–Langmuir model of the collision free low-pressure column (see section 6.3) hydrodynamically and kinetically. They were the first to find the kinetic $\varepsilon^{8/9}$ scaling. The same scaling was obtained by Riemann (1978, 1979) for the transition layer of the charge exchange model. (The results are shown in section 6.2, see figure 12). The transition layer of the 'regular case' with the expected $\varepsilon^{6/7}$ scaling has, apparently, not yet been tackled.

To avoid confusion with other 'transition regions' (and even different notations) it should be observed that a complete plasma–boundary analysis may involve various further model zones: in the plasma region (scale Λ) it may be convenient to neglect ion inertia and/or it may be necessary to account for additional processes outside the intrinsic presheath (scale L). On the sheath side a further model zone is formed, if high

voltages are applied: considering the limit $\chi \gg 1$ we recover from equation 9 the 'Child–Langmuir law'

$$\chi^{3/2} = \frac{3}{4} y_0 (\xi + \text{constant})^2 \quad (104)$$

describing the (unipolar) ion sheath. Its thickness

$$d \sim \chi^{3/4} \lambda_D \quad (105)$$

may finally cause the need to consider a 'thick sheath' accounting for collisions, geometry etc, thus 'inviting' the introduction of further model zones and transition regions (Lam 1967).

A list of different notation used in literature is presented in table 1. For comparison, also accounted for are the corresponding regions of the continuum model $\lambda_D \gg \lambda$. Note that in all cases the Debye length λ_D refers locally to the sheath edge and not to the plasma centre. If the electron/ion density of the boundary layer is related by $n = 0(n_0 L / \Lambda)$ to the plasma density n_0 , we have $\lambda_D \sim (\Lambda / L)^{1/2} \lambda_{D0}$. This 'enhancement' of the Debye length should be observed in estimating or interpreting the sheath thickness of collision dominated plasmas (Metze *et al* 1989, Valentini 1989). The continuum case $L = \lambda_D$ results in a sheath thickness $L = \lambda_D \sim \Lambda^{1/3} \lambda_{D0}^{2/3}$.

6. Special problems

Since the solution of Poisson's equation in the sheath can be reduced to straightforward integrations, if the sheath edge distribution is known, the analysis of special problems is essentially concerned with the presheath solution. This analysis is seriously complicated by the self-consistency problem arising from the simultaneous determination of the ion distribution function and of the quasi-neutral potential variation. In the following the problems in general are not considered, or the solution methods: rather specific aspects in the light of general conclusions on the Bohm criterion and sheath formation are discussed.

6.1. Geometrical presheath: spherical probes

Probe theory is a wide field, and an introduction is given by Allen (1974); more comprehensive reviews are given by Kagan and Perel (1964), Chen (1965), Swift and Schwar (1970) and Chung *et al* (1975). This review concerns only the plasma–sheath transition of collisionless ($\lambda \gg R$), sufficiently negative probes. The basic ion collection mechanism is described by the cold-ion model of Allen *et al* (1957), which was used in section 2.4 to illustrate the space-charge formation. The thin ($\sim \lambda_D$) sheath covering the probe surface is surrounded by an extended ($L \sim R$) presheath region accelerating the ions. At Bohm's velocity v_B the presheath ends in a sheath edge with linear field singularity ($b = 1$, see equations (73) and (103)). As a consequence, the ion (saturation) current to the probe is determined by the plasma density and electron temperature and does not depend on the

Table 1. Nomenclature and scaling of various boundary layer model zones.

$\lambda_D \ll \lambda$	Plasma, presheath		Transition layer	Debye	Sheath ion
	Λ	L		$\lambda_D = \lambda_{D0}(\Lambda/L)^{1/2}$	$\chi_w^{3/4} \lambda_D$
Characteristic extension			$\lambda_D^{4/5} L^{1/5} \dagger$		
Boyd (1951)	normal extra sheath region		abnormal extra sheath region		sheath
Ecker <i>et al</i> (1962, 1965, 1966)	zone of mobility limited motion		zone of inertia limited motion		sheath
Lam (1965)	} quasineutral region		1. transition layer	2. transition layer	sheath
Lam (1967)			transsonic layer	thin collisionless layer	(collisionless) thick sheath
Su (1967)	diffusion dominated region	kinetic neutral region	transition region	collisionless sheath	
Franklin and Ockenden (1970)		outer region	intermediate region		inner region
Franklin (1976)		plasma	presheath		sheath
Chodura (1982)§	plasma presheath	magnetic presheath			sheath
Chodura (1986, 1988)§	presheath	magnetic sheath			electrostatic (Debye) sheath
Main (1987)			← presheath →		
Zawaideh <i>et al</i> (1990)			← sheath →		
$\lambda_D \gg \lambda$	Plasma			Debye	Sheath ion
Characteristic extension	Λ			$\lambda_D = \lambda_{D0}^{2/3} \Lambda^{1/3}$	¶
Su and Lam (1963)	} quasineutral region			transitional region	(ion) sheath
Cohen (1963)				thin space charge sheath	ion sheath
Lam (1967)				merged sheath edge layer	(mobility) thick sheath
Blank (1968)		plasma region			sheath region

† Scaling of the fluid approach, in kinetic approaches slightly different, cf equation (103).
 ‡ Ion inertia neglected.
 § See section 2.6.
 || No plain classification possible.
 ¶ Depending on the collision model.

probe voltage. (A weak dependence is due to the finite sheath thickness (cf equation (105)) increasing the effective probe radius.)

This simple model is essentially complicated if a finite ion temperature is taken into account. The angular momentum connected with the thermal motion acts like a repulsive (centrifugal) force and hinders a part of the ions attracted by the electric field from reaching the probe surface; moreover there may exist trapped ions on 'planetary' orbits. The pioneer work tackling this problem is due to Bernstein and Rabinowitz (1959). In this famous investigation the thermal motion is represented by monoenergetic ions with random direction. An extension to a full Maxwellian ion distribution was given by Laframboise (1966). The

numerical results for finite λ_D/R show a considerable influence of the ion energy on the probe current. Lam (1965) supplemented the monoenergetic model by a systematic asymptotic theory $\lambda_D/R \rightarrow 0$ distinguishing four different model regions (see section 5, table 1). Despite using the kinetic approach he obtained the linear field singularity ($b = 1$) and the $\epsilon^{4/5}$ scaling of the transition layer typical for a fluid approach (see equation (103)). This is obviously due to the artificial monoenergetic ion assumption. Parrot *et al* (1982) presented a presheath analysis accounting for a full Maxwellian ion distribution. Their numerical results seem to confirm the square root field singularity ($b = \frac{1}{2}$, cf equations (73) and (103)) expected for a kinetic analysis with a regular source.

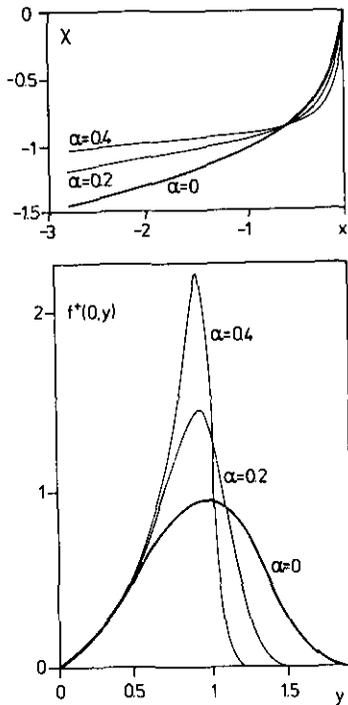


Figure 11. Presheath potential variation and sheath edge ion distribution function for the 'cold' C-X model (equation (106)) of the collisional presheath ($\alpha = 0$). The curves $\alpha > 0$ account for specular ion reflection at the wall.

6.2. Collisional presheath

The collision dominated plasma is usually described in terms of diffusion and mobility. For the quasi-neutral plasma this approach breaks down near the boundary with a potential tending to infinity (Schottky 1924). This is also true if the decreasing mobility in the high field of the boundary layer is accounted for (Boyd 1951, Frost 1957). To avoid this breakdown Boyd (1951) used the Bohm criterion to define a cut-off, where he merged the quasi-neutral mobility controlled region ('extra sheath') with the collision free sheath. Of course, this crude approach can yield only qualitative results. Persson (1962) was apparently the first to recognize the crucial role of ion inertia also in the sheath transition of the collision dominated plasma. His fluid approach results in the appropriate sheath edge structure. A rigorous analysis of the inhomogeneous, inertia influenced boundary layer (presheath), which has a typical extension of an ion mean free path, should, however, use kinetic methods.

As discussed in section 4.2 half kinetic methods based on discontinuous Maxwellian distributions (e.g. Chekmarev *et al* 1983, 1984) are not suitable to describe the sheath edge correctly. The full kinetic treatment accounting for the self-consistency problem depends on a sufficiently tractable ion collision term. A suitable model distinguished both by physical relevance and by mathematical simplicity is provided by symmetric charge-exchange (C-X) with neutral atoms. Two reasonable specifications of the collision term

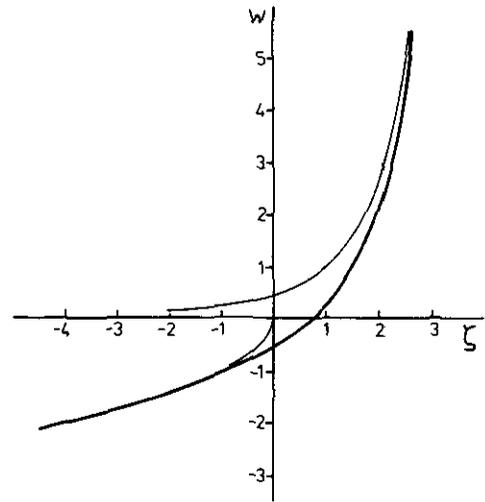


Figure 12. Intermediate transition layer solution (bold full curve) of the 'cold' C-X model (106) matching the presheath and sheath approximations (fine full curves). For ζ and w see equation (108).

(source function, cf equations (46) and (76)) allow one-dimensional ion kinetics (see, e.g. Biehler *et al* 1988).

(a) C-X with cold neutrals, constant mean free path:

$$s^+(\chi, y) = y^{1/2} [j_i \delta(y) - f^+(\chi, y)] \quad s^- = 0 \quad (106)$$

(b) C-X with constant collision frequency:

$$s^\pm(\chi, y) = n_i(\chi) f_N(y) - f_i^\pm(\chi, y). \quad (107)$$

Model (a) was treated first with numerical methods and iterative approximations by Bakhst *et al* (1969). Since there are always ions undergoing collisions (resulting in zero velocity!) near the sheath edge these authors claimed that Bohm's criterion could not be satisfied, and a monotonic sheath potential could only be obtained if a finite initial field and collisions were considered in the sheath.

In a later investigation the self-consistent presheath problem was solved analytically (Riemann 1977, 1981); the resulting potential variation and sheath edge ion distribution function are plotted in figure 11 ($\alpha = 0$). In contrast to the above claim and in accordance with the discussion in section 3 the analytic results proved that the kinetic Bohm criterion holds exactly with the equality sign. The potential variation starts from a weak plasma field (due to the ion friction) and runs into a linear sheath edge field singularity ($b = 1$, see equations (73) and (103)), which is typical for the δ singularity in the source function (106). According to (99) and (103) the intermediate scale of the transition layer linking the presheath and sheath is given by the transformation

$$x = \epsilon^{8/9} \zeta \quad \chi = \epsilon^{4/9} w. \quad (108)$$

The zero-order basic equation describing the transition layer and numerical approximations have been given by Riemann (1978, 1979). Figure 12 shows the intermediate solution together with the leading terms

(97) of the sheath solution and (98) of the presheath solution. Physically, the transition layer is characterized by the first space-charge formation due to the high density contribution of *slow* ions emerging from collisions. In comparison with that, the loss of fast ions due to collisions can be neglected. Therefore, collisions on the intermediate scale have the same effect as ionization processes. This is the intrinsic reason why the scaling (equation (108)) agrees with that of the collisionless Tonks–Langmuir model (Franklin and Ockendon 1970, see section 6.3).

In section 4.5 it has been seen that specular ion reflection without emission has no influence on the Bohm criterion. This fact was used to construct an approximation for the charge–exchange presheath with reflecting walls (Riemann 1985). To exhibit the effect more distinctly figure 11 shows results for extremely high reflection coefficients, $\alpha = 0.2$ and 0.4 . With increasing α the ion current to the wall and the driving electric field in the plasma decrease. Nevertheless, to fulfil the Bohm criterion, the field immediately in front of the sheath edge must be increased. The resulting steep (nearly collisionless) potential variation is reflected in a sharp peak of the sheath edge ion distribution function.

The C – X model (b), which is suitable to account for a finite neutral gas temperature, was first investigated by Ecker and co-workers (1966) and subsequently by Shcherbinin (1972), but a self-consistent numerical solution was only obtained by Biehler *et al* in 1988. The results show a decrease of the presheath potential drop with increasing neutral temperature, which can be understood from the contribution of the thermal motion to the velocity average required by the Bohm criterion. All potential curves run into a sheath edge field singularity. The type of the singularity was not discussed by Biehler *et al* and the resolution of the numerical grid was not sufficient to verify the expectation $b = 1/2$ (see equations (73) and (103)).

6.3. Ionizing presheath: the Tonks–Langmuir model

The famous model of Tonks and Langmuir (1929) is the oldest problem of the plasma–sheath transition, the one most investigated, and—in a certain sense—the most important problem. The whole plasma is not only *influenced* by its boundary, but the whole plasma *is* the presheath. The model refers to the collision free ($\lambda \gg R$) low-pressure column. Physically it is characterized by the free fall of ions originating from ionization of (cold) neutrals. The problem was investigated in cylindrical and in plane geometry. The plane model, originally designed to elucidate the basic mechanism in a simplified analysis, has attracted new interest as a simple model for the one-dimensional plasma flow in magnetic flux tubes (Emmert *et al* 1980, Stangeby 1984b).

Tonks and Langmuir treated the problem kinetically, introduced the subdivision in separate plasma and sheath regions and solved the plasma

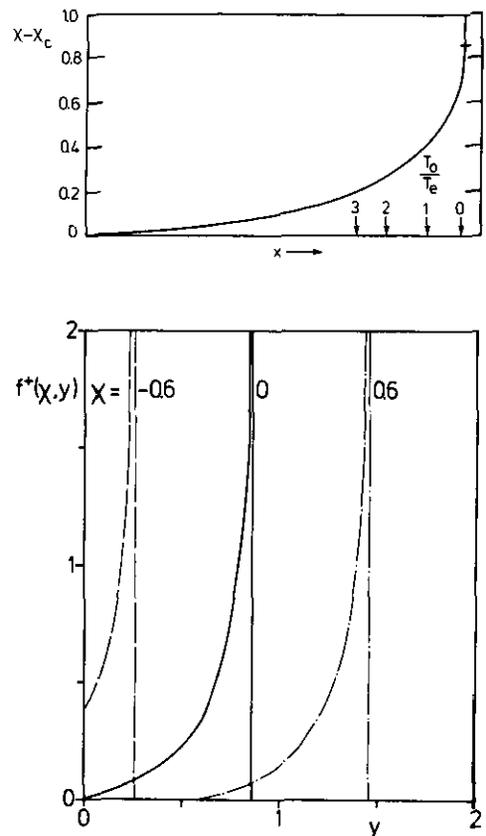


Figure 13. Presheath potential variation and ion distribution function for the ‘cold’ Tonks–Langmuir model (109), $\sigma = \text{const}$. The arrows refer to Emmert’s ‘warm’ plasma model (equation (110)) $T_0/T_e > 0$.

equation by series expansion. Harrison and Thompson (1959) found the closed analytic solution in plane geometry and Caruso and Cavaliere (1962) re-investigated the plane problem with emphasis on a systematic two-scale formalism. Self (1963) and Parker (1963) presented numerical solutions (in plane and cylindrical geometry) for finite λ_D/L avoiding the subdivision in plasma and sheath regions and Self (1965) obtained numerical solutions to the asymptotic problem $\lambda_D/L \rightarrow 0$ in various geometries. Woods (1965) and Kino and Shaw (1966) showed that a simple fluid approach is suitable to describe basic features of the system with reasonable agreement. This provided a basis for numerous subsequent investigations accounting for various additional effects. The quasi-neutral Harrison–Thompson solution, which is based on the singular source function (cf equations (46) and (76))

$$s^+(\chi, y) = \sigma(\chi) \delta(y^{1/2}) \quad s^- = 0 \quad (109)$$

is (for $\sigma = \text{constant}$) presented in figure 13. The potential curve starts with zero field at the centre of the symmetric plasma (only one half-space is shown) and ends with a sheath edge field singularity. The ion distribution function shows a singularity representing the

ions generated in the plasma centre. This singularity disappears in more realistic geometries (Self 1965). Despite the completely different shape of the sheath edge ion distribution (cf figures 11 and 13) the plasma sheath transition has the same structure as that of the cold C - X model (cf (103) and discussion in section 6.2)). The electric field shows a linear singularity ($b = 1$), the sheath edge transition layer is characterized by a $\varepsilon^{8/9}$ scaling (Franklin and Ockendon 1970) and the kinetic Bohm criterion is fulfilled marginally.

The relevance of the Bohm criterion for the model was questioned by Auer (1961) with arguments based on the seeming contradiction of \leq and \geq signs resulting from the presheath and sheath considerations. Formally such 'contradictions' are resolved if the equality sign in Bohm's criterion is observed (cf Section 2.5). From specific physical considerations (already indicating the role of a transition layer) Auer's criticism was discussed by Franklin (1962).

A surprising aspect entered the discussion of the Tonks-Langmuir model (and of the plasma-sheath transition in general), when Emmert *et al* (1980) presented their investigation of a 'finite ion temperature' plasma based on the regular source function

$$s^+(\chi, y) = s^-(\chi, y) = \sigma(\chi)y^{1/2} \exp\left(-\frac{T_e}{T_0}y\right). \quad (110)$$

Emmert and co-workers were able to reduce the resulting 'warm' plasma equation by a similarity transformation to the 'cold' plasma equation of Harrison and Thompson. By this transformation the old ($T_0 = 0$) potential curve was cut off at a new ($T_0 > 0$) sheath edge as indicated by small arrows in figure 13. Obviously this result was the first example of a sheath edge without field singularity. As shown by Bissel (1987), the kinetic Bohm criterion is again fulfilled with the equality sign.

Physically, the factor $y^{1/2}$ in equation (110) appears to be somewhat artificial. Bissel and Johnson (1987) presented numerical results for a Maxwellian source

$$s^+(\chi, y) = s^-(\chi, y) = \sigma(\chi) \exp\left(-\frac{T_e}{T_0}y\right) \quad (111)$$

again showing the usual sheath edge singularity. The type of the singularity was not discussed, the plots seem to confirm the expectation $b = \frac{1}{2}$ (cf equations (73) and (103)). For a detailed discussion of the source models (equations (110) and (111)) and for a comparison with fluid theories reference is made to the review by Bissel *et al* (1989).

The analysis made by Bissel and Johnson used the equality form of the kinetic Bohm criterion as a *boundary condition*. Scheuer and Emmert (1988b) criticized this *imposition* of the Bohm criterion and reinvestigated the Maxwellian source problem without this boundary condition and reproduced the results of Bissel and Johnson. Simultaneously, they *verified* (within the numerical accuracy) the equality form of Bohm's criterion and left the reason for the equality both for this

problem and for Emmert's model as 'an interesting open question'. From the discussion of equations (76) and (77) in section 3.3 we know the reason. Interestingly, the equality sign can be attributed in the usual way to a field singularity only for the Maxwellian source (equation (111)) with $s(0, 0) \neq 0$. For Emmert's source (equation (110)) it is exclusively due to the symmetry $s^+ = s^-$ and no longer holds if this symmetry is disturbed in any way. This can be seen explicitly from an asymmetric modification of Emmert's model accounting for a superimposed plasma drift (van den Berg *et al* 1991): in contrast to the symmetric case the upstream facing sheath-edge shows a field singularity ($b = 1$ in this case). At the downstream facing sheath-edge the field remains finite, but Bohm's criterion is oversatisfied.

Another example is a generalization of Emmert's model to the flow in magnetic flux tubes with an open field configuration $dB/dx < 0$ (Sato *et al* 1989, Hussein and Emmert 1990). Again, the field remains finite and the Bohm criterion is oversatisfied. The same is true if Emmert's model is supplemented by a collision term of the form (76) with $\nu(0) = 0$ (Scheuer and Emmert 1988a). If, on the other hand, a more realistic collision model with $\nu(0) \neq 0$ is used (Koch and Hitchon 1989) or if the problem is described hydrodynamically (Scheuer and Emmert 1990) the sheath edge again exhibits the field singularity and Bohm's criterion holds marginally.

7. Summary

The potential distortion caused by a negative wall is shielded by a positive space-charge layer ('sheath'), whose thickness is characterized by the (local) electron Debye length λ_D . Usually the Debye length is the smallest characteristic length and the 'thin' sheath is planar and collision free. Necessary conditions for the formation of a thin shielding sheath is the validity of Bohm's criterion, which demands—in its simplest form—that the ions enter the sheath region with at least the velocity of ion acoustic waves (section 2.1). To fulfil this condition the ions must be pre-accelerated in a quasi-neutral 'presheath' region (section 2.2), which is, apart from the crucial role of ion inertia, dominated by (at least) one of the following processes:

- (a) geometric current concentration (sections 2.3 and 6.1);
- (b) collisional ion friction (sections 2.3 and 6.2);
- (c) ionization (sections 2.3 and 6.3);
- (d) magnetic ion deflection (section 2.6).

The mechanism of the presheath acceleration can be illustrated from a simple fluid model (section 2.3). The magnetic presheath remains poorly understood; an additional condition of supersonic plasma flow along the magnetic field lines postulated in literature is questioned (section 2.6). Depending on the presheath mechanism the characteristic extension L of the presheath is determined by the system geometry, by the ion mean free path, the ionization length or by the

ion gyroradius. In the case of a collision-free bounded plasma the presheath may extend over the whole plasma.

Except for very special cases, the sheath edge, i.e. the presheath–sheath interface, is indicated by a formal singularity of the presheath electric field, which is closely related to the marginal validity (equality form) of the Bohm criterion (sections 2.3, 3.3 and 4.4). Because of this singularity the plasma and sheath cannot be smoothly matched without introducing an additional transition layer (sections 2.4 and 5).

Bohm's original criterion refers to the simplified model of monoenergetic cold ions. It can be generalized to its kinetic form accounting for the full ion distribution function (section 4.1). The usual way to derive this kinetic form was criticized with mathematical reasons. A rigorous kinetic analysis, however, confirms the result (section 3.2). The kinetic Bohm criterion depends decisively on the contribution of slow ions and demands that the sheath-edge ion distribution function tends to zero sufficiently rapidly for vanishing ion energy. Distribution functions with artificial cut-offs (e.g. truncated Maxwellian distributions) cannot account for this contribution and give no reliable results (section 4.2).

The decisive influence of slow ions results in a delicate dependence of the sheath edge structure on the formation of slow ions by elementary processes. This is clearly seen from the type of sheath edge singularity and from the scaling of the sheath edge transition layer (section 5, equation (103)). For vanishing collision frequency and ionization rate at zero ion energy, the field edge singularity can disappear; in such cases the Bohm criterion may be oversatisfied (section 3.3 and 6.3). In all other cases the sheath edge exhibits the usual singularity and the Bohm criterion holds marginally. The equality form of the criterion may then be used as a boundary condition both for the sheath and for the presheath (sections 4.4 and 6.3).

The kinetic analysis generalizing Bohm's original criterion (section 3) is not restricted to absorbing walls, but can also account for reflecting and emitting walls. Specular ion reflection has no, or only indirect, influence on the Bohm criterion. By diffuse reflection and ion emission (contact ionization) the requirements of the criterion are relaxed and can, in principle, even be cancelled (section 4.5). This relaxation, however, does not alter the above statements concerning the marginal validity of the sheath condition and its relation to a sheath edge field singularity.

Arguments questioning the relevance or the validity of the Bohm criterion are in most cases based on a misinterpretation of the accurate meaning or on a confusion with respect to the model regions (section 2.5). Bohm's criterion refers to the sheath edge, which is uniquely defined (only) in the asymptotic limit $\lambda_D/L \rightarrow 0$. It definitely does not refer to a (more or less arbitrary) 'sheath boundary' where such an analysis begins to use Poisson's equation. Due to the asymptotic limit it is restricted to collisionless sheaths but not to collisionless plasmas. The

name 'criterion' (and particularly the terms 'criterion for a stable sheath' and 'criterion for a monotonic potential variation') are somewhat unfortunate if not misleading: Bohm's criterion governs the local structure of the solution at the sheath edge and cannot guarantee the global monotonic nature or stability of the sheath. (If a monotonic nature is required to exclude trapped ion orbits, additional conditions depending on the specific properties of the system must be considered.)

The Bohm criterion expresses a necessary condition for an electrostatic sheath field fading away in the plasma region, or considered from the other end, a necessary condition for an electrostatic potential growing up to fulfil the boundary condition at the wall. An oversatisfied Bohm criterion results in an exponential sheath potential variation; in the usual case of the marginal validity of the criterion the sheath potential decays according to some power law (sections 2.1 and 3.2). This decay refers to the (strong) electrostatic field on the 'small' Debye-scale λ_D of the sheath and does not exclude a penetrating (weak) quasi-neutral field on the 'large' scale L of the presheath. The violated Bohm criterion in the presheath region does not hinder the build-up of a monotonically growing (presheath) potential accelerating a subsonic plasma flow. On the contrary, this is established exactly by the presheath mechanism in order to fulfil the Bohm criterion at the sheath edge. The existence of oscillatory solutions in this region is no contradiction, but indicates the stability of the quasi-neutral solution with respect to electrostatic disturbances (sections 2.5 and 2.6).

The singularity characterizing the plasma sheath transition is no sign of an oversimplification, but indicates the transition to a 'smaller' scale governed by different physical processes. Of course, the singularity can be resolved if Poisson's equation is used everywhere—just as the discontinuity of the material wall can be resolved, if the whole system is described on an atomic scale using Schrödinger's equation. This uniform description, however, is hardly suitable to deepen the physical insight. With present computers it is no evidence of superior mathematical skill to produce uniformly valid solutions starting from Poisson's equation—on the contrary, in most cases it is far easier to obtain numerical solutions for special finite values $\lambda_D/L > 0$ than to find the asymptotic solution $\lambda_D/L = 0$. The value of asymptotic methods is based to a great extent on universal structures revealed by the limiting process. The universal structures of the wall sheath formation are closely related to the equality form of Bohm's criterion.

Acknowledgments

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Appendix 1. Expansion of the ion density

To evaluate the singular integral

$$J(\chi) = \int_0^\infty \frac{\bar{f}(y)}{(y + \chi)^{1/2}} dy \tag{A1}$$

in equation (54) we represent $\bar{f}(y)$ in the form

$$\bar{f}(y) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{-sy} sK(s) ds \tag{A2}$$

originating from Laplace-transformation technique (γ is a suitable constant). Interchanging the sequence of integrations and using the expansion (Abramowitz and Stegun 1967, p 302 and 297)

$$\begin{aligned} \int_0^\infty \frac{e^{-sy}}{(y + \chi)^{1/2}} dy &= \left(\frac{\pi}{s}\right)^{1/2} e^{s\chi} \operatorname{erfc}(s\chi)^{1/2} \\ &= \left(\frac{\pi}{s}\right)^{1/2} \sum_{\nu=0}^\infty \frac{(-1)^\nu}{(\nu/2)!} (s\chi)^{\nu/2} \end{aligned} \tag{A3}$$

we obtain

$$J(\chi) = \pi^{1/2} \sum_{\nu=0}^\infty \frac{(-1)^\nu}{(\nu/2)!} G\left(\frac{\nu+1}{2}\right) \chi^{\nu/2} \tag{A4}$$

with

$$G(n) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} s^n K(s) ds. \tag{A5}$$

The characteristic function $G(n)$ is related to the derivatives and moments of the distribution \bar{f} , from equations (A2) and (A5) we find

$$\bar{f}^{(m)}(0) = (-1)^m G(m+1) \quad m = 0, 1, 2, \dots \tag{A6}$$

and

$$\int_0^\infty y^m \bar{f}(y) dy = m! G(-m) \quad m \neq -1, -2, \dots \tag{A7}$$

Since for negative m the moments (A7) do not necessarily exist, one applies the more general relation

$$m! G(-m) = \frac{(-1)^k m!}{(m+k)!} \int_0^\infty y^{m+k} \bar{f}^{(k)}(y) dy \tag{A8}$$

obtained formally by partial integration of equation (A7). Especially with $m = -n - \frac{1}{2}$ and $k = n$ this yields

$$G\left(\frac{2n+1}{2}\right) = \frac{(-1)^n}{(-1/2)!} \int_0^\infty y^{-1/2} \bar{f}^{(n)}(y) dy. \tag{A9}$$

From (A4), (A6) and (A9) we obtain

$$J(\chi) = \sum_{\nu=0}^\infty \alpha_\nu \chi^{\nu/2}$$

with

$$\begin{aligned} \alpha_{2n} &= \frac{(-1)^n}{n!} \int_0^\infty y^{-1/2} \bar{f}^{(n)}(y) dy \\ \alpha_{2n+1} &= \frac{(-1)^{n+1} \pi^{1/2}}{(n + \frac{1}{2})!} \bar{f}^{(n)}(0). \end{aligned} \tag{A10}$$

Note that the coefficients with an even index are

exactly the coefficients of the formal Taylor series. They can conveniently be combined with the expansion coefficients of the second integral in equation (54) which have the same form. Observing the identity

$$\frac{1 + \alpha}{1 - \alpha} \bar{f}(y) + \frac{2}{1 - \alpha} g(y + \chi_w) = f^+(0, y) + f^-(0, y) \tag{A11}$$

then results in the density expansion given in equation (56).

Appendix 2: Integration of Boltzmann's equation

Integrating Boltzmann's equation (46) along the characteristics $\chi - y = \eta$ of the LHS (lines of constant total energy $-\eta$) yields

$$\begin{aligned} f_i^\pm(\chi, \chi - \eta) &= \pm \int_0^\chi \frac{k'(\psi)}{(\psi - \eta)^{1/2}} \\ &\quad \times s^\pm(\psi, \psi - \eta) d\psi + h^\pm(\eta) \end{aligned} \tag{A12}$$

where the integration 'constants' h^\pm , which depend on η , must be determined from the boundary conditions. From (70) we obtain

$$h^-(\eta) = \alpha h^+(\eta) + g(\chi_w - \eta). \tag{A13}$$

A second relation is obtained from the boundary condition (48) in the turning points $\eta = \chi$ which results in

$$h^+(\eta) = h^-(\eta) + \int_\eta^0 \frac{k'(\eta)}{(\psi - \eta)^{1/2}} s(\psi, \psi - \eta) d\psi \tag{A14}$$

with

$$s(\chi, y) = s^+(\chi, y) + s^-(\chi, y). \tag{A15}$$

Strictly speaking a further boundary condition is needed relating f_i^+ and f_i^- at 'the other end' of the plasma, because (48) and (A14) are restricted to those 'orbits' which have a turning point in the plasma, i.e. to $\eta > \chi_{\min}$ if χ_{\min} designates the potential minimum in a bounded plasma. (Observe further that the 'orbits' $\eta = \text{constant}$ are not necessarily the real ion orbits, because $S = C + R$ contains phase space convection terms, see equation (42).) Because we are only interested in slow ions near the sheath edge let us restrict ourselves to kinetic energies $y < \chi - \chi_{\min}$ which are correctly represented by equation (A14).

Solving (A13) and (A14) for $h^\pm(\eta)$ we obtain from (A12)

$$\begin{aligned} f_i^{\pm}(\chi, \chi - \eta) &= \frac{+}{(-)} \int_0^\chi \frac{k'(\psi)}{\sqrt{\psi - \eta}} s^{\pm}(\psi, \psi - \eta) d\psi \\ &\quad + \frac{1(\alpha)}{1 - \alpha} \int_\eta^0 \frac{k'(\psi)}{\sqrt{\psi - \eta}} s(\psi, \psi - \eta) d\psi \\ &\quad + \frac{1}{1 - \alpha} g(\chi_w - \eta). \end{aligned} \tag{A16}$$

At the sheath edge especially $\chi = 0$, $\eta = -y$ which yields

$$f_1^{\pm}(0, y) = \frac{1(\alpha)}{1-\alpha} \int_0^y \frac{k'(-\psi)}{\sqrt{y-\psi}} s(-\psi, y-\psi) d\psi + \frac{1}{1-\alpha} g(\chi_w - \eta) \quad (\text{A17})$$

or (see equations (55) and (57))

$$\bar{f}(y) = \int_0^y \frac{k'(-\psi)}{\sqrt{y-\psi}} s(-\psi, y-\psi) d\psi \quad (\text{A18})$$

$$f(y) = \frac{1+\alpha}{1-\alpha} \bar{f}(y) + \frac{2}{1-\alpha} g(\chi_w + y).$$

Using the ansatz (72) and (73) we find to lowest order in y

$$\int_0^y \frac{k'(-\psi)}{\sqrt{y-\psi}} s(-\psi, y-\psi) d\psi \rightarrow s_0(0) k_0 y^{a+b+(1/2)} \times \int_0^1 t^b (1-t)^{a-(1/2)} dt \quad (\text{A19})$$

resulting in equation (74).

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