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GENERAL THEOREM ABOUT GRAVITATIONAL LENSING: EXTENDED RELATION AMONG THE NUMBERS OF IMAGES WITH DIFFERENT PARITIES

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ABSTRACT

The known theorem of gravitational lensing that describes the relationship between the numbers of images with different parities is valid in the case in which the density profile of the lensing object is smooth. In this paper, using the complex representation of the lens equation, we reformulate the known theorem. Our formulation not only gives the known theorem for the smooth mass density profile but also gives new results for singular mass densities such as point masses model and the multipole expansion model. In order to see how the theorem works in a real event, we give the numerical fittings of quadruple images of PG 1115+080 by the regular elliptical density model and by the singular multipole expansion model.

Subject heading: gravitational lensing

1. INTRODUCTION

Gravitational lensing provides us with a new tool to investigate massive structures in the universe, because gravitational light deflection is insensitive to the nature and physical states of the deflector except for the mass distribution. More than 20 multiple-imaged quasars have been discovered (Keeton & Kochanek 1996; see also Schneider, Ehlers, & Falco 1992 for general reviews). Assuming a simple galaxy model for deflectors of these lens systems, we can extract the mass distribution of the deflectors by numerical fitting of the image configurations. An elliptical potential (or density profile) or a circular potential plus shear that has a small number of free parameters is often used (Kochanek 1991; Wambsganss & Paczyński 1994; Witt, Mao, & Schechter 1994).

Recently, Keeton, Kochanek, & Seljak (1997) found that no quadruple lens system is well fitted by simple models such as singular isothermal density ellipsoids, although such models provide a reasonable statistical model for the whole observed lens system. Further, they found numerically that an additional shear term can drastically reduce χ^2 in fitting. However, the amplitudes of the additional shears required to produce the good fits are larger than their estimates for typical external shear perturbations. This suggests that the origin of the extra angular structure must be intrinsic to the primary lens galaxy in most cases. Although their estimates for external shear perturbations are not definitive, the suggestion implies that it is necessary to examine various lens models.

Besides simple phenomenological models such as the elliptical density models, we have the multipole expansion (MPE) model, which is obtained by expanding the mass distribution of a lens with respect to an impact parameter (Yoshida & Omote 1988; Kakigi, Okamura, & Fukuyama 1995; Fukuyama, Kakigi, & Okamura 1997). The MPE model is less popular than phenomenological models in lensing analysis but is very powerful and respects a universal structure of the mass distribution of a lens object (Fukuyama et al. 1997).

Thus, many lensing models with different parameters can fit the observed data to some degree. In addition, the number of degrees of freedom defined by the number of fitted observed data minus the number of fitting parameters is small at present. This situation will improve with better observational data. However, at least at this stage, it is dangerous to deduce a conclusive statement simply by having reproduced the restricted observed data.

For individual lens systems, it is important to investigate not only the simple models usually used but also various other lens models. Simultaneously, it becomes important to study any general theorem that is universally applicable to these various models.

In order to understand the properties of gravitational lens systems that are independent of lens models, we should examine the general properties of the gravitational lens equation. At present, there are two general theorems about gravitational lensing (MacKenzie 1985; Schneider et al. 1992) that address the numbers and magnifications of lensed images. These general theorems were established under conditions wherein the lens has a smooth density profile and a finite total mass. Since MPE models have a singular density profile, these theorems are not applicable to them.

In this paper, we prove a theorem about the numbers of images that is applicable to MPE models as well as to lens models with a regular density profile.

2. THE KNOWN THEOREM

Gravitational lens systems are characterized by the Fermat potential, ϕ,

$$\phi(\boldsymbol{x}, \boldsymbol{y}) = \tfrac{1}{2}|\boldsymbol{x} - \boldsymbol{y}|^2 - \psi(\boldsymbol{x}) , \tag{1}$$

$$\psi(\boldsymbol{x}) = \frac{1}{\pi}\int d^2x'\kappa(\boldsymbol{x}')\ln|\boldsymbol{x} - \boldsymbol{x}'| , \tag{2}$$

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where $\boldsymbol{x}$ and $\boldsymbol{y}$ are image position and source position, respectively, and κ is surface mass density of the lens object. Since the Fermat potential is closely related to the light travel time of a light ray, the lens equation is given by the extrema of ϕ,

$$\phi_i(\boldsymbol{x}, \boldsymbol{y}) = 0 \ . \tag{3}$$

Here and hereafter the subscripts i, j of ϕ denote the derivatives with respect to the image positions x^i, x^j such as $\phi_i \equiv \partial\phi/\partial x^i$ and $\phi_{ij} \equiv \partial^2\phi/\partial x^i \partial x^j$.

For fixed $\boldsymbol{y}$, the Fermat potential $\phi(\boldsymbol{x}, \boldsymbol{y})$ defines an arrival-time surface. Ordinary images of the source, where $\phi_i = 0$ and the Jacobian ($\equiv$determinant of ϕ_{ij}) $\neq 0$, are located at local extrema and saddle points of the arrival time surface. Thus, ordinary images are classified, by eigenvalues of the Jacobian matrix ϕ_{ij}, into three types: (I) both eigenvalues of the Jacobian matrix are positive, (II) one eigenvalue is positive and the other negative, and (III) both eigenvalues are negative.

Therefore, type I and type III images have positive parity, and type II images have negative parity.

The known theorem about the relation between the numbers of images of each type states

$$n_{\mathrm{I}} - n_{\mathrm{II}} + n_{\mathrm{III}} = 1 \ , \tag{4}$$

where n_{I}, n_{II}, and n_{III} are the number of images of types I, II, and III, respectively (MacKenzie 1985; Schneider et al. 1992). Hereafter, we call equation (4) the number theorem.

The theorem is proved, under the condition that the lens object has a finite total mass and its surface density profile is smooth, by using the index theorem for critical points in two-dimensional flat or closed curved surfaces, as is well known to mathematicians (Eguchi, Gilkey, & Hanson 1980). This can be summarized as follows: The image positions of the source are given by critical points P_j of the vector field ϕ_i at which the vector field vanishes, $\phi_i(P_j) = 0$. So let us consider in general a vector field on the surface S. We introduce the concept of index that is the winding number of the vector ϕ_i when the independent variable $\boldsymbol{x}$ goes once around P_i in two-space. We get the following index theorem,

$$\sum_i (\text{index of the vector at } P_i) = \chi(S) \ , \tag{5}$$

where $\chi(S)$ is the Euler characteristic of S that is estimated by the simplicial complex K homeomorphic to S as $\chi(S) = \chi(K) = \sum_{n=0}^{2} (-1)^n \alpha_n$, where α_n is the number of n-simplex. Thus, for a sphere χ is 2, and for flat space, 1. Returning to gravitational lensing, since an image of type I or III has a positive index and an image of type II has a negative index, we finally obtain equation (4).

This theorem does not hold in cases in which the vector field has singular points, that is, in cases in which the lens object has a singular density profile. The Fermat potential in an MPE model is given by

$$\phi(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}(\boldsymbol{x} - \boldsymbol{y})^2 - M \ln|\boldsymbol{x}| + \frac{q_i x^i}{|\boldsymbol{x}|^2} + Q_{ij} \frac{x^i x^j}{|\boldsymbol{x}|^4} + \cdots \ , \tag{6}$$

where M, q_i, and Q_{ij} are the normalized total mass, dipole moment, and quadrupole moment, respectively. Therefore, equation (4) does not hold for MPE models.

In order to incorporate models with a singular density profile such as MPE models, we reformulate the index theorem and extend the number theorem.

3. FORMULATION

Before proceeding to the concrete physical situation, we begin with the mathematical preparations. Under the thin lens approximation, the lens equation defines a map from a two-dimensional source plane into a two-dimensional image plane. So it is useful to define these two-dimensional positions as complex numbers.

Let $z = x^1 + ix^2$ and $z_s = y^1 + iy^2$ and use the Wirtinger derivatives,

$$\partial_z \equiv \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$
$$\partial_{\bar{z}} \equiv \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{7}$$

where $\bar{z}$ is the complex conjugate of z. Then the Fermat potential equation (1) and the deflection potential equation (2) are rewritten as

$$\phi(z, z_s) = \tfrac{1}{2}|z - z_s|^2 - \psi(z) \ , \tag{8}$$

$$\psi(z) = \frac{1}{\pi}\int d^2z' \kappa(z') \ln|z - z'| \ , \tag{9}$$

where the integral measure d^2z means $d\Re(z)d\Im(z)$. The lens equation is given by

$$\phi_{\bar{z}} \equiv \partial_{\bar{z}}\phi(z, z_s) = 0 \ . \tag{10}$$

Using the mean curvature κ and the shear vector (γ^1, γ^2), the Jacobian matrix ϕ_{ij} is parameterized as

$$(\phi_{ij}) = \begin{pmatrix} 1 - \kappa - \gamma^1 & -\gamma^2 \\ -\gamma^2 & 1 - \kappa + \gamma^1 \end{pmatrix}, \tag{11}$$

$$\kappa \equiv \tfrac{1}{2}\,\Delta\psi \ , \tag{12}$$

$$\gamma^1 \equiv \tfrac{1}{2}(\partial_1^2 - \partial_2^2)\psi \ , \tag{13}$$

$$\gamma^2 \equiv \partial_1 \partial_2 \psi \ . \tag{14}$$

In the complex notation, the Jacobian matrix, the mean curvature, and the shear vector are written as

$$(\phi_{\alpha\beta}) \equiv \begin{pmatrix} \dfrac{\partial^2\phi}{\partial z\,\partial z} & \dfrac{\partial^2\phi}{\partial z\,\partial \bar{z}} \\ \dfrac{\partial^2\phi}{\partial \bar{z}\,\partial z} & \dfrac{\partial^2\phi}{\partial \bar{z}\,\partial \bar{z}} \end{pmatrix} = M(\phi_{ij})\,{}^tM \tag{15}$$

with

$$M \equiv \begin{pmatrix} \dfrac{\partial x^1}{\partial z} & \dfrac{\partial x^2}{\partial z} \\ \dfrac{\partial x^1}{\partial \bar{z}} & \dfrac{\partial x^2}{\partial \bar{z}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \tag{16}$$

and ${}^tM \equiv$ transposed matrix of M,

$$\kappa = 2\partial_z \partial_{\bar{z}} \psi \ , \tag{17}$$

$$\gamma \equiv \gamma^1 + i\gamma^2 = 2\,\partial_{\bar{z}}^2 \psi \ . \tag{18}$$

Subsequently, we reformulate the index theorem, incorporating pole terms. Let us consider a complex function $f(z)$

of a complex variable z that is not necessarily holomorphic. The function f plays the role of the vector field in the index theorem. The index of f is defined by

$$\text{Ind}\ (f) \equiv \frac{1}{2\pi i}\oint_{C_z} d\ln[f(z)]\ . \tag{19}$$

This measures the variation of the phase of f along the closed curve C_z, which goes counterclockwise. When C_z does not enclose any zero points or poles, the index vanishes. If C_z encloses some zero points or poles, $\{P_i\}$, then the index is given by the summation of the contributions from each point P_i:

$$\text{Ind}\ (f) = \sum_i \text{Ind}\ (f;\ P_i)\ , \tag{20}$$

$$\text{Ind}\ (f;\ P_i) = \frac{1}{2\pi i}\oint_{C_z^{(i)}} d\ln[f(z)]\ , \tag{21}$$

where $C_z^{(i)}$ is the circle enclosing the point P_i. In the context of the gravitational lenses, f is replaced by $\phi_{\bar z}$, and $\phi_{\bar z} = 0$ is the lens equation. Therefore, ordinary images are zero points of order 1. If the lens object has finite total mass and extended regular mass density, then we have a pole only at infinity. However, in the case of MPE models and models with singular mass density such as point masses, other poles are produced (see eq. [6]).

From the form of equation (20), we should pay attention to each contribution. Let us consider the map $\omega = f(z)$ in the neighborhood of P_i. Then the Ind $(f;\ P_i)$ is rewritten as

$$\text{Ind}\ (f;\ P_i) = \frac{1}{2\pi i}\oint_{C_\omega} d\ln\omega\ , \tag{22}$$

where $C_\omega = \omega(C_z^{(i)})$. The value of the right-hand side is the (algebraic) number of times that we wind around C_ω when we go around $C_z^{(i)}$ in the counterclockwise direction.

First, we will consider the zero point case and proceed to the case of poles. Hereafter, we will set the origin of complex z-plane at the zero point P_i and we use capital letters for the value of any function g, for example, $g^* = g(P_i)$.

3.1. *The Case of a Zero Point of Order 1*

In the neighborhood of a zero point, ω can be written as

$$\omega = zF_z + \bar z F_{\bar z} + O(|z|^2)\ , \tag{23}$$

where the subscripts z and $\bar z$ of f mean partial derivatives with respect to themselves (in the sense of the Wirtinger derivatives eq. [7]), for example,

$$f_{z^k\bar z^l} \equiv \frac{\partial^{k+l}}{\partial z^k\,\partial\bar z^l} f\ .$$

We define μ as

$$\mu \equiv \frac{F_{\bar z}}{F_z}\ .$$

Then the Jacobian D of the mapping from z to ω in this case is given by

$$D = |f_z^*|^2 - |f_{\bar z}^*|^2 = |f_z^*|^2(1 - |\mu|^2)\ . \tag{24}$$

That is, $\omega = f(z)$ retains (reverses) the direction of rotation if $|\mu| < 1$ ($|\mu| > 1$).

The winding number of C_ω is one for the point of order 1 and

$$\text{Ind}\ (f;\ P_i) = \begin{cases} 1 & \text{for } |\mu| < 1 \\ -1 & \text{for } |\mu| > 1\ . \end{cases} \tag{25}$$

3.2. *The Case of a Zero Point of Higher Orders*

About a zero point, ω is expanded as a polynomial of z and $\bar z$. However, the winding number is determined by the lowest power term, as is easily checked.

For simplicity, we consider the case where the dominant part in the neighborhood of the zero point consists only of the term

$$\omega = \frac{F_{z^k\bar z^l}}{k!\,l!}\, z^k\bar z^l + (\text{subdominant terms})\ . \tag{26}$$

We call this type of zero point a (k, l)-type critical point for later use. Although equation (26) restricts the form of f, as we see later, for the case of point masses and MPE models, it is enough to examine equation (26).

Then the index of f is given by

$$\text{Ind}\ (f;\ P_i) = k - l\ . \tag{27}$$

Therefore, from § 3.1, we can regard first order zero points with $|\mu| < 1$ as (1, 0)-type and ones with $|\mu| > 1$ as (0, 1)-type.

3.3. *The Case of Pole Terms*

As was mentioned, besides the zero points, the poles of f also contribute to Ind (f). In the neighborhood of a pole of f we define g by $g \equiv 1/f$, then

$$\text{Ind}\ (f;\ P_i) = -\text{Ind}\ (g;\ P_i) \tag{28}$$

comes from the definition of equation (19).

Thus, as in the case of zero points, we can regard the pole whose dominant term has the form of equation (26) as a (k, l)-type critical point with negative integers k and l.

4. APPLICATION TO GRAVITATIONAL LENSING

We have completed the mathematical preparations and will proceed to the application to gravitational lensing. In this case, the derivative of Fermat potential corresponds to f in the previous section,

$$f = \phi_{\bar z}\ . \tag{29}$$

We assume

$$\lim_{|z|\to\infty}\left|\frac{\Psi_{\bar z}}{z}\right| = 0\ , \tag{30}$$

and this condition holds for models with finite total mass. For the large impact parameter, $\phi_{\bar z}$ becomes

$$\phi_{\bar z} = \tfrac{1}{2}(z - z_s) + o(|z|)\ . \tag{31}$$

That is, the point at infinity becomes a pole. Noting that the counterclockwise loop C_z ($|z| \gg 1$) is viewed as clockwise from the infinite point, we obtain the contribution of the infinite point to the index as

$$\text{Ind}\ (\phi_{\bar z}) = 1\ . \tag{32}$$

Diminishing the radius of the circle C_z, it is entangled with the zero points and poles. Then equation (32) must be equal to the sum of these contributions.

As we have mentioned, a zero point of order 1 in the neighborhood of the origin is classified as (1, 0)-type for $|\mu| < 1$ and (0, 1)-type for $|\mu| > 1$ by the contribution to the index. Thus, we can say that the contribution comes from the term

$$\omega \sim \frac{C_{k,l}}{k!\,l!}\, z^k z^l ,$$

where $C_{k,l}$ is a constant and positive (negative) k and l correspond to the zero points (poles). We denote the number of (k, l)-type critical points as $n^{(k,l)}$. Then, from the arguments of § 3, we get

$$1 = \mathrm{Ind}\,(\phi_{\bar{z}}) = \sum_{k,l} (k-l) n^{(k,l)} . \tag{33}$$

For the extended lens models that have no poles except at infinity, we obtain

$$1 = n^{(1,0)} - n^{(0,1)} = n_+ - n_- = n_{\mathrm{I}} + n_{\mathrm{III}} - n_{\mathrm{II}} , \tag{34}$$

where n_+ (n_-) is the number of images with positive (negative) parity. Therefore, $n_+ = n_{\mathrm{I}} + n_{\mathrm{II}}$ and $n_- = n_{\mathrm{II}}$.

For point masses, we have a pole of the $(0, -1)$-type for every point mass, and therefore

$$1 = n_{\mathrm{I}} + n_{\mathrm{III}} - n_{\mathrm{II}} + (\text{the number of point masses}) . \tag{35}$$

For an MPE model up to 2^k-pole, Fermat's potential is given by (Kakigi et al. 1995)

$$\phi = \frac{1}{2}|z - z_s|^2 - \frac{Q_0}{2}\ln(z\bar{z}) + \sum_{n=1}^{k} \frac{1}{2n}\left(\frac{Q_n}{z^n} + \frac{\bar{Q}_n}{\bar{z}^n}\right), \tag{36}$$

and the vector field becomes

$$\phi_{\bar{z}} = \frac{1}{2}(z - z_s) - \frac{Q_0}{2\bar{z}} - \frac{1}{2}\sum_{n=1}^{k} \frac{\bar{Q}_n}{\bar{z}^{n+1}} , \tag{37}$$

where Q_n is 2^nth pole moment:

$$Q_n \equiv \frac{1}{\pi}\int d^2 z' \kappa(z') z'^n . \tag{38}$$

This gives rise to a pole of the $(0, -k-1)$-type and

$$\begin{aligned} 1 &= n^{(1,0)} - n^{(0,1)} + (k+1) n^{(0,-k-1)} \\ &= n_{\mathrm{I}} + n_{\mathrm{III}} - n_{\mathrm{II}} + k + 1 . \end{aligned} \tag{39}$$

Finally, we will give concrete examples of this number theorem.

As examples, we perform numerical fitting of the quadruple images of PG 1115+080 (Young et al. 1981; Christian, Crabtree, & Waddell 1987; Kristian et al. 1993) by a model with softened power-law surface density and by an MPE model. Although we can extract various types of physical information, such as the total mass of the lens object, the Hubble constant, and so on (as recent references, see Schechter et al. 1997; Keeton & Kochanek 1997), here we show only image positions predicted by the two models, since we are interested in checking the number theorem.

Figure 1 shows the critical and caustic lines of the Fermat potential and the image positions predicted by a model with softened power-law surface density (Asano & Fukuyama 1996). Critical lines are those on which the determinant of the Jacobian matrix, equation (15), vanishes. So the parities of images are reversed across a critical line. Images outside the outermost critical line have positive parities. Hence, critical lines are useful for determining the parities of images.

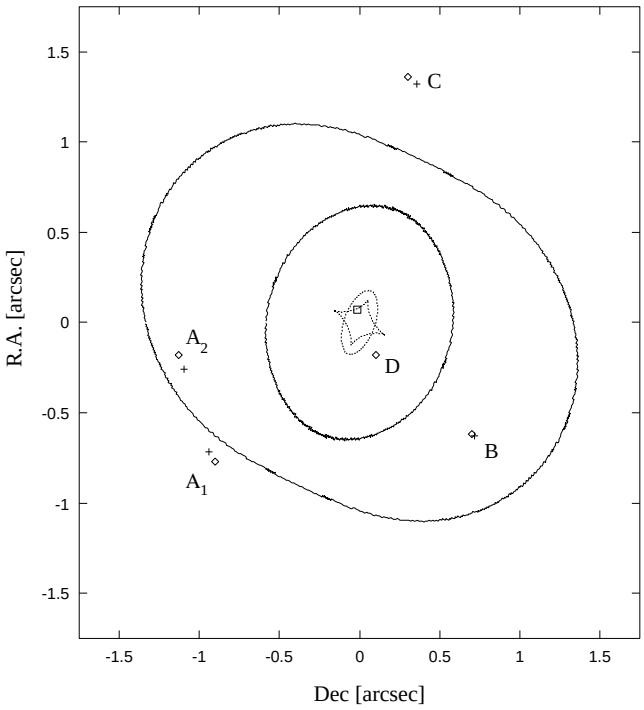


FIG. 1.—The observed image positions (*crosses*) and numerically fitted ones (*diamonds*) predicted by a model with softened power-law surface mass density. The square shows source position. The solid and dotted lines are the critical line and caustics, respectively.

From the positions of the critical lines, we know that A_1 and C belong to n_{I}, A_2 and B to n_{II}, and D to n_{III}. So the calculated numbers of images satisfy the original theorem equation (4).

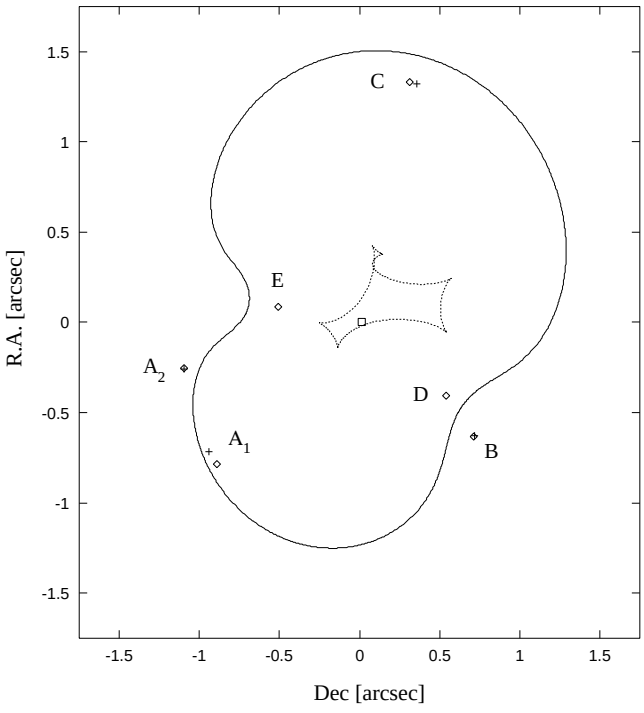


FIG. 2.—Same as Fig. 1, except fitted by an MPE model

On the other hand, Figure 2 shows the result of fitting of the same lens system by an MPE model including up to quadrupole terms (Fukuyama et al. 1997). A_2 and B belong to n_I and A_1, C, D, and E belong to $n_{\rm II}$. Thus, these satisfy equation (39) for the case of quadrupole, $k = 2$.

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