Mathematics of the USSR-Sbornik

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To cite this article: V A Galaktionov *et al* 1986 *Math. USSR Sb.* **54** 421

Матем. Сборник
Том 126(168) (1985), Вып. 4

Math. USSR Sbornik
Vol. 54(1986), No. 2

ON ASYMPTOTIC "EIGENFUNCTIONS" OF THE CAUCHY PROBLEM FOR A NONLINEAR PARABOLIC EQUATION

UDC 517.956

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ABSTRACT. The asymptotic ($t \to +\infty$) behavior of solutions of the Cauchy problem is studied for the semilinear parabolic equation

$$u_t = \Delta u - u^\beta, \quad t > 0,\ x \in R^N; \qquad u(0,x) = u_0(x) \geqslant 0, \quad x \in R^N,$$

where $\beta = \text{const} > 1$ and $u_0(x) \to 0$ as $|x| \to +\infty$. The existence is established of an infinite collection (a continuum) of distinct self-similar solutions of the form $u_A(t,x) = (T+t)^{-1/(\beta-1)}\theta_A(\xi)$, $\xi = |x|/(T+t)^{1/2}$, where the function $\theta_A > 0$ satisfies an ordinary differential equation. Conditions for the asymptotic stability of these solutions are established. It is shown that for $\beta \geqslant 1 + 2/N$ there exist solutions of the problem whose behavior as $t \to +\infty$ is described by approximate self-similar solutions (ap.s.-s.s.'s) $u_a(t,x)$ which in the case $\beta > 1 + 2/N$ coincide with a family of self-similar solutions of the heat equation $(u_a)_t = \Delta u_a$, while for $\beta = 1 + 2/N$ and $u_0 \in L^1(R^N)$ the ap.s.-s.s. has the form

$$u_a = \left[(T+t)\ln(T+t)\right]^{-N/2} c_N \exp\left(-|x|^2/4(T+t)\right),$$

where $c_N = (N/2)^{N/2}(1 + 2/N)^{N^2/4}$.

Figures: 2. Bibliography: 78 titles.

§1. Introduction

1. *Formulation of the problem.* In this paper we study the asymptotic behavior (as $t \to +\infty$) of solutions of the Cauchy problem for a semilinear parabolic equation which describes the diffusion of heat in a medium with a nonlinear volumetric energy sink

$$\mathbf{B}(u) \equiv u_t - \Delta u + u^\beta = 0, \qquad t > 0,\ x \in R^N, \tag{1.1}$$

$$u(0,x) = u_0(x) \geqslant 0, \qquad x \in R^N;\ u_0 \in C(R^N),\ \sup u_0 < \infty. \tag{1.2}$$

Here $t \geqslant 0$ and $x \in R^N$ are, respectively, the time and space coordinates, $\Delta = \sum_1^N \partial/\partial x_i^2$ is the Laplace operator, and $\beta > 1$ is a fixed constant. Regarding the initial function u_0 in (1.2) it is assumed that $u_0(x) \to 0$ as $|x| \to +\infty$. A bounded solution of problem (1.1), (1.2) exists, is unique, and is a classical solution in any region of the form $[\tau, +\infty) \times R^N$, $\tau > 0$ (see, for example, [1]). Moreover, it is not hard to show that for $\beta > 1$ the solution $u = u(t,x)$ is strictly positive in $R^1_+ \times R^N$ if $u_0 \not\equiv 0$.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35K55, 35K15; Secondary 35K05, 35B35.

0025-5734/86 $1.00 + $.25 per page

Before proceeding to an exposition of the main results, we shall say a few words about the physical meaning ascribed to the present investigation. Here we limit ourselves to a brief explanation of the concept of an "eigenfunction" of a nonlinear dissipative medium; for more details see the survey given below (§1.3) and the literature cited there.

In an unbounded heat-conducting medium with nonlinear absorption in which the diffusion of heat is described by equation (1.1) an initial temperature perturbation $u(0, x) = u_0(x) \geqslant 0$, $u_0 \not\equiv 0$, of rather arbitrary form is introduced from without. What is the subsequent fate of this perturbation? It turns out that, after the elapse of a certain time, as a rule some stable spatially inhomogeneous formation develops in the medium which is a time-dependent thermal structure with its own laws of evolution for large t. To each of these laws there corresponds a set of attraction $\mathscr{W}$ in the space of initial functions, and the "laws" of evolution for large times do not depend on specific features of the initial perturbation (i.e., they are the same for any $u_0 \in \mathscr{W}$). In other words, using the terminology of [2] and [3], it may be said that a given nonlinear medium has its own collection of stable "eigenfunctions" (abbreviated e.f., and we henceforth omit the quotation marks) which determine the asymptotic behavior of solutions of problem (1.1), (1.2) for rather arbitrary initial perturbations u_0. We emphasize that for (1.1) we consider the Cauchy problem without boundary conditions which makes it possible to speak precisely of an e.f. of the medium not subject to any external influence. Of course, this formulation is also natural from the point of view of the qualitative theory of partial differential equations (what else if not the equation itself should determine the asymptotic behavior of solutions of the problem). However, we wish to especially mention that the concept of an e.f. of a nonlinear medium plays an important role in the description of complex time-dependent physical, chemical, biological, and other processes, in particular, the process of morphogenesis in active biological media (regarding this, see [2], [4], [5], and the brief survey given below in §1.3).

There arises the question of what is the collection of e.f. of the nonlinear medium (1.1) in question and the set of attraction in the space of initial perturbations (1.2) corresponding to each e.f. In other words, we pose the problem of the constructive description of an attractor of the nonlinear parabolic evolution equation (1.1).

2. *The main results.* A considerable part of the paper is devoted to the investigation of one family of special solutions of equation (1.1)—radially symmetric self-similar solutions of the form

$$u_A(t, x) = (T + t)^{-1/(\beta-1)}\theta_A(\xi), \qquad \xi = |x|/(T + t)^{1/2}, \tag{1.3}$$

where $T \geqslant 0$ is an arbitrary constant, while the function $\theta_A(\xi) > 0$ is determined by integration of the ordinary differential equation obtained after substitution of (1.3) into the original equation (1.1):

$$\mathbf{A}_R(\theta_A) \equiv \Delta_\xi\theta_A + \theta_A'\xi/2 + \theta_A/(\beta - 1) - \theta_A^\beta = 0, \qquad \xi > 0, \tag{1.4}$$

where Δ_ξ is the Laplace operator; in the radially symmetric case

$$\Delta_\xi = \xi^{1-N}(d/d\xi)(\xi^{N-1}(d/d\xi)).$$

It has the obvious homogeneous solution

$$\theta_A(\xi) = \theta_H \equiv (\beta - 1)^{-1/(\beta-1)}, \qquad \xi \geqslant 0, \tag{1.5}$$

and all other solutions must satisfy the boundary conditions

$$\theta_A'(0) = 0, \qquad \theta_A(+\infty) = 0. \tag{1.6}$$

Because of the first of these, the function (1.3) is a solution of (1.1) everywhere in $R_+^1 \times R^N$.

In [2] we established the existence of an infinite collection of distinct nontrivial self-similar functions $\theta_A(\xi) > 0$ defined for any $\xi \geqslant 0$. It is shown that the structure of the family $\{\theta_A\}$ is different in the cases $\beta \geqslant 1 + 2/N$ and $1 < \beta < 1 + 2/N$, and in the final analysis this difference is what determines the principal special features of the asymptotic behavior of solutions of the problem for $\beta \geqslant 1 + 2/N$ and $\beta \in (1, 1 + 2/N)$ (the meaning of this conclusion is explained at the end of the paper).

In §3 we establish that each self-similar solution, generally speaking, is the "center of attraction" of a large set of solutions of problem (1.1), (1.2) corresponding to distinct "non-self-similar" initial distributions $u_0(x)$, i.e., the particular solutions $u_A(t, x)$ are asymptotically stable under perturbations of the "self-similar" initial function $u_A(0, x)$ which are not too "large" and do not take $u_0(x)$ out of the set of attraction $\mathscr{W}_A$. Thus, the functions u_A are the desired (self-similar) e.f. of the nonlinear medium and are elements of an attractor of the evolution equation. Here there thus arises a situation which has already become common in which the e.f. are suitable self-similar (invariant) solutions of the parabolic equation which as a result pass from the class of simply individual special solutions of the equation into a class of global attractors of a large set of rather arbitrary solutions (see the bibliography in [6]–[8]). We mention also that the self-similar laws of development of the process are frequently also preserved under rather "strong perturbations" of the nonlinear operator of the parabolic equation in question; this makes it possible in a number of cases to construct a complete system of so-called approximate self-similar solutions (ap.s.-s.s.'s) of the equation with coefficients of rather arbitrary form (see [6], [7], and [9]–[12]); however, it often happens that the "generating" self-similar solutions lie in a family of invariant solutions of an equation which at first glance has nothing in common with the equation considered; such "degenerate" ap.s.-s.s.'s, which satisfy an equation of lower order than the original equation, were constructed in [9], [7], and [10]. Nontrivial ap.s.-s.s.'s also exist for problem (1.1), (1.2) at the "critical" value of the parameter $\beta = 1 + 2/N$ and also for some initial functions u_0 in the case $\beta > 1 + 2/N$.

It is shown in §3 that for $\beta \in (1, 1 + 2/N)$ the self-similar solutions exhaust the entire set of radially symmetric e.f.(1)

Here the structure of the set of attraction $\mathscr{W}_A$ corresponding to the solution (1.3) with a fixed self-similar function $\theta_A(\xi)$ is determined by means of upper $\theta_+(\xi)$ and lower $\theta_-(\xi)$ ($\theta_+ \geqslant \theta_-$ in R_+^1) solutions of (1.4) on the basis of which the existence of a given solution of problem (1.4), (1.6) is proved in §3 (we note that generally the asymptotic stability of the solutions (1.3) proved in §3 makes it possible to simultaneously establish certain properties of solutions of problem (1.4), (1.6) which are rather difficult to obtain directly from an analysis of the ordinary differential equation).

(1) The question of the existence of self-similar solutions (1.3) which are not radially symmetric, where $\xi = x(T + t)^{-1/2} \in R^N$, remains open. We mention that $\xi \in R^N$ does not belong to the class of equations indicated in [13], all solutions of which are symmetric relative to some point (see (1.4)).

In the case $\beta \geqslant 1 + 2/N$ classes of solutions of problem (1.1), (1.2) are distinguished which evolve as $t \to +\infty$ according to "non-self-similar laws" different than in (1.3). For $\beta > 1 + 2/N$ the asymptotic behavior of the majority of them is described by a family of distinct self-similar solutions of the heat equation without a sink

$$v_t = \Delta v, \qquad t > 0,\ x \in R^N, \tag{1.7}$$

but, as already mentioned above, there also exist solutions whose asymptotic evolution is described by nontrivial ap.s.-s.s.'s. For the case $\beta = 1 + 2/N$ in §3 a family of solutions of problem (1.1), (1.2) is distinguished whose asymptotic behavior as $t \to +\infty$ is described by the ap.s.-s.s.

$$u_a(t, x) = [(T + t)\ln(T + t)]^{-N/2} g_*(\xi), \qquad \xi = |x|/(T + t)^{1/2}. \tag{1.8}$$

It differs from the self-similar solution (1.3) (for $\beta = 1 + 2/N$) by the additional logarithmic factor. The function $g_*(\xi)$ is generally uniquely determined (see §3). These results sharpen the corresponding conclusions of [14]. We note that in [14], in addition, sufficient conditions are obtained for the asymptotic stability of the homogeneous self-similar solution (1.3) with function (1.5) which, as we show, are also necessary.

The ap.s.-s.s. (1.8) satisfies the parabolic equation

$$\partial u_a/\partial t = \Delta u_a - (N/2) u_a/(T + t)\ln(T + t). \tag{1.9}$$

It is interesting to trace how the structure of the attractor of the original parabolic equation (1.1) changes depending on the change of the parameter $\beta > 1$ (in the case of radially symmetric solutions). For $\beta \in (1, 1 + 2/N)$ the attractor consists entirely of a one-parameter family of solutions of the type (1.3), and its dimension is hence equal to 1. For $\beta > 1 + 2/N$, as already mentioned, the e.f. additionally include a manifold of self-similar solutions of the heat equation (1.7) without a sink, which under particular conditions on the initial function $u_0(x)$ is inconsequential as $t \to +\infty$. Because of the linearity of (1.7), the latter family is already two-dimensional. The value of the parameter $\beta = 1 + 2/N$ is thus "critical"; on passing through it, the type of the equation for one collection of e.f. changes ((1.1) goes over into (1.7)), and their structure and also, apparently, the dimension of the attractor also change. Here at the point $\beta = 1 + 2/N$ a new asymptotics arises—the ap.s.-s.s. (1.8) satisfying equation (1.9) which is notably different from the original equation.

To a large extent the investigation of the asymptotic behavior of solutions of problem (1.1), (1.2) is based on results and representations developed in [2], [3], and [15]–[17] (see also the bibliography in [2], [4], [5], and [7]) which are devoted to the analysis of unbounded self-similar e.f. (regimes with peaking) corresponding to the quasilinear heat equation with a source

$$u_t = \nabla(u^\sigma \nabla u) + u^\beta, \qquad t > 0,\ x \in R^N, \tag{1.10}$$

where $\sigma \geqslant 0$ and $\beta > 1$ are constants and $\nabla(\cdot) = \operatorname{grad}_x(\cdot)$ (see also [18], where global self-similar solutions of (1.10) for $\beta > 1 + 2/N$ were considered).

Remark. In §§2 and 3 we have tried to use as infrequently as possible the semilinear structure of equation (1.1), i.e., the possibility of reducing problem (1.1), (1.2) to an equivalent integral equation by inverting the operator $\partial/\partial t - \Delta$. Therefore, the majority of our results carry over without major alteration to the case of the quasilinear equation

$$u_t = \Delta u^{\sigma+1} - u^\beta, \qquad t > 0,\ x \in R^N, \tag{1.11}$$

where $\beta > \sigma + 1$ and $\sigma > 0$ (some specific possibilities of such a generalization are discussed in the text of §§2 and 3). The construction of upper and lower solutions of (1.1) should be coordinated with the space-time structure of its self-similar solutions which in this case have the form

$$u_A(t, x) = (T + t)^{-1/(\beta-1)}\theta_A(\xi), \qquad \xi = |x|/(T + t)^{\beta-(\sigma+1)/2(\beta-1)}. \tag{1.12}$$

For the function $\theta_A \geqslant 0$ a degenerate ordinary differential equation is obtained which admits compactly supported generalized solutions. Here there also occurs a "critical" value of the parameter equal to $\beta = \sigma + 1 + 2/N$, and the structure of the family $\{\theta_A\}$ and the asymptotic properties of solutions of the partial differential equation differ in an essential manner in the cases $\beta \geqslant \sigma + 1 + 2/N$ and $\beta \in (\sigma + 1, \sigma + 1 + 2/N)$ (we note that this same "critical" value β also occurs in the case of the equation with a source (1.8) [2], [16]; it thus characterizes some general laws of evolution of global solutions of (1.9) and (1.8) for $\beta > \sigma + 1 + 2/N$). The quasilinear equation (1.11) will be discussed in somewhat more detail in the survey below.

3. *A brief survey of results of the investigation of time-dependent solutions of nonlinear heat equations with a sink.* We first note the role which in our view the study of time-dependent solutions of nonlinear parabolic equations (in particular, they include (1.10), (1.1), and (1.11)) plays in the theory of dissipative structures and synergetics which have been developed intensively in recent times (see [19]–[24]). A large number of dissipative structures arising in open thermodynamic systems due to the interaction, actually, of three "conflicting" processes are now known and have been studied in some detail; these three processes are diffusion (a dissipative process) and, we emphasize especially, processes of emission and absorption of "energy", i.e., the action of volumetric sources and sinks. As a result, after the elapse of a certain time stable steady-state or quasi-steady-state periodic structures of complex form may arise in a medium (for example, in the familiar "Brusselator" model [19], [21], [24]), periodic travelling waves (bores in [25]), etc. We especially note that in all such problems a localization effect of the evolution processes occurs over particular lengths or segments of the medium in each of which they proceed as if independently. It can be asserted that localization in some form is a necessary prerequisite for the occurrence of complex structures in a dissipative medium. Moreover, estimation of the size of the region of localization (it may frequently not be arbitrary, and is determined by intrinsic properties of the medium) makes it possible to predict in advance the e.f. which can arise and stably evolve in the region in question (the process of self-organization in the medium [2]).

From our point of view, to understand the "laws" of interaction of the processes of emission and absorption of "energy", which in the final analysis form the spectrum of the dissipative structures, it is first necessary to study them in the simplest models individually, i.e., to separately consider an equation with a source and an equation with a sink. On this path there immediately arises the concept of essentially time-dependent dissipative structures whose amplitude and spatial profile vary rapidly with time. Here it is appropriate to draw an analogy, for example, with the generally accepted conclusions that in a neighborhood of a bifurcation point of steady-state, quasi-steady-state, or periodic solutions of infinite-dimensional nonlinear evolution equations the number of degrees of freedom of the system is reduced to small finite values which generally bears witness to the existence of general laws of development of instabilities in various nonlinear media (see

[26]–[28]). We believe that the same generality must be contained in elementary time-dependent models of "nonlinear heat conduction" + a "source" (or + a "sink"), pertaining now to the character of occurrence of essentially time-dependent processes, i.e. the development of various rapid (or even singular in time) processes have intrinsic common features that are evident at the qualitative level in certain simple models. This generality is undoubtedly present in processes occurring in "superfast" regimes with peaking (see [2], [5], and the bibliography given there).

For comparison we indicate below in very compressed form the range of problems and the main directions of investigation of regimes with peaking which arise in nonlinear media with volumetric energy emission—a process opposite to absorption.

3.1. *On equations with a source.* Properties of time-dependent solutions of quasilinear parabolic equations with volumetric energy emission, for example, of the form (1.10) have been studied in a large number of papers; see the surveys in [5] and [2]. It is now possible to speak of establishing certain general laws of the occurrence of such processes. In particular, of great interest from the viewpoint of applications is the study of thermal regimes with peaking to which there correspond unbounded solutions of (1.10) which exist for a finite time with an amplitude which grows without bound over a finite interval of change of time: $\sup_x u(t,x) \to +\infty$ as $t \to T_0^- < +\infty$ ($t = T_0$ is the time of peaking). It was established that in the Cauchy problem for (1.10) for values of the parameters of the nonlinear medium $\beta \geqslant \sigma + 1$ unbounded solutions are localized in space in the sense that $u(t,x) \to +\infty$ as $t \to T_0^-$ in a bounded region (for $\beta > \sigma + 1$—the *LS*-regime—on a set of measure zero, and for $\beta = \sigma + 1$ on a fundamental "length" depending only on the quantity $\sigma > 0$; the profile of the nonmonotone initial perturbation $u_0(x)$ here plays no role). The effect of localization for $\beta \geqslant \sigma + 1$ in the final analysis determines the possibility that complex structures (e.f.) appear in the medium, evolving in a regime with peaking. Investigations have shown that e.f. with a large number of extremal points on the spatial profile are self-similar solutions of (1.10) of the form (cf. (1.3))

$$u_A(t,x) = (T_0 - t)^{-1/(\beta-1)}\theta_A(\xi), \qquad \xi = x/(T_0 - t)^m, \tag{1.13}$$

where $m = [\beta - (\sigma + 1)]/2(\beta - 1)$, and the function $\theta_A(\xi) \geqslant 0$ satisfies the elliptic equation

$$\nabla_\xi\left(\theta_A^\sigma \nabla_\xi \theta_A\right) - m \sum_{i=1}^{N} \frac{\partial \theta_A}{\partial \xi_i}\xi_i - \frac{1}{\beta - 1}\theta_A + \theta_A^\beta = 0, \qquad \xi \in R^N, \tag{1.14}$$

$\theta_A(\xi) \to 0$ as $|\xi| \to +\infty$. In the recent note [29] it was shown that equation (1.14) for $\beta > \sigma + 1$ in the multidimensional case has a particular collection of solutions which are not radially symmetric of very diverse form, for example, a solution effectively localized in a region forming a star (regarding previous investigations of equation (1.14) see [2], [3], [5], [15], and [17]). In accordance with (1.13) each of these solutions determines a time-dependent dissipative structure evolving as $t \to T_0^-$ in a regime with peaking with its own "architecture" of the spatial profile, its own region of localization, and "laws" intrinsic only to it of combining elementary structures into more complex structures.

The effect of heat localization occurs also in problems for an equation without a source, for example, of the form

$$u_t = \left(u^\sigma u_x\right)_x, \qquad t > 0,\ x > 0, \tag{1.15}$$

where the volumetric energy emission is modelled by the action on the boundary of the half-space $x > 0$ of a regime with peaking (see [30] and the bibliography given there). The special methods developed for investigating nonlinear parabolic equations make it possible to carry out a rather detailed investigation of unbounded solutions of a heat equation with a source of the general form

$$u_t = \nabla(k(u)\nabla u) + Q(u), \tag{1.16}$$

where $k \geqslant 0$ and $Q \geqslant 0$ are practically arbitrary functions (the coefficient of thermal conductivity and the power of volumetric energy sources respectively); see [5]. This concerns primarily 1) conditions for the occurrence of regimes with peaking, 2) conditions on k and Q for the occurrence of the effect of heat localization, and 3) in some generality it is possible also to resolve the question of a "fine" space-time structure of unbounded solutions—e.f. of the nonlinear medium (1.16).

Nevertheless, a number of important problems in the theory of regimes with peaking in heat-conducting media remain unsolved even with regard to equation (1.11) with power nonlinearities. Here we single out two of them. The first consists in a constructive description of the entire collection of e.f. of the nonlinear medium, i.e., all possible solutions of the elliptic equation (1.14) (the results obtained in [29] were for the most part based on qualitative and numerical methods). In this connection we note that (1.14) does not admit an equivalent variational formulation; therefore, variational methods of studying nonlinear elliptic problems in R^N developed in recent years (see, for example, [31]) are apparently not applicable to it. The second problem consists in investigating asymptotic stability as $t \to T_0^-$ of all e.f.; this makes it possible to carry out "evolutional selection" of the e.f. of the nonlinear medium. The difficulties occurring here are, in general, connected with the singularity of the solution in time and its instability in the usual sense under perturbations of the initial function. It must be said that this question remains open also in other problems for nonlinear equations and systems of Schrödinger type, hyperbolic equations, etc. (regarding this, see, for example, [5]). We note that a problem of the same sort arises for equations (1.1) or (1.11) with absorption for values of $\beta \in (0, 1)$ when a singular process of complete cooling in finite time is possible; see below.

From the point of view indicated above the properties of time-dependent solutions of parabolic equations with absorption have not been studied in as much detail as in the case of equations with sources. This pertains mainly to the Cauchy problem, in which the concept of an e.f. of the nonlinear medium arises in a natural way. To a considerable extent this stimulated our work. At the same time, a large amount of literature has been devoted to the investigation of the equations with absorption (1.1) and (1.11) and also the equation of general form

$$u_t = \nabla(k(u)\nabla u) - Q(u), \qquad k \geqslant 0, Q \geqslant 0, \tag{1.17}$$

and a number of important and interesting results have been obtained; we shall now briefly consider some of them.

3.2. *A nonlinear heat equation with a sink.* The authors of [32] first called attention to the interesting property of localization of thermal perturbations in media with absorption. In particular, they constructed a simple special generalized solution of the Cauchy problem for the equation with a linear sink

$$u_t = \Delta u^{\sigma+1} - u, \qquad t > 0, x \in R^N,$$

which has the form

$$u(t,x) = \exp(-t)[x_0(t)]^{-N}\left[\frac{\sigma}{2(2+N\sigma)}\left(a^2 - \frac{|x|^2}{x_0^2(t)}\right)^+\right]^{1/\sigma}, \tag{1.18}$$

where $a > 0$ is an arbitrary constant and

$$x_0(t) = \{[1 - \exp(-\sigma t)]/\sigma\}^{1/(2+N\sigma)}, \qquad t > 0 \tag{1.19}$$

(in (1.18) we have used the notation $(p)^+ = \{p \text{ if } p \geqslant 0;\ 0 \text{ if } p < 0\}$). At each time $t > 0$ this solution is compactly supported in x; the support of the generalized solution is a ball of radius $R(t) = ax_0(t)$. Moreover, it follows from (1.19) that $R(t)$ increases, and $R(t) \to a\sigma^{-1/(2+N\sigma)} < +\infty$ as $t \to +\infty$, i.e., thermal perturbations penetrate a finite distance and are localized in the region $\{|x| < a\sigma^{-1/(2+N\sigma)}\}$ during all time. Other examples of localized solutions of equation (1.17) and of (1.1), $\beta \in (0,1)$, in the one-dimensional case were also presented in [32]. In the subsequent paper [33] localized solutions of (1.11), $N = 1$, for $\beta \neq 1$ were constructed numerically.

In [34] existence and uniqueness theorems were then proved for a solution of the Cauchy problem for degenerate parabolic equations (1.17), $N = 1$, the validity of theorems for comparing solutions was established, and the concepts of generalized upper and lower solutions (super- and subsolutions; see also [35]) were introduced. On the basis of this machinery in [34] sufficient (and close to necessary) conditions for the localization of thermal perturbations were obtained by comparison with steady-state solutions. For equation (1.11), $N = 1$, they look as follows: if $\beta < \sigma + 1$, then compactly supported thermal perturbations are localized; for $\beta > \sigma + 1$ there is no localization, and the heat wave penetrates arbitrarily far (as was subsequently shown in [36], this also occurs for $\beta = \sigma + 1$). For equation (1.17), $N = 1$, of general form the sufficient condition for localization has the form [34]

$$\int_0^1 \left[\int_0^\eta Q(\varphi^{-1}(\zeta))\,d\zeta\right]^{-1/2} d\eta < +\infty, \tag{1.20}$$

where φ^{-1} is the function inverse to $\varphi(u) = \int_0^u k(s)\,ds$, $u \geqslant 0$. For (1.11) with $N = 1$, (1.20) is the necessary and sufficient condition for localization of compactly supported solutions.

Moreover, in [34] the curious effect of complete cooling in finite time was studied: if the function $Q(u)$, $u > 0$, satisfies the condition

$$\int_0^1 \frac{d\eta}{Q(\eta)} < +\infty, \tag{1.21}$$

then for any bounded initial perturbation $u_0(x) \geqslant 0$ there exists a $T > 0$ such that $u(t,x) \equiv 0$ for all $t \geqslant T$ (this result was obtained by comparison of the solution of the Cauchy problem with the spatially homogeneous solution $U(t)$ satisfying the ordinary differential equation $U'(t) = -Q(U(t))$; for (1.1) and (1.11) condition (1.21) implies that $\beta \in (0,1)$). Differential properties of generalized solutions of (1.11), $N = 1$, were studied in [37]; it was shown that a generalized solution $u(t,x)$ possesses a bounded derivative $(u^\lambda)_x$ where $\lambda = \max\{\sigma, (\sigma + 1 - \beta)/2\}$ (this result is sharp).

Conditions for the motion or immobility during finite time of a compactly supported front of a heat wave described by equation (1.11), $N = 1$, were obtained in [35] by constructing upper and lower solutions. Conditions for the movement of a heat wave forward (a "heating wave") or backward (a "cooling wave") were determined in [38]. In all cases the character of the motion of the front depends on the asymptotic behavior of $u_0(x)$ in a neighborhood of the initial point of degeneration. Regarding the generalization of the results of [35] to the case of equation (1.17) of general form, see [39]. Possible regimes of motion of a heat wave and also the effect of complete cooling are very graphically illustrated by the following solution of (1.11) for $\beta = 1 - \sigma$, $\sigma \in (0,1)$; see [38] and [39]:

$$u_A(t,x) = \psi(t)\theta(\xi), \qquad \xi = |x|/\varphi(t),$$

where $\psi(t) \geqslant 0$ and $\varphi(t) \geqslant 0$ are, respectively, the amplitude and width of the thermal structure:

$$\psi^\sigma(t) = a_0 t^{-N\sigma/(2+N\sigma)}\left(A - b_0 t^{2(1+N\sigma)/(2+N\sigma)}\right)^+,$$

$$\varphi^2(t) = c_0 t^{2/(2+N\sigma)}\left(A - b_0 t^{2(1+N\sigma)/(2+N\sigma)}\right)^+.$$

Here $A > 0$ is an arbitrary constant,

$$a_0 = \left[\frac{2(2+N\sigma)}{\sigma}\right]^{N\sigma/(2+N\sigma)}, \qquad b_0 = \frac{\sigma^2}{4(1+N\sigma)}\left[\frac{2(2+N\sigma)}{\sigma}\right]^{2(1+N\sigma)/(2+N\sigma)},$$

$$c_0 = \left[\frac{2(2+N\sigma)}{\sigma}\right]^{2/(2+N\sigma)}$$

and the function $\theta(\xi) \geqslant 0$ is compactly supported: $\theta(\xi) = [(1-\xi^2)^+]^{1/\sigma}$. The size of the support of the given generalized solution varies in time in a nonmonotone manner. On the interval $(0, t_*)$, where

$$t_* = \left[A/(2+N\sigma)b_0\right]^{(2+N\sigma)/2(1+N\sigma)},$$

the width of the structure $\varphi(t)$ increases, the surface of the heat "front" then begins to move backward toward the origin $x = 0$, and finally at time

$$t = T_0 = (A/b_0)^{(2+N\sigma)/2(1+N\sigma)}$$

$\psi(t)$ and $\varphi(t)$ vanish simultaneously ($u_A(T_0, x) \equiv 0$), i.e., complete cooling ensues. The self-similar solution is localized; at any time the diameter of its support does not exceed $2\varphi(t_*) < +\infty$.

Theorems on the existence and uniqueness of generalized solutions of boundary value problems for (1.17), $N = 1$, are proved in [36]. Conditions for the localization of boundary regimes in a medium with absorption are also obtained there (see also [38] and [41]). Regarding localization in the case of equations with absorption containing additional terms, for example,

$$u_t = (u^{\sigma+1})_{xx} - (u^\lambda)_x - u^\beta, \qquad \lambda \geqslant 1,$$

see [42] and [43]. We note that here localization may be one-sided depending on the relation between the parameters σ, β, and λ; in particular, if $\lambda < \min\{\sigma + 1, \beta\}$, then there is localization on the left [42].

We note that, somewhat earlier than [32], in [44] there was constructed a localized special solution of another quasilinear equation with linear absorption

$$u_t = \left(|u_x|^\sigma u_x\right)_x - u, \qquad \sigma > 0 \tag{1.22}$$

(see also [45] and [46]). Subsequently, various (mainly qualitative) investigations of localization and the character of motion of heat fronts in the case of nonlinearities more general than in (1.11) and (1.22) were carried out in [47]–[49] and elsewhere.

We return to the one-dimensional equation (1.17). Naturally, localization of thermal perturbations is possible only in the case where the heat wave has a finite front, i.e., perturbations in the medium propagate with finite speed. The necessary and sufficient condition for the presence of this property obtained in [50] has the form

$$\int_0^1 \left\{ \frac{k(\eta)\,d\eta}{\eta + \int_0^\eta \frac{k(\zeta)Q(\zeta)\,d\zeta}{\zeta(1+[1+k(\zeta)Q(\zeta)/\zeta]^{1/2})}} \right\} < +\infty \tag{1.23}$$

(it is easy to verify that if the localization condition (1.20) holds, then the integral in (1.23) also converges). Some rather delicate results for equation (1.17) of general form for $N = 1$ were obtained in [51]. They pertain to conditions for the absence of the effect of complete cooling (what happens if (1.21) is not satisfied?) and to answering the question of the closeness of (1.20) to a necessary condition for localization.

For the degenerate equation (1.17) with several space variables existence and uniqueness theorems, various types of comparison theorems, etc. were established in [43], [52], and [53] (see also [54] and the bibliographies of these papers). An interesting effect directly connected with the property of complete cooling in finite time was discovered in [55], where equation (1.17) for $k \equiv 1$ was considered. It was shown that under the condition

$$\int_0^1 \frac{d\eta}{(\eta Q(\eta))^{1/2}} < +\infty \tag{1.24}$$

(if Q is a monotone function, then (1.21) follows from (1.24)), any solution of the Cauchy problem with an initial function $u_0(x) > 0$ in R^N not having compact support such that $u_0(x) \to 0$ as $|x| \to +\infty$ at some time $t > 0$ becomes compactly supported in x.

Questions connected with the asymptotic behavior as $t \to +\infty$ of time-dependent solutions of equations of the type (1.11) considered in a bounded domain $\Omega \subset R^N$ were studied in [56] and [57]. A problem for the equation $u_t = \Delta(|u|^\sigma u) - |u|^{\beta-1}u$ (the "parabolic" continuation of (1.11) into the region of negative values of u) with Neumann conditions on the boundary, $(\partial/\partial n)(|u|^\sigma u) = 0$, $t > 0$, $x \in \partial\Omega$, was analyzed in [56]. It was shown that in the case $\beta > \sigma + 1$ as $t \to +\infty$ stabilization to the spatially homogeneous solution $y(t) = [(\beta - 1)t]^{-1/(\beta-1)}$ always occurs in the sense that

$$u(t,x)/y(t) \to c \quad \text{as } t \to +\infty,$$

where c takes one of the values -1, 0, $+1$ depending on the initial perturbation $u_0(x) \equiv u(0,x)$ in Ω. Here the condition $\beta > \sigma + 1$ is essential; in the case $\beta = \sigma + 1$ the equation has an infinite collection of spatially inhomogeneous solutions in separated variables of the form $u(t,x) = y(t)X_l(x)$ (of course, the functions X_l do not have constant sign in Ω). In the boundary value problem for (1.11) with the Dirichlet condition $u(t,x) = 0$ on $\partial\Omega$, studied in detail in [57], stabilization may take place according to different laws depending on the magnitude of the parameters β and σ. In the case $\beta > \sigma + 1$ for large t the sink is inconsequential compared with the diffusion operator, and the asymptotics of the process is described by the self-similar solution $v_A = (1+t)^{-1/\sigma}g(x)$ of the equation $v_t = \Delta v^{\sigma+1}$ where $g(x) > 0$ in Ω is such that $\Delta g^{\sigma+1} + (1/\sigma)g = 0$ in Ω, $g = 0$ on $\partial\Omega$. If $\beta = \sigma + 1$, then stabilization takes place in accordance with the exact self-similar solution u_A of (1.11), $u_A = (1+t)^{-1/\sigma}f(x)$, where $f(x) > 0$ satisfies the problem $\Delta f^{\sigma+1} + (1/\sigma)f - f^{\sigma+1} = 0$ in Ω, $f = 0$ on $\partial\Omega$. If $1 \leqslant \beta < \sigma + 1$

(for these values localization of perturbations occurs in the Cauchy problem; see (1.20)) the diffusion term of the equation is inconsequential as $t \to +\infty$. As a result, as $t \to +\infty$ the solution of the problem converges asymptotically everywhere in $\operatorname{supp} u(+\infty, x)$ to the spatially homogeneous solution $y(t)$ ($y'(t) = -y^\beta(t)$), i.e.,

$$t^{1/(\beta-1)}u(t,x) - (\beta-1)^{-1/(\beta-1)} \to 0 \quad \text{as } t \to +\infty,$$

for any $x \in \operatorname{supp} u(+\infty, x)$; for all other $x \in \Omega$, $u(t,x) \equiv 0$. Estimates of the rate of stabilization were obtained in [56] and [57].

The asymptotic behavior of solutions of the Cauchy problem for (1.11) has been studied to less extent. In addition to [14], mentioned earlier (see also §3 of the present paper), and the result of [57] on the stabilization to a homogeneous state for $1 \leqslant \beta < \sigma + 1$, the sharp estimates of the speed of motion of heat fronts obtained in [39], [58], and [59] also tell us about the asymptotic properties of solutions of (1.11), $N = 1$; see also the bibliography there. In [39] and [58] it was established that in the case $\beta > \sigma + 3$ the size of the support of a compactly supported solution of the Cauchy problem can be estimated according to the formula $\operatorname{supp} u(t,x) \sim t^{1/(\sigma+2)}$, $t \to +\infty$, which can be associated with the convergence of $u(t,x)$ to the known self-similar solution $v_A = t^{-1/(\sigma+2)}f(xt^{-1/(\sigma+2)})$ of the equation $v_t = (v^{\sigma+1})_{xx}$. This means that for $\beta > \sigma + 3$ in the Cauchy problem the sink in the equation is inconsequential as $t \to +\infty$ (the conclusions of [52] bear witness to this). For $\sigma + 1 < \beta < \sigma + 3$ in [59] the estimate

$$\operatorname{supp} u(t,x) \sim t^{(\beta-(\sigma+1))/2(\beta-1)}, \qquad t \to +\infty,$$

is obtained, which can be assessed as evidence of the asymptotic stability of the self-similar solution (1.12). Finally, if $\beta = \sigma + 1$, then $\operatorname{supp} u(t,x) \sim \ln t$ for large t [59] (in particular, $\operatorname{supp} u \to R^1$ as $t \to +\infty$ and for $\beta = \sigma + 1$ there is no localization). In this case equation (1.11), $N = 1$, $\beta = \sigma + 1$, has a self-similar solution of the form

$$u_A(t,x) = (1+t)^{-1/\sigma} f(\eta), \qquad \eta = x - \alpha \ln(1+t),\ \alpha = \text{const},$$

which under the assumption of stability provides the required estimate.

In conclusion we mention that the following question remains open: What is the structure of singular solutions of (1.11) with absorption under conditions of complete cooling in finite time when $\beta \in (0,1)$? In this case there exist completely damped localized self-similar solutions of the form (1.13), $u_A \equiv 0$ for $t \geqslant T_0$, where the (compactly supported) function θ_A satisfies the elliptic equation (cf. (1.14))

$$\Delta_\xi \theta_A^{\sigma+1} - m \sum_{i=1}^{N} \frac{\partial \theta_A}{\partial \xi_i}\xi_i + \frac{1}{\beta-1}\theta_A - \theta_A^\beta = 0, \qquad \xi \in R^N, \tag{1.25}$$

and, as usual, $\theta_A(\xi) \to 0$ as $|\xi| \to +\infty$. It is important to note that this equation admits nonmonotone solutions, while solutions (1.12) without damping ($\beta > 1$) cannot be nonmonotone (this follows directly from the equation for the function θ_A). Here there arise fundamental problems of investigating the manifold of solutions of the nonlinear elliptic problem (1.25), and thus of determining the spectrum of possible e.f. of the nonlinear medium with absorption, and also of proving their asymptotic stability.

§2. On radially symmetric self-similar solutions

This section is entirely devoted to investigating solutions of the boundary value problem (1.4), (1.6) for an ordinary differential equation. It is convenient to simultaneously consider the family of Cauchy problems for the same equation

$$\mathbf{A}_R(\theta) = 0, \qquad \xi > 0;\ \theta'(0) = 0,\ \theta(0) = \mu, \tag{2.1}$$

where the constant $\mu > 0$ plays the role of a parameter. It is necessary to find values $\mu > 0$ to which there corresponds a function $\theta = \theta(\xi; \mu)$ positive in R^1_+ satisfying the condition $\theta(+\infty; \mu) = 0$. The function $\theta(\xi; \mu)$ will then be the desired "self-similar" function $\theta_A(\xi)$. From the form of the operator $\mathbf{A}_R$ we immediately obtain

LEMMA 1. *For all* $0 < \mu < (\beta - 1)^{-1/(\beta-1)} = \theta_H$ *the solution of problem* (2.1) *is strictly decreasing everywhere where it is positive. If* $\mu \geqslant \theta_H$, *then* $\theta(+\infty; \mu) > 0$.

A formal asymptotic analysis of equation (1.4) as $\xi \to +\infty$ (i.e. as $\theta_A \to 0^+$) gives the following asymptotics of possible solutions of problem (1.4), (1.6); there is either the "power" asymptotics

$$\theta_A(\xi) = C\xi^{-2/(\beta+1)} + \cdots, \qquad \xi \to +\infty;\ C > 0, \tag{2.2}$$

or the "exponential" asymptotics

$$\theta_A(\xi) = D\xi^{1/(\beta-1)-N/2}\exp(-\xi^2/4) + \cdots, \qquad \xi \to +\infty;\ D > 0. \tag{2.3}$$

It can be said that (2.3) is a limit case of the "power" asymptotics (2.2), i.e., (2.2) "goes over" into (2.3) as $C \to 0^+$ (this conclusion will be justified below). The existence of solutions of the type (2.2) and (2.3) is established by analysis of equation (1.4) in a neighborhood of $\xi = +\infty$ based on fixed-point theorems of continuous transformations; in the same way the corresponding asymptotics for the derivatives $\theta_A'(\xi)$ and $\theta_A''(\xi)$ are obtained, and local solvability of the Cauchy problem for all sufficiently small $\xi > 0$ is proved.

Investigation of problem (1.4), (1.6) is carried out by different methods in the cases $\beta \geqslant 1 + 2/N$ and $\beta \in (1, 1 + 2/N)$.

1. *The case* $\beta \geqslant 1 + 2/N$. It is shown below that for $\beta \geqslant 1 + 2/N$ there exists only one infinite set of functions with asymptotics of type (2.2) (this is established in the same way as in subsection 2 by constructing upper and lower solutions of (1.4); in addition, this will be proved in §3 by a different method).

THEOREM 1. *Let* $\beta \geqslant 1 + 2/N$. *Then for all* $\mu \in (0, \theta_H)$ *the solution of problem* (2.1) *is strictly positive in* R^1_+, $\theta(+\infty; \mu) = 0$, *and it cannot have "exponential" asymptotics of the type* (2.3).

PROOF. We multiply (2.1) by ξ^{N-1} and integrate both sides over the interval $(0, \xi)$. As a result, we obtain

$$\xi^{N-1}\theta'(\xi) + \frac{1}{2}\theta(\xi)\xi^N = \int_0^\xi \eta^{N-1}\theta(\eta)\left[\frac{N}{2} - \frac{1}{\beta - 1} + \theta^{\beta-1}(\eta)\right] d\eta. \tag{2.4}$$

For $\beta \geqslant 1 + 2/N$ we have $N/2 - 1/(\beta - 1) \geqslant 0$, and hence the right side of (2.4) is strictly positive. Suppose $\theta = \theta(\xi; \mu)$ vanishes at some point $\xi = \xi^* > 0$. Then obviously $\theta'(\xi^*; \mu) \leqslant 0$, the left side of (2.4) for $\xi = \xi^*$ is nonpositive, and (2.4) is impossible. Thus, $\theta(\xi; \mu) > 0$ in R^1_+ and by monotonicity $\theta(+\infty; \mu) = 0$.

The second assertion of the theorem also follows from (2.4). Indeed, if $\theta(\xi; \mu)$ had "exponential" asymptotics and $\theta(\xi; \mu) = o(\xi^{-\alpha})$ as $\xi \to +\infty$ for all $\alpha > 0$ (then it is not hard to show that the derivative θ' would also have this property; see [18]), then (2.4) for $\xi = +\infty$ would give a contradiction.

REMARK. It will be shown in §3 that for $\beta > 1 + 2/N$ a solution with a fixed constant $C > 0$ in the expansion (2.2) is unique.

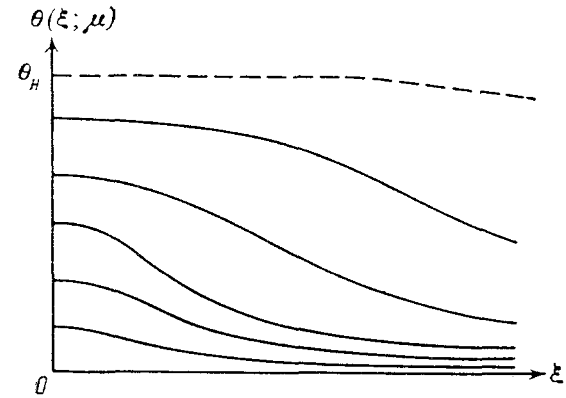

FIGURE 1
The case $\beta \geqslant 1 + 2/N$

For $\beta \geqslant 1 + 2/N$ the solutions $\theta(\xi;\mu)$ are easily ordered according to the magnitude of the parameter μ.

THEOREM 2. *Suppose $\beta \geqslant 1 + 2/N$. Then the solution $\theta(\xi;\mu)$ depends monotonically on the parameter $\mu \in (0, \theta_H)$, i.e., if $0 < \mu_1 < \mu_2 < \theta_H$, then*

$$\theta_1 = \theta(\xi;\mu_1) < \theta(\xi;\mu_2) = \theta_2 \quad \textit{everywhere in } R^1_+. \tag{2.5}$$

PROOF. We set $z = \theta_2 - \theta_1$. We suppose otherwise: there exists a $\xi = \xi_*$ such that $z(\xi_*) = 0$ for $z > 0$ on $(0, \xi_*)$, and hence $z'(\xi_*) \leqslant 0$. From (2.4) for θ_1 and θ_2 we then easily obtain

$$\xi_*^{N-1} z'(\xi_*) = \int_0^{\xi_*} \eta^{N-1} z(\eta) \left\{ \frac{N}{2} - \frac{1}{\beta - 1} \right. \\ \left. + \beta \int_0^1 [\rho\theta(\eta;\mu_1) + (1-\rho)\theta(\eta;\mu_2)]^{\beta-1} d\rho \right\} d\eta > 0,$$

which contradicts the condition $z'(\xi_*) \leqslant 0$.

Figure 1 shows the approximate schematic representation of the behavior of the functions $\theta = \theta(\xi;\mu)$ for different values of the parameter $\mu \in (0, \theta_H)$.

2. *The case* $\beta \in (1, 1 + 2/N)$. Here the picture of the behavior of the solutions corresponding to different values of μ is somewhat different than shown in Figure 1.

2.1. We first consider problem (2.1) for sufficiently small $\mu > 0$. We start by proving the following simple assertion.

LEMMA 2. *For all*

$$0 < \mu \leqslant (1/(\beta - 1) - N/2)^{1/(\beta-1)}, \qquad 1 < \beta < 1 + 2/N, \tag{2.6}$$

there does not exist an everywhere positive solution of problem (2.1) *with asymptotics* (2.2) *or* (2.3).

PROOF. We suppose otherwise: let $\theta(\xi;\mu) > 0$ in R^1_+ for some μ in the set (2.6). Passing to the limit in (2.4) as $\xi \to +\infty$ and using the fact that $\theta\xi^N \to 0$ and $\theta'\xi^{N-1} \to 0$ as $\xi \to +\infty$, we obtain

$$\int_0^{+\infty} \eta^{N-1}\theta(\eta;\mu)\left[\frac{N}{2} - \frac{1}{\beta-1} + \theta^{\beta-1}(\eta;\mu)\right] d\eta = 0.$$

However, because of the monotonicity of $\theta(\xi;\mu)$ in ξ this integral is strictly less than the expression

$$\int^{+\infty} \eta^{N-1}\theta(\eta;\mu)\left[\frac{N}{2}-\frac{1}{\beta-1}+\mu^{\beta-1}\right]d\eta,$$

which is nonpositive if (2.6) holds; this leads to a contradiction.

It is not hard to show that under the conditions of the lemma the solution $\theta(\xi;\mu)$ vanishes at some point $\xi=\xi_\mu<+\infty$. This involves some rather cumbersome computations (see, for example, [18]). Instead of this, we prove the simpler

LEMMA 3. *Suppose $\beta\in(1,1+2/N)$. Then there is a value $\mu_1\in(0,\theta_N)$ such that for all $\mu\in(0,\mu_1)$ the solution of problem* (2.1) *vanishes at some point $\xi=\xi_\mu<+\infty$.*

The proof proceeds by "linearizing" equation (2.1) relative to the trivial solution $\theta\equiv 0$. We first extend the equation into the region of negative values of θ, for example, in the following manner: in place of (2.1) we consider the equation

$$(\xi^{N-1}\theta')'/\xi^{N-1}+\theta'\xi/2+\theta/(\beta-1)-|\theta|^{\beta-1}\theta=0,\qquad \xi>0 \tag{2.7}$$

(in the region $\theta\geqslant 0$ it coincides with the original equation). We now set

$$f_\mu(\xi)=\theta(\xi;\mu)/\mu,\qquad \xi>0,$$

where $\theta(\xi;\mu)$ is the solution of (2.7) with conditions (2.1). Then obviously

$$f_\mu(0)=1,\qquad f_\mu'(0)=0 \tag{2.8}$$

and $f_\mu(\xi)$ satisfies the equation

$$\mathbf{F}_R(f_\mu)\equiv(\xi^{N-1}f_\mu')'/\xi^{N-1}+f_\mu'\xi/2+f_\mu/(\beta-1)=\mu^{\beta-1}|f_\mu|^{\beta-1}f_\mu \tag{2.9}$$

with the small parameter $\mu^{\beta-1}$ in front of the nonlinear "perturbation" term on the right side. We consider the corresponding linear problem for the value $\mu=0$:

$$\mathbf{F}_R(f_0)=0,\qquad \xi>0;\ f_0(0)=1,\ f_0'(0)=0. \tag{2.10}$$

By the change

$$\xi=2(-\eta)^{1/2},\qquad \eta<0, \tag{2.11}$$

(2.10) reduces to a boundary value problem for the degenerate hypergeometric equation

$$f_0''\eta+(N/2-\eta)f_0'-f_0/(\beta-1)=0,\qquad \eta<0.$$

PROPOSITION (see [60]). *If $N/2\geqslant 1/(\beta-1)$, then $f_0(\eta)>0$ for all $\eta<0$. If $N/2<1/(\beta-1)$, i.e. $\beta<1+2/N$, then $f_0(\eta)$ has at least one root for $\eta<0$.*

Lemma 3 follows from this proposition and the continuous dependence of a solution of the ordinary differential equation (2.9) on the small parameter $\mu^{\beta-1}$.

2.2. We begin to increase the parameter μ; suppose now (2.6) is not satisfied. We prove the following assertion.

THEOREM 3. *For $\beta\in(1,1+2/N)$ problem* (2.1) *has at least one positive solution with exponential asymptotics.*

The proof is based on constructing suitable upper and lower solutions θ_+ and θ_- of (2.1) which we seek in the "exponential" form

$$\theta_\pm(\xi)=A_\pm\exp(-\alpha_\pm\xi^2),\qquad \xi>0\ (\theta_\pm'(0)=0), \tag{2.12}$$

where $A_{\pm}$ and $\alpha_{\pm}$ are positive constants which remain to be determined. It is not hard to show that

$$\mathbf{A}_R(\theta_{\pm}) = A_{\pm}\exp(-\alpha_{\pm}\xi^2)\big[\alpha_{\pm}(4\alpha_{\pm}-1)\xi^2 + 1/(\beta-1) - 2N\alpha_{\pm} - A_{\pm}^{\beta}\exp(-\alpha_{\pm}\xi^2(\beta-1))\big],$$

and hence θ_+ is an upper solution (i.e., $\mathbf{A}_R(\theta_+) \leqslant 0$ in R^1_+) if

$$\alpha_+ < \frac{1}{4}, \qquad A_+^{\beta-1} \geqslant \left(\frac{1}{\beta-1} - 2N\alpha_+\right)\exp\left[\frac{\beta-1}{1-4\alpha_+}\left(\frac{1}{\beta-1} - 2N\alpha_+\right)\right], \tag{2.13}$$

and, correspondingly, θ_- is a lower solution ($\mathbf{A}_R(\theta_-) \geqslant 0$ in R^1_+) if, for example,

$$\alpha_- = 1/4, \qquad A_-^{\beta-1} \leqslant 1/(\beta-1) - N/2. \tag{2.14}$$

We note that it follows from (2.14) that an "exponential" lower solution exists only in the case where $1/(\beta-1) - N/2 > 0$, i.e. for $\beta < 1 + 2/N$.

For arbitrary values $A_{\pm}$ and $\alpha_{\pm}$ satisfying (2.13) and (2.14) respectively, everywhere in R^1_+ there is the inequality $\theta_-(\xi) < \theta_+(\xi)$, and hence (see, for example, [61]) problem (2.1) has at least one positive solution $\theta(\xi;\mu)$ such that $\theta_- \leqslant \theta \leqslant \theta_+$ in R^1_+. From the last inequality we obtain the "exponential" character of the asymptotics of this solution (for example, $\theta = o(\exp(-(1/4-\varepsilon)\xi^2))$ as $\xi \to +\infty$ for any $\varepsilon \in (0, 1/4)$).

REMARK. It is possible to refine the spatial structure of the solution by setting in place of (2.12)

$$\theta_{\pm}(\xi) = A_{\pm}\left(a_{\pm}^2 + \xi^2\right)^{\delta_{\pm}}\exp(-\xi^2/4), \qquad \xi > 0,$$

and it is not hard to show that for the lower solution we can take $\delta_- = 1/(\beta-1) - N/2$, while a suitable upper solution exists for any $\delta_+ > 1/(\beta-1) - N/2$. Thus,

$$\theta(\xi;\mu) = o\left\{\xi^{1/(\beta-1)-N/2+\varepsilon}\exp(-\xi^2/4)\right\}, \qquad \xi \to +\infty, \tag{2.12$'$}$$

for any $\varepsilon > 0$ (for $\varepsilon = 0$ the asymptotics (2.12′) coincides with (2.3)).

We now show that the solution with "exponential" asymptotics is the smallest among all positive solutions of problem (1.4), (1.6).

THEOREM 4. *Let $\beta \in (1, 1 + 2/N)$, and set*

$$\mu_* = \sup\mathscr{M} = \sup\left\{\mu > 0 \,|\, \theta(\xi;\mu) = 0 \text{ for some } \xi = \xi_\mu < +\infty\right\}. \tag{2.15}$$

Then $\mu_*(0, \theta_H)$, and the function $\theta_A^*(\xi) \equiv \theta(\xi;\mu_*)$—the solution of problem (1.4), (1.6)*—has "exponential" asymptotics and is minimal among all other solutions of it.*

PROOF. By Lemma 3 the set $\mathscr{M}$ is nonempty, and by Theorem 3 it is bounded above. Hence, $\sup\mathscr{M}$ exists, $\mu_* \in (0, \theta_H)$, and $\theta(\xi;\mu_*) > 0$ in R^1_+. The last follows from the continuous dependence of a solution of problem (2.1) on the parameter μ on an arbitrary compact set of R^1_+. The second part of the theorem will be proved in §3.

From Theorem 4 we immediately obtain the

COROLLARY. *Let $\beta \in (1, 1 + 2/N)$. Then for any $\mu \in (\mu_*, \theta_H)$ the functions $\theta(\xi;\mu)$ are strictly positive and are solutions of problem* (1.4), (1.6).

2.3. We now proceed to the investigation of solutions with "power" asymptotics. Such solutions exist, as is shown below, for any values $\beta > 1$.

THEOREM 5. *For $\beta > 1$ problem* (1.4), (1.6) *has an infinite set of solutions with "power" asymptotics* (2.2).

PROOF. This time we seek upper and lower solutions of (2.1) in the form

$$\theta_\pm(\xi) = A_\pm\left(a_\pm^2 + \xi^2\right)^{-1/(\beta-1)}, \qquad \xi > 0\ \left(\theta'_\pm(0) = 0\right). \tag{2.16}$$

It is not hard to verify that

$$\mathbf{A}_R(\theta_\pm) = A_\pm\left(a_\pm^2 + \xi^2\right)^{-(2\beta-1)/(\beta-1)}\left\{\left(\frac{a_\pm^2 - 2N}{\beta - 1} - A_\pm^{\beta-1}\right)a^2 + \left[\frac{a_\pm^2 - 2N}{\beta - 1} + \frac{4\beta}{(\beta-1)^2} - A_\pm^{\beta-1}\right]\xi^2\right\}. \tag{2.17}$$

Therefore, $\mathbf{A}_R(\theta_+) \leqslant 0$ in R^1_+, i.e., $\theta_+(\xi)$ is an upper solution if

$$A_+^{\beta-1} \geqslant \left(a_+^2 - 2N\right)/(\beta-1) + 4\beta/(\beta-1)^2. \tag{2.18}$$

Similarly, $\mathbf{A}_R(\theta_-) \geqslant 0$ in R^1_+, and $\theta_-(\xi)$ is a lower solution if

$$a_-^2 > 2N, \qquad A_-^{\beta-1} \leqslant \left(a_-^2 - 2N\right)/(\beta-1). \tag{2.19}$$

It remains to distinguish pairs of functions θ_- and θ_+ such that $\theta_- \leqslant \theta_+$ in R^1_+. Between them there then lies a positive solution of problem (1.4), (1.6). We fix an arbitrary $C > 0$ satisfying the condition

$$C^{\beta-1} > (4/(\beta-1))(\beta/(\beta-1) - N/2), \tag{2.20}$$

and we set $A_+ = A_- = C$. The functions (2.16) then have the same asymptotics

$$\theta_\pm(\xi) = C\xi^{-2/(\beta-1)} + \cdots, \qquad \xi \to +\infty.$$

Obviously $\theta_+ \geqslant \theta_-$ in R^1_+ if $a_+ \leqslant a_-$. We choose

$$a_+^2 = (\beta-1)\left[C^{\beta-1} - \frac{4}{\beta-1}\left(\frac{\beta}{\beta-1} - \frac{N}{2}\right)\right], \qquad a_-^2 = 2N + (\beta-1)C^{\beta-1} > 2N.$$

Then $a_- \geqslant a_+ > 0$ (the last inequality is ensured by condition (2.20)), and (2.18) and (2.19) are satisfied. Hence, there exists a solution $\theta_A(\xi)$ of (1.4), (1.6) such that $\theta_-(\xi) \leqslant \theta_A(\xi) \leqslant \theta_+(\xi)$ in R^1_+, and it has the asymptotics (2.2).

REMARK. It follows from (2.20) that for $N = 1$ and $N = 2$ to each arbitrary value $C > 0$ in the expansion (2.2) there corresponds at least one solution of problem (1.4), (1.6).

Figure 2 shows the approximate behavior of solutions of problem (2.1) for different $\mu \in (0, \theta_H)$ in the case $\beta \in (1, 1 + 2/N)$. We note that here distinct curves $\theta = \theta(\xi; \mu)$ may, generally speaking, intersect, but this can occur only at those points where θ is sufficiently small. Namely, the following assertion holds; it is proved in the same way as Theorem 2.

THEOREM 6. *Let $\beta \in (1, 1 + 2/N)$. Then in the region*

$$\theta > \left\{(1/\beta)(1/(\beta-1) - N/2)\right\}^{1/(\beta-1)}$$

the solutions $\theta(\xi; \mu)$ depend on the parameter μ in monotone fashion.

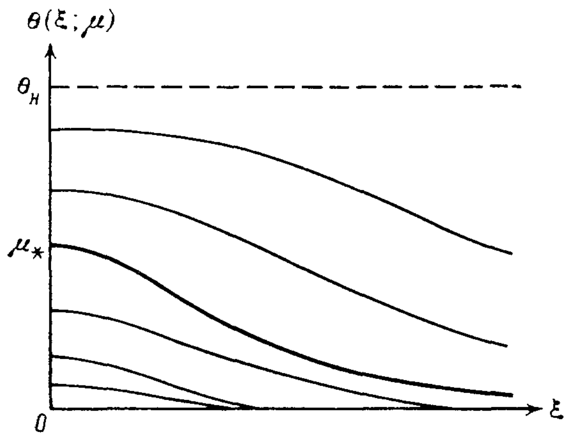

FIGURE 2

The case $\beta \in (1, 1 + 2/N)$. The heavy line designates the "self-similar" function $\theta_A^*(\xi) \equiv \theta(\xi; \mu_*)$ with "exponential" asymptotics.

In conclusion we present a solution of problem (1.4), (1.6) which can be represented in explicit form [62]. Let $\beta = 2$. Then the functions

$$\theta_A^{\pm}(\xi) = A^{\pm}/\left(a^{\pm} + \xi^2\right)^2 + B^{\pm}/\left(a^{\pm} + \xi^2\right), \qquad \xi > 0,$$

where

$$A^{\pm} = 48\left(-N - 14 \mp 10(1 + N/2)^{1/2}\right),$$

$$B^{\pm} = 24\left(2 \pm (1 + N/2)^{1/2}\right), \qquad a^{\pm} = 2\left(N + 14 \pm 10(1 + N/2)^{1/2}\right)$$

satisfy (1.4). Of them $\theta^+(\xi)$ is strictly positive in R^1_+ for any integer $N \geqslant 1$. We note that for $N = 1$ the quantity $\beta = 2$ lies in the interval $(1, 1 + 2/N) \equiv (1, 3)$; for $N = 2$ we have $\beta = 2 = 1 + 2/N$, while for all $N \geqslant 3$ we have $\beta = 2 > 1 + 2/N$.

§3. Eigenfunctions of the nonlinear problem

In this section we investigate the asymptotic behavior of solutions of problem (1.1), (1.2) and, in particular, prove the stability of self-similar solutions. Here one of the main problems consists in distinguishing in the space of initial functions regions of attraction corresponding to each e.f. of the nonlinear problem in question (as shown in §2, there are infinitely many self-similar solutions which are the principal "candidates" for the role of e.f.).

In §2 the self-similar functions $\theta_A(\xi) > 0$ were ordered by introducing the parameter $\mu = \theta_A(0) \in (0, \theta_H)$. The set of attraction corresponding to the self-similar solution(²)

$$u_A(t, x; T) = (T + t)^{-1/(\beta - 1)}\theta(\xi; \mu), \qquad \xi = x/(T + t)^{1/2}, \tag{3.1}$$

where $T = \text{const} > 0$, we denote by $\mathscr{W}_\mu$ (naturally, we are interested in nontrivial sets $\mathscr{W}_\mu$ consisting not only of self-similar initial distributions $u_A(0, x; T)$). By asymptotic stability of the self-similar solution (3.1), as usual (see, for example, [6], [7], and the bibliography there), we mean the convergence as $t \to +\infty$ of the self-similar representation of the

(²) Here $\xi \in R^N$; the self-similar functions constructed in §2 depend on the single variable $|\xi|$.

solution of the original problem (1.1), (1.2) corresponding to some value $T > 0$,

$$\begin{aligned}\theta_T(t,\xi) &= \mathbf{P}_T(u(t,x))(t,\xi)\\ &= (T+t)^{1/(\beta-1)}u\big(t,\xi(T+t)^{1/2}\big), \qquad t>0,\ \xi\in R^N, \end{aligned}\tag{3.2}$$

to the corresponding self-similar function $\theta_A = \theta(\xi;\mu)$. The quantity T in (3.2) is determined on the basis of the form of the initial function $u_0 \in \mathscr{W}_\mu$. It is obvious that the normalization (3.2) of the self-similar solution (3.1) gives precisely the function $\theta(\xi;\mu)$.

Below we shall need the following elementary upper and lower bounds for a solution of problem (1.1), (1.2).

LEMMA 4. *Suppose $\beta > 1$. Then everywhere in $R^1_+ \times R^N$*

$$u(t,x) \leqslant (T_0+t)^{-1/(\beta-1)}, \qquad T_0 = [\theta_H/\sup u_0]^{\beta-1}, \tag{3.3}$$

and for any $t_0 > 0$ everywhere in $[t_0, +\infty) \times R^N$

$$u(t,x) \geqslant (4\pi t)^{-N/2}F_0\exp(-|x|^2/4t), \tag{3.4}$$

where the constant $F_0 = F_0(t_0)$ has the form

$$F_0 = \int_{R^N} \exp\left(-\frac{|\zeta|^2}{4t_0}\right)u_0(\zeta)\,d\zeta > 0.$$

1. *The case $\beta > 1 + 2/N$.*

1.1. *Stability of self-similar e.f.* For $\beta > 1 + 2/N$ it is especially simple to determine conditions for the stability of self-similar solutions. Below we prove theorems on the stabilization $\theta \to \theta_A$ as $t \to +\infty$ which is uniform with respect to $\xi \in R^N$.

THEOREM 7. *Suppose that for $\beta > 1 + 2/N$ there exists a $T > 0$ such that*

$$u_0(x) - u_A(0,x;T) \in L^1(R^N). \tag{3.5}$$

Then

$$\|\theta_T(t,\xi) - \theta_A(\xi)\|_{C_\xi(R^N)} = O(t^{-N/2+1/(\beta-1)}), \tag{3.6}$$

$$\|\theta_T(t,\xi) - \theta_A(\xi)\|_{L^1_\xi(R^N)} = O(t^{-N/2+1/(\beta-1)}) \underset{t\to+\infty}{\to} 0. \tag{3.6$'$}$$

PROOF. We set

$$w_0^+(x) = \max\{u_0(x), u_A(0,x;T)\}, \qquad w_0^-(x) = \min\{u_0(x), u_A(0,x;T)\}$$

and denote by $w^\pm(t,x)$ solutions of (1.1) with the initial conditions $w^\pm(0,x) = w_0^\pm(x)$, $x \in R^N$. It is obvious that $w^+ \geqslant u_A(t,x;T)$ and $w^- \leqslant u_A(t,x;T)$. We consider the function $z^+ = w^+ - u_A \geqslant 0$ which satisfies the problem

$$\begin{gathered} z_t^+ = \Delta z^+ - (w^+)^\beta + u_A^\beta \leqslant \Delta z^+, \qquad t>0,\ x\in R^N, \\ z^+(0,x) = z_0^+(x) \equiv w_0^+(x) - u_A(0,x;T) \in L^1(R^N). \end{gathered}\tag{3.7}$$

From this we immediately obtain

$$\begin{aligned} z^+(t,x) &\leqslant \frac{1}{(4\pi t)^{N/2}}\int_{R^N}\exp\left(-\frac{|y|^2}{4t}\right)z_0^+(x+y)\,dy \\ &\leqslant \frac{1}{(4\pi t)^{N/2}}\|z_0^+\|_{L^1(R^N)} = O(t^{-N/2}). \end{aligned}\tag{3.8}$$

The lower bound for $z^-(t,x)$ is derived in a similar way. Now, using the facts that $z^- \leqslant u - u_A \leqslant z^+$ and

$$\|u - u_A\|_{C_x(R^N)} = (T+t)^{-1/(\beta-1)}\|\theta_T(t,\xi) - \theta_A(\xi)\|_{C_\xi(R^N)}, \tag{3.9}$$

we arrive at (3.6).

The second estimate is obtained even more simply. In analogy to the foregoing, from the equations for the functions $z^\pm$ we have

$$\|u(t,x) - u_A(t,x;T)\|_{L^1_x(R^N)} \leqslant \|u_0(x) - u_A(0,x;T)\|_{L^1_x(R^N)}.$$

From this, since

$$\|u - u_A\|_{L^1_x(R^N)} = (T+t)^{-1/(\beta-1)+N/2}\|\theta_T - \theta_A\|_{L^1_\xi(R^N)},$$

we obtain (3.6′).

REMARK. It is necessary to observe that in spite of the heat potential occurring in (3.8) this theorem can be completely carried over without difficulty to the case of the quasilinear equation (1.9). Here in place of (3.8) we use the following estimate of the solution of the Cauchy problem for the equation $p_t = \Delta p^{\sigma+1}$, $\sigma > 0$, presented, for example, in [63]:

$$\sup_{x\in R^N} p(t,x) \leqslant (c/t^k)\|p(0,x)\|^{2k/N}_{L^1(R^N)},$$

where $k = N/(2+N\sigma)$ and $c > 0$ is a constant not depending on the function $p(0,x)$ (in this case it is easy to verify that $\theta_T \to \theta_A$ as $t \to +\infty$ in the norms of $C(R^N)$ and $L^1(R^N)$ holds for $\beta > \sigma + 1 + 2/N$).

The next theorem gives what is apparently the optimal set $\mathscr{W}_\mu$ of stability of the self-similar solution (3.1).

THEOREM 8. *For $\beta > 1 + 2/N$ the set of attraction $\mathscr{W}_\mu$ corresponding to the given self-similar solution* (3.1) *has the form*

$$\begin{aligned}\mathscr{W}_\mu = \{u_0 \geqslant 0 \mid &\text{there exists } T > 0 \text{ such that}\\ &u_0(x) - u_A(0,x;T) = o(|x|^{-2/(\beta-1)}) \text{ as } |x| \to +\infty\}.\end{aligned} \tag{3.10}$$

REMARK. That the set (3.10) is optimal follows from Theorem 5.

PROOF. We consider the first estimate (3.8) for the function $z^+(t,x)$. In the present case, generally speaking, $z_0^+(x) \notin L^1(R^N)$, but because of the condition $u_0 \in \mathscr{W}_\mu$ there is a monotonically decreasing function $\varphi = \varphi(|x|) = o(|x|^{-2/(\beta-1)})$, $|x| \to +\infty$, such that $z_0^+(x) \leqslant \varphi(|x|)$ in R^N. Suppose for simplicity that $\sup_x z_0^+(x) = z_0^+(0)$. From (3.8) we then have

$$\sup_x z(t,x) = z^+(0,x) = O\left(t^{-N/2}\int_0^{+\infty} y^{N-1}\exp\left(-\frac{y^2}{4t}\right)\varphi(y)\,dy\right).$$

From this estimate and an analogous one for z^-, using (3.9), we obtain

$$\|\theta_T - \theta_A\|_{C_\xi(R^N)} = O\left[t^{1/(\beta-1)}\int_0^{+\infty} \eta^{N-1}\exp\left(-\frac{\eta^2}{4}\right)\varphi(t^{1/2}\eta)\,d\eta\right],$$

and, as is not hard to see, the right side tends to zero as $t \to +\infty$.

A curious corollary follows immediately from Theorem 8.

COROLLARY 1. *Let $\beta > 1 + 2/N$, and fix an arbitrary $C > 0$. Then problem* (1.4), (1.6) *can have at most one solution with given leading term in the "power" asymptotics* (2.2).

We emphasize that here the uniqueness theorem for a solution of an ordinary differential equation is proved on the basis of an analysis of solutions of a corresponding partial differential equation.

1.2. "*Non-self-similar*" *e.f.* (*approximate solutions*). Of course, Theorem 8 remains valid also in the case where $u_A \equiv 0$, i.e., for $\theta_A(\xi) \equiv 0$. Therefore, the set of attraction $\mathscr{W}_0$ corresponding to the self-similar function $\theta_A(\xi) = \theta(\xi; 0) \equiv 0$ has the form

$$\mathscr{W}_0 = \left\{ u_0 \geqslant 0 \,|\, u_0(x) = o\left(|x|^{-2/(\beta-1)}\right),\ |x| \to +\infty \right\}. \tag{3.11}$$

Thus, if for $\beta > 1 + 2/N$ we have $u_0 \in \mathscr{W}_0$, then

$$\theta_T(t, \xi) \to 0 \quad \text{as } t \to +\infty \tag{3.12}$$

uniformly with respect to $\xi \in R^N$. We observe that this simultaneously proves the following familiar assertion (see Theorem 1).

COROLLARY 2 OF THEOREM 8. *For $\beta > 1 + 2/N$ problem* (1.4), (1.6) *has no solution $\theta_A(\xi)$ with the "exponential" asymptotics* (2.3).

Indeed, otherwise there would exist a self-similar solution $u_A(t, x; T)$ such that $u_A(0, x; T) \in \mathscr{W}_0$. Then $\mathbf{P}_T(u_A(t, x; T)) \equiv \theta_A(\xi) > 0$ in R^N, which contradicts (3.12).

Thus, if $u_0 \in \mathscr{W}_0$, then the solution $u(t, x)$ evolves as $t \to +\infty$ according to non-self-similar laws. It will be shown below that the asymptotic behavior of "almost all" such solutions is determined by the space-time structure of a family of self-similar solutions of the heat equation

$$v_t = \Delta v, \qquad t > 0,\ x \in R^N, \tag{3.13}$$

i.e., in other words, if $u_0 \in \mathscr{W}_0$, then in the case $\beta > 1 + 2/N$ in the original equation (1.1) the nonlinear energy sink u^β becomes inconsequential as $t \to +\infty$ by comparison with the diffusion term. We note that such a situation is rather typical for quasilinear parabolic equations. In particular, on the basis of this particular "asymptotic degeneracy" of certain terms of an equation large classes of stable ap.s.-s.s.'s of various nonlinear heat equations with quite arbitrary (nonpower) coefficients were constructed in [6], [7], and [9]–[12].

We shall need a one-parameter family of self-similar solutions of (3.13) which has the form

$$v_A(t, x; T) = (T + t)^{-\alpha} f_a(\eta), \qquad \eta = |x|/(T + t)^{1/2}, \tag{3.14}$$

where $\alpha > 0$ is a parameter. Substitution of (3.14) into (3.13) leads to the following problem for the function $f_a > 0$:

$$\eta^{1-N}\left(\eta^{N-1} f_a'\right)' + f_a'\eta/2 + \alpha f_a = 0, \qquad \eta > 0, \tag{3.15}$$

$$f_a'(0) = 0, \qquad f_a(+\infty) = 0. \tag{3.16}$$

Conditions for the solvability of problem (3.15), (3.16) and some required properties of its solutions are indicated in the next assertion.

PROPOSITION (see [60]). *For any $\alpha \in (0, N/2)$ problem* (3.15), (3.16) *has a positive solution $f_a(\eta)$ in R^1_+, and if $\alpha \in (0, N/2)$, then*

$$f_a(\eta) = M\eta^{-2\alpha} + \cdots, \qquad \eta \to +\infty;\ M = \text{const} > 0. \tag{3.17}$$

If $\alpha = N/2$, then the only suitable solution is the function

$$f_a = f_a^*(\eta) = M\exp(-\eta^2/4), \qquad \eta > 0;\ M > 0. \tag{3.18}$$

For $\alpha > N/2$ positive solutions do not exist.

We denote by $f_T(t,\eta)$ (here $\eta \in R^N$, in contrast to (3.14)) the self-similar representation of the solution $u(t,x)$ coordinated with the form of the self-similar solution (3.14) of (3.13):

$$f_T(t,\eta) = (T+t)^{\alpha} u\big(t, \eta (T+t)^{1/2}\big), \qquad t > 0,\ \eta \in R^N. \tag{3.19}$$

Just as earlier in Theorem 8, in the case $u_0 \in \mathscr{W}_0$ the asymptotics of the solution of problem (3.15), (3.16) determined by the choice of the parameter $\alpha \in (0, N/2]$ in (3.14) depends on the specific form of the initial function. In particular, two cases here are essentially different: $u_0 \notin L^1(R^N)$ and $u_0 \in L^1(R^N)$. We first consider the case $u_0 \notin L^1(R^N)$, to which there correspond values $\alpha < N/2$. We note that in the proof of our theorem the possibility of inverting the operator $\partial/\partial t - \Delta$ is not used, i.e., this assertion can be reformulated without difficulty in application to the quasilinear equation (1.11).

THEOREM 9. *Suppose that $\beta > 1 + 2/N$ and, moreover,*

$$\beta(N-2) \leqslant N, \tag{3.20}$$

i.e., $\beta \in (1 + 2/N, +\infty)$ for $N = 1$ or $N = 2$ and $\beta \in (1 + 2/N, N/(N-2))$ for $N \geqslant 3$. Suppose there exist $\alpha \in (1/(\beta - 1), N/2)$ and positive constants T, M, and A such that

$$u_0(x) - v_A(0, x; T) \in L^1(R^N), \tag{3.21}$$

$$u_0(x) \leqslant A v_A(0, x; T), \qquad x \in R^N. \tag{3.22}$$

Then $\|f_T(t,\eta) - f_a(\eta)\|_{L^1_\eta(R^N)} \to 0$ as $t \to +\infty$.(3)

PROOF. We set $z = v_A - u$. For z we then obtain the equation

$$z_t = \Delta z + u^{\beta}, \qquad t > 0,\ x \in R^N;\ z(0,x) \in L^1(R^N). \tag{3.23}$$

We set $z^+ = \max\{0, z\} \geqslant 0$, $z^- = -\min\{0, z\} \geqslant 0$, $t \geqslant 0$ and $x \in R^N$. Obviously

$$\|z(t)\|_{L^1(R^N)} = \|z^+(t)\|_{L^1(R^N)} + \|z^-(t)\|_{L^1(R^N)},$$

and by (3.21) $z^+(0,x) \in L^1(R^N)$. From (3.23) it follows immediately that

$$\frac{d}{dt}\|z^+(t)\|_{L^1(R^N)} \leqslant \int_{R^N} u^{\beta}(t,x)\,dx, \qquad \frac{d}{dt}\|z^-(t)\|_{L^1(R^N)} \leqslant 0 \tag{3.24}$$

(these inequalities are derived under the natural assumptions $\Delta|z|(t,\cdot) \in L^1(R^N)$, $z \in C^1(R^1_+; R^N)$). From (3.22) on the basis of the maximum principle we conclude that $u \leqslant A v_A$ in $R^1_+ \times R^N$, and from the first estimate (3.24) we then obtain

$$\begin{aligned}\frac{d}{dt}\|z^+(t)\|_{L^1(R^N)} &\leqslant A^{\beta}\int_{R^N} v_A^{\beta}(t,x;T)\,dx \\ &= (T+t)^{-\alpha\beta + N/2} A^{\beta} \|f_a\|^{\beta}_{L^{\beta}(R^N)}, \qquad t > 0.\end{aligned} \tag{3.25}$$

Using (3.17) it is not hard to verify that for any $\alpha > 1/(\beta - 1)$ and all β satisfying (3.20) we have $f_a \in L^{\beta}(R^N)$. Then (3.25) implies that

$$\frac{d}{dt}\|z^+(t)\|_{L^1(R^N)} \leqslant \mathrm{const} \cdot (T+t)^{N/2 - \alpha\beta}, \qquad t > 0. \tag{3.26}$$

From the second estimate of (3.24) we obtain

$$\|z^-(t)\|_{L^1(R^N)} \leqslant \mathrm{const}, \qquad t > 0. \tag{3.27}$$

(3) An estimate of the rate of convergence will be obtained in the proof.

Finally, noting that

$$\|z(t)\|_{L^1(R^N)} = (T+t)^{-\alpha+N/2}\|f_T(t,\eta) - f_a(\eta)\|_{L^1_\eta(R^N)}, \tag{3.28}$$

from (3.27) and (3.28) under the condition $\alpha < N/2$ we obtain

$$\|f_T(t,\eta) - f_a(\eta)\|_{L^1_\eta(R^N)} = \begin{cases} O(t^{1-\alpha(\beta-1)}), & \alpha < (N+2)/2\beta, \\ O(t^{\alpha-N/2}\ln t), & \alpha = (N+2)/2\beta, \\ O(t^{\alpha-N/2}), & \alpha > (N+2)/2\beta. \end{cases}$$

Hence, if $\alpha \in (1/(\beta-1), N/2)$, then $\|f_T - f_a\|_{L^1(R^N)} \to 0$ as $t \to +\infty$.

REMARK 1. This theorem is not altogether "optimal" with regard to the admissible values of the parameter β (this refers to condition (3.20)). However, if (3.20) is not satisfied, then it is convenient to carry out the proof of convergence $f_T \to f_a$ as $t \to +\infty$, in the norm of $L^{p+1}(R^N)$, choosing the quantity $p > 0$ from the condition $p > N/2\alpha\beta - 1$. Assuming that $u_0(x) - v_A(0,x;T) \in L^{p+1}(R^N)$, it is then not difficult to show in a similar way that $\|f_T - f_a\|_{L^{p+1}(R^N)} \to 0$ as $t \to +\infty$ for $\alpha \in (1/(\beta-1), N/2)$ (here the conditions $z = v_A - u \in C^1(R^1_+; L^{p+1}(R^N))$ and $z^{(p+1)/2} \in H^1(R^N)$ are required).

REMARK 2. In Theorem 9 we obtained the asymptotics of solutions $u(t,x)$ corresponding to initial functions of the form (this follows from (3.21))

$$u_0(x) \sim |x|^{-2\alpha}, \qquad |x| \to +\infty, \tag{3.29}$$

for $\alpha \in (1/(\beta-1), N/2)$. The restriction $\alpha > 1/(\beta-1)$ is indeed essential, since for $\alpha = 1/(\beta-1)$ the corresponding e.f. are to be sought in the class of self-similar solutions (3.1). If $\alpha \in (0, 1/(\beta-1))$, then, as shown in [14], $\theta_T(t,\xi)$ tends to the spatially homogeneous solution $\theta_A(\xi) \equiv \theta_H = (\beta-1)^{-1/(\beta-1)}$, and this conclusion is valid for any $\beta > 1$. For $\alpha \in (1/(\beta-1), N/2)$ all functions of (3.29) have infinite initial "energy": $u_0 \notin L^1(R^N)$. This also holds for $\alpha = N/2$, but here the e.f. does not have "self-similar" structure as in (3.1) and (3.14). This is manifest, for example, in the fact that the heat equation (3.13) with initial function

$$v(0,x) = v_0(x) \sim |x|^{-N}, \qquad |x| \to +\infty, \tag{3.30}$$

has, as is not difficult to show, the non-self-similar asymptotics

$$v(t,x) \to ct^{-N/2}\ln t\exp(-|x|^2/4t), \qquad t \to +\infty,\ c = \text{const} > 0. \tag{3.31}$$

Thus, the case $\alpha = N/2$ in (3.29) is "critical" in the sense that here problem (1.1), (1.2), $\beta > 1 + 2/N$, has a nontrivial ap.s.-s.s. compatible with the asymptotics (3.31) of the corresponding linear problem (3.13), (3.30) without a sink.

The case $u_0 \in L^1(R^N)$, $u_0 \not\equiv 0$, is considered in [14], where a theorem is formulated on the convergence as $t \to +\infty$ of the self-similar representation (3.19) corresponding to the value $\alpha = N/2$ to the function $c^* f_a^*(\eta)$, where $c^* > 0$ is a constant depending on the initial function. We note that with certain additional conditions on $u_0(x)$ this result can be obtained in another way without invoking the integral equation for the function $u(t,x)$ which is apparently used in [14]. Thus, an analogous assertion for $\beta > \sigma + 1 + 2/N$ is valid for the quasilinear equation (1.11). Without considering this in detail, we note only that the main problem here consists in the following: it is necessary to prove that the limit function $f_a(\eta)$ to which the function $f_T(t,\eta)$ converges as $t = t_i \to +\infty$ (it satisfies (3.13) because of relatively easily derived estimates of the solution of problem (1.1), (1.2))

1) is unique, i.e., does not depend on the choice of the sequence $\{t_i\}$, and

2) is nonzero.

Under these conditions we immediately obtain $f_a(\eta) = c^* f_a^*(\eta) > 0$ in R^N (at least this will hold for the functions $u_0 = u_0(|x|)$).

The first follows directly from the monotonicity of $\|u(t,x)\|_{L^1_x(R^N)} \equiv \|f_T(t,\eta)\|_{L^1_\eta(R^N)}$ (here we take into account that distinct curves (3.18) cannot intersect). The second is obtained without difficulty by constructing a special type of lower solution of equation (1.1). Namely, we set

$$u_-(t,x) = \psi(t)(T+t)^{-N/2}\exp\left[-|x|^2/4(T+t)\right], \tag{3.32}$$

where the function $\psi(t) > 0$ will be indicated below. Then u_- will be a lower solution of (1.1), i.e., $\mathbf{B}(u^-) \leqslant 0$ in $R^1_+ \times R^N$ if

$$\psi'(t) \leqslant -\psi^\beta(t)(T+t)^{-N(\beta-1)/2}, \qquad t > 0.$$

It is not hard to see that for $\beta > 1 + 2/N$ we can take for $\psi(t)$ a function bounded away from zero uniformly with respect to all $t \geqslant 0$, for example, of the form

$$\psi(t) = \psi_0 + A(T+t)^{-\gamma}, \qquad t > 0;\ \psi_0 > 0,\ A > 0, \tag{3.32'}$$

where $\gamma = (N/2)(\beta - 1) - 1 > 0$ and $\gamma A > (\psi_0 + AT^{-\gamma})^\beta$. Therefore, under the corresponding conditions on $u_0(x)$ we have $u(t,x) \geqslant u_-(t,x)$ in R^N, i.e., $f_T(t,\eta) > \psi_0 \exp(-|\eta|^2/4)$ for any $t > 0$ and $\eta \in R^N$. We note that it is not possible to simply set $\psi(t) = \psi_0$ (this is easily verified, because then (3.32) will not be a lower solution); a small "correction" has therefore been introduced in (3.32′) which nevertheless "compensates" the nonlinear term u_-^β in the inequality $\mathbf{B}(u_-) \leqslant 0$ which the lower solution must satisfy. This actually means that this term of the equation is inconsequential as $t \to +\infty$ by comparison with the diffusion term.

REMARK 3. The family of self-similar solutions (3.14) used in Theorem 9 is a two-parameter family: each function f_a is characterized, first, by the value of the parameter α in problem (3.15), (3.16) and, second, as any solution of a linear equation, by the magnitude of the constant factor M in (3.17). It follows from condition (3.21) that the last parameter is also essential. From this we can conclude that for $\beta > 1 + 2/N$ the dimension of the attractor of the parabolic equation in question is not less than two.

2. *The case* $\beta \in (1, 1 + 2/N]$. *Stability of self-similar e.f.* Some of the results obtained below are valid for any $\beta > 1$. Here the investigation is broken into several steps.

For convenience we first formulate the following simple lemma, which follows from the maximum principle.

LEMMA 5. *Let* $\theta_+(\xi) \in C^2(R^N)$ *be any upper solution* (2.12) *or* (2.16) *of equation* (2.1). *Suppose that for some* $T > 0$

$$u_0(x) \leqslant T^{-1/(\beta-1)}\theta_+(x/T^{1/2}), \qquad x \in R^N. \tag{3.33}$$

Then for the solution of problem (1.1), (1.2) *there is the estimate*

$$u(t,x) \leqslant (T+t)^{-1/(\beta-1)}\theta_+\left(x/(T+t)^{1/2}\right), \qquad t > 0,\ x \in R^N, \tag{3.34}$$

REMARK. The lemma remains valid if θ_+ is replaced by a corresponding lower solution θ_- of (2.1) and the inequality signs in (3.33) and (3.34) are reversed. Here the function θ_- is not required to be C^2-smooth in the entire space R^N; it suffices that $\theta_- \in C^2$ everywhere where $\theta_- > 0$, i.e., $\theta_-(\xi)$ can here also be a compactly supported function.

2.1. *An auxiliary assertion.* At the first stage we use an approach to investigating the asymptotic behavior of solutions of parabolic equations proposed for investigating other problems in [64] and [65] (it was applied in [14]).

We consider the one-parameter family of functions([4])

$$u_k(t,x) = k^{1/(\beta-1)}u(kt, k^{1/2}x), \qquad t>0,\ x\in R^N, \tag{3.35}$$

where $k>0$ is an arbitrary constant. Each of the functions u_k satisfies the original equation (1.1):

$$\mathbf{B}(u_k)=0, \qquad (t,x)\in R^1_+\times R^N, \tag{3.36}$$

$$u_k(0,x) = k^{1/(\beta-1)}u_0(k^{1/2}x), \qquad x\in R^N. \tag{3.37}$$

LEMMA 6. *Let $\beta\in(1,1+2/N]$, and suppose condition* (3.33) *holds. Then for all sufficiently large k*

$$u_k(t,x)\leqslant c_1, \qquad t\geqslant\tau>0,\ x\in R^N, \tag{3.38}$$

$$\|u_k(t)\|^2_{L^2(R^N)}\leqslant c_2, \qquad \int_\tau^T\|\nabla u_k(s)\|^2_{L^2}\,ds\leqslant c_3, \tag{3.39}$$

$$\int_\tau^T\|(u_k)_t(s)\|^2_{L^2(R^N)}\leqslant c_4, \qquad \|\nabla u_k(\tau)\|^2_{L^2}\leqslant c_5, \tag{3.40}$$

where the positive constants c_i do not depend on k.

The first two estimates follow directly from (3.33); the remaining estimates are easily derived from (3.36) and the preceding estimates (see, for example, [64]).

On the basis of the Sobolev imbedding theorems [66], from Lemma 6 we obtain the following assertion.

THEOREM 10. *From any monotone sequence $k_j\to+\infty$ it is possible to select a subsequence $k_i\to+\infty$ such that*

$$u_k(t,x)\to w(t,x), \qquad k=k_i\to+\infty, \tag{3.41}$$

the convergence is uniform on any compact set in $[\tau,+\infty)\times R^N$, and the limit function $w(t,x)$ satisfies equation (1.1).

If in (3.41) we set $t=1$, $k_i=T+t_i$ for sufficiently large i and $x=\xi$, we obtain the

COROLLARY. *Under the above assumptions,*

$$\theta_T(t_i,\xi)\to w(1,\xi), \qquad \xi=x/(T+t_i)^{1/2}, \quad \text{as } t_i\to+\infty, \tag{3.42}$$

uniformly on each compact set in R^N.

Here θ_T is the self-similar representation (3.2) of the solution $u(t,x)$. The function w in (3.41) and (3.42) may, generally speaking, depend on the choice of sequence k_j in Theorem 10 (we note that it is so far unknown whether w is a self-similar solution of (1.1)). However, for $\beta\in(1,1+2/N)$ all possible functions w have a common property which emerges on the basis of the analysis of solutions of the "self-similar" equation (1.4) carried out in §2.

([4]) The family (3.35) actually makes it possible to determine the self-similar representation (3.2) in another way, although, of course, the methods used below are applicable for a direct analysis of the asymptotic behavior of the self-similar representation (3.2) as $t\to+\infty$.

THEOREM 11. *Let* $\beta \in (1, 1 + 2/N)$. *If* $u_0(x) \not\equiv 0$, *then* $w(1, \xi) \not\equiv 0$.

PROOF. We assume with no loss of generality that $u_0(0) > 0$. It is then not hard to see that there exist sufficiently small $T > 0$ and $\mu_1 > 0$ such that

$$u_0(x) \geqslant T^{-1/(\beta-1)}\theta\left(|x|/T^{1/2}; \mu_1\right), \qquad |x| < T^{1/2} \operatorname{supp} \theta(\zeta; \mu_1),$$

where $\theta(\xi; \mu_1)$ is a solution of (2.1). Therefore (see the remark to Lemma 5),

$$u(t, x) \geqslant (T + t)^{-1/(\beta-1)}\theta\left(|x|/(T + t)^{1/2}; \mu_1\right),$$

$$t > \theta,\ |x| < (T + t)^{1/2} \cdot \operatorname{supp} \theta(\xi; \mu_1),$$

and hence

$$\theta_T(t, \xi) \geqslant \theta(\xi; \mu_1) > 0, \qquad |\xi| < \operatorname{supp} \theta(\xi; \mu_1)$$

for all $t > 0$.

REMARK. For $\beta > 1 + 2/N$ (see Theorem 9) and also in the case $\beta = 1 + 2/N$ Theorem 11 ceases to hold. This can be related to the absence for $\beta \geqslant 1 + 2/N$ of "compactly supported" solutions $\theta(\xi; \mu)$ of problem (2.1) (Theorem 1).

2.2. *Stability of self-similar solutions.* We first show that independence of the limit function w in (3.41) from the choice of sequence k_j guarantees its "self-similarity".

LEMMA 7. *Let*

$$u_k(t, x) \to w(t, x) \quad \text{as } k \to +\infty. \tag{3.43}$$

Then

$$w(t, x) = t^{-1/(\beta-1)}\theta(\xi), \qquad \xi = x/t^{1/2}, \tag{3.44}$$

where $\theta(\xi) = w(1, \xi)$.

Indeed, from (3.43) it follows immediately that

$$\gamma^{1/(\beta-1)}w\left(\gamma t, \gamma^{1/2}x\right) \equiv w(t, x)$$

for any $\gamma > 0$. Setting $\gamma = t^{-1}$, we arrive at (3.44).

Thus, if (3.43) holds, then w is self-similar, and $\theta(\xi)$ satisfies the elliptic equation

$$\mathbf{A}(\theta) \equiv \Delta_\xi \theta + \frac{1}{2} \sum_{i=1}^{N} \frac{\partial \theta}{\partial \xi_i} \xi_i + \frac{1}{\beta - 1}\theta - \theta^\beta = 0, \qquad \xi \in R^N, \tag{3.45}$$

which in the radially symmetric case coincides with equation (1.4) considered in §2.

It is now necessary to proceed to an investigation which would make it possible to specify the form of the radially symmetric, "self-similar" limit function $\theta = \theta_A(|\xi|)$ in (3.44) and also the set of attraction $\mathscr{W}_\mu$ ($\mu = \theta_A(0)$) corresponding to it in the space of initial functions. For this it is convenient to go over to the equation for the self-similar representation (3.2). Introducing a new "time" by the formula $\tau = \ln(1 + t/T)$, we find that the function $\theta_T = \theta_T(\tau, \xi)$ satisfies the following parabolic problem:

$$\partial\theta_T/\partial\tau = \mathbf{A}(\theta_T), \qquad (t, \xi) \in R^1_+ \times R^N, \tag{3.46}$$

$$\theta_T(0, \xi) = T^{1/(\beta-1)}u_0(T^{1/2}\xi), \qquad \xi \in R^N. \tag{3.47}$$

Comparing (3.46) and (2.1), we see that in the new notation the question of asymptotic stability of the self-similar solutions (3.1) reduces to the problem of the stability and construction of regions of attraction of steady-state solutions of (3.46). Starting from

Lemma 7, it is here convenient to determine conditions for "monotonicity in k" of the function sequences $u_k(t, x)$ in (3.35) which is equivalent to a condition of monotonicity in τ of the self-similar representation $\theta_T(\tau, \xi)$. These conditions are given by

LEMMA 8. *Let $\theta_+(\xi)$ (respectively, $\theta_-(\xi)$) be any upper (lower) solution of equation* (2.1) *(see* (2.12) *and* (2.16)), *i.e.,* $\mathbf{A}_R(\theta_+) \leqslant 0$ ($\mathbf{A}_R(\theta_-) \geqslant 0$) *everywhere in R^N. Then the solution of equation* (3.46) *with initial function $\theta_T(0, \xi) = \theta_+(\xi)$ (respectively, $\theta_T(0, \xi) = \theta_-(\xi)$) is nonincreasing (nondecreasing) in τ in R^N:*

$$\partial\theta_T(\tau, \xi)/\partial\tau \leqslant 0 \qquad (\partial\theta_T/\partial\tau \geqslant 0) \quad \text{in } R^1_+ \times R^N. \tag{3.48}$$

The lemma follows from the maximum principle. Indeed, the function $z = (\theta_T)_\tau$ satisfies the following "linear" parabolic equation:

$$z_\tau = \Delta_\xi z + \frac{1}{2}\sum_{i=1}^N \frac{\partial z}{\partial \xi_i}\xi_i + \left(\frac{1}{\beta - 1} - \beta\theta^{\beta-1}\right)z \quad \text{in } R^1_+ \times R^N,$$

where $z(0, \xi) = \mathbf{A}_R(\theta_+(\xi)) \leqslant 0$ in R^N. Hence, $z \leqslant 0$ everywhere in $R^1_+ \times R^N$. Assertions similar to Lemma 8 were established for a special type of parabolic equations in [61] and [67], and in the general case in [68] and [69], where they were used to prove theorems for comparing solutions of parabolic equations with different nonlinear operators. We note that the smoothness condition on the function $\theta_-(\xi)$ ensuring the critical property of the solution $(\theta_T)_\tau \geqslant 0$ in $R^1_+ \times R^N$ can be relaxed; it suffices that the inequality $\mathbf{A}_R(\theta_-) \geqslant 0$ be satisfied only where $\theta_- > 0$ (see [68] and [69]). Therefore, for $\theta_-(\xi)$ it is possible to take, for example, a compactly supported function whose smoothness may be violated at those points where $\theta_- = 0$.

From Lemmas 7 and 8 we obtain

THEOREM 12. *Suppose that $\beta > 1$ and the radially symmetric function $u_0(x)$ in* (1.2) *is such that*

$$\theta_T(0, \xi) = T^{1/(\beta-1)}u_0(T^{1/2}\xi)$$

is an upper or lower solution of equation (2.1). *Then there exists a radially symmetric "self-similar" function $\theta_A(|\xi|)$ satisfying* (1.4) *and* (1.6) *such that*

$$\theta_T(\tau, \xi) \to \theta_A(|\xi|) \quad \text{as } \tau \to +\infty,$$

uniformly on each compact set in R^N.

REMARK. Of course, the entire set of upper and lower solutions of (2.1) is not exhausted by those functions $\theta_\pm$ indicated in §2 (see (2.12) and (2.16)). For example, to each "self-similar" function there corresponds an entire family of such functions

$$\theta_\pm(\xi) = A_\pm\theta_A(|\xi|),$$

where $A_+ > 1$ and $A_- < 1$ are constants. Indeed, in this case

$$\mathbf{A}_R(\theta_\pm(\xi)) = A_\pm\theta_A^\beta(|\xi|)(1 - A_\pm^{\beta-1}),$$

i.e.,

$$\mathbf{A}_R(\theta_+) < 0, \qquad \mathbf{A}_R(\theta_-) > 0 \quad \text{in } R^N.$$

We now proceed to the construction of the set of attraction $\mathscr{W}_\mu$ corresponding to the given function $\theta_A = \theta(|\xi|; \mu)$. This can be done by means of the next assertion, which follows from Lemma 8.

LEMMA 9. *Let* θ_+, $\theta_- \in C^2(R^N)$, *respectively, be upper and lower radially symmetric solutions of equation* (2.1) *to which there corresponds one unique "self-similar" function* $\theta_A = \theta(|\xi|; \mu)$ *such that*

$$\theta_-(|\xi|) \leqslant \theta_A(|\xi|) \leqslant \theta_+(|\xi|), \qquad \xi \in R^N.$$

Then the set

$$\mathscr{H}_\mu = \{u_0 \geqslant 0 \mid \textit{there exists } T > 0 \textit{ such that } \theta_-(|\xi|) \leqslant T^{1/(\beta-1)} u_0(T^{1/2}\xi) \leqslant \theta_+(|\xi|),\ \xi \in R^N\}$$

belongs to $\mathscr{W}_\mu$.

Thus, the problem of distinguishing the sets $\mathscr{W}_\mu$ is closely related to the problem of the uniqueness classes for solutions $\theta_A(|\xi|)$ of problem (1.4), (1.6).

Below we shall investigate in more detail the set $\mathscr{W}_A^*$ corresponding to the "minimal" function $\theta_A^*(|\xi|) = \theta(|\xi|; \mu_*)$ indicated in Theorem 4 (§2).

THEOREM 13. *Let* $\beta \in (1, 1 + 2/N)$. *Suppose the initial function* $u_0(x) \not\equiv 0$ *is such that*

$$u_0(x) \leqslant T^{-1/(\beta-1)} \theta_A^*(|x|/T^{1/2}), \qquad x \in R^N. \tag{3.49}$$

Then $\theta_T(\tau, \xi) \to \theta_A^*(|\xi|)$ *as* $\tau \to +\infty$ *uniformly on each compact set in* R^N.

PROOF. From the corollary to Theorem 10 it follows that $\theta_T(\tau, \xi) \to w(1, \xi)$ as $\tau = \tau_i \to +\infty$, and by (3.49) $w(1, \xi) \leqslant \theta_A^*(|\xi|)$ in R^N. Proceeding as in the proof of Theorem 11, it is then not hard to find constants $\mu_1 \in (0, \mu_*)$ and $\tau_0 \geqslant 0$ such that

$$\theta_-(\xi) = \theta(|\xi|; \mu_1) \leqslant \theta_T(\tau_0, \xi) \leqslant \theta_A^*(|\xi|), \qquad \xi \in R^N.$$

However, by Lemma 8 the solution $\theta_T^-(\tau, \xi)$ radially symmetric in ξ of (3.46) corresponding to the initial distribution $\theta_T^-(\tau_0, \xi) = \theta(|\xi|; \mu_1)$ is nondecreasing in τ, and hence it follows from Theorem 4 (§2) that $\theta_T^-(\tau, \xi) \to \theta_A^*(|\xi|)$ as $\tau \to +\infty$. Hence $\theta_T(\tau, \xi)$ also stabilizes to $\theta_A^*(|\xi|)$ as $\tau \to +\infty$.

REMARK. Thus, the "self-similar" function $\theta_A^*(|\xi|)$ is stable below. If θ_A^* is the unique solution of (1.4) with "exponential" asymptotics, then it is also stable above, and the set of attraction $\mathscr{W}_A^*$ in this case can be defined, for example, in the following manner (see Remark 2 to Theorem 12):

$$\mathscr{W}_A^* = \{u_0(x) \geqslant 0 \mid \text{there exist } T > 0 \text{ and } A \geqslant 1 \text{ such that } 0 < u_0(x) \leqslant T^{-1/(\beta-1)} A \theta_A^*(|x|/T^{1/2}),\ x \in R^N\}.$$

We note that from Theorem 13 we immediately obtain the following assertion, which completes the proof of Theorem 4 in §2.

COROLLARY. *For* $\beta \in (1, 1 + 2/N)$ *the "self-similar" function* $\theta = \theta_A^*(|\xi|)$ *is the minimal solution in* R^N *among all nonnegative solutions* (*including radially symmetric solutions*) *of the elliptic equation* (3.45).

3. *The "critical" case* $\beta = 1 + 2/N$, $u_0 \in L^1(R^N)$. *An approximate self-similar solution.* The stability of nontrivial self-similar solutions (3.1) for $\beta = 1 + 2/N$ was studied in the preceding subsection. We recall that by Theorem 5 to these solutions there corresponds infinite "energy". Below we consider the question of the evolution of initial perturbations $u_0(x) \in L^1(R^N)$, and we restrict ourselves to the analysis of the case where

$$u_0(x) = o\{\exp(-\gamma|x|^2)\}, \qquad |x| \to +\infty, \tag{3.50}$$

for some constant $\gamma > 0$. It will be shown that here the following self-similar representation of the solution $u(t, x)$ is "stabilizing":

$$g_T(t,\xi) = \mathbf{Q}_T(u) \equiv [(T+t)\ln(T+t)]^{N/2}u\big(t,\xi(T+t)^{1/2}\big), \tag{3.51}$$

where $T > 1$ is a constant. This implies that in the case where $g_T(t, \xi)$ stabilizes as $t \to +\infty$ to some bounded function $g_*(\xi) \not\equiv 0$, the asymptotic properties of $u(t, x)$ are described by an ap.s.-s.s. $u_a(t, x)$ of the form

$$u(t,x) \to u_a(t,x) = [(T+t)\ln(T+t)]^{-N/2}g_*\big(x/(T+t)^{1/2}\big). \tag{3.52}$$

As compared with (3.1) there is an additional logarithmic factor in (3.52).

We shall first show that for large t the solution $u(t, x)$ is bounded below in R^N by a function having the structure of the ap.s.-s.s. in (3.52). This assertion refines one of the results of [14].

LEMMA 10. *Suppose $\beta = 1 + 2/N$ and $u_0 \not\equiv 0$. Then for any $T > 1$ there exist $\tau > 0$ and $A \in (0, (N/2)^{N/2})$ such that everywhere in $[\tau, +\infty) \times R^N$*

$$u(t,x) \geqslant u_-(t,x) = A[(T+t)\ln(T+t)]^{-N/2}\exp[-|x|^2/4(T+t)]. \tag{3.53}$$

Because of the estimate (3.4), for the proof it suffices to show that the function u_- in (3.53) is a lower solution of (1.1). This follows directly from the form of the equation for the function g_T which will be obtained below.

We shall now bound $u(t,x)$ above by a function close in form for large t to the ap.s.-s.s. (3.52).

LEMMA 11. *Suppose $\beta = 1 + 2/N$ and the initial function u_0 satisfies condition* (3.50). *Then for any $T > e^2$ there exist constants $a > 0$ and $H > 0$ such that*

$$u(t,x) \leqslant u_+(t,x) = H[(T+t)\ln(T+t)]^{-N/2}\exp\left\{-\frac{|x|^2}{4(T+t)[1+a\ln^{-1}(T+t)]^2}\right\} \tag{3.54}$$

everywhere in $R^1_+ \times R^N$.

PROOF. We shall show that u_+ in (3.54) is an upper solution of (1.1) if the constant $H = H(a, T) > 0$ is sufficiently large. Choosing a and H large, with (3.50) taken into account we then arrive at (3.54). Thus, we shall determine conditions under which

$$\mathbf{B}(u_+) \geqslant 0 \quad \text{in } R^1_+ \times R^N. \tag{3.55}$$

We introduce the notation $\tau = (T + t)$ and $\varphi(\tau) = \tau^{1/2}[1 + a\ln^{-1}\tau]$. It is then not hard to verify that (3.55) is equivalent to the inequality

$$\frac{1}{2}|\xi|^2\left(\frac{1}{2} - \varphi\varphi'\right) - \frac{N}{2} + \frac{N}{2}\frac{\varphi^2}{\tau}(1+\ln^{-1}\tau) - \frac{H^{2/N}\varphi^2}{\tau\ln\tau}\cdot\exp\left(-\frac{1}{2N}|\xi|^2\right) \leqslant 0, \quad x \in R^N, \tag{3.56}$$

where we have introduced the notation $|\xi|^2 = |x|^2/\tau$. We have

$$\varphi(\tau)\varphi'(\tau) = \frac{1}{2}\left(1 + \frac{a}{\ln\tau}\right)\left(1 + \frac{a}{\ln\tau} - \frac{2a}{\ln^2\tau}\right) \geqslant \frac{1}{2} + \frac{a_1}{\ln\tau};$$

$$\frac{N}{2}\frac{\varphi^2(\tau)}{\tau}\left(1 + \frac{1}{\ln\tau}\right) = \frac{N}{2}\left(1 + \frac{a^2}{\ln^2\tau} + \frac{2a}{\ln\tau}\right)\left(1 + \frac{1}{\ln\tau}\right) \leqslant \frac{N}{2} + \frac{a_2}{\ln\tau},$$

where

$$a_1 = a_1(a, T) = (a/2)(1 - 2/\ln T) > 0,$$
$$a_2 = a_2(a, T) = (N/2)\{1 + 2a + (a^2 + 2a)/\ln T + a^2/\ln T\}.$$

Using these estimates, we find that (3.56) is clearly satisfied if

$$-a_1|\xi|^2/2 + a_2 - H^{2/N}\exp(-|\xi|^2/2N) \leqslant 0.$$

This inequality is obvious for sufficiently large $H > 0$, for example, if

$$H^{2/N} \geqslant Na_1 \max\{1, \exp(a_2/Na_1 - 1)\}.$$

The lemma is proved.

From the last two lemmas it follows that if (3.50) holds and $u_0 \not\equiv 0$, then the "self-similar" representation (3.51) (T is large) for all large t is bounded above and below:

$$A\exp(-|\xi|^2/4) \leqslant g_T(t, \xi) \leqslant H\exp\left\{-|\xi|^2/4\left(1 + \frac{a}{\ln(T+t)}\right)^2\right\}. \tag{3.57}$$

Hence, the same holds also for the possible limit function $g_*(\xi)$, and the estimates here have the form

$$A\exp(-|\xi|^2/4) \leqslant g_*(\xi) \leqslant H\exp(-|\xi|^2/4) \quad \text{in } R^N. \tag{3.57'}$$

Setting $\tau = \ln(1 + t/T)$, it is not hard to derive the parabolic equation for the function $g_T = g_T(\tau, \xi)$:

$$\frac{\partial g_T}{\partial \tau} = \Delta_\xi g_T + \frac{1}{2}\sum_{i=1}^{N}(g_T)_{\xi_i}\xi_i + \frac{N}{2}g_T + \frac{1}{\tau + \ln T}\left(\frac{N}{2}g_T - g_T^{1+2/N}\right). \tag{3.58}$$

From this equation on the basis of the pointwise estimates (3.57) and also other integral estimates of g_T (to derive them it is necessary to first bring the differential terms on the right side of (3.58) to divergence form) we find that the function $g_T(\tau, \xi)$ stabilizes as $\tau = \tau_i \to +\infty$ (τ_i is a subsequence of an arbitrary sequence $\tau_j \to +\infty$, $j \to +\infty$), uniformly on each compact set in R^N to a solution of the steady-state equation corresponding to $\tau = +\infty$, i.e., $g_*(\xi)$ satisfies the equation

$$\Delta_\xi g_* + \frac{1}{2}\sum_{i=1}^{N}(g_*)_{\xi_i}\xi_i + \frac{N}{2}g_* = 0, \qquad \xi \in R^N. \tag{3.59}$$

In the radially symmetric case ($u_0 = u_0(|x|)$) it has the form (3.14), $\alpha = N/2$, and hence

$$g_*(\xi) = M\exp(-|\xi|^2/4), \qquad M = \text{const}, \tag{3.60}$$

where because of (3.57') $M \in [A, H]$; here the constants A and H indicated in Lemmas 10 and 11 depend on the form of the initial function u_0.

We note that Lemma 10 follows immediately from the form of equation (3.58), since the function $q(\xi) = \mathbf{Q}_T(u_-(t, x)) \equiv A\exp(-|\xi|^2/4)$ satisfies (3.59), and, since $(N/2)q - q^{1+2/N} \geqslant 0$ in R^N for $A \in (0, (N/2)^{N/2}]$, $q(\xi)$ is a lower solution of (3.58).

The mere fact of the stabilization of $g_T(\tau, \xi)$ as $\tau = \tau_i \to +\infty$ is not hard to prove by the same method as in §3.2.1. Namely, defining the family of functions

$$u_k(t, x) = (k\ln k)^{N/2}u(kt, k^{1/2}x), \qquad t > 0,\ x \in R^N,\ k > 1, \tag{3.61}$$

in accordance with (3.51), each of which satisfies the equation

$$(u_k)_t = \Delta u_k - (\ln k)^{-1}u_k^{1+2/N}, \tag{3.62}$$

by means of estimates analogous to those in Lemma 6 we can establish that for the function sequence $\{u_k\}$ an assertion of the type of Theorem 10 holds, i.e.

$$u_k(t,x) \to w(t,x), \qquad k = k_i \to +\infty,$$

uniformly on each compact set in R^N, where the function w satisfies (3.62) for $k = +\infty$, i.e.,

$$w_t = \Delta w, \qquad t > 0,\ x \in R^N. \tag{3.63}$$

Hence,

$$g_T(t_i, \xi) \to w(1,\xi), \qquad \xi = x/(T+t_i)^{1/2}, \quad \text{as } t_i \to +\infty.$$

It is curious that the "logarithmic" deviation in the ap.s.-s.s. (3.52) (or in (3.62)) from the self-similar dependence (3.1) does not change the property of invariance of the limit function $w(t,x)$, i.e., just as before, the following result holds here (see Lemma 7).

LEMMA 12. *Let*

$$u_k(t,x) \to w(t,x) \quad \textit{as } k \to +\infty. \tag{3.64}$$

Then the function w is a self-similar solution of equation (3.63):

$$w(t,x) = t^{-N/2} g(\xi), \qquad \xi = x/t^{1/2}. \tag{3.65}$$

PROOF. Condition (3.64) implies that

$$(k \ln k)^{N/2} u(kt, k^{1/2}x) \to w(t,x) \quad \text{as } k \to +\infty. \tag{3.66}$$

We multiply both sides of (3.66) by $\delta^{N/2}$, $\delta = \text{const} > 0$, make the substitution $t \to \delta t$, $x \to \delta^{1/2} x$, and set $k' = k\delta$. We then obtain

$$(\ln k/\ln k')^{N/2} (k' \ln k')^{N/2} u\left(k't, (k')^{1/2} x\right) \to \delta^{N/2} w(\delta t, \delta^{1/2} x).$$

However, by (3.64) the left side converges as $k' \to +\infty$ to $w(t,x)$, i.e., $w(t,x) = \delta^{N/2} w(\delta t, \delta^{1/2} x)$ for any $\delta > 0$, whence we obtain (3.65).

In conclusion we show that under the conditions of Lemma 12 the limit function $g_*(\xi)$ does not depend on the initial function $u_0(|x|)$ and is unique in the following sense.

THEOREM 14. *Suppose $\beta = 1 + 2/N$ and $u_0 = u_0(|x|) \not\equiv 0$ satisfies* (3.50). *Suppose that uniformly on each compact set in R^N*

$$g_T(\tau,\xi) \to g_*(\xi) \quad \textit{as } \tau \to +\infty.$$

Then

$$g_*(\xi) = (N/2)^{N/2} (1 + 2/N)^{N^2/4} \exp(-|\xi|^2/4), \qquad \xi \in R^N. \tag{3.67}$$

PROOF. As shown earlier, under these assumptions only functions of the form (3.60) where $M \in [A, H]$ can be limit functions, i.e., it remains to prove that in the present case only one value of the constant M is possible, namely,

$$M = (N/2)^{N/2} (1 + 2/N)^{N^2/4}, \tag{3.68}$$

which thus does not depend on the initial function. Integrating (3.58) over R^N, which is possible because of (3.57), we get

$$\begin{aligned}(d/d\tau)\|g_T(\tau)\|_{L^1(R^N)} &= G(g_T)(\tau)/(\tau + \ln T)\\ &\equiv (1/(\tau + \ln T))\left\{(N/2)\|g_T(\tau)\|_{L^1(R^N)} - \|g_T(\tau)\|^{1+2/N}_{L^{1+2/N}(R^N)}\right\}, \qquad \tau > 0.\end{aligned} \tag{3.69}$$

Because of the uniform boundedness of $\|g_T(\tau)\|_{L^1(R^N)}$ in τ (see (3.57)), from (3.69) there follows the condition of convergence of the integral

$$\left|\int_{\tau_0}^{+\infty} \frac{G(g_T)(\tau)\,d\tau}{\tau + \ln T}\right| < +\infty \tag{3.70}$$

for all $\tau_0 > 0$. However, under the conditions of the theorem $G(g_T)(\tau) \to G(g_*)$ as $\tau \to +\infty$. From this we necessarily obtain the condition

$$G(g_*) \equiv (N/2)\|g_*\|_{L^1(R^N)} - \|g_*\|_{L^{1+2/N}(R^N)}^{1+2/N} = 0 \tag{3.71}$$

(otherwise the integral in (3.70) diverges). Substituting the function g_* of (3.60) into (3.71), we arrive at (3.68) and hence at (3.67), which completes the proof.

We note that under the conditions of Theorem 14 the ap.s.-s.s. (3.52)

$$u_a(t,x) = [(T+t)\ln(T+t)]^{-N/2}(N/2)^{N/2}(1+2/N)^{N^2/4}\exp[-|x|^2/4(T+t)],$$

which describes the behavior of solutions of the problem for large t, satisfies the equation

$$\frac{\partial u_a}{\partial t} = \Delta u_a - \frac{N}{2}\frac{u_a}{(T+t)\ln(T+t)}, \qquad t > 0,\ x \in R^N,$$

which differs in an essential way from the original equation (1.1).

4. *Concluding remarks.* We emphasize that in many of the cases considered the evolution properties of solutions of problem (1.1), (1.2) can in a certain sense be predicted and explained by analyzing the structure of the family $\{u_A\}$ of its self-similar solutions of the type (3.1) or, equivalently, the family of solutions $\{\theta(\xi;\mu), 0 < \mu < \theta_H\}$ of the "self-similar" equation (1.4); this structure is shown schematically in Figures 1 and 2. As we mentioned earlier, the functions $\theta(\xi;\mu)$ are either steady-state solutions of equation (3.46) which the self-similar representation of the solution $u(t,x)$ satisfies or critical functions if $\theta(\xi;\mu)$ vanishes somewhere (we note that the latter is a general property of a large class of quasilinear parabolic equations; see [68] and [69]). Thus, the presence in the family $\{\theta(\xi;\mu)\}$ of "compactly supported" functions (the case $\beta \in (1, 1+2/N)$) guarantees that the limit distribution $\theta_T(+\infty,\xi)$ is nontrivial and thus belongs to the class of self-similar functions being considered (Theorem 11). Conversely, if there are no "compactly supported" functions in the family $\{\theta(\xi;\mu)\}$ ($\beta \geqslant 1+2/N$), then this indicates that from sufficiently small initial functions lying as $|\xi| \to +\infty$ "below" any steady-state solution $\theta(\xi;\mu)$ stabilization takes place to the trivial solution (Theorem 8 for $u_A \equiv 0$; Lemma 11). This means that here there exist e.f. not belonging to the family $\{\theta(\xi;\mu)\}$ which should therefore be sought in another set of "non-self-similar" limit functions (subsections 1.2 and 3). Thus, the structure of the continuous "field" of particular solutions $\{u_A\}$ of equation (1.1) makes it possible to distinguish some very general asymptotic properties of all its possible solutions.

These ideas are close in meaning to the conclusions obtained on the basis of the method of stationary states [70]–[72], within whose framework the most general property of localization of unbounded solutions (regimes with peaking) of nonlinear parabolic problems was determined by means of a similar analysis of a continuous "field" of functions consisting of solutions constructed of the corresponding steady-state problem. In this connection we also mention [73] and [74], where an approach is presented for investigating the effect of localization of the action of boundary regimes with peaking on a medium with nonlinear thermal conductivity which is actually equivalent to a kind of "approximation" of the boundary LS-regime with peaking on a continuous "field" of localized

S-regimes to each of which there corresponds its own very simple self-similar solution. Here the LS-regime has no such simple solution demonstrating the property of localization.

The generality of these considerations makes it possible to conclude that the construction of a complete continuous family ("field") of any particular, sufficiently simple (for example, self-similar or invariant) solutions of a nonlinear parabolic equation makes it possible by the "approximation" indicated above to make a well-grounded judgement about many important evolution properties of the process in question.

In conclusion we note that by means of the self-similar solutions (1.3) constructed in §2 it is possible to illustrate the results of [75] where, in particular, conditions for the solvability of the Cauchy problem for (1.1) with initial condition $u(0, x) = \delta(x)$, δ the delta function, were studied. It was established that for $\beta \geqslant 1 + 2/N$ it has no solution; more precisely, its "solution" is $u(t, x) \equiv 0$, so that $\lim_{t \to 0^+} u(t, x) \equiv 0 \neq \delta(x)$ in R^N. All positive self-similar solutions (1.3), $T = 0$, in this case have an initial function of the form

$$u_A(0^+, x) = C|x|^{-2/(\beta-1)}, \qquad x \neq 0;\ C = \text{const} > 0, \tag{3.72}$$

so that $u_0(0^+, x) \notin L^1(R^N)$ which correctly agrees with the conclusion of [75].

In the case $\beta \in (1, 1 + 2/N)$ the self-similar solutions (1.3), $T = 0$, satisfy either (3.72) if $\theta_A(\xi)$ is a function with the "power" asymptotics (2.2), or the initial condition

$$u_A(0^+, x) = A_0\delta^l(x), \qquad x \in R^N;\ l = 2/(\beta - 1)N > 1 \tag{3.73}$$

(in particular, $u_A(0^+, x) = 0$ in $R^N \setminus \{0\}$) if the function $\theta_A = \theta_A^*(\xi)$ has the "exponential" asymptotics (2.3). In (3.73) the constant A_0 is equal to

$$A_0 = \left\{\int_{R^N} [\theta_A^*(\xi)]^{1/l} d\xi\right\}^l < +\infty.$$

For $\beta \in (1, 1 + 2/N)$, in both cases $u_A(0^+, x) \notin L^1(R^N)$.

Thus, (3.72) and (3.73) characterize the degree of singularity at the point $x = 0$ of the initial function for which the Cauchy problem for (1.1) has a classical (and nontrivial) solution in $R^1_+ \times R^N$.

REMARK. After the present paper was submitted, we learned of the existence of [76], in which some results of §§2 and 3 pertaining to the analysis of the case $\beta > 1 + 2/N$ (the case $\beta \leqslant 1 + 2/N$ was not considered) were obtained by a somewhat different method. The new result of [76] is mainly the proof of the existence of an infinite-dimensional set of asymptotically stable self-similar solutions of equation (1.1) of the form (1.3) which are not radially symmetric, where the functions $\theta_A(\xi) > 0$ satisfy the elliptic equation $A(\theta_A) = 0$ in R^N (the operator A is defined in (3.45)). We note that the methods used in §3 (the case $\beta > 1 + 2/N$) are applicable for the proof of stability of such self-similar solutions. We also mention [77], in which the results of [75] are generalized to the case of the quasilinear equation (1.11), and, moreover, for $1 < \beta < \sigma + 1 + 2/N$ the asymptotics of the solution of the Cauchy problem $u(0, x) = \delta(x)$ as $t \to 0$ is studied. The results of [14] are presented in more detail in [78].

The authors are grateful to L. A. Peletier for sending the preprints of [76] and [77].

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Received 23/JULY/84

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Translated by J. R. SCHULENBERGER