Mathematics of the USSR-Sbornik

SOURCES OF CURVATURE OF A VECTOR FIELD

To cite this article: Ju A Aminov 1969 *Math. USSR Sb.* **9** 199

View the article online for updates and enhancements.

You may also like

- A research-informed graphical tool to visually approach Gauss' and Stokes' theorems in vector calculus
L Hahn, S A Blaue and P Klein
- A geometric approach to contact Hamiltonians and contact Hamilton–Jacobi theory
Katarzyna Grabowska and Janusz Grabowski
- On the resolution of singularities of one-dimensional foliations on three-manifolds
J. C. Rebelo and H. Reis

This content was downloaded from IP address 18.119.131.178 on 05/05/2024 at 08:30

Mat. Sbornik
Tom 80 (122) (1969), No. 2

Math. USSR Sbornik
Vol. 9 (1969), No. 2

SOURCES OF CURVATURE OF A VECTOR FIELD

Ju. A. AMINOV

UDC 513.736.35

Abstract. It is known that for a vector field in three-dimensional space we can introduce the concepts of curvature and mean curvature. In the present article we derive integral formulas for these concepts; these formulas allow us to decide whether a vector field has, for example, singularities in a domain. We explain the influence of the modulus of the curvature of a vector field on the magnitude of its nonholonomity.

We also consider the question of the influence of the curvature of a family of surfaces on the distortion of the enveloping space for a given size of domain.

Bibliography 5 items.

Introduction

The present article is concerned with the theory of families of surfaces and vector fields. It is well known (see, for example, [1] or [2]) that for a vector field n in three-dimensional space we can introduce the analogs of the gaussian curvature and the mean curvature of a surface; we denote them by K and H respectively; and the magnitude of K will be called the total curvature. Under the assumption that there are no singular points of n in the interior of a three-dimensional domain V, we established in [5] that

$$\int_V K dv = \int_{\psi(\partial V)} (\mathbf{x}\mathbf{n})\, d\sigma, \tag{1}$$

where ψ is the mapping of ∂V onto the unit sphere σ by the unit vector field n ($d\sigma$ is understood to have a sign), and x is the radius vector of a point of V. In §1 of the present article we use (1) to introduce, in a natural way, sources of curvature and their strength. We prove (§2) an analog of (1) that is a generalization to higher dimensions and, in addition, is constructed for all symmetric functions of the principal curvatures; we then derive bounds for the integrals, over an n-dimensional domain, of symmetric functions of the principal curvatures of algebraic fields. In §3 we consider the influence of $|K|$ on the size of the (three-dimensional) domain of existence of the field and on the magnitude $\rho = \frac{1}{2}(\mathbf{n} \operatorname{curl} \mathbf{n})$ of its nonholonomity by taking $K < 0$ and by assuming that the field is constant along a fixed direction. In §4 and §5 we study the influence of the curvature of a family of surfaces $\phi(u^1, u^2, u^3) = \text{const}$ on the distortion of the space when the size of the domain of definition is fixed. The point is that we can intuitively regard the curvature of a family of surfaces (or of a vector field) as a loading on a domain in three-dimensional space $\mathfrak{R}$; under the action of this loading the metric of the enveloping space will be changed to some large, critical metric; that is, the euclidean space becomes distorted, and the magnitude of the distortion will be determined to some extent by the loading. This influence can be compared with the distortion of a disc. Under the influence of internal stresses in the disc the relative positions of points and the distances between them are altered; the external result of this is that the disc is bent and leaves the shape of a three-dimensional plane. We characterize the distortion of $\Re$ by Q_0. Let R be the scalar Ricci curvature and let $K_{\Re}$ be the curvature of a two-dimensional element of area in $\Re$. Then $Q_0 = \max |R/2 + K_{\Re}|$, where the maximum is taken over all two-dimensional elements of area at a point and then over all points of a sphere of unit radius. We shall find a lower bound for the amount of distortion of the space. To do this we establish: 1) a generalized-divergent form of the mean curvature of a hyperplane of the family $\phi(u^1, \cdots, u^{n+1}) = \text{const}$ lying in a Riemannian space; it will follow from this, for example, that for any compact, orientable Riemannian space $\Re^{n+1}$ without a boundary, and for any family of level hypersurfaces in $\Re^{n+1}$, we have

$$\int\limits_{\Re^{n+1}} H dv = 0$$

under the assumption that the set of singular points has zero hypersurface measure; 2) an upper bound for the ratio S/v of the surface area of the unit sphere (without apolar points) to its volume in terms of the curvature of two-dimensional elements of area and scalar Ricci curvature in $\Re^3$ (see (21)). An estimate of the distortion Q_0 of $\Re^3$ is given by the inequalities (22) of Theorem 3.

In conclusion I express my sincere thanks to N. V. Efimov, under whose supervision the present article was written.

§1. Sources of curvature

Let M be the set of singular points of the vector field $\mathbf{n}$ contained in the domain of definition of $\mathbf{n}$, let F_ϵ be an ϵ-neighborhood of M and let ∂F_ϵ be its boundary, oriented so that for each connected component the normal is directed into the interior of F_ϵ. Suppose that the following limit exists:

$$\lim_{\varepsilon \to 0} \int\limits_{\psi(\partial F_\varepsilon)} (\mathbf{xn})\, d\sigma = Q.$$

If $Q > 0$ ($Q < 0$) we shall say that M is a *source* (*sink*) of curvature and shall call Q the *strength* of the source or sink. Let us show that Q is independent of the choice of origin of the coordinate system. Let ξ_1, ξ_2 and ξ_3 be components of $\mathbf{n}$. When the origin is translated by a vector $\mathbf{a}$, Q is changed by $\lim_{\epsilon \to 0} \int_{\psi(\partial F_\epsilon)} (\mathbf{an}) d\sigma$. Hence it is sufficient to show that, for any surface ∂F_ϵ,

$$\int\limits_{\psi(\partial F_\varepsilon)} \xi_i \, d\sigma = 0.$$

But this obviously follows from the fact that ∂F_ϵ is closed and that ξ_i is an odd function relative to the plane $\xi_i = 0$ in (ξ_1, ξ_2, ξ_3)-space.

With regard for the singular points in the interior of V we now rewrite (1) as (the integral on the left-hand side is improper)

$$\int\limits_V K dv = \int\limits_{\psi(\partial V)} (\mathbf{xn})\, d\sigma + Q.$$

Thus, when the field on ∂V is fixed, the greater the strength of the sources, the more the large integral curvature is confined to the interior of V.

Consider an isolated singular point of M. If for any sufficiently small ϵ we have

$$\int_{\psi(\partial F_\varepsilon)} |d\sigma| < C,$$

where C is a positive constant, then the strength of the point source of curvature M is zero. In fact, since Q is independent of the choice of origin we can locate it at the point of M. Because $|(\mathbf{xn})| < \epsilon$ on the sphere ∂F_ϵ, we have

$$\left| \int_{\psi(\partial F_\varepsilon)} (\mathbf{xn})\, d\sigma \right| \leqslant \varepsilon C,$$

and, when we take the limit as $\epsilon \to 0$, we find that $Q = 0$. Hence, for example, it follows that the isolated singular points of an algebraic field have zero strength.

We next consider the question of an extremum of

$$J = \int_{\psi(dV)} (\mathbf{xn})\, d\sigma \tag{2}$$

where ∂V is a fixed closed surface. In view of (1) this problem is related to that of extremizing the integral of the complete curvature K. Note that for any $K_0 > 0$ we can construct, in a given domain V, a regular field $\mathbf{n}$ (see §3) whose complete curvature K is such that $|K| \geq K_0$; that is, (2) cannot be bounded from above or from below. However, we have

Theorem 1. *Let ∂V be a sphere. In the class of fields homotopic to the normals to ∂V, the field of normals gives an extreme value of* (2).

Let us find those fields for which the first variation of (2) vanishes. Suppose that the variation is given by the vector field $\epsilon\mathbf{w}$. Since we are varying in a class of unit vectors, we have that $2(\mathbf{nw}) = -\epsilon(\mathbf{ww})$. We split ∂V into domains D_1 and D_2 with a common curve Γ, and in each of D_1 and D_2 we introduce its parameters (u, v). When we use

$$d\sigma = (\mathbf{n}_u\mathbf{n}_v\mathbf{n})\, du\, dv,$$

we easily find that the coefficient of ϵ in the integrand for J is equal to

$$(\mathbf{xw})(\mathbf{n}_u\mathbf{n}_v\mathbf{n}) + (\mathbf{xn})\{(\mathbf{w}_u\mathbf{n}_v\mathbf{n}) + (\mathbf{n}_u\mathbf{w}_v\mathbf{n}) + (\mathbf{n}_u\mathbf{n}_v\mathbf{w})\}.$$

We have

$$\begin{aligned} (\mathbf{xn})(\mathbf{w}_u\mathbf{n}_v\mathbf{n}) + (\mathbf{xn})(\mathbf{n}_u\mathbf{w}_v\mathbf{n}) &= \frac{\partial}{\partial u}(\mathbf{xn})(\mathbf{wn}_v\mathbf{n}) \\ - \frac{\partial}{\partial v}(\mathbf{xn})(\mathbf{wn}_u\mathbf{n}) - (\mathbf{x}_u\mathbf{n})(\mathbf{wn}_v\mathbf{n}) &+ (\mathbf{x}_v\mathbf{n})(\mathbf{wn}_u\mathbf{n}) - (\mathbf{xn}_u)(\mathbf{wn}_v\mathbf{n}) \\ &+ (\mathbf{xn}_v)(\mathbf{wn}_u\mathbf{n}) + 2(\mathbf{xn})(\mathbf{wn}_u\mathbf{n}_v). \end{aligned} \tag{3}$$

If we integrate by parts we reduce the integrals over D_1 and D_2 of the first two terms on the right-hand side of (3) to contour integrals of the form $\int_\Gamma (\mathbf{xn})(\mathbf{w}\, d\mathbf{nn})$ which depend on the orientation of Γ. Because the same orientations of D_1 and D_2 induce opposite orientations on Γ, the sum of these contour integrals is zero.

We set

$$\mathbf{v} = [\mathbf{x}_v\mathbf{n}_u] - [\mathbf{x}_u\mathbf{n}_v].$$

When we use the relation $[[\mathbf{ab}]\mathbf{c}] = (\mathbf{ac})\mathbf{b} - \mathbf{a}(\mathbf{bc})$, we have that

$$\begin{aligned} -(\mathbf{x}_u\mathbf{n})(\mathbf{wn}_v\mathbf{n}) + (\mathbf{x}_v\mathbf{n})(\mathbf{wn}_u\mathbf{n}) &= -([[[\mathbf{x}_u\mathbf{n}_v]\mathbf{n}]\mathbf{n}]\mathbf{w}) \\ + ([[[\mathbf{x}_v\mathbf{n}_u]\mathbf{n}]\mathbf{n}]\mathbf{w}) = ([[\mathbf{vn}]\mathbf{n}]\mathbf{w}) &= (\mathbf{vn})(\mathbf{nw}) - (\mathbf{vw}). \end{aligned} \tag{4}$$

Similarly we obtain that

$$(\mathbf{x}\mathbf{n}_v)(\mathbf{w}\mathbf{n}_u\mathbf{n}) - (\mathbf{x}\mathbf{n}_u)(\mathbf{w}\mathbf{n}_v\mathbf{n}) = ((\mathbf{x}\mathbf{n}_v)\mathbf{n}_u - (\mathbf{x}\mathbf{n}_u)\mathbf{n}_v, \mathbf{n}, \mathbf{w})$$
$$= -([[\mathbf{n}_u\mathbf{n}_v]\mathbf{x}]\mathbf{n}]\mathbf{w}) = (\mathbf{n}_u\mathbf{n}_v\mathbf{w})(\mathbf{x}\mathbf{n}) - (\mathbf{n}_u\mathbf{n}_v\mathbf{n})(\mathbf{x}\mathbf{w}). \tag{5}$$

When we use (3), (4) and (5), we find that for the first variation

$$\delta J = \int_{\partial V} \{(\mathbf{v}\mathbf{n})(\mathbf{n}\mathbf{w}) - (\mathbf{v}\mathbf{w}) + 4(\mathbf{x}\mathbf{n})(\mathbf{n}_u\mathbf{n}_v\mathbf{w})\}\, du\, dv.$$

Because, with accuracy up to ϵ, $(\mathbf{n}\mathbf{w}) = 0$ and $[\mathbf{n}_u\mathbf{n}_v] = \lambda\mathbf{n}$, we can assume that the integrand in δJ is equal to $-(\mathbf{v}\mathbf{w})$. If $[\mathbf{n}_u\mathbf{n}_v] \neq 0$ then $\mathbf{w}$ can be represented as

$$\mathbf{w} = \alpha\mathbf{n}_u + \beta\mathbf{n}_v.$$

where α and β are arbitrary functions of (u, v). We obtain that

$$(\mathbf{v}\mathbf{w}) = (\mathbf{x}_v\mathbf{n}_u\mathbf{n}_v)\beta - (\mathbf{x}_u\mathbf{n}_v\mathbf{n}_u)\alpha.$$

Hence for an extremal field $\mathbf{n}$ we have $(\mathbf{x}_u\mathbf{n}_u\mathbf{n}_v) = (\mathbf{x}_v\mathbf{n}_u\mathbf{n}_v) = 0$; that is, either $\mathbf{n}$ is orthogonal to ∂V or $[\mathbf{n}_u\mathbf{n}_v] = 0$. If $[\mathbf{n}_u\mathbf{n}_v] = 0$ at a point, then the set of points where $\mathbf{n} \perp \partial V$ does not contain interior points, and in this case $\mathbf{n}$ is not homotopic to the field of normals to ∂V. Hence everywhere $[\mathbf{n}_u\mathbf{n}_v] \neq 0$ and $\mathbf{n} \perp \partial V$.

We next show that, if a normal to ∂V is directed outwards, then $\delta^2 J < 0$ in a neighborhood of the field of normals. For simplicity we take the origin of the coordinate system to be at the center of the sphere ∂V. When we write out the terms in ϵ^2 we establish that the integrand in the expression for δJ contains terms in ϵ, because we have that $2(\mathbf{n}\mathbf{w}) = -\epsilon(\mathbf{w}\mathbf{w})$. Then we obtain that

$$\delta^2 J = \int_{\partial V} (\mathbf{x}\mathbf{w})\{(\mathbf{w}_u\mathbf{n}_v\mathbf{n}) + (\mathbf{n}_u\mathbf{w}_v\mathbf{n}) + (\mathbf{n}_u\mathbf{n}_v\mathbf{w})\}\, du\, dv$$

$$+ \int_{\partial V} (\mathbf{x}\mathbf{n})\{(\mathbf{w}_u\mathbf{w}_v\mathbf{n}) + (\mathbf{n}_u\mathbf{w}_v\mathbf{w}) + (\mathbf{w}_u\mathbf{n}_v\mathbf{w}) - 2(\mathbf{w}\mathbf{w})(\mathbf{n}_u\mathbf{n}_v\mathbf{n})\}\, du\, dv.$$

Because, with accuracy up to ϵ, $(\mathbf{n}\mathbf{w}) = 0$ and $\mathbf{x}$ is parallel to $\mathbf{n}$, we have that $(\mathbf{x}\mathbf{w}) = 0$, and so the first integral vanishes. We transform the second integral. We have

$$(\mathbf{x}\mathbf{n})(\mathbf{w}_u\mathbf{w}_v\mathbf{n}) = \frac{\partial}{\partial u}\frac{1}{2}(\mathbf{x}\mathbf{n})(\mathbf{w}\mathbf{w}_v\mathbf{n}) + \frac{\partial}{\partial v}\frac{1}{2}(\mathbf{x}\mathbf{n})(\mathbf{w}_u\mathbf{w}\mathbf{n})$$
$$-\frac{1}{2}(\mathbf{x}\mathbf{n})(\mathbf{w}\mathbf{w}_v\mathbf{n}_u) - \frac{1}{2}(\mathbf{x}\mathbf{n})(\mathbf{w}_u\mathbf{w}\mathbf{n}_v). \tag{6}$$

Since $[\mathbf{n}_u\mathbf{n}_v] \neq 0$, when we neglect the terms containing ϵ we find that

$$\mathbf{w} = \alpha\mathbf{n}_u + \beta\mathbf{n}_v, \quad \mathbf{w}_u = \alpha_u\mathbf{n}_u + \alpha\mathbf{n}_{uu} + \beta_u\mathbf{n}_v + \beta\mathbf{n}_{vu},$$
$$\mathbf{w}_v = \alpha_v\mathbf{n}_u + \alpha\mathbf{n}_{uv} + \beta_v\mathbf{n}_v + \beta\mathbf{n}_{vv},$$

where α and β are arbitrary functions of (u, v). Hence

$$(\mathbf{n}_u\mathbf{w}_v\mathbf{w}) + (\mathbf{w}_u\mathbf{n}_v\mathbf{w}) = \alpha^2(\mathbf{n}_{uu}\mathbf{n}_v\mathbf{n}_u) + 2\alpha\beta(\mathbf{n}_u\mathbf{n}_{uv}\mathbf{n}_v) + \beta^2(\mathbf{n}_u\mathbf{n}_{vv}\mathbf{n}_v). \tag{7}$$

Let L, M and N be the coefficients in the second fundamental form of the unit sphere σ, oriented so that the normal is directed outwards, and let $d\sigma$ be the element of area of σ. Then, when we take (6) and (7) into account, we obtain that

$$\delta^2 J = -\int_{\Psi(\partial V)} (\mathbf{x}\mathbf{n})\left\{\frac{3}{2}(L\alpha^2 + 2M\alpha\beta + N\beta^2) + 2(\mathbf{w}\mathbf{w})\right\} d\sigma.$$

Hence ψ is the gaussian spherical mapping of ∂V. Because $L\alpha^2 + 2M\alpha\beta + N\beta^2 = -(\mathbf{w}\mathbf{w})$ (with accuracy up to ϵ^2), we have that $\delta^2 J < 0$, and so the field of normals gives a maximum of (2) (provided the normal to ∂V is directed outwards).

Corollary 1. *If the field* $\mathbf{n}$ *on a sphere is homotopic to the field of inward drawn normals of this sphere, then*

$$\int_{\psi(\partial V)} (\mathbf{xn})\, d\sigma \geqslant -4\pi R,$$

where R *is the radius of the sphere.*

Corollary 2. *If the index of a singular point of the field is unity, then the strength of the point source of curvature at this point is greater than or equal to zero.*

In fact, if ∂F_ϵ is the sphere of radius ϵ with center at the singular point, then, by Corollary 1 to Theorem 1, the integral (2) is bounded from below by $-4\epsilon\pi$. When we take the limit as $\epsilon \to 0$ we find that $Q \geq 0$.

Remark to Theorem 1. If ∂V is a smooth convex surface with nonzero gaussian curvature, then, in the class of fields homotopic to the field of normals to ∂V, $\delta J = 0$ only for the field of normals.

§2. The integral curvatures of an n-dimensional field

We are going to consider an n-dimensional vector field $\mathbf{n}$ in n-dimensional euclidean space; in particular it can be the field of normals to a family of hypersurfaces. Let $\lambda_1, \cdots, \lambda_n$ be the principal curvatures of the field; that is, the eigenvalues of the equation $d\mathbf{n} = \lambda d\mathbf{x}$. Since $\mathbf{n}$ is a unit vector, one of these eigenvalues, for example λ_n, is zero. When we use the equation $d\mathbf{n} = \lambda d\mathbf{x}$ we obtain the following expression for the symmetric functions of $\lambda_1, \cdots, \lambda_{n-1}$:

$$S^k = \sum_{i_1 \ldots i_k = 1}^{n-1} \lambda_{i_1} \ldots \lambda_{i_k} = \sum_{i_1 \ldots i_k = 1}^{n} \begin{vmatrix} \xi_{i_1 x_{i_1}} & \cdots & \xi_{i_1 x_{i_k}} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \xi_{i_k x_{i_1}} & \cdots & \xi_{i_k x_{i_k}} \end{vmatrix}, \tag{8}$$

that is, S^k is equal to the sum of the principal minors of the kth order matrix $\|\xi_{ix_j}\|$. If we substitute $\lambda_1 \cdots \lambda_{n-1} = S^{n-1}$ for K in (1), it is easy to establish that (1) holds for the product of all the principal curvatures.

We next prove a formula which concerns all the symmetric functions S^k. We write

$$d\xi_{i_2} \wedge d\xi_{i_3} \wedge \cdots \wedge d\xi_{i_{k+1}} = d\sigma_{i_2 \ldots i_{k+1}},$$

$$dx_{i_{k+2}} \wedge \cdots \wedge dx_{i_n} = dv_{i_{k+2} \ldots i_n},$$

$\epsilon^{1 \cdots n}_{i_1 \cdots i_n} = \epsilon_\nu$ is the Kronecker symbol and $\nu = (i_1 \cdots i_n)$. Then, if we assume that there are no singular points of the field in the interior of V, we have that

$$\int_V S^k dv = \int_{\partial V} \sum_\nu \varepsilon_\nu x_{i_1} d\sigma_{i_2 \ldots i_{k+1}} \wedge dv_{i_{k+2} \ldots i_n}; \tag{9}$$

that is, the value of the integral of S^k over the volume is given in terms of the boundary values of $\mathbf{n}$.

Equation (9) is a natural generalization of (1); for simplicity we shall prove (9) for the second symmetric function of the principal curvatures of a field in four-dimensional euclidean space. According to (8), in this case we have

$$S^2 = 2\left\{\begin{vmatrix} \xi_{1x_1} \xi_{1x_2} \\ \xi_{2x_1} \xi_{2x_2} \end{vmatrix} + \begin{vmatrix} \xi_{2x_2} \xi_{3x_2} \\ \xi_{2x_3} \xi_{3x_3} \end{vmatrix} + \cdots\right\},$$

and (9) can be written as

$$\int_V S^2 dv = \int_{\partial V} x_1 (d\xi_2 \wedge d\xi_3 \wedge dx_4 + d\xi_2 \wedge dx_3 \wedge d\xi_4 + dx_2 \wedge d\xi_3 \wedge d\xi_4) + \dots,$$

where on the right-hand side we have not written out the terms that would be obtained if we were to replace x_1 by x_2, x_3 and x_4. Consider the integral over ∂V. Let us take, for example, the first term in the integrand. We have that

$$x_1 d\xi_2 \wedge d\xi_3 \wedge dx_4 = x_1 [(\xi_{2x_1}\xi_{3x_2} - \xi_{2x_2}\xi_{3x_1})\, dx_1\, dx_2\, dx_4$$
$$+ (\xi_{2x_1}\xi_{3x_3} - \xi_{2x_3}\xi_{3x_1})\, dx_1\, dx_3\, dx_4 + (\xi_{2x_2}\xi_{3x_3} - \xi_{2x_3}\xi_{3x_2})\, dx_2\, dx_3\, dx_4].$$

We transform the integral over the boundary into a volume integral by using the Gauss-Ostrogradskiĭ Theorem. The expression we have written out gives the following contribution to the integral over V:

$$\begin{vmatrix} \xi_{2x_2} & \xi_{2x_3} \\ \xi_{3x_2} & \xi_{3x_3} \end{vmatrix} + x_1 \left\{ \frac{\partial}{\partial x_3} \begin{vmatrix} \xi_{2x_1} & \xi_{2x_2} \\ \xi_{3x_1} & \xi_{3x_2} \end{vmatrix} - \frac{\partial}{\partial x_2} \begin{vmatrix} \xi_{2x_1} & \xi_{2x_3} \\ \xi_{3x_1} & \xi_{3x_3} \end{vmatrix} + \frac{\partial}{\partial x_1} \begin{vmatrix} \xi_{2x_2} & \xi_{2x_3} \\ \xi_{3x_2} & \xi_{3x_3} \end{vmatrix} \right\}.$$

It is easy to see that the coefficient of x_1 in this expression is zero. We similarly treat all the terms in the integrand of the integral over the boundary; when we combine them we find that each principal minor $(\xi_{ix_i}\xi_{jx_j} - \xi_{ix_j}\xi_{jx_i})$ occurs twice in the integral over V; that is, the integrand of this integral is equal to S^2, as we were required to prove.

We next derive an estimate for the integral of S^k for an algebraic field defined in a cube with side a. A field $\mathbf{n}$ is said to be *algebraic of order* m if for some function $\lambda \neq 0$ the components $(A_1, \cdots, A_n)$ of $\lambda\mathbf{n}$ are polynomials in $x_1, \cdots, x_n$ of degree not higher than m. For simplicity we consider the estimate of the integral of S^2 for a regular field in four-dimensional space. For this, in view of (9), it is sufficient to estimate integrals of the form

$$\int_{\partial V} | d\xi_\alpha \wedge d\xi_\beta \wedge dx_\gamma |, \quad \alpha \neq \beta \neq \gamma \neq \alpha,$$

where ∂V is the boundary of the cube, arranged so that its edges are along the axes x_1, x_2, x_3 and x_4. To be specific we put $\alpha = 2$, $\beta = 3$ and $\gamma = 4$. We take a face of the cube with the coordinates (x_i, x_j, x_4) in it, $i \neq j \neq 4 \neq i$, and denote it by T_{ij}. We consider another three-dimensional space with the coordinates (ξ_2, ξ_3, x_4) and the mapping ψ: $(x_i, x_j, x_4) \to (\xi_2, \xi_3, x_4)$ defined by

$$\xi_2 = \xi_2(x_i, x_j, x_4), \quad \xi_3 = \xi_3(x_i, x_j, x_4), \quad x_4^* = x_4.$$

We are going to find the maximum number of inverse images of a point $(\xi_2^0, \xi_3^0, x_4^{*0}) \in \psi(T_{ij})$. We can write $A_2 = \xi_2^0\lambda$ and $A_3 = \xi_3^0\lambda$, where $\lambda^2 = A_1^2 + A_2^2 + A_3^2 + A_4^2 \neq 0$ is a polynomial of degree not greater than $2m$. Hence we have the equation

$$A_2^2 - (\xi_2^0)^2 \lambda^2 = 0, \qquad A_3^2 - (\xi_3^0)^2 \lambda^2 = 0,$$

in which the variables other than x_i and x_j are fixed. We denote the left-hand sides of these equations by $P(x_i, x_j)$ and $Q(x_i, x_j)$. By a well-known theorem the number of points at which $P = Q = 0$ and $J\left(\frac{P,\, Q}{x_i,\, x_j}\right) \neq 0$ is not greater than $(2m)^2$. By a straightforward calculation we find that, at the points where $P = Q = 0$,

$$J\left(\frac{P, Q}{x_i, x_j}\right) = \xi_2\xi_3\lambda^4 J\left(\frac{\xi_2, \xi_3, x_4^*}{x_i, x_j, x_4}\right).$$

Hence the image of the points where $P = Q = J\left(\frac{P,\, Q}{x_i,\, x_j}\right) = 0$ has measure zero in the space of points (ξ_2, ξ_3, x_4). Thus, except for a set of measure zero, the number of inverse images of points of $\psi(T_{ij})$ is not greater than $(2m)^2$. Since $\xi_2^2 + \xi_3^2 \leq 1$ and $|x_j| \leq a$, we have that

$$\int_{T_{ij}} |d\xi_2 \wedge d\xi_3 \wedge dx_4| \leqslant 4\pi a m^2.$$

When we use (9) we obtain that

$$\left|\int_V S^2 dv\right| \leqslant C a^2 m^2,$$

where C is an absolute constant. For an arbitrary symmetric function S^k of the principal curvatures and a regular algebraic field of order m in the n-dimensional cube V of side a we obtain that

$$\left|\int_V S^k dv\right| \leqslant C(n, k)\, m^k a^{n-k}.$$

Corollary. *If for an algebraic field of order m defined in a cube of side a we have $|S^k| \geq S_0 > 0$, then*

$$a \leqslant \frac{m \sqrt[k]{C(n,k)}}{\sqrt[k]{S_0}}.$$

§3. The complete curvature of a field and the magnitude of nonholonomity

In what follows we put $\rho = \frac{1}{2}(\mathbf{n}\operatorname{curl}\mathbf{n})$. This quantity characterizes the nonholonomity of the family of elements of area orthogonal to the field $\mathbf{n} = (\xi_1, \xi_2, \xi_3)$. For its geometrical interpretation see [4].

Let us consider the asymptotic lines of the field. We can define the asymptotic directions by the equation $[\mathbf{n} d\mathbf{n}] = \mu d\mathbf{x}$, where $d\mathbf{x}$ is the displacement of a point of the space. If we take the coordinate axes at a point so that the x_3-axis is in the direction of $\mathbf{n}$, then it is easy to prove that μ is defined by

$$\mu^2 - 2\rho\mu + K = 0.$$

The line along which $[\mathbf{n} d\mathbf{n}] = \mu d\mathbf{x}$ is an asymptotic line of the field; here $\mathbf{n}$ turns out to be its binormal and μ its torsion. Thus along an asymptotic line we have

$$\xi_2 d\xi_3 - \xi_3 d\xi_2 = (\rho \pm \sqrt{\rho^2 - K})\, dx_1,$$

$$\xi_3 d\xi_1 - \xi_1 d\xi_3 = (\rho + \sqrt{\rho^2 - K})\, dx_2,$$

$$\xi_1 d\xi_2 - \xi_2 d\xi_1 = (\rho \pm \sqrt{\rho^2 - K})\, dx_3. \tag{10}$$

Let us put $\xi_2 = r\cos\phi$ and $\xi_3 = r\sin\phi$; then we can write the first of the equations (10) as

$$r^2 d\varphi = (\rho \pm \sqrt{\rho^2 - K})\, dx_1. \tag{11}$$

Next we give an example of a field, regular in its bounded domain of definition, for which $|K|$ is arbitrarily large uniformly over all the domain. It is sufficient to take the domain of definition to be a cube. Let $1 \leq x_i \leq 2$, $i = 1, 2, 3$. We put

$$n \parallel (A_1, A_2, A_3), \quad A_1 = x_1 \cos\lambda x_2, \quad A_2 = x_1 \sin\lambda x_2 \quad A_3 \equiv 1,$$

where λ is a parameter. Then, by (2) of [5], we have that

$$K = \frac{A_{1x_1}A_{2x_2} - A_{1x_2}A_{2x_1}}{(1 + A_1^2 + A_2^2)^2} = \frac{x_1\lambda}{(1 + x_1^2)^2}.$$

If $\lambda > 0$, then $K \geq \lambda/25$; if $\lambda < 0$, then $K \leq -|\lambda|/25$. Our assertion follows since $|\lambda|$ is arbitrary.

We next establish

Theorem 2. *Let a regular field* **n**, *constant along a fixed direction* **v**, *be defined in a ball of radius R. Suppose that* $K \leq -K_0 < 0$ *and* $|\rho| \leq \rho_0$. *Then*

$$R \leqslant \frac{38}{\sqrt{2}\,(\sqrt{K_0+\rho_0^2}-\rho_0)}.$$

This theorem generalizes a theorem of N. V. Efimov [3] concerning the projection onto a square of a surface of negative curvature; it is proved by the same methods as Efimov's theorem and so we give, in the main, only an outline of the proof. The theorem shows that if we increase $|K|$ the modulus of the magnitude of nonholonomity is also increased, at least at separate points.

Proof. We show that $(\mathbf{nv}) \neq 0$. Let $\mathbf{P} = \{(\mathbf{n}_{x_2}\mathbf{n}_{x_3}\mathbf{n}), (\mathbf{n}_{x_3}\mathbf{n}_{x_1}\mathbf{n}), (\mathbf{n}_{x_1}\mathbf{n}_{x_2}\mathbf{n})\}$.

As was shown in [5], the curvature K of the field is given by

$$K = (\mathbf{nP}).$$

Let us indicate the geometrical significance of **P**. We choose the coordinate axes so that at some point of M the x_3-axis is in the direction of **n**. Then $\xi_1 = \xi_2 = \xi_{3x_i} = 0$. In these coordinates **P** has the form

$$\mathbf{P} = \left\{\begin{vmatrix} \xi_{1x_2} & \xi_{1x_3} \\ \xi_{2x_2} & \xi_{2x_3}\end{vmatrix}, \begin{vmatrix} \xi_{1x_3} & \xi_{1x_1} \\ \xi_{2x_3} & \xi_{2x_1}\end{vmatrix}, \begin{vmatrix} \xi_{1x_1} & \xi_{1x_2} \\ \xi_{2x_1} & \xi_{2x_2}\end{vmatrix}\right\}.$$

We determine a direction tangential to the line $\mathbf{n} = \text{const}$ through the point of M. For the displacement $d\mathbf{r} = (dx_1, dx_2, dx_3)$ along the line $\mathbf{n} = \text{const}$ we have

$$\xi_{1x_1}\,dx_1 + \xi_{1x_2}\,dx_2 + \xi_{1x_3}\,dx_3 = 0, \qquad \xi_{2x_1}\,dx_1 + \xi_{2x_2}\,dx_2 + \xi_{2x_3}\,dx_3 = 0.$$

Hence we obtain that $d\mathbf{r} = \lambda\mathbf{P}$, $\lambda \neq 0$; that is, **P** is the tangent vector (but, in general, not a unit vector) to the line $\mathbf{n} = \text{const}$. Thus, because $K \neq 0$ and $\mathbf{P}\|\mathbf{v}$, we have $(\mathbf{nv}) \neq 0$. We draw the great circle orthogonal to **v** on the sphere. We now choose the coordinate axes so that the x_3-axis is directed along **v** and take the origin of the coordinate system at the center of the sphere. Since $(\mathbf{nv}) \neq 0$ we can write **n** as

$$\mathbf{n} = \{p(1+p^2+q^2)^{-\frac{1}{2}},\ q(1+p^2+q^2)^{-\frac{1}{2}},\ -(1+p^2+q^2)^{-\frac{1}{2}}\},$$

where the functions $p = \xi_1/\xi_3$ and $q = \xi_2/\xi_3$ only depend on x_1 and x_2, because **n** is independent of x_3. From the formula for the total curvature of the field (see [5]) we have that

$$K = \frac{p_{x_1}q_{x_2} - p_{x_2}q_{x_1}}{(1+p^2+q^2)^2}, \qquad \rho = \frac{p_{x_2} - q_{x_1}}{2\,(1+p^2+q^2)}. \tag{12}$$

In what follows all lines will be considered in the square T: $x_3 = 0$, $-R/\sqrt{2} \leq x_i \leq R/\sqrt{2}$, $i = 1, 2$. Since $K \neq 0$, when we use (12) we obtain that q varies monotonically along the line $p = \text{const}$. As in [3] we consider "chains"; that is, lines formed in a definite way from parts of asymptotic lines. Along the chain constructed with respect to the x_1-axis, q (and hence $\phi = \operatorname{arctg} q$) varies monotonically. When we use (11) in the same way as in [3] we obtain that the absolute value of the projection of the x_ith chain onto the x_i-axis is less than $2\pi/(\sqrt{\rho_0^2 + K_0} - \rho_0)$. We prove our theorem by contradiction. Let $R > 38/\sqrt{2}(\sqrt{\rho_0^2 + K_0} - \rho_0)$ (then the side of T is greater than $19/(\sqrt{\rho_0^2 + K_0} - \rho_0)$), and let T_1 be a square in the interior of T whose sides are at a distance $(2\pi + \epsilon)/(\sqrt{\rho_0^2 + K_0} - \rho_0)$, $0.01 \geq \epsilon > 0$, from those of T. We consider the mapping of T_1 onto the unit sphere by the vector field **n**. If we bear in mind that $\sqrt{K} \geq \sqrt{K_0 + \rho_0^2} - \rho_0$, for the area σ of the image of T_1 we have

$$\sigma = \left| \iint\limits_{T_1} \frac{p_{x_1}q_{x_2} - p_{x_2}q_{x_1}}{(1+p^2+q^2)^{3/2}} dx_1\, dx_2 \right| \geqslant \iint\limits_{T_1} |K|\, dx_1\, dx_2$$
$$\geqslant (\sqrt{\rho_0^2 + K_0} - \rho_0)^2 \iint\limits_{T_1} dx_1\, dx_2 = (19 - 4\pi - 2\varepsilon)^2 > 6 \cdot 2\pi.$$

It follows from here that on the unit sphere we can find a point that is covered by the image of T_1 at least six times, and hence in T_1 there are at least six points at which (p, q) takes the same value (p_0, q_0). We then develop the discussion with the lines $p = p_0$ and $q = q_0$ and "chains", and, as in [3], we arrive at a contradiction.

Remark. According to the theorem, when the modulus of the curvature of the field under consideration is large and its nonholonomity is small, singularities of the field are inevitable in each sphere of radius $R \geq R_0$, where R_0 is a constant.

§4. A generalized-divergent form of the mean curvature

Suppose that a family of level hypersurfaces $\phi(u^1, \cdots, u^{n+1}) = \text{const}$ is defined in a Riemannian space with the metric $ds^2 = g_{ij}du^i du^j$ $(i, j = 1, \cdots, n+1)$. Then for the mean curvature H of a hypersurface we have

$$H = \frac{1}{n\sqrt{g}} \sum_{i=1}^{n+1} \frac{\partial}{\partial u^i} \left(\frac{g^{i\alpha}\varphi_{u^\alpha}\sqrt{g}}{\sqrt{g^{\alpha\beta}\varphi_{u^\alpha}\varphi_{u^\beta}}} \right) = \frac{1}{n} \operatorname{Div} \mathbf{n}, \tag{13}$$

where $g = \det \|g_{ij}\|$ and $\mathbf{n}$ is the normal vector. The expression (13) is similar to the expression for the geodesic curvature of a line on a surface.

For simplicity we shall prove (13) in the three-dimensional case. Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be basis vectors, let (du^1, du^2, du^3) be the tangent vector to the surface and let $d\mathbf{x} = \mathbf{e}_i du^i$ be the displacement vector of a point of the space. The half-sum of the eigenvalues of the equation $D\mathbf{n} = \lambda d\mathbf{x}$ is equal to H. It is easy to check that the vector with components $\phi^j = g^{j\alpha}\phi_{u^\alpha}$, $j = 1, 2, 3$, is orthogonal to the tangent plane of the surface. Suppose that the normalized unit vector has components ξ^i; that is, that $\mathbf{n} = \xi^i \mathbf{e}_i$. Then

$$D(\xi^i \mathbf{e}_i) = (D_k \xi^i \mathbf{e}_i)\, du^k = \left(\frac{\partial \xi^i}{\partial u^k} \mathbf{e}_i + \xi^i D_k \mathbf{e}_i \right) du^k = \left(\frac{\partial \xi^j}{\partial u^k} + \xi^i \Gamma^j_{ki} \right) \mathbf{e}_j du^k.$$

Hence we can write $D\mathbf{n} = \lambda d\mathbf{x}$ as

$$\left(\frac{\partial \xi^j}{\partial u^k} + \xi^i \Gamma^j_{ki} \right) \mathbf{e}_j du^k - \lambda\, \mathbf{e}_j\, du^j = 0.$$

We obtain the characteristic equation

$$\lambda^3 - \lambda^2 \left(\frac{d\xi^i}{\partial u^i} + \xi^\alpha \Gamma^i_{i\alpha} \right) + \lambda(\ldots) - \det \left\| \frac{\partial \xi^i}{\partial u^k} + \Gamma^i_{k\alpha}\xi^\alpha \right\| = 0.$$

This is of the third degree, but, since $\mathbf{n}$ is a unit vector, $\det \|\partial \xi^i / \partial u^k + \Gamma^i_{k\alpha}\xi^\alpha\| = 0$, and it reduces to a quadratic equation. Hence

$$2H = \frac{\partial \xi^i}{\partial u^i} + \Gamma^i_{i\alpha}\xi^\alpha.$$

Next we express ξ^i in terms of the derivatives of ϕ. We consider the vector $\phi^j \mathbf{e}_j$. Its norm is equal to

$$\sqrt{g_{ij}\varphi^i\varphi^j} = \sqrt{g_{ij}g^{j\alpha}\varphi_{u\alpha}g^{i\beta}\varphi_{u\beta}} = \sqrt{g^{\alpha\beta}\varphi_{u\alpha}\varphi_{u\beta}}.$$

Hence

$$\xi^i = \frac{g^{i\alpha}\varphi_{u\alpha}}{\sqrt{g^{\alpha\beta}\varphi_{u\alpha}\varphi_{u\beta}}}.$$

When we multiply ξ^i by $\sqrt{g}$ and divide by $\sqrt{g}$ we obtain that

$$2H = \frac{1}{\sqrt{g}}\frac{\partial}{\partial u^i}\frac{g^{i\alpha}\varphi_{u\alpha}\sqrt{g}}{\sqrt{g^{\alpha\beta}\varphi_{u\alpha}\varphi_{u\beta}}} + \frac{g^{\alpha\beta}\varphi_{u\beta}}{\sqrt{g^{\gamma\delta}\varphi_{u\gamma}\varphi_{u\delta}}}\Gamma^i_{i\alpha} - \frac{g^{i\alpha}\varphi_{u\alpha}\sqrt{g}}{\sqrt{g^{\alpha\beta}\varphi_{u\alpha}\varphi_{u\beta}}\,2g^{3/2}}\frac{\partial g}{\partial u^i}.$$

As is well known, $\Gamma^i_{i\alpha} = (\partial g/\partial u^\alpha)/2g$; therefore the last two terms cancel and we have proved (13). It is obvious that the condition that the field $\mathbf{n}$ be holonomic is not essential.

Let μ be the unit outward drawn normal to ∂V. If we integrate (13) over the volume V, we obtain

$$\int_V H dv = \frac{1}{2}\int_{\partial V} (\mu \mathbf{n})\, dS. \tag{14}$$

Exactly as in §1, by using (14) we can define the strength of sources of mean curvature. If $\mathbf{n}$ is defined in a compact orientable Riemannian space $\mathfrak{R}^{n+1}$ without a boundary, then the integral of the mean curvature of the field over $\mathfrak{R}^{n+1}$ is equal to the strength Q_{me} of these sources:

$$\int_{\mathfrak{R}^{n+1}} H dv = Q_{\mathrm{me}}. \tag{15}$$

In particular, when the surface measure of the singular points is zero, $Q_{\mathrm{me}} = 0$. If the field is defined in a compact orientable space $\mathfrak{R}^3$ without a boundary, then, by using (15) and the inequality $H^2 + \rho^2 \geq K_e - \rho^2$, where $K_e = \lambda_1\lambda_2$ is the outer curvature of the field and $\rho = \frac{1}{2}(\mathbf{n}\,\mathrm{curl}\,\mathbf{n})$, we obtain that, for any field for which the strength of the sources of mean curvature is zero,

$$\min_{\mathfrak{R}^3}(K_e - \rho^2) \leqslant 0.$$

§5. The influence of the curvature of a family of level surfaces on the distortion of the enveloping space

Suppose that a family of surfaces $\phi(x^1, x^2, x^3) = \text{const}$, with outer curvature $K_e \geq K_0 > 0$, is defined in a sphere of unit radius in three-dimensional Riemannian space. First of all note that in Riemannian space it is impossible, in general, to derive an upper bound for the radius of a sphere in which such a family exists for an arbitrary K_0, since this holds for a family of surfaces in euclidean space. In fact, in Lobačevskiĭ space, a family of geodesically parallel spheres whose outer curvature K_e is greater than unity is regular everywhere except for a single point. Hence we can take a sphere of arbitrarily large radius that is filled regularly by a family of surfaces with $K_e > 1$. At the same time we can show that a bound on the radius of a sphere in Lobačevskiĭ space is possible if $K_e > 1 + \epsilon$ ($\epsilon > 0$ is a constant). By taking this remark into account we come to the problem under consideration from another point of view: we fix the radius of the sphere and study the influence of the outer curvature of the family of surfaces on the distortion of the enveloping space. Since $H^2 \geq K_e \geq K_0$, by using (14) we obtain that

$$K_0 \leqslant \frac{S}{2v}, \tag{16}$$

where $S(v)$ is the surface area (volume) of the sphere.

Next we find a bound for the ratio S/v for the unit sphere in Riemannian space in terms of the maximum modulus of the scalar Ricci curvature and of the curvature of two-dimensional elements of area. We have

$$-g\frac{R}{2} = g_{11}R_{3232} - g_{12}R_{1323} - g_{13}R_{1232} - g_{12}R_{1323}$$
$$+ g_{22}R_{1313} - g_{23}R_{3121} - - g_{13}R_{1232} - g_{23}R_{3121} + g_{33}R_{2121}.$$

Suppose that there is a family of geodesically parallel surfaces with the help of which we can define semigeodesic coordinates in the space. The line element is written as

$$ds^2 = g_{ij}dx^i dx^j + (dx^3)^2, \quad i, j = 1, 2.$$

We take the coordinates so that on a fixed surface $x^3 = \text{const}$ of the family we have $g_{12} = 0$. Then, on this surface,

$$-\frac{R}{2}W = \frac{g_{11}}{W}R_{3232} + \frac{g_{22}}{W}R_{1313} + \frac{1}{W}R_{2121},$$

where $W = \sqrt{g_{11}g_{22} - g_{12}^2}$. $R_{2121}/W^2 = K_{\mathfrak{R}}$ is the curvature of the two-dimensional element of area tangential to our fixed surface. We put

$$\frac{g_{11}}{W}R_{3232} + \frac{g_{22}}{W}R_{1313} = T.$$

By a straightforward calculation we find that

$$R_{3232} = \frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^3 \partial x^3} - \frac{1}{4}\left[\frac{1}{g_{11}}\left(\frac{\partial g_{21}}{\partial x^3}\right)^2 + \frac{1}{g_{22}}\left(\frac{\partial g_{22}}{\partial x^3}\right)^2\right],$$

$$R_{1313} = \frac{1}{2}\frac{\partial^2 g_{11}}{\partial x^3 \partial x^3} - \frac{1}{4}\left[\frac{1}{g_{11}}\left(\frac{\partial g_{11}}{\partial x^3}\right)^2 + \frac{1}{g_{22}}\left(\frac{\partial g_{21}}{\partial x^3}\right)^2\right].$$

$$\frac{\partial^2 W}{\partial x^3 \partial x^3} = \frac{g_{11}}{2W}\frac{\partial^2 g_{22}}{\partial x^3 \partial x^3} + \frac{g_{22}}{2W}\frac{\partial^2 g_{11}}{\partial x^3 \partial x^3} - \frac{g_{11}}{4Wg_{22}}\left(\frac{\partial g_{22}}{\partial x^3}\right)^2$$
$$-\frac{g_{22}}{4Wg_{11}}\left(\frac{\partial g_{11}}{\partial x^3}\right)^2 - \frac{1}{2W}\frac{\partial g_{22}}{\partial x^3}\frac{\partial g_{11}}{\partial x^3} + \frac{1}{W}\left[\frac{\partial g_{11}}{\partial x^3}\frac{\partial g_{22}}{\partial x^3} - \left(\frac{\partial g_{12}}{\partial x^3}\right)^2\right].$$

When we note that $K_e = (2W)^{-2}[\partial g_{11}/\partial x^3 \cdot \partial g_{22}/\partial x^3 - (\partial g_{12}/\partial x^3)^2]$, we obtain that

$$T - \frac{\partial^2 W}{\partial x^3 \partial x^3} = -2K_e W.$$

Hence for the scalar Ricci curvature we have

$$-\frac{R}{2}W = \frac{\partial^2 W}{\partial x^3 \partial x^3} - 2K_e W - K_{\mathfrak{R}}W = \frac{\partial^2 W}{\partial x^3 \partial x^3} - 2K_i W + K_{\mathfrak{R}}W,$$

where K_i is the inner curvature of the surface $x^3 = \text{const}$. Because the formula is written in an invariant form, the assumption that $g_{12} = 0$ on the surface is not essential. We denote the displacement along the normal to the surface by r and write our result as

$$\frac{\partial^2 W}{\partial r^2} = 2K_i W - \left(\frac{R}{2} + K_{\mathfrak{R}}\right)W. \tag{17}$$

Now suppose that the family of surfaces $r = \text{const}$ is a family of closed surfaces of spherical type, contracting to a point, and that the family covers a domain of three-dimensional space simply. We integrate both sides of (17) over the surface of a sphere of radius r. Let $S(r)$ $(v(r))$ be the surface area (volume) of the sphere; then

$$\frac{d^2S(r)}{dr^2}=8\pi-\int\limits_{r=\mathrm{const}}\left(\frac{R}{2}+K_{\Re}\right)dS.$$

We put $Q_0=\max|R/2+K_{\Re}|$, where the maximum is taken over all the two-dimensional elements of area at a point and then over all points of a sphere of unit radius. We can rewrite the equation for $S(r)$ as

$$\frac{d^2S}{dr^2}=8\pi+QS,$$

where $Q=-\int((R/2+K_{\Re})/S)dS$. Obviously $|Q|\le Q_0$. As in the case of a family of parallel spheres in euclidean space, the initial conditions for $S(r)$ are $S(0)=S'(0)=0$. We compare the solution of (18) with the solutions of the equations

$$\frac{d^2S}{dr^2}=Q_0S+8\pi, \tag{19}$$

$$\frac{d^2S}{dr^2}=-Q_0S+8\pi. \tag{20}$$

Let $y_1(r)$ and $y_2(r)$ be a fundamental system for (18) such that $y_1(0)y_2'(0)-y_1'(0)y_2(0)=1$. We put $K(r,\tau)=y_1(\tau)y_2(r)-y_1(r)y_2(\tau)$ and similarly form the functions $K_+(r,\tau)$ for (19) and $K_-(r,\tau)$ for (20). Since with respect to τ these functions satisfy (18), (19) and (20), and they have the same initial data at $\tau=r$, when $\tau\le r$ we have

$$K_-(r,\tau)\leqslant K(r,\tau)\leqslant K_+(r,\tau).$$

It is easy to find that, when $\tau\le r$,

$$K_-(r,\tau)=\frac{\sin\sqrt{Q_0}(r-\tau)}{\sqrt{Q_0}},\qquad K_+(r,\tau)=\frac{e^{\sqrt{Q_0}(r-\tau)}+e^{-\sqrt{Q_0}(r-\tau)}-2}{2\sqrt{Q_0}}.$$

By using the method of variation of parameters we find that, for the solution of (18),

$$S(r)=8\pi\int\limits_0^r K(r,\tau)\,d\tau.$$

Thus we have the following bound for $S(1)/v(1)$:

$$\frac{S(1)}{v(1)}\leqslant\frac{\int\limits_0^1 K_+(1,\tau)\,d\tau}{\int\limits_0^1\int\limits_0^r K_-(r,\tau)\,d\tau\,dr}\leqslant\frac{\sqrt{Q_0}\,(e^{\sqrt{Q_0}}+e^{-\sqrt{Q_0}}-2)}{2(\sqrt{Q_0}-\sin\sqrt{Q_0})}. \tag{21}$$

By expanding $e^{\sqrt{Q_0}}$, $e^{-\sqrt{Q_0}}$ and $\sin\sqrt{Q_0}$ as power series in Q_0 we establish that for $Q_0<20$ the following must hold:

$$\frac{S(1)}{v(1)}\leqslant\frac{3}{2}\left(\frac{1+Q_0e^{20}/12}{1-Q_0/20}\right).$$

From here and from (16) we find that when $K_0>3/2$ the enveloping space must be distorted; that is, $Q_0>0$. More precisely, for $Q_0>20$ we have that

$$Q_0\geqslant\frac{K_0-\dfrac{3}{2}}{\dfrac{K_0}{20}+\dfrac{e^{20}}{8}}.$$

When $Q_0 \geq \pi/2$ we see that

$$\operatorname{ch} \sqrt{Q_0} \geqslant \frac{\pi - 2}{\sqrt{\frac{\pi}{2}}} K_0 + 1.$$

Thus we have proved

Theorem 3. *Suppose that a family of surfaces* $\phi(u^1; u^2, u^3) = \text{const}$, *with outer curvature* $K_e \geq K_0 > 0$, *is defined in a sphere of unit radius in the metric of a Riemannian space; further, suppose that the set of points where* $\phi_{u^1} = \phi_{u^2} = \phi_{u^3} = 0$ *has zero surface measure. Let* $Q_0 = \max |R/2 + K_{\mathfrak{R}}|$ *over all two-dimensional elements of area and all points of the sphere. Then*

$$Q_0 \geqslant \frac{K_0 - \frac{3}{2}}{\frac{K_0}{20} + \frac{e^{20}}{8}} \quad \textit{when } Q_0 < 20$$

$$\operatorname{ch} \sqrt{Q_0} \geqslant \frac{\pi - 2}{\sqrt{\frac{\pi}{2}}} K_0 + 1 \quad \textit{when } Q_0 \geqslant \frac{\pi}{2}. \tag{22}$$

Remark. Consider (17) when the family of parallel surfaces lies in euclidean space; let the surfaces be closed and have genus p. By integrating both sides of (17) over a fixed surface we obtain that

$$\frac{d^2 S}{dr^2} = 8\pi (1 - p);$$

that is, the second derivative of the area depends only on the genus of the surface. It is clear from this that for torus-type surfaces the change in area is proprotional to the displacement along the normal.

Received 26 JULY 68

BIBLIOGRAPHY

[1] D. M. Sincov, *On the system of integral curves of the Pfaffian equation* $Pdx + Qdy + Rdz = 0$, Uč. Zap. Naučno-Issled. Kaf. Ukrain. Otd. Mat. 3 (1928), 107–148. (Russian)

[2] S. S. Bušgens, *The geometry of a vector field*, Izv. Akad. Nauk SSSR Ser. Mat. 10 (1946), 73–96. (Russian) MR 8, 90.

[3] N. V. Efimov, *Investigation of a single-valued projection of a surface of negative curvature*, Dokl. Akad. Nauk SSSR 93 (1953), 609–611. (Russian) MR 15, 740.

[4] V. V. Vagner, *On the geometrical interpretation of the curvature vector of a non-holonomic* V_3^2 *in the three dimensional Euclidean space*, Mat. Sb. 4 (46) (1938), 339–356.

[5] Ju. A. Aminov, *Divergence properties of the curvatures of a vector field and of a family of surfaces*, Mat. Zametki 3 (1968), 103–111. (Russian) MR 37 #828.

Translated by:
L. Longdon