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Asymptotic distribution of the eigenvalues of systems of Navier-Stokes type

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In a bounded Lipschitz domain $\Omega \subset \mathbf{R}^n$ we consider the eigenvalue problem of the form

$$(1)\qquad \begin{cases} Au+F^*p=tBu, \\ Fu=0. \end{cases}$$

We justify a formula for the asymptotic expansion of the positive and negative spectra of (1) with some estimate of the remainder. In the case $B = I$, the asymptotic expansion of (1) was established in [1] without an estimate of the remainder. As in [1], the problem (1) is treated in variational form; we use a modification of the method of approximate spectral projection in [2]-[6] and some ideas from [1].

Suppose that $m > r \geqslant 0$, $l > l_1 > 0$ are integers, $m_j \in \{0, 1, \dots, m\}$ ($j = 1, \dots, l_1$), and that $H^s(\Omega)$ is a Sobolev space. We put

$$L(\Omega)=L_2(\Omega)^l, \quad W_m=H^m(\Omega)^l, \quad X(\Omega)=\prod_{1\leqslant j\leqslant l_1} H^{m_j}(\Omega).$$

Let $F = (F_{ij})_{i=1,\dots,l_1,\ j=1,\dots,l}$, where

$$F_{ij}=F_{ij}(x, D)=\sum_{|\alpha|\leqslant m-m_i} f_{ij}^{\alpha}(x)D_x^{\alpha}, \quad f_{ij}^{\alpha}\in C^{m_i}(\bar{\Omega}).$$

We put $F' = (F'_{ij})_{i=1,\dots,l_1,\ j=1,\dots,l}$, where

$$F'_{ij}(x, \xi)=\sum_{|\alpha|=m-m_i} f_{ij}^{\alpha}(x)\xi^{\alpha},$$

and assume that

$$(2)\quad \forall\,(x, \xi)\in \bar{\Omega}\times(\mathbf{R}^n\setminus 0)\cup \partial\Omega\times(\mathbf{C}^n\setminus 0)\ \operatorname{rank} F'(x, \xi)=l_1.$$

We consider the two forms

$$A(u, v)=\sum_{\substack{1\leqslant i, j\leqslant l\\ |\alpha|, |\beta|\leqslant m}} \langle a_{ij}^{\alpha\beta}D^{\alpha}u_i, D^{\beta}v_j\rangle, \quad B(u, v)=\sum_{\substack{1\leqslant i, j\leqslant l\\ |\alpha|, |\beta|\leqslant r}} \langle b_{ij}^{\alpha\beta}D^{\alpha}u_i, D^{\beta}v_j\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in L_2, and we assume that for all α, β, i, j

$$(3)\qquad a_{ij}^{\alpha\beta},\ b_{ij}^{\alpha\beta}\in L^{\infty}(\Omega), \quad a_{ij}^{\alpha\beta}=\overline{a_{ji}^{\beta\alpha}}, \quad b_{ij}^{\alpha\beta}=\overline{b_{ji}^{\beta\alpha}}.$$

Suppose further that there is a $\sigma \in (0, 1]$ such that for all i, j,

$$(4)\qquad a_{ij}^{\alpha\beta}\in \mathrm{Lip}_{\sigma}(\Omega), \quad |\alpha|=|\beta|=m, \qquad b_{ij}^{\alpha\beta}\in \mathrm{Lip}_{\sigma}(\bar{\Omega}), \quad |\alpha|=|\beta|=r,$$

$$(5)\qquad \partial^{\gamma}f_{ij}^{\alpha}\in \mathrm{Lip}_{\sigma}(\bar{\Omega}), \quad |\gamma|=m_i, \qquad |\alpha|=m-m_i.$$

We put $a' = (a'_{ij})$, $b' = (b'_{ij})_{i,j=1,\dots,l}$, where

$$a'_{ij}(x, \xi)=\sum_{|\alpha|=|\beta|=m} a_{ij}^{\alpha\beta}(x)\xi^{\alpha+\beta}, \quad b'_{ij}(x, \xi)=\sum_{|\alpha|=|\beta|=r} b_{ij}^{\alpha\beta}(x)\xi^{\alpha+\beta}.$$

It follows from (3) that A and B are continuous Hermitian forms on W_m. Suppose that $W \subset W_m$ is a subspace, $C_0^{\infty}(\Omega)^l \subset W$, and that there are $C_0 > 0$ and $C_1 \geqslant 0$ such that for all $u \in W$

$$(6)\qquad c_0\|u\|_{W_m}^2\leqslant A(u, u)+C_1\|Fu\|_{X(\Omega)}^2.$$

We put $V_1 = \{u \in W,\ Fu = 0\}$ and denote the closure of V_1 in $L(\Omega)$ by L_1. Let A_0 and $D(A_0)$ be the positive definite self-adjoint operator in L_1 associated with the form A. Since $m > 2r$, the form B determines an operator B_0, $D(B_0) = D(A_0)$, that is compact with respect to A_0, so that the problem

$$(7)\qquad A_0u=tB_0u, \quad u\in D(A_0)$$

has a discrete spectrum. Let $N_{\pm}(t)$ be the collection of eigenvalues (taking account of multiplicity) of (7) lying in $[0, t)$ for $+$ and in $(-t, 0]$ for $-$.

Theorem. *For every $\varepsilon > 0$*

$$N_{\pm}(t) = t^{ns}(c_{\pm} + O(t^{-\gamma+\varepsilon})), \tag{8}$$

where $s = 1/2(m - r)$, $\gamma = ns\sigma/(\sigma + n(3\sigma + 1))$ and the constants $c_{\pm}$ are defined as follows.

It was shown in [1] that under the condition (6) the form $\langle a'(x, \xi)\cdot, \cdot\rangle_{\mathbf{C}^l}$ is positive definite on $V_{x\xi} = \operatorname{Ker} F'(x, \xi) \subset \mathbf{C}^l$ for all $(x, \xi) \in \overline{\Omega} \times (\mathbf{R}^n \setminus 0)$, so that the problem

$$\langle a'(x, \xi)u, v\rangle_{\mathbf{C}^l} = t\langle b'(x, \xi)u, v\rangle_{\mathbf{C}^l}, \quad u \in V_{x\xi}, \quad \forall v \in V_{x\xi},$$

has the real spectrum $\{t_1, t_2, \ldots, t_{l-l_1}\}$, and we put

$$\omega_{\pm}(x, \xi) = \sum_{0 \leqslant \pm t_k(x, \xi) \leqslant 1} 1, \quad c_{\pm} = (2\pi)^{-n} \int_{\Omega \times \mathbf{R}^n} \omega_{\pm}(x, \xi)\, dx\, d\xi.$$

Remark. If (5) is discarded and (4) is replaced by the condition that the corresponding coefficients be continuous, then (8) is valid with $o(1)$ replacing $O(t^{-\gamma+\varepsilon})$. If $B = I$, this is the result of [1].

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