Soviet Physics Uspekhi

REVIEWS OF TOPICAL PROBLEMS

Topological phases in quantum mechanics and polarization optics

To cite this article: S I Vinitskiĭ *et al* 1990 *Sov. Phys. Usp.* **33** 403

Topological phases in quantum mechanics and polarization optics

S. I. Vinitskiĭ, V. L. Derbov, V. N. Dubovik, B. L. Markovski, and Yu. P. Stepanovskiĭ

Joint Institute for Nuclear Research, Dubna; N. G. Chernyshevskiĭ State University, Saratov; Physicotechnical Institute, Academy of Sciences of the Ukrainian SSR, Kharkov

Usp. Fiz. Nauk. **160**, 1–49 (June 1990)

A mathematical treatment is presented of the known Berry, Wilczek-Zee, Aharonov-Anandan, and Pancharatnam topological phases, and simple illustrative examples of their quantum mechanics are presented. The continuity and connection is traced among the various phases, while filling the gap involved with the forgotten works of S. M. Rytov and V. V. Vladimirskiĭ in polarization optics. A set of current experiments in polarization optics where the topological phases are measured is discussed in detail. Additional information on recently obtained results involving manifestations of geometrical phases in quantum mechanics and other fields of physics is contained in the Appendix and in the studies cited there.

1. INTRODUCTION

Recently M. Berry was able to pose sharply the question, with a number of examples of quantum mechanics, of the conditions for appearance of a topological phase of the wave function describing the Schrödinger evolution with a time-dependent Hamiltonian. This aroused great interest of both theoreticians and experimentalists working in the most varied fields of physics. The given problematics has deep roots in our century (see the historical section of the Appendix to this review) and is associated with contemporary mathematical methods (see Secs. 2.9, 2.10, and the discussion in the Conclusion–Sec. 4). It turned out that the consequences of the existence of topological phases in quantum-mechanical systems were experimentally observed long ago. In this connection it suffices to recall the half-integral orbital momenta of molecules (Van Vleck, 1929) and the phases of the wave function of a charged particle in the field of a monopole or a solenoid (Dirac, 1931; Aharonov-Bohm, 1959). However, only recently has the purposeful study of geometrical (topological) phases begun.

Let us try to trace the logic and chronology of the discussions that have led to the current understanding of topological phases in quantum mechanics and polarization optics. As is well known, quantum-mechanical systems are described by state vectors or density matrices that respectively satisfy the Schrödinger and Liouville equations. The evolution of an initial state is very simple when the Hamiltonian operator does not depend explicitly on the time, since the equations of motion give rise to a phase that depends linearly on the time. However, for Hamiltonians of general form, the pattern does not become simpler than for general linear dynamical systems. In this case the solution acquires a phase that contains information of nondynamical character (see Sec. 2.6). For example, if we examine the class of Hamiltonian evolutions $H(\lambda)$, $\lambda \in \Re^n$, of the state vectors of the quantum system while assigning the explicit time dependence $\{\lambda(\tau)\}$, it turns out that the phase of the wave function is the sum of two terms $\gamma + \delta$. One component γ of the phase owes its existence to the fact that the parameter space Λ is not simply connected. In other words, there are contours in parameter space that cannot be contracted continuously into a point. This component γ of the phase is of geometrical character, since it does not depend on the details of the time dependence of the parameters. That is, it does not depend on the dynamics fixed by the individual Hamiltonian, but reflects the properties of a family of Hamiltonians. This construction, within the framework of the adiabatic approximation, leads to the so-called Abelian phase of Berry (see Secs. 2.1–2.4). A more general situation arises when the parametrically assigned family of Hamiltonians possesses a degenerate discrete spectrum. Then the geometrical phase factors, which are incremented in the time of evolution of the state vectors corresponding to a degenerate eigenvalue, fix the non-Abelian phase of Berry or the Wilczek-Zee phase (see Secs. 2.5–2.6). We easily note that, in essence, a time-dependent family of Hamiltonians is not necessarily fixed parametrically, since in deriving the geometrical phase one uses only the fact of existence of suitable closed contours $\{\lambda(0)\} = \{\lambda(\tau_c)\}$ in the parameter space Λ. This circumstance enabled Aharonov and Anandan to proceed further and to isolate the topological phase of the wave function without assuming adiabaticity, and with weaker restrictions on the family of Hamiltonians. For these purposes they treat a class of time-dependent Hamiltonians upon which they impose the condition of cyclic evolution. That is, the state vectors must transform into themselves within the time of evolution apart from a unitary arbitrary factor. The geometrical phase obtained here (the Aharonov-Anandan phase) is discussed in Secs. 2.7–2.11. It was further noted that the state space itself possesses a Kähler metric and that transport of states along a geodesic in this metric gives rise to a phase of topological character–the Pancharatnam phase of Sec. 2.13. Similar properties of phases in quantum mechanics can be discerned also in its "algebraic formulation", i.e., if we fix in a self-consistent way the time-dependent algebra of the observables and characterize the state vectors with quantum numbers that correspond to a commuting subalgebra. We shall mention one model of this type in Sec. 2.14. Section 2.12 also presents a model of a topological phase that is incremented in a nonlinear system describing the propaga-

 0038-5670/90/060403-26$01.00

tion of light in a Kerr medium (see also Sec. 3.10). Here the reason for appearance of the phase involves the existence of invariants of the evolution.

Since, as we have already noted, the appearance of topological phases is a property of any dynamical system, then polarization optics in the appropriate parametrization offers a new field for seeking these objects. The phases associated with the rotation of the polarization vector upon displacement along bent light rays–a phenomenon predicted already by S. M. Rytov (1938) and V. V. Vladimirskiĭ (1941), is a prototype of the phases of Berry and Pancharatnam. Sections 3.1 and 3.2 are devoted to this problem.

This review will pay special attention to the overall pattern of topological phases in problems of quantum mechanics and optics. In our presentation the quantum-mechanical examples are methodological and illustrative in type as compared with the problems from polarization optics (Secs. 3.3–3.11). The latter have independent value, and also serve as a source of deep analogies and graphic models. In this review we have examined far from all aspects of the problematics of topological phases. We have tried to present additional information on recently published studies containing new results in the Appendix (see Sec. 5).

2. TOPOLOGICAL PHASES IN QUANTUM MECHANICS

In going from the theory of Planck and Einstein to the quantum mechanics of de Broglie, Heisenberg, Born, Schrödinger, and Dirac, a large role was played by the adiabatic hypothesis of P. Ehrenfest (1913); any state defined from the standpoint of quantum mechanics transforms under an adiabatic change in the parameters of the system again into a definite state with the same quantum numbers, which thus are adiabatic invariants. Within the framework of quantum mechanics the adiabatic hypothesis of Ehrenfest was proved for nondegenerate quantum-mechanical systems in 1928 by M. Born and V. A. Fock.[1] Since then this problem seemed exhausted. However, in 1984 M. Berry[2] again drew attention to the adiabatic theorem. He pointed out that the wave function of a system under a cyclic variation of the parameters in the general case acquires a phase factor that contains, besides the dynamical phase, also an additional topological phase.

2.1. The adiabatic approach of Born and Fock

In their study[1] Born and Fock, in solving the Schrödinger equation ($\hbar = 1$)

$$i\dot{\psi} = H(\tau)\psi \tag{2.1}$$

with the Hamiltonian $H(\tau)$, which varies slowly with time, introduced the complete set of orthonormalized eigenfunctions $\varphi_n(t)$ of the Hamiltonian

$$H(\tau)\varphi_n(\tau) = E_n(\tau)\varphi_n(\tau) \tag{2.2}$$

under the assumption that the spectrum $E_n(\tau)$ is not degenerate. They chose the phases of the functions $\varphi_n(\tau)$ so that the condition was satisfied that

$$(\varphi_n, \dot{\varphi}_n) = 0. \tag{2.3}$$

Further they represented the solution of the Schrödinger equation (2.1) with the initial condition $\psi(0) = \varphi_n(0)$ in the form

$$\psi_n(\tau) = \sum_m c_m^{(n)}(\tau)\varphi_m(\tau)\exp\left(-i\int_0^\tau E_m(\tau')\,d\tau'\right). \tag{2.4}$$

Upon taking account of (2.3), they obtained the following equation for determining the quantities $c_m \equiv c_m^{(n)}$:

$$\dot{c}_n = \sum_{m\neq n} c_m(\varphi_n, \dot{\varphi}_m)\exp\left[i\int_0^\tau (E_n(\tau') - E_m(\tau'))\,d\tau'\right]. \tag{2.5}$$

Upon using Eq. (2.2), the expression for (φ_n, φ_m) for $m\neq n$ can be rewritten in the form

$$(\varphi_n, \dot{\varphi}_m\Delta\tau_{nm}) = -\frac{(\varphi_n, \dot{H}\Delta\tau_{nm}\varphi_m)}{E_n - E_m}, \tag{2.6}$$

Here $\Delta\tau_{nm}$ is the time of transition between states. The evolution of the wave function of (2.4) is considered adiabatic if the right-hand side of (2.6) is much smaller in modulus than unity. Then the system of equations (2.5) implies that in the adiabatic limit we have $c_m \equiv c_m^{(n)} = \delta_{nm}$. That is, the function

$$\varphi_n(\tau)\exp\left(-i\int_0^\tau E_n(\tau')\,d\tau'\right)$$

satisfies the Schrödinger equation (2.1), while the remaining arbitrariness in the phase $\varphi_n(\tau)$ is fixed by the condition (2.3).

2.2. The adiabatic phase of Berry

The study of Born and Fock presupposed an explicit dependence of the Hamiltonian H on the time τ. Following Berry, we shall assume that the Hamiltonian H depends on the time τ via a set of functions $\{\lambda^j(\tau)\}_{j=1}^N = \lambda(\tau), \tau\in[0,\tau_c]$. Let us introduce the basis of the orthonormalized eigenfunctions $\chi_n(\lambda)$ of the instantaneous Hamiltonian $H = H(\lambda)$:

$$H(\lambda)\chi_n(\lambda) = E_n(\lambda)\chi_n(\lambda), \tag{2.7}$$

with arbitrary, yet fixed phases. Under the assumption that the spectrum $E_n(\lambda)$ is not degenerate, these functions at each instant of time $\tau\in[0,\tau_c]$ coincide, apart from a gauge transformation, with the eigenfunctions of (2.2):

$$\varphi_n(\tau) = \mathcal{U}(\lambda)\chi_n(\lambda) = \exp(i\Theta^n(\lambda))\chi_n(\lambda). \tag{2.8}$$

The condition (2.3) gives rise to the equation for the phase

$$\dot{\Theta}^n = i(\chi_n, \dot{\chi}_n) = A_j^n\dot{\lambda}^j. \tag{2.9}$$

Here

$$A_j^n = i\left(\chi_n, \frac{\partial\chi_n}{\partial\lambda^j}\right) \tag{2.10}$$

has the meaning of the "induced gauge field", since in gauge transformations

$$\chi_n \to \exp(i\alpha_n(\lambda))\chi_n \tag{2.11}$$

the quantity A_j^n transforms like the vector potential in electrodynamics:

$$A_j^n \to A_j^n + \frac{\partial\alpha_n}{\partial\lambda^j}. \tag{2.12}$$

The solution of Eq. (2.9) has the form

$$\Theta^n(\tau_c) = \int_0^{\tau_c}\dot{\Theta}^n d\tau = \int_C A_j^n d\lambda^j = \Theta^n(\lambda(\tau_c)) - \Theta^n(\lambda(0)). \tag{2.13}$$

Now let us examine the closed contour C in the space of the parameters λ, i.e., $\lambda(\tau_c) = \lambda(0)$. Then the change in phase $\Theta^n(\tau)$ in the time τ_c characterizes the evolution of the function $\varphi_n(\tau)$ with respect to the initial condition $\varphi_n(0)$:

$$\varphi_n(\tau_c) = \exp(i\Theta^n(\tau_c))\,\varphi_n(0). \tag{2.14}$$

As is implied by (2.13), the phase $\Theta^n(\tau_c)$, which is equal to

$$\Theta^n(\tau_c) = \Theta^n_c = \oint_C A^n_j\,d\lambda^j, \tag{2.15}$$

does not depend on the time of evolution, but is determined only by the closed contour C along which the quantum system having the Hamiltonian $H(\lambda)$ evolves.

Upon using the Stokes theorem, we can write the expression for the phase Θ^n_c, which is customarily called the *Berry phase*:

$$\Theta^n \equiv \Theta^n_c = \int_\Sigma F^n_{jk}\,ds^{jk}, \tag{2.16}$$

Here d^{jk}_s is an element of the orientable surface Σ extended along the contour C, while the quantity

$$F^n_{jk} = \frac{\partial A^n_k}{\partial\lambda^j} - \frac{\partial A^n_j}{\partial\lambda^k} \tag{2.17}$$

is the "gauge-field tensor" of (2.10). The gauge invariance of the tensor F with respect to (2.12) explicitly implies that the Berry phase does not depend on the choice of the basis of (2.7), and is unambiguously defined by the relationship (2.16).

We note that we can treat the Born-Fock condition (2.3), which fixes the phases of the basis (2.7) as the condition of horizontality in the projective Hilbert fibration over the space of rays employed in the study of B. Kostant.[3] Thus, with account taken of the parametric dependence of the function $\chi(\lambda)$, a natural topological interpretation of the Berry phase arises as the element of the holonomy group $\mathscr{U}(1)$ of the connectivity of (2.10) (see B. Simon[4]).

2.3. The Berry phase in the evolution of spin 1/2 in a magnetic field

Let us illustrate the concepts introduced above with the model, which has already become classic, of the evolution of spin 1/2 in a magnetic field that depends on the time τ. Here the Hamiltonian is a 2×2 matrix depending on the three parameters $\{\lambda^1,\lambda^2,\lambda^3\} = \lambda(\tau)$. Let us write it in a more general form:[5]

$$H = (\mathbf{B}\lambda) + (\lambda G\sigma). \tag{2.18}$$

Here G is a certain nondegenerate 3×3 matrix, and the σ are the standard Pauli matrices, $\mathbf{B}\in\Re^3$.

Let us make the replacement in the parameter space $\lambda\to\mathbf{p}$:

$$\lambda = G^{-1}\mathbf{p}. \tag{2.19}$$

The eigenvalues $\varepsilon_\pm$ of the instantaneous Hamiltonian $H = H(\mathbf{p})$, which as usual we find from the equation

$$\det(H - \varepsilon) = 0, \tag{2.20}$$

and the eigenvectors $\varphi_\pm$ are respectively equal to

$$\varepsilon_\pm = \mathbf{B}G^{-1}\mathbf{p} \pm |\mathbf{p}|, \tag{2.21}$$

$$\varphi_\pm = \frac{1}{(2|\mathbf{p}|)^{1/2}}\begin{pmatrix} \pm\dfrac{p_-}{(|\mathbf{p}| \mp p_3)^{1/2}} \\ (|\mathbf{p}| \mp p_3)^{1/2}\end{pmatrix}. \tag{2.22}$$

Here we have $p_\pm = p_1 \pm ip_2$. Let us introduce the standard spherical coordinates $\{|\mathbf{p}|,\vartheta,\varphi\}$ in the parameter space $\mathbf{p}\in\Re^3$. Then the eigenvectors will acquire the form

$$\varphi_+ = \begin{pmatrix} \cos\dfrac{\theta}{2}\cdot e^{-i\varphi} \\ \sin\dfrac{\theta}{2}\end{pmatrix}, \quad \varphi_- = \begin{pmatrix} -\sin\dfrac{\theta}{2}\cdot e^{-i\varphi} \\ \cos\dfrac{\theta}{2}\end{pmatrix}. \tag{2.23}$$

The components of the induced gauge field (2.10) in this case equals

$$\mathbf{A}_+ = i\langle\varphi_+\nabla_p\varphi_+\rangle = (A^p_+,\ A^\varphi_+,\ A^\theta_+) = \left(0,\ -\frac{i}{2|\mathbf{p}|}\operatorname{ctg}\frac{\theta}{2},\ 0\right),$$

$$\mathbf{A}_- = i\langle\varphi_-\nabla_p\varphi_-\rangle = (A^p_-,\ A^\varphi_-,\ A^\theta_-) = \left(0,\ -\frac{i}{2|\mathbf{p}|}\operatorname{tg}\frac{\theta}{2},\ 0\right). \tag{2.24}$$

The corresponding form of the curvature (2.17) has the form

$$\mathbf{F}_\pm = \operatorname{rot}\mathbf{A}_\pm = \mp\frac{\mathbf{p}}{2|\mathbf{p}|^3}. \tag{2.25}$$

According to the definition (2.16), the Berry phase equals

$$\Theta_\pm = \int_\Sigma \operatorname{rot}\mathbf{A}_\pm\,ds. \tag{2.26}$$

The final expression for $\Theta_\pm$ stems from the known formula[6] on the Gaussian mapping induced by replacing the variables of (2.19):

$$\Theta_\pm = \mp\frac{1}{2}\operatorname{sign}(\det G)\,\Omega_C; \tag{2.27}$$

Here Ω_c is the solid angle:

$$\Omega_C = \int_{\mathscr{S}^2}\frac{d\Omega}{|\mathbf{p}|^2}. \tag{2.28}$$

The latter is subtended by the contour C lying on the sphere $\mathscr{S}^2$ of radius $|\mathbf{p}|$ with its center at the origin of coordinates of the parameter space $\mathbf{p}\in\Re^3$, which coincides with the starting origin in the parameter space $\lambda\in\Re^3$. In particular, for the contour C lying in the plane $\lambda_3 = 0$, we find the solid angle $\Omega_c = 2\pi$, i.e.,

$$\Theta_\pm = \mp\operatorname{sign}(\det G)\,\pi. \tag{2.29}$$

Let us go to the particular parametrization of the Hamiltonian of (2.18)

$$H = \varkappa\mathbf{H}\mathbf{s}, \tag{2.30}$$

which describes the evolution of the wave function of a particle of spin 1/2 in an external slowly varying magnetic field $\mathbf{H}(\tau)$, where $\mathbf{s}$ is the spin operator, and κ is a constant that includes the gyromagnetic ratio. At the initial instant of time let the projection of the spin $m = \pm 1/2$ on the direction of the magnetic field $\mathbf{t} = \mathbf{H}/|\mathbf{H}|$ be fixed:

$$i\dot\psi = \varkappa\mathbf{H}\mathbf{s}\psi. \tag{2.31}$$

Under a cyclic variation of the magnetic field $\mathbf{H}(\tau_c) = \mathbf{H}(0)$ the function $\varphi_m(\tau)$, according to (2.14), acquires a geometrical factor:

$$\varphi_m(\tau_c) = \exp(i\Theta^m)\varphi_m(0). \tag{2.32}$$

Here the topological phase Θ^m is calculated from Eq. (2.16). In line with (2.27) and (2.28) (for $B=0, G=\varkappa H\mathbf{1}$) it equals

$$\Theta^m = -m\Omega(C), \quad m=\pm\frac{1}{2}. \tag{2.33}$$

Here $\Omega(C)$ is the solid angle that the vector $\mathbf{t}$ describes on the unit sphere while passing along the closed contour C in the parameter space $\mathbf{t}\in\mathscr{S}^2$. Bitter and Dubbers[7] have experimentally measured the phase of (2.33) for $m=1/2$ for polarized neutrons in a helical magnetic field.

2.4. The Berry phase in the evolution of an arbitrary spin in a magnetic field

Let us study a generalization of Eq. (2.33) for a particle having a general spin in the slowly varying magnetic field $\mathbf{H}(\tau)$. Here we shall use the method of the movable Frenet reference system $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ for calculating the Berry phase Θ^m. The equations for the eigenfunctions of the instantaneous Hamiltonian $H(\tau)$ have the form

$$(\mathbf{st})\varphi_m = m\varphi_m. \tag{2.34}$$

Here the projection of the spin m is an adiabatic invariant, while the phases φ_m are fixed by the Born-Fock-Simon condition (2.3). Upon differentiating (2.34) with respect to the time τ, we have the identity

$$(\mathbf{s}\dot{\mathbf{t}})\varphi_m + (\mathbf{st}-m)\dot{\varphi}_m = 0. \tag{2.35}$$

We find from (2.35), owing to the properties of the spin operator, that φ_m is a superposition of the form

$$\dot{\varphi}_m = \alpha\varphi_{m+1} + \beta\varphi_m + \gamma\varphi_{m-1}, \tag{2.36}$$

since $(\mathbf{t}\dot{\mathbf{t}})=0$. Then Eq. (2.3) implies that $\beta=0$, i.e.,

$$\dot{\varphi}_m = \alpha\varphi_{m+1} + \gamma\varphi_{m-1}. \tag{2.37}$$

Further, as a result of identity transformations, we obtain

$$\dot{\varphi}_m = -(\mathbf{st}-m)(\mathbf{s}\dot{\mathbf{t}})\varphi_m. \tag{2.38}$$

Finally, with account taken of the commutation relationships of the spin operator

$$(\mathbf{st})(\mathbf{s}\dot{\mathbf{t}}) - (\mathbf{s}\dot{\mathbf{t}})(\mathbf{st}) = i[\mathbf{t}\dot{\mathbf{t}}]\,\mathbf{s}$$

we find the evolution equation in the customary notation

$$\dot{\varphi}_m = -i\mathbf{s}[\mathbf{t}\dot{\mathbf{t}}]\varphi_m. \tag{2.39}$$

Now let us introduce the contour $\lambda=\lambda(s)$ in the parameter space $\lambda=\{\lambda_1,\lambda_2,\lambda_3\}$ on which the instantaneous Hamiltonian $H(\lambda)$ depends, such that the tangential vector

$$\mathbf{t} = \frac{d\boldsymbol{\lambda}}{ds} \tag{2.40}$$

coincides with $\mathbf{H}/|\mathbf{H}|$, while the vectors of the normal $\mathbf{n}$ and the binormal $\mathbf{b}$ are defined in the standard way for the curve $\lambda(s)$ with the natural parametrization along the arc length s.

Upon replacing the differentiation with respect to τ in (2.39) with differentiation with respect to the arc length s and using the Frenet formulas

$$\frac{d\mathbf{t}}{ds}=\frac{\mathbf{n}}{R}, \quad \frac{d\mathbf{n}}{ds} = -\frac{\mathbf{t}}{R}-\frac{\mathbf{b}}{T}, \frac{d\mathbf{b}}{ds}=\frac{\mathbf{n}}{T}, \tag{2.41}$$

where R and T are the curvature and torsion of the curve $\lambda(s)$, we obtain the equation

$$\frac{d\varphi_m}{ds} = -i\frac{(\mathbf{sb})}{R}\varphi_m; \tag{2.39'}$$

Here $\mathbf{b}/R$ is the angular velocity of precession of the spin with respect to the stationary system of coordinates. The angular velocity of precession of the Frenet trihedron with respect to this stationary system is $(\mathbf{b}/R)-(\mathbf{t}/T)$. Therefore in the stationary system associated with the Frenet trihedron, the spin precesses with the angular velocity $\mathbf{t}/T$. Hence we have

$$\frac{d\varphi_m}{ds} = -i\frac{(\mathbf{st})}{T}\varphi_m = -i\frac{m}{T}\varphi_m. \tag{2.42}$$

Upon assuming by analogy with (2.32) that

$$\varphi_m(s) = \exp(i\Theta^m(s))\varphi_m(0),$$

we find from (2.42) the equation for the phase $\Theta(s)$:

$$\frac{d\Theta^m}{ds} = -\frac{m}{T}. \tag{2.43}$$

Then we have the following expression for the Berry phase of (2.15):

$$\Theta^m(\tau_c) \equiv \Theta^m(s_c) = -m\int_0^{s_c}\frac{ds}{T}. \tag{2.44}$$

Upon using the known Gauss-Bonnet result (see, e.g., Ref. 6),

$$\int_0^{s_c}\frac{ds}{T} = \Omega(C), \tag{2.45}$$

we obtain the generalization of the Berry formula (2.33) for an arbitrary spin s having the projection m (cf. the results of Ref. 8):

$$\Theta^m = -m\Omega(C). \tag{2.33'}$$

We note that the presented derivation remains in force also when treating the motion of a massless particle having an arbitrary spin, and having the helicity m, if the helicity is an adiabatic invariant.

2.5. The non-Abelian adiabatic phase of Wilczek and Zee

Born and Fock proved the adiabatic theorem under the condition that the spectrum of the Hamiltonian is not degenerate and does not become degenerate in the process of evolution. Their proof can be generalized also to the case in which the spectrum of the Hamiltonian is degenerate, but in the process of evolution the states pertaining to different energy levels do not mix.[1] According to the adiabatic theorem, the additional quantum numbers a that characterize the degenerate state φ_{na} do not vary. Here the system does not necessarily return to the initial state, but is subjected to a certain unitary transformation that generalizes the Abelian $U(1)$ transformation of Berry in (2.14). Thus, for example, if the state is degenerate in angular momentum, then the projection of the angular momentum in adiabatic evolution does not vary, but the axis of quantization can change in direction. A non-Abelian generalization of the Berry phase was found by F. Wilczek and A. Zee,[9] who showed how non-Abelian gauge fields that generalize the vector potential of (2.10) arise in describing the adiabatic evolution of simple quantum systems.

As Fock[10] showed, the orthonormalized eigenfunctions φ_{na} always can be made subject to a condition that generalizes the condition (2.3):

$$(\varphi_{na}, \dot{\varphi}_{nb}) = 0 \tag{2.46}$$

for any a and b. Observance of the condition (2.46) has the result that in the adiabatic evolution the eigenfunctions

$$\varphi_{na}(\tau) \exp\left(-i\int_0^\tau E_n(\tau')\,d\tau'\right) \tag{2.47}$$

satisfy the Schrödinger equation. Upon introducing the basis set of orthonormalized eigenfunctions $\chi_{na}(\lambda)$ of the instantaneous Hamiltonian $H(\lambda)$, one can determine the non-Abelian gauge potential

$$(A_i^n)_{ab} = i\left(\chi_{nb}, \frac{\partial\chi_{na}}{\partial\lambda^i}\right), \tag{2.48}$$

that generalizes the vector potential of (2.10). Transformation to another basis in (2.48)

$$\chi'_{na} = \Lambda_{ab}\chi_{nb} \tag{2.49}$$

leads to a non-Abelian gauge transformation (for simplicity of notation we shall omit the matrix indices)

$$A_i^{n'} = \Lambda A_i^n \Lambda^{-1} + \frac{\partial\Lambda}{\partial\lambda^i}\Lambda^{-1}. \tag{2.50}$$

Now the solutions of the Schrödinger equation $\varphi_{na}(\tau)$ are associated with the basis functions by the unitary transformation

$$\varphi_{na}(\tau) = \mathcal{U}_{ab}^n(\lambda(\tau))\,\chi_{nb}(\lambda(\tau)). \tag{2.51}$$

As a result Eq. (2.9) is generalized as follows:

$$\dot{\mathcal{U}}_{ab}^n = -\mathcal{U}_{ab}^n (A_i^n)_{ca}\dot{\lambda}^i. \tag{2.9'}$$

Yet instead of (2.13) we obtain the non-Abelian generalization of the Berry factor in the form of the P-exponential

$$\mathcal{U}^n(C_{\tau_c}) = P\exp\left(i\oint A_i^n\,d\lambda^i\right). \tag{2.52}$$

The stress tensor F_{ij} corresponding to the vector potential (2.48) is derived in the usual way as in gauge theories:

$$F_{ij}^{ab} = \partial_i A_j^{ab} - \partial_j A_i^{ab} + [A_i A_j]^{ab} \quad (i, j = \overline{1, N}). \tag{2.53}$$

At present there is an extensive literature pertaining to non-Abelian generalizations of the Berry phase (see, e.g., Refs. 11–16).

Spin 1/2 *and isospin* 1/2 *in an external field.* As an example illustrating the general theory presented in this section, let us study the Hamiltonian[11]

$$H = B_{i\alpha}\sigma^i \otimes \sigma^\alpha. \tag{2.54}$$

Here we shall assume the matrix $B_{i\alpha}$ to be parametrized by the three-dimensional vector of the adiabatic parameters $B_{i\alpha} = \varepsilon_{i\alpha k}\lambda_k$. Upon rewriting the Hamiltonian explicitly as a 4×4 matrix, we find its eigenvalues: $E_1 = -2|\lambda|$, $E_2 = 0$ (doubly degenerate), and $E_3 = 2|\lambda|$. The degeneracy of the levels is constant in parameter space except for the point $\lambda = 0$, and thus we have $\Lambda \sim \Re^3\setminus\{0\}$.

The eigenvectors are respectively equal to

$$\begin{aligned}
&\langle E_1| = \frac{1}{2|\lambda|}(\omega, -|\lambda| + \rho, |\lambda| + \rho, \bar{\omega}),\\
&\langle E_{2,1}| = c_0(-2\rho, \omega, \omega, 0),\\
&\langle E_{2,2}| = \frac{c_0}{|\lambda|}(-\omega^2, -\rho\omega, -\rho\omega, \lambda^2 + \lambda_3^2),\\
&\langle E_3| = \frac{1}{2|\lambda|}(\omega, |\lambda| + \rho, -|\lambda| + \rho, \bar{\omega}),
\end{aligned} \tag{2.55}$$

where we have

$$c_0 = [2(\lambda^2 + \lambda_3^2)]^{-1/2}, \quad \rho = i\lambda_3, \quad \omega = \lambda_2 - i\lambda_1.$$

The existence of a global basis (i.e., fixed at each point of the manifold λ) is guaranteed (see Ref. 5) by the fact that the homotopic group $\pi_2[\mathrm{U}(4)/\mathrm{U}(2)]$ is trivial. For the degenerate level E_2 the calculations of the connectivity A and the curvature F_{ij}^{ab} are performed by the standard formulas (2.48) and (2.53). Consequently the "components of the curvature" are defined as

$$-\varepsilon_{ikl}F_{kl}^{ab} = (b_i)^{ab} \quad (a, b = 1, 2).$$

Direct calculations yield the following expressions for the vector quantities $\mathbf{b}$:

$$b_l = \frac{\lambda_l}{|\lambda|(\lambda^2 + \lambda_3^2)}\begin{pmatrix} -2i\lambda_3 & \dfrac{(\lambda_2 - i\lambda_1)^2}{|\lambda|} \\ -\dfrac{(\lambda_2 + i\lambda_1)^2}{|\lambda|} & 2i\lambda_3 \end{pmatrix}. \tag{2.56}$$

We see that the curvature is regular everywhere in Λ. As $|\lambda| \to \infty$, b_l behaves as $1/|\lambda|^2$, which corresponds to a non-Abelian magnetic monopole (see, e.g., Ref. 17). It is interesting to calculate the topological invariant that "reads out" the monopole charge. For this purpose we shall denote the matrix part of Eq. (2.56) as $\Delta(n)$ and introduce the parametrization $\hat{\beta} = \beta\sigma$:

$$b_l = -i\lambda_l\frac{\Delta(n)}{|\lambda|^3} = -\frac{i\lambda_l}{|\lambda|^3}\beta^\alpha\sigma.$$

In the case of a monopole we have $\beta^2 \to \mathrm{const}$. Then we can introduce the coordinates $\mathbf{e}(\lambda) = \beta/|\beta|$ on the unit sphere and determine the regular mapping $\mathcal{S}^2 \to \mathcal{S}^2$. Let us calculate its degree of mapping[6] by the standard formula:

$$m[\mathbf{e}] = \frac{1}{4\pi}\int_S \varepsilon_{ikl}e^l\frac{\partial e^k}{\partial\theta}\frac{\partial e^l}{\partial\varphi}\,d\theta\,d\varphi. \tag{2.57}$$

Upon substituting the unit vector

$$\mathbf{e}(\lambda) = (\lambda^2 + \lambda_3^2)^{-1}\begin{pmatrix} 2\lambda_1\lambda_2 \\ \lambda_1^2 - \lambda_2^2 \\ 2|\lambda|\lambda_3 \end{pmatrix}$$

into the formula (2.57) for the topological charge and transforming to spherical coordinates in the integral for $m[\mathbf{e}]$, we find $m[\mathbf{e}] = -2$, which corresponds to the two-monopole case.[18]

2.6. Factorization of the solutions of linear evolution systems and the geometrical phase of Berry

Up to now we have been treating the Berry phase that arises within the framework of the adiabatic approach.[2)] In line with the existence of a geometrical phase that accompanies the cyclic evolution of isolated systems,[19] the question arises of how to distinguish the Berry factor in the overall solution of the evolution equations. We shall assume that the continuous time derivative exists of the set of functions that we have chosen $\{|n,k(t)\rangle\}$, which form a complete orthonormalized basis in the Hilbert space of solutions of the arbitrary linear evolution equation $i\partial_\tau\psi = H\psi$ with self-conjugate H. Then we can isolate from the Cauchy operator for this equation a component of geometrical origin in the form of an operator coefficient. This part of the Cauchy operator in the cyclic evolution of the basis $|n,k(\tau)\rangle$ acquires the form of an Abelian (2.14) or non-Abelian (2.52) phase factor.

Let us introduce the complete orthonormal basis of functions $|n,k\rangle$ and represent the solution of the evolution equation $i\partial_\tau\psi = H\psi$ in general form:

$$\psi(t) = \sum c_{nk}(t)\,|nk,\ \tau\rangle \equiv \sum c_M\,|M(\tau)\rangle.$$

Here n is the principal quantum number. The formal solution of the systems of equations for the coefficients of the expansions c_{nk} is given by the expression

$$\left\langle nk \left| T\exp\left(-i\int_0^\tau H(\tau')\,d\tau'\right)\right| \psi(0)\right\rangle. \tag{2.58}$$

Let us introduce the evolution operators $\overleftarrow{U}$ and $\overrightarrow{U}$, which satisfy the equations

$$i\frac{d\overleftarrow{U}(b,a)}{db} = H(b)\overleftarrow{U}(b,a),$$

$$i\frac{d\overrightarrow{U}(a,b)}{db} = \overrightarrow{U}(a,b)\,H(b),\quad \overrightarrow{U}(a,a) = \overleftarrow{U}(b,b) = 1;$$

Here $\overleftarrow{U}$ and $\overrightarrow{U}$ are respectively the left and right evolution operators written in the form of ordered exponentials.

The following statements are proved:

1) One can represent the solution (2.58) of the Cauchy problem for the evolution equation with the initial condition $\psi(0) = \varphi(x)$ in the form

$$\psi(\tau,x) = \overleftarrow{U}_{-A}(\tau,0)\,\overleftarrow{U}_{-\hat{E}}(\tau,0)_{NM}\,\langle M(x,0)|\varphi(x)\rangle\,|N(x,\tau)\rangle,$$

where the matrices A and E are fixed by the relationships

$$A \equiv A_{NM} = \left\langle N\frac{dM}{d\tau}\right\rangle,$$

$$\hat{E} \equiv \overrightarrow{U}_A(0,\tau)_{nk;mj}\,\langle mj|H|mj\rangle\,\overleftarrow{U}_{-A}(\tau,0)_{mj;nl}.$$

In the language of ordered exponentials one can write the solution in the usual form

$$\psi(x,\tau) = \left[P\exp\left(-\int_0^\tau A\right)\right]_{NM}\left(P\exp\int_0^\tau\hat{E}\right)_{ML} \times \langle L(0)|\varphi\rangle\,|N(\tau,x)\rangle. \tag{2.58$'$}$$

2) Let the Hamiltonian H be such that the transition probabilities $\langle N(\tau)|M(\tau')\rangle \equiv Q_{NM}$ can be represented in quasidiagonal form:

$$\gamma_N\delta_{nm} + \delta_{n-1,m}a_N + \delta_{n,m-1}b_N.$$

Here the nondiagonal terms are small in comparison with the diagonal terms: $|a_N/\gamma_N|, |b_N/\gamma_N| \ll 1, \forall N$. Then we can expand the expression (2.58′) in a series according to perturbation theory in the order of smallness of the terms containing a_n and b_n (for details see Ref. 20).

Now let us examine the evolution of a system for which the Hamiltonian H and the basis $|n,k(\tau)\rangle$ associated with this Hamiltonian depend on the time via the parameters $\lambda(\tau)$ so that in the time $\tau\in[0,\tau_c]$ the state vector returns to its initial position. Then, in solving the Cauchy problem (2.58′) for the evolution equation, the first factor acquires the form of the non-Abelian Berry phase of (2.52). If we can neglect the nondiagonal terms in Q_{NM}, then we obtain the Abelian phase. As we see, for the probabilities of transitions Q_{NM} of general form, the evolution of a mixed initial state will include nonadiabatic processes.

The results of this section generalize the postadiabatic approximation proposed in Ref. 21, and agree with the results of Ref. 22.

2.7. The geometrical phase of Aharonov and Anandan

After the publication of the studies of Berry[2] and Simon[4] the problem immediately arose of studying topological phases outside the framework of the adiabatic approximation. In the mathematical formulation of Simon, where the natural structure is a Hilbert fiber space with a basis of adiabatic parameters of finite dimension, the gauge transformation (2.11) and the connectivity of (2.10) are described by operators in the Hilbert fibers. This circumstance complicates the analysis of the topological structure of the solution space. Therefore, by using the method of induced fibrations (see, e.g., Ref. 6, Sec. 27), we can reduce Simon's formulation to studying universal fiber spaces of the type of complex projective spaces, where one uses a space of rays as the basis. The properties of these spaces enabled deriving classification theorems[5] and led to a generalization of the Berry phase, which consists in the following.

Let us study an isolated quantum system that is described by the state vector $\psi(\tau)$ and the Hamiltonian $H(\tau)$ in the time interval $[0,\tau_c]$. Let us introduce the new state $\Phi(\tau)$, while separating out the dynamical phase

$$\psi(\tau) = \exp\left(-i\int_0^\tau Q(\tau')\,d\tau'\right)\Phi(\tau) = \exp(-i\delta)\,\Phi(\tau). \tag{2.59}$$

When account is taken of the identity $\mathrm{Re}\langle\psi d\psi\rangle = 0$, the latter equals

$$\delta = \int_0^\tau Q = \int_0^\tau \mathrm{Re}\frac{\langle\psi H\psi\rangle}{\langle\psi\psi\rangle}\,d\tau'. \tag{2.60}$$

Then the Schrödinger equation for the state vector $|\Phi\rangle$ acquires the form

$$i\frac{d\Phi}{d\tau} = (H-Q)\,\Phi. \tag{2.61}$$

Averaging with respect to $\langle\Phi|$ leads us to the equation

$$\mathrm{Im}\left\langle\Phi\frac{d\Phi}{d\tau}\right\rangle = 0, \tag{2.62}$$

which coincides in form with the condition of parallel transport (2.3) that appears in the adiabatic pattern, but is written for state vectors having the "mixed spectrum" (2.61).

Let us introduce the concept of cyclic evolution of the quantum system on the interval $[0,\tau_c]$, while understanding thereby the following: during the time of evolution τ_c

along the curve s in the state space $\mathscr{H}$, the vector $\psi(\tau_c)$ returns to the initial state apart from the phase ϕ. That is, we have

$$|\psi(\tau_c)\rangle = e^{i\phi}|\psi(0)\rangle, \qquad \|\psi(\tau_c)\| = \|\psi(0)\|, \tag{2.63}$$

if the curve s is projected on the closed contour C in the ray space. Here the phase φ characterizes the gauge transformation necessary for closing s in $\mathscr{H}$. Let us define the gauge $\psi \to \psi e^{-if(\tau)} = \tilde{\psi}$ such that the periodicity condition is obeyed for $\tilde{\psi}$ in the interval $[0,\tau_c]$: $\tilde{\psi}(\tau_c) = \tilde{\psi}(0)$. Then we obtain the following expression for the gauge transformation generated by the function f:

$$f(\tau_c) - f(0) = \phi \,(\mathrm{mod}\, 2\pi m), \quad m \in \mathscr{Z}. \tag{2.64}$$

Upon using Eqs. (2.60) and (2.61), we write the condition of cyclicity (2.63) in the form

$$\psi(\tau_c) = \exp(i\gamma)\exp\left(-i\int_0^{\tau_c} Q\,dt\right)\psi(0) = \exp(i\phi)\psi(0), \tag{2.65}$$

where we have

$$\gamma = i\int_0^{\tau_c}\langle\tilde{\psi}\dot{\tilde{\psi}}\rangle\, d\tau = i\oint_C\langle\tilde{\psi}|d\tilde{\psi}\rangle. \tag{2.66}$$

We note that the total phase ϕ of the wave function (2.65) is determined by the sum of the geometrical phase γ and the dynamical phase δ:

$$\phi = \gamma - \delta. \tag{2.67}$$

Here, at first glance, the geometrical phase γ introduced by Aharonov and Anandan[19] coincides in form with the adiabatic phase of Berry in (2.15). However, it has been derived without the assumption of adiabaticity and is realized in projective fibrations over the basis of rays $\mathscr{R} = \mathscr{H}/\sim$,[3] whereas the adiabatic phase of Berry is interpreted as an element of the holonomy group of a Hilbert fibration with the basis of the parametric space Λ that defines the Hamiltonian of an open quantum system. In the following three sections we shall convince ourselves that the existence of a Kähler metric in the projective space leads to a natural parallel transport along a geodesic, and this transport does not involve the dynamics.

2.8. A model example of the complex projective space $\mathscr{C}\mathscr{P}_n$ –direct approach

Let us take up in greater detail a model quantum-mechanical example[23] having a self-conjugate matrix Hamiltonian H and a vector state $\{z_\alpha\}_{\alpha=0}^n \in \mathscr{C}_{n+1}$. The Schrödinger equation has the standard form

$$i\dot{z}_\alpha = H_\alpha^\beta z_\beta, \quad \tau \in [0, \tau_c], \; z^\alpha(\tau_c) = e^{i\phi}z^\alpha(0). \tag{2.68}$$

Let us require the condition of cyclic evolution of the state vector $\mathbf{z}(\tau)$ in the form (2.63). According to Eq. (2.60), the dynamical phase equals

$$\delta = \int_0^{\tau_c}\frac{\bar{z}_\alpha H^{\alpha\beta}z_\beta}{|\mathbf{z}|^2}\,d\tau.$$

We shall find the geometrical phase γ according to (2.66). To do this, let us construct ψ and $\bar{\psi}$ explicitly. Let the state vector ψ be given in the form

$$|\psi\rangle = \frac{1}{(|\mathbf{z}|^2)^{1/2}}\begin{pmatrix} z_0 \\ \vdots \\ z_n\end{pmatrix}.$$

Let us transform to projective coordinates, while assuming, e.g., that $\forall\tau\in[0,\tau_c]$. In these coordinates the state vector looks as follows:

$$|\psi\rangle = \frac{\exp(i\arg z_0)}{(1+|w|^2)^{1/2}}\begin{pmatrix} 1 \\ w_1 \\ \vdots \\ w_n\end{pmatrix}, \tag{2.69}$$

where $\omega_i = z_i/z_0$. The arbitrary phase $\arg z_0$ reflects the explicit $U(1)$-gauge freedom in transforming to projective coordinates. We note that the condition of cyclicity (2.63) as applied to the non-normalized state $|\psi\rangle$ implies that $\|\psi(0)\| = \|\psi(\tau_c)\|$. However, in the language of projective coordinates $\mathbf{w}$, the condition (2.63) does not lead to additional restrictions.

The condition of periodicity for the wave function ψ yields an equation for the phases

$$-\frac{i}{2}\ln\frac{z_0(\tau_c)}{\bar{z}_0(\tau_c)} - f(\tau_c) = -\frac{i}{2}\ln\frac{z_0(0)}{\bar{z}_0(0)} - f(0),$$

or, if we take account of the restrictions on the gauge function f in (2.64),

$$-\frac{i}{2}\ln\frac{z_0(\tau_c)\bar{z}_0(0)}{\bar{z}_0(\tau_c)z_0(0)} = \phi. \tag{2.70}$$

We have the following expression from Eq. (2.67) for the geometrical phase

$$\gamma = \phi + \delta = -\frac{i}{2}\ln\frac{z_0(\tau_c)\bar{z}_0(0)}{\bar{z}_0(\tau_c)z_0(0)} + \int_0^{\tau_c}\mathrm{Re}\frac{\bar{\mathbf{z}}H\mathbf{z}}{|\mathbf{z}|^2}\,d\tau. \tag{2.71}$$

Upon using the identity $\mathrm{Re}(\bar{\mathbf{z}}H\mathbf{z}) = (i/2)(\bar{\mathbf{z}}d\mathbf{z} - \mathbf{z}d\bar{\mathbf{z}})$, which is fulfilled by the solutions of Eq. (2.68) for γ, we obtain the expression

$$\gamma = -\frac{i}{2}\ln\frac{z_0(\tau_c)\bar{z}_0(0)}{\bar{z}_0(\tau_c)z_0(0)} + \frac{i}{2}\int_0^{\tau_c}\frac{\bar{\mathbf{z}}\,d\mathbf{z} - \mathbf{z}\,d\bar{\mathbf{z}}}{|z_0|^2 + |\mathbf{z}|^2}.$$

Upon assuming the existence of derivatives of $|\psi(\tau)\rangle$ in the neighborhood of τ_c, we find the 1-form that gives rise to the phase γ:

$$\dot{\gamma}\,d\tau = -\frac{i}{2}\left(\frac{dz_0}{z_0} - \frac{d\bar{z}_0}{\bar{z}_0}\right) + \frac{i}{2}\frac{\bar{z}_0 dz^0 - z_0 d\bar{z}_0}{|z_0|^2(1+|\mathbf{w}|^2)} + \frac{i}{2}\frac{\bar{z}_i dz^i - z_i d\bar{z}^i}{|z_0|^2(1+|\mathbf{w}|^2)}.$$

Now let us contract on the sphere $\mathscr{S}^{2n+1}: |\mathbf{z}|^2 = 1$ of normalized solutions of Eq. (2.68). Upon taking account of the rules of calculations with dependent differentials for the form $\dot{\gamma}d\tau$, we obtain

$$\dot{\gamma}\,d\tau = \frac{i}{2}\frac{\bar{w}_i dw^i - w_i d\bar{w}^i}{1+|\mathbf{w}|^2}. \tag{2.72}$$

The electromagnetic tensor corresponding to the potential $\gamma(\cdot)$ has the form[24]

$$F = i\begin{pmatrix} 0 & -\frac{1}{1+|\mathbf{w}|^2}+\frac{\mathbf{w}\otimes\bar{\mathbf{w}}}{(1+|\mathbf{w}|^2)^2} \\ \frac{1}{1+|\mathbf{w}|^2}-\frac{\bar{\mathbf{w}}\otimes\mathbf{w}}{(1+|\mathbf{w}|^2)^2} & 0 \end{pmatrix}. \tag{2.73}$$

2.9. Geometrical construction of the Aharonov-Anandan phase

Now let us perform, following Ref.25, the geometrical construction of the Aharonov-Anandan phase in terms of projective spaces, and then proceed to concrete parametrizations.

Let us study an initial Hilbert space of states $\mathscr{H}$ as a linear vector fiber space $E(,,\pi)$ having a basis of a space of rays $\mathscr{R}$ and a standard scalar product $\langle\cdot,\cdot\rangle$. Locally any element of the fiber space $|\psi\rangle$ is parametrized in the form

$$|\psi\rangle=\xi|w\rangle. \tag{2.74}$$

Here $\xi\in\mathscr{C}_1$ is the coordinate of the fiber, while $|w\rangle$ is an element of the basis $\mathscr{R}$ (Fig. 1). Let us isolate in the tangential space TE the vertical subspace $T_V E$, which realizes the partitioning $TE = T_H E + T_V E$, where $T_H E$ is the horizontal subspace of TE. To do this, we introduce the "vertical" variation of the vector $|d_V\psi\rangle$ according to

$$d_V|\psi\rangle=\omega|w\rangle; \tag{2.75}$$

Here the 1-form ω, as usual, is fixed by the condition of orthogonality of the horizontal subspace $T_H E$ to the fiber (see Fig. 1). That is, we have

$$\langle\psi d_H\psi\rangle = 0. \tag{2.76}$$

Upon introducing the local coordinates of (2.74) into the condition (2.76), we obtain the two equations:[3)]

$$\begin{gathered}\xi(\langle w|\,d\bar\xi+\bar\xi\langle dw|-\omega\langle w|)|w\rangle=0,\\ \bar\xi(d\xi\langle w|w\rangle+\xi\langle w|dw\rangle-\omega\langle w|w\rangle)=0,\end{gathered} \tag{2.77}$$

which are coordinated and which determine the form ω as follows:

$$\omega = d\xi+\xi\frac{\langle w\,dw\rangle}{\langle ww\rangle}=d\xi+\theta\xi. \tag{2.78}$$

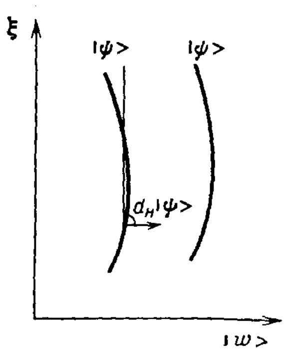

FIG. 1. Separation of the horizontal subspaces in a linear vector fibration.

Hence the parallel transport is determined by the condition $\omega = 0$. Correspondingly the components of the horizontal vectors $|d_H\psi\rangle$ equal:

$$d_H\psi=\xi|dw\rangle-\xi\frac{\langle w\,dw\rangle}{\langle ww\rangle}|w\rangle. \tag{2.79}$$

In the gauge transformation G_η, the form $d_H\psi$ transforms covariantly, i.e., as an extended derivative

$$G_\eta|d_H\psi\rangle=\eta|d_H\psi\rangle. \tag{2.80}$$

In other words, the horizontal subspace is determined by the vanishing of the form ω or by the fact that the vertical variations are expressed in terms of the horizontal ones:

$$\frac{d\xi}{\xi}=-\frac{\langle w\,dw\rangle}{\langle ww\rangle}=-\theta. \tag{2.81}$$

Thus we obtain from the geometrical construction the rule of parallel transport (2.76) or (2.81).

The linear complex vector fiber space that we have introduced admits a Hermitian metric k in the fiber, which is given in the form

$$k=|\xi|^2h(w,\bar w). \tag{2.82}$$

Here $h\neq 0$ is a real function of the complex arguments W and $\bar w$.

The introduced metric k must be coordinated with the partitioning of the tangential space into $T_V E$ and $T_H E$, which reduces to the condition of covariant constancy

$$dk=0. \tag{2.83}$$

Here d amounts to outer differentiation: $d=\partial+\bar\partial$. The condition (2.83) in the local coordinates of (2.74) acquires the form

$$(\bar\xi\,d\xi+\xi\,d\bar\xi)h+|\xi|^2dh=0. \tag{2.84}$$

Here the substitution of the vertical differentials from (2.81) yields an equation that expresses the condition of covariant constancy of the function

$$dh-(\theta+\bar\theta)h=0. \tag{2.85}$$

Here, under the assumption that the complex form θ is a (1,0)-form, we obtain the following expressions for the θ-form:

$$\theta=\partial\ln\bar h,\quad \bar\theta=\bar\partial\ln h. \tag{2.86}$$

The form of the connectivity, which corresponds to the real vector potential, is usually chosen in the form of the difference between the forms θ and $\bar\theta$:

$$A=\frac{i}{2}(\theta-\bar\theta)=-\operatorname{Im}\theta=i\theta-\frac{i}{2}d\ln h. \tag{2.87}$$

It corresponds to the 2-form of the curvature F obtained by the standard rule:

$$F=dA. \tag{2.88}$$

If we express F in terms of the metric h, then we have

$$F=\frac{i\bar\partial\wedge\partial h}{h}=\frac{i}{h}\left(\bar\partial\wedge\partial h-\frac{\bar\partial h\wedge\partial h}{h}\right). \tag{2.89}$$

The equation for the geodesics in the metric k by definition[24] has the form

$$|d_Hd_H\psi\rangle=0. \tag{2.90}$$

2.10. A model example of the complex projective space $\mathscr{C}\mathscr{P}_n$ –canonical approach

Let us turn again to the example of a linear fibration over $\mathscr{C}\mathscr{P}_n$ from the previous section. We shall show how, independently of the dynamics defined by the Hamiltonian H, one can derive the Aharonov-Anandan phase. To do this we shall study the metric $k = \langle\psi\psi\rangle$ and $TE \sim \mathscr{C}_{n+1}$. It induces the metric in the horizontal subspaces $\langle d_H\psi d_H\psi\rangle$. According to Eq. (2.68) it is defined in local coordinates (the map U_0, for example) in the form

$$k = |\xi|^2(1+|\mathbf{w}|^2).$$

Hence we have $h = 1+|\mathbf{w}|^2$ and we obtain from Eq. (2.86) the following expression for A:[4)]

$$A = \frac{i}{2}\frac{\bar{\mathbf{w}}\,d\mathbf{w} - \mathbf{w}\,d\bar{\mathbf{w}}}{1+|\mathbf{w}|^2}, \tag{2.72'}$$

which coincides with that found earlier (2.72). Direct calculation of the 2-form of the curvature dA yields the quantity

$$F = \frac{i}{1+|\mathbf{w}|^2}\left(\sum_i d\bar{w}_i \wedge dw_i - \frac{\sum_i w_i d\bar{w}_i \wedge \sum_j \bar{w}_j dw_j}{1+|\mathbf{w}|^2}\right). \tag{2.73'}$$

If we isolate the coefficients of the 2-form F in the "basis" $d\bar{\mathbf{w}}\wedge d\mathbf{w}$ then the coincidence of (2.73) with (2.73') becomes obvious. We note that the form (2.73') determines the so-called Fubini-Study metric (see, e.g., Ref. 24). This metric in the local coordinates from U_0 determines the element of length of the form

$$ds^2 = \left[\frac{1}{1+|\mathbf{w}|^2}\,1 - \frac{\bar{\mathbf{w}}\otimes\mathbf{w}}{(1+|\mathbf{w}|^2)^2}\right]\frac{d\mathbf{w}}{dl}\otimes\frac{d\mathbf{w}}{dl}\,dl^2. \tag{2.91}$$

The form that we have found of the connectivity A (or of the curvature F) allows us to conclude that, in the discussed model of a linear vector fiber space, the canonical connectivity completely determines the Aharonov-Anandan phase.[26]

2.11. Parametrization of rays in terms of the density matrix. Example: spin 1/2

In a number of cases it is convenient to make use of an explicit parametrization of the ray space by using the elements of the density matrix of pure states

$$\rho = |\psi\rangle\langle\psi|, \quad \text{or} \quad \tilde{\rho} = \frac{\rho}{\operatorname{tr}\rho}. \tag{2.92}$$

Evidently ρ does not vary upon gauge transformation, while $\tilde{\rho}$ is invariant also upon multiplying ψ by a function of the time. In the parametrization (2.92) the metric in the space of density matrices differs from the induced metric $\langle d_H\psi d_H\psi\rangle$ by an inessential coefficient and is defined, according to Ref. 27, by the formula

$$(\operatorname{tr}\rho)^{-1}\operatorname{tr}(d\rho\, d\rho) - \frac{1}{2}\operatorname{tr}^2 d\rho = k. \tag{2.93}$$

We can convince ourselves by direct substitution of (2.92) into (2.93) that one obtains as a result an expression proportional to (2.73').

Let us return to the classical example of evolution of spin 1/2 in a magnetic field in a formulation using the projective fibration $\mathscr{C}\mathscr{P}_1$. As a typical representative of a ray (ψ) we shall choose (for the projection $m = 1/2$):

$$|w\rangle = e^{i\varphi/2}U(R(\theta,\varphi))\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle. \tag{2.94}$$

Here $R(\theta,\varphi)$ defines the rotation by the angle θ with respect to the y axis and by the angle φ with respect to the z axis. We shall parametrize the density matrix ρ as follows:

$$\rho = \frac{1}{2}(1+\mathbf{p}\sigma). \tag{2.95}$$

Here

$$\mathbf{p} = \operatorname{Sp}(\rho\sigma) = \langle w\sigma w\rangle$$

is the polarization vector.

Upon using the parametrization (2.94) we find that for spin 1.2 we have

$$z_1 = \cos\frac{\theta}{2}, \quad z_2 = \sin\frac{\theta}{2}\cdot e^{i\varphi}.$$

Upon using the general formula (2.81) for calculating the form of the connectivity θ, with account taken of the normalization, we have

$$\theta = \bar{z}_1 dz_1 + \bar{z}_2 dz_2 \quad (|z|^2 = 1),$$
$$i\sin^2\frac{\theta}{2}\,d\varphi = \frac{i}{2}(1-\cos\theta)\,d\varphi,$$

while θ is fixed.

If the closed contour C in ray space (or in the polarization space $\mathbf{p}$) realizes a cyclic evolution, the integral of the 1-form of the connectivity θ leads to the following Aharonov-Anandan phase:

$$\gamma = -\frac{1}{2}\oint_C (1-\cos\theta)\,d\varphi. \tag{2.96}$$

Then, using again (2.81), we obtain a formula that expresses the geometrical phase γ of Aharonov and Anandan in terms of the solid angle described by the end of the polarization vector on the sphere $\mathscr{S}^n$ subtended by the contour C:

$$i\gamma = \log\frac{\xi(\tau_c)}{\xi(0)} = -\frac{i}{2}\Omega(C). \tag{2.97}$$

We recall the condition of cyclicity (2.63). It implies that $\xi(\tau_c) = e^{i\phi}\xi(0)$ and $|\xi(\tau_c)| = |\xi(0)|$. Hence the total phase equals the geometrical phase

$$\log\frac{\xi(\tau_c)}{\xi(0)} = i\phi = i\gamma. \tag{2.98}$$

2.12. The geometrical phase for nonlinear evolution equations

In certain cases the methods involved in determining the Aharonov/Anandan phase are applicable to nonlinear evolution equations. One can introduce the class of such equations, as in Ref. 28, by the method of Lagrange functions. In order not to complicate the understanding of the mechanism of origin of the geometrical phase, let us restrict the treatment here to nonlinearities of rational form.

1) Let the equation

$$id_\tau\psi = \mathscr{H}(\psi, \bar{\psi}, \nabla\psi, \nabla\bar{\psi}) \tag{2.99}$$

be invariant with respect to a global gauge:

$$\psi \to e^{-i\alpha}\psi. \tag{2.100}$$

2) Let the solution of Eq. (2.99) satisfy the condition of cyclicity over the time of evolution

$$\psi(\tau_c) = e^{i\phi}\psi(0).$$

Just as in (2.63), we shall reduce by the gauge transformation the cyclicity for ψ to periodicity for the function $\bar{\psi} = e^{-if}\psi{:}\bar{\psi}(\tau_c) = \bar{\psi}(0)$. We can derive from Eq. (2.99) the invariant $q = \int \bar{\psi}\psi \mathrm{d}v$. In fact, from the equation

$$(\mathrm{i}\mathrm{d}_\tau\bar{\psi},\ \psi) = (\psi,\ \mathscr{H}) - (\mathscr{H},\ \psi), \tag{2.101}$$

while taking account of the rationality of the Hamiltonian $\mathscr{H}$ and its global gauge invariance:

$$\mathscr{H}(e^{i\alpha}\psi,\ e^{-i\alpha}\bar{\psi},\ e^{i\alpha}\nabla\psi,\ e^{-i\alpha}\nabla\bar{\psi}) = e^{i\alpha}\mathscr{H}(\psi,\ \bar{\psi},\ \nabla\psi,\ \nabla\bar{\psi}),$$

we can easily convince ourselves that the right-hand side of Eq. (2.101) vanishes, as was to be proved. Now, upon choosing the local gauge

$$\tilde{\psi}(\tau) = e^{-if(\tau)}\psi(\tau),$$

we arrive at the equation

$$\begin{aligned}\gamma = i\frac{\langle\psi\,\mathrm{d}\psi\rangle}{\langle\psi\psi\rangle} &= \int \dot{f}\,\mathrm{d}\tau + \frac{1}{\langle\psi\psi\rangle}\int(\tilde{\psi},\ \tilde{\mathscr{H}})\,\mathrm{d}\tau \\ &= f(\tau_c) - f(0) + \frac{1}{\langle\psi\psi\rangle}\int(\tilde{\psi},\ \tilde{\mathscr{H}})\,\mathrm{d}\tau \\ &= \phi + \frac{1}{\langle\psi\psi\rangle}\int(\tilde{\psi},\ \tilde{\mathscr{H}})\,\mathrm{d}\tau.\end{aligned}$$

That is, we reproduce the Aharonov-Anandan formula (2.67) for the total phase $\phi{:}\phi = \gamma - \delta$, where the dynamical phase δ, in accordance with Eqs. (2.59) and (2.60), is defined by the integral

$$\delta = \frac{1}{\langle\psi\psi\rangle}\int(\tilde{\psi},\ \tilde{\mathscr{H}})\,\mathrm{d}\tau.$$

2.13. The geometrical phase of Pancharatnam

Up to now we have been treating situations of adiabatic or cyclic evolutions corresponding to return to the initial ray in the time of evolution. However, we can pose the question of comparing the relative phase of two different rays (ψ_1) and (ψ_2), which corresponds to rejecting the assumption of cyclic evolution. Pancharatnam[29] defined the phase difference between the states $|\psi_1\rangle$ and $|\psi_2\rangle$ as follows:

$$e^{i\chi} = \frac{\langle\psi_1\psi_2\rangle}{|\langle\psi_1\psi_2\rangle|}. \tag{2.102}$$

Now let us denote $\pi(|\psi_1\rangle)$ and $\pi(|\psi_2\rangle)$ respectively as (ψ_1) and (ψ_2), where π is the projection on the basis of the projective vector fiber space. Following the authors of Refs. 27 and 30, let us determine the phase difference χ_p obtained by parallel transport of the state vector $|\psi_1\rangle$ along the geodesic that connects (ψ_1) with (ψ_2) in the basis of the fiber space. We note the proof that the phase χ_p so defined coincides with the phase difference of the state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$. In other words, if the contour of evolution is not closed in state space and in projective space, then we are studying a "contour" consisting of broken lines, including the contour of the dynamic evolution l_d, the contour of the shortest geodesic l_g in the basis between (ψ_1) and (ψ_2), and the vertical lift $\xi(\tau_c)$ (Fig. 2). The proof is conducted in several steps. The first step consists in proving that the parallel transport of the vector field $|\psi(\tau)\rangle$ along the dynamical contour does not contribute to the geometrical phase. The second step consists in calculating the parallel transport on the shortest geodesic. The third step consists in closing the contours with the vertical lift. Finally, the proof is concluded by calculating the contribution from the transport along the contour C while using the explicit form of the equations (2.90) for the geodesic.

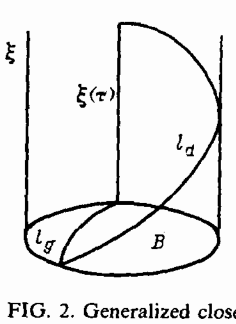

FIG. 2. Generalized closed contour in linear vector fibrations. l_d—dynamical path, l_g—segment of the shortest geodesic, $\xi(\tau)$—lift.

We note that the Pancharatnam phase finds application in describing the procedure of measuring in agreement with the Copenhagen interpretation of quantum mechanics and, more concretely, in measurements performed by the filtration method. If there is a mixture of two or more pure states, then to select one of them, it suffices to absorb the rest of the states (e.g., by using a polarizer having a fixed polarizing plate). In other words, if

$$|\psi\rangle = P_1|\psi\rangle + P_2|\psi\rangle,$$

where P_1 and P_2 are projectors on the states $|\xi_1\rangle$ and $|\xi_2\rangle$, respectively, to arrive at the state $|\xi_1\rangle$, the filtering instrument of the experimentalist must absorb the state $|\xi_2\rangle$. The phase of the amplitude of the obtained state

$$|\psi_1\rangle = |\xi_1\rangle\langle\xi_1|\psi\rangle = |\xi_1\rangle|\langle\xi_1|\psi\rangle|e^{i\chi},$$

evidently is expressed in terms of the Pancharatnam phase χ of (2.102).

2.14. The geometrical phase in the presence of invariants

The existence of invariants in an evolving quantum system enables one to solve the equations of motion if one knows the eigenfunctions of the invariants, and thus to isolate in explicit form the dynamical from the geometrical phase. We recall that the dynamical equation for any invariant operator $I(\tau)$ in the Heisenberg representation has the form[31]

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = \frac{\partial I}{\partial\tau} + i[I,\ H] = 0 \quad (I^+ = I). \tag{2.103}$$

If we introduce the complete orthonormal set of eigenfunctions $\{\varphi_{na}\}$ for the fixed quantum numbers a and eigenvalues λ_n for the operator I, then the solution of the Schrödinger equation $(i\partial_\tau - H)\psi = 0$ with the initial condition $\psi(0) = \psi_n(x)$ differs in phase from $\varphi_{na}(\tau)$:[32]

$$\psi_n(\tau) = e^{i\phi_n}\varphi_{na}(\tau). \tag{2.104}$$

Here the phase ϕ is defined by the expression

$$\phi_n = \int_{\tau_0}^{\tau}\bar{\varphi}_{na}(\tau')(i\partial_\tau - H)\varphi_{na}(\tau')\,\mathrm{d}\tau' \tag{2.105}$$

and is the difference between the geometrical and the dynamical phases.

Thus the Schrödinger equation for the function $\psi_n(\tau)$

is replaced by Eq. (2.103). Let us seek the solution of the latter under the assumption that H and I are constructed of a set of N operators $\{Q_i\}_{i=1}^{N}$ in the form of linear combinations[33]

$$I(\tau)=\sum_i \alpha_i(\tau)\,O_i,\quad H=\sum_i h_iO_i. \tag{2.106}$$

For the parameters $\alpha_i(\tau)$, Ref. 103 yields a system of first-order differential equations

$$\dot{\alpha}_i+i\sum g_{ij}\alpha_j=0. \tag{2.107}$$

Here the g_{ij} are defined functions of the time such that the evolution of the algebra generated by $\{Q_\alpha\}$ is closed

$$[HO_i]=\sum g_{ij}O_j.$$

Let us study again the academic example of a particle of spin 1/2 in the constant magnetic field $\mathscr{H}=\text{const}$. The Hamiltonian is $H=-\mu\mathscr{H}\sigma_3$. Evidently the topological phase is not of adiabatic type, since the external field can be arbitrarily large. According to the general formula (2.106) we shall seek the invariant I in the form

$$I(\tau)=\alpha_1\sigma_1+\alpha_2\sigma_2+\alpha_3\sigma_3.$$

For the functions $\{\alpha_i\}$ we have the following equations of motion:

$$\dot{\alpha}_3=0,\quad \dot{\alpha}_1=2\mu\mathscr{H}\alpha_2,\quad \dot{\alpha}_2=-2\mu\alpha_1\mathscr{H}. \tag{2.107'}$$

From this we find the combinations constant in time

$$\alpha_3,\ \alpha_1^2+\alpha_2^2.$$

Let us transform to the spherical parametrization

$$\alpha_3=\cos\theta,\quad \alpha_1=\sin\theta\cdot\cos\varphi,\quad \alpha_2=\sin\theta\cdot\sin\varphi$$

(where θ is the integral of motion) and select $\varphi(\tau)=2\mu\mathscr{H}\tau;\tau\in(0,\pi/\mu\mathscr{H})$. Then the eigenvectors of the invariant I have the form

$$|\varphi_+\rangle=\begin{pmatrix}\cos\dfrac{\theta}{2}\\ \sin\dfrac{\theta}{2}\cdot e^{i\varphi}\end{pmatrix},\quad |\varphi_-\rangle=\begin{pmatrix}\sin\dfrac{\theta}{2}\\ -\cos\dfrac{\theta}{2}\cdot e^{i\varphi}\end{pmatrix}.$$

Then, upon calculating the geometrical phase

$$\Phi^g=i\int_C\langle\varphi_\pm\partial_\tau\varphi_\pm\rangle,$$

we obtain

$$\Phi^g=-\pi(1\mp\cos\theta).$$

The dynamical phase is $\Phi^d=\mp\pi\cos\theta$.

Evidently the condition $\partial I/\partial\tau\approx 0$ corresponds to transition to the adiabatic case. That is, I becomes an adiabatic invariant. Here the equation of motion (2.103) implies that the eigenfunctions of the operators I and H coincide. Then the geometrical phase

$$i\int_C\langle\varphi_\pm\partial_\tau\varphi_\pm\rangle\,d\tau$$

coincides also in meaning with the adiabatic Berry phase of (2.16). The example that we have studied again illustrates the fact noted in Sec. 2 of the "independence" of the geometrical phase from the dynamics.

We note that an algebraic construction of (2.106) of special form was recently proposed in which the configuration space itself is the homogeneous space G/H. Here the invariants I are constructed as elements of the Cartan subalgebra K, while the eigenfunctions of the invariants are expressed in terms of coherent states (the corresponding references are given in Sec. 5).

3. TOPOLOGICAL PHASES IN POLARIZATION OPTICS

3.1. The Berry phase of helical photons and parallel transport of the linear-polarization vector in a sequence of glancing reflections

One can speak of circularly polarized electromagnetic waves as a set of photons of definite helicity. As is known, the helicity of a photon is the eigenvalue of the projection of the spin of the photon on its momentum, and it can acquire the values ± 1. The corresponding problem for the eigenvalues has the form

$$\frac{\mathbf{sp}}{\mathbf{p}}\mathbf{u}_\pm=\pm\mathbf{u}_\pm,\quad \mathbf{u}_\pm\in L^2, \tag{3.1}$$

Here $\{\mathbf{s}_i\}_{kl}=-i\varepsilon_{ikl}$ is the spin operator of the photon, while ε_{ikl} is the completely antisymmetric tensor $\varepsilon_{123}=1$.[34] Thus, on the one hand, the complete wave function of a free photon

$$\psi_\pm=\mathbf{u}_\pm\exp[\hbar^{-1}(ipx-i\mathscr{E}\tau)],\quad \mathscr{E}=|\mathbf{p}|,$$

with a definite helicity satisfies the following equation, which stems from (3.1):

$$i\frac{\partial\psi_\pm}{\partial\tau}=\pm\operatorname{rot}\psi_\pm. \tag{3.2}$$

On the other hand, if we stay completely within the framework of classical electrodynamics, we can represent Maxwell's equations for electromagnetic waves in a medium with a constant permittivity ε and magnetic susceptibility μ in a form analogous to (3.2):

$$i\frac{\partial\psi_\pm}{\partial\tau}=\pm\frac{1}{(\varepsilon\mu)^{1/2}}\operatorname{rot}\psi_\pm,\quad \operatorname{div}\psi_\pm=0, \tag{3.3}$$

Here we have $\psi_\pm=\varepsilon^{1/2}\mathbf{E}\pm i\mu^{1/2}\mathbf{H}$. The equations (3.3) describe the propagation of circularly polarized electromagnetic waves. If the medium is inhomogeneous, these equations must be supplemented with terms that do not conserve circular polarization and which contain the gradients of ε and μ. Under the condition of smallness of these gradients we can consider the circular polarization (or respectively the helicity of the photons) to be approximately conserved. In this sense the helicity is an adiabatic invariant of the given process.

Moreover, the helicity of photons, just like the helicity of any massless particle, is a relativistic invariant. As is known, the state of a massless particle with a definite helicity is fully determined by the momentum of the particle.[5)] If the momentum of the particle is subjected to a sequence of intrinsic Lorentz transformations and returns to its initial value, then the particle also returns to the initial state, whose wave function is determined apart from a phase factor. It was shown in Ref. 35 that in the case of continuous transformations this geometrical phase coincides with the Berry phase.[36] In contrast to these studies, we shall now study a discrete sequence of Lorentz transformations that transforms $\mathbf{p}_1$ into $\mathbf{p}_2$, $\mathbf{p}_2$ into $\mathbf{p}_3$, ..., and $\mathbf{p}_n$ again into $\mathbf{p}_1$. Each transformation $\mathbf{p}_i\to\mathbf{p}_{i+1}$ amounts to a Lorentz boost

$\mathscr{E}_i \to \mathscr{E}_{i+1}$ and a rotation $\hat{k}_i \to \hat{k}_{i+1}$, where $\hat{k} = \mathbf{p}/\mathscr{E}$, while $\mathscr{E}$ is the energy of the particle. A Lorentz boost does not alter the phase of the wave function. Hence it suffices to study the sequence of rotations $\hat{k}_1 \to \hat{k}_2 \to ... \to \hat{k}_n \to \hat{k}_1$. In the rotation $\hat{k}_i \to \hat{k}_{i+1}$ any vector $\mathbf{a}_i$ orthogonal to $\hat{k}_i$ is transformed according to the formula

$$\mathbf{a}_{i+1} = \mathbf{a}_i - 2(\mathbf{a}_i, \hat{k}_{i+1}) \frac{\hat{k}_i + \hat{k}_{i+1}}{(\hat{k}_i + \hat{k}_{i+1})^2}. \tag{3.4}$$

We can put the discrete Lorentz transformations of (3.4) into correspondence with the transformation of the polarization vector of the electromagnetic wave upon glancing reflection at the boundary of two media

$$\mathbf{e}_2 = \mathbf{e}_1 - 2(\mathbf{e}_1, \hat{k}_2) \frac{\hat{k}_1 + \hat{k}_2}{(\hat{k}_1 + \hat{k}_2)^2}. \tag{3.5}$$

Here $\hat{k}_1$ and $\hat{k}_2$ are unit vectors in the direction of the incident and the reflected waves. The relationship (3.5) is a consequence of the Fresnel formulas,[38] which in the case of total internal reflection at the boundary of two dielectrics or reflection from an ideal metallic mirror can be written in the form

$$\begin{gathered} \mathbf{e}_2 = \mathbf{e}_1 - (\mathbf{e}_1, \hat{k}_2)(\alpha\hat{k}_1 + \beta\hat{k}_2)[1 - (\hat{k}_1, \hat{k}_2)^2]^{-1}, \\ \alpha = e^{i\varkappa} - (\hat{k}_1, \hat{k}_2), \quad \beta = 1 - e^{i\varkappa}(\hat{k}_1, \hat{k}_2). \end{gathered} \tag{3.6}$$

In glancing incidence at which the angle of incidence approaches $\pi/2$, the quantity x approaches zero, and we obtain (3.5) from (3.6).

Thus we have shown that the sequence of Lorentz transformations finds an analogy in optics in the case of a sequence of glancing reflections of rays. Equation (3.5) has a simple geometrical meaning involving parallel transport of the vector $\mathbf{e}$ along the contour formed by the points $\hat{k}_i$ on the unit sphere and the arcs of the great circles that connect these points (Fig. 3).

As an example let us examine three successive reflections $\hat{k}_1 \to \hat{k}_2 \to \hat{k}_3 \to \hat{k}_1$. Then Eq. (3.5) defines a parallel transport of the polarization vector $\mathbf{e}$ along the contour of the spherical trihedron with angles α_1, α_2, and α_3. The angle of rotation Θ of the vector $\mathbf{e}$ here equals Ω, where

$$\Omega = \sum_{i=1}^{3} (\pi - \alpha_i).$$

Apart from a term 2π we have $\Omega = \pi - \alpha_1 - \alpha_2 - \alpha_3 = -S_\Delta$, where S_Δ is the area of the spherical triangle. This implies that the polarization vector has rotated counterclockwise by the angle Θ, which equals the solid angle Ω bounded by the spherical triangle that the end of the vector $\hat{k}$ describes on the unit sphere. Now we shall replace the triangle with a polygon with an arbitrarily large number of sides. Evidently the parallel transport of the vector $\mathbf{e}$ along the sides of this polygon is determined by the formula, which stems from (3.5):

$$\delta\mathbf{e} = -(\mathbf{e}, \delta\hat{k})\,\hat{k}, \tag{3.7}$$

It is accompanied by rotation of the polarization vector by the angle $\Theta = \Omega$. This rotation of the linear-polarization vector, which can be observed experimentally, enables one actually to measure the Berry phase acquired by the helical photon owing to the sequence of glancing reflections of circularly polarized light.

Actually, let the wave vector $\mathbf{k}$ at the instant of time $\tau = 0$ be directed along the z axis, while the linear-polarization vector $\mathbf{e}(0) = \mathbf{e}$ is directed along the x axis:

$$\mathbf{e}(0) = \frac{1}{2}(\mathbf{e}_+ + \mathbf{e}_-),$$

where

$$\mathbf{e}_\pm = \mathbf{e}_1 \pm i\mathbf{e}_2$$

are the circular-polarization vectors. For a cyclic variation of the wave vector $\mathbf{k}$: $\mathbf{k}(\tau_c) = \mathbf{k}(0)$,

$$\mathbf{e}(\tau_c) = \frac{1}{2}(\mathbf{e}_+ e^{-i\Omega} + \mathbf{e}_- e^{i\Omega}) = \mathbf{e}_1 \cos\Omega + \mathbf{e}_2 \sin\Omega.$$

Thus the topological Berry phase that appears for circularly polarized photons corresponds to rotation of the linear-polarization vector by the angle

$$\Theta = \Omega. \tag{3.8}$$

As we have seen, by measuring the rotation of the polarization vector of the light after a sequence of glancing reflections, we can observe the corresponding Berry phase. Thus, in the frequently mentioned paper[39] a sequence of three reflections of light at an angle of incidence of 45° is treated. In this case the rotation of the angle of polarization by 90° is fortuitous, and there is no relation to the Berry phase, since in this case the Fresnel formulas (3.6) do not lead to the rule of parallel transport (3.5). If the polarization vector of the initial ray studied in Ref. 39 is rotated by 45° with respect to that discussed there, then no rotation of the polarization vector owing to three reflections will be observed.

3.2. The law of parallel transport of Rytov and the phase of Vladimirskiĭ in the geometrical optics of inhomogeneous media

The discovery of the law of parallel transport of the vectors $\mathbf{e} = \mathbf{E}/E$ and $\mathbf{h} = \mathbf{H}/H$ of the electric and the magnetic field that characterize the polarization of an electromagnetic wave in a medium having a slowly varying refractive index goes back to the study of S. M. Rytov "On the transition from wave to geometrical optics" published in 1938.[40] S. M. Rytov showed that, for a light ray having the form of a nonplanar curve, a rotation of the vector $\mathbf{e}$ and $\mathbf{h}$ occurs with respect to the natural trihedron $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ formed by the vectors of the tangent $\mathbf{t}$, the normal $\mathbf{n}$, and the binormal $\mathbf{b}$ to the curved ray:

$$\begin{aligned} \mathbf{e} &= \mathbf{n}\cos\varphi + \mathbf{b}\sin\varphi, \\ \mathbf{h} &= -\mathbf{n}\sin\varphi + \mathbf{b}\cos\varphi. \end{aligned}$$

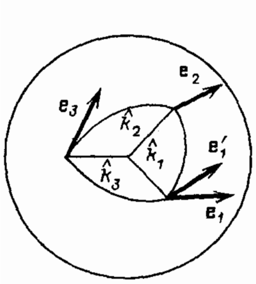

FIG. 3. Parallel transport of the polarization vector $\mathbf{e}$ along the contour that the wave vector $\hat{k}$ describes on the unit sphere upon triple reflection.

Here the derivative of the angle of rotation φ with respect to the arc length s equals the torsion of the curve

$$\frac{d\varphi}{ds}=\frac{1}{T}. \tag{3.9}$$

Here T is the radius of torsion of the ray. This formula is known as "Rytov's law"[41] and can be derived from the condition of transverseness of the light wave:

$$(\mathbf{et})=0, \quad \mathbf{e}^2=1. \tag{3.10}$$

Upon differentiating this relationship we have

$$(\dot{\mathbf{e}}\mathbf{t})+(\mathbf{e}\dot{\mathbf{t}})=0, \quad (\mathbf{e}\dot{\mathbf{e}})=0. \tag{3.11}$$

Let us expand $\dot{\mathbf{e}}$ in the complete set of vectors orthogonal to $\mathbf{e}$,

$$\dot{\mathbf{e}}=\alpha\mathbf{t}+\beta[\mathbf{te}]. \tag{3.12}$$

We shall assume that the medium is not gyrotropic, i.e., that $\beta=0$. Upon using (3.11) we obtain "Rytov's law"

$$\dot{\mathbf{e}}=-(\mathbf{e}\dot{\mathbf{t}})\,\mathbf{t}, \tag{3.13}$$

which coincides with the law of parallel transport (3.7) of the vector $\delta\mathbf{e}$. Upon differentiating with respect to the arc length s and using the Frenet formulas (3.6), we arrive at Eq. (3.9).

As was noted in a subsequent study by V. V. Vladimirskiĭ;[6)] (Ref. 30) "although the instantaneous angular velocity of the trihedron $\mathbf{t}$, $\mathbf{e}$, and $\mathbf{h}$ is always directed perpendicular to the ray, and hence no rotation of the field vectors about the ray occurs, the plane of polarization will not in the general case return to its initial position every time that the tangent ray coincides with its initial direction, since the axis of rotation $\mathbf{b}$ all the time changes its orientation in space if the ray possesses torsion" (Fig. 4). The study of V. V. Vladimirskiĭ "On the rotation of the plane of polarization in a curved light ray" was completed in 1941 and remained practically unknown. However, it was precisely there that a global (topological) effect was predicted on the basis of Rytov's law: "the angle of rotation Θ of the plane of polarization of a light ray whose path in an optically inhomogeneous medium amounts to a nonplanar curve equals the integral of the Gaussian curvature over the region bounded by the contour C that the end of the vector $\mathbf{t}$ describes on the unit sphere (Fig. 5). The angle Θ equals the solid angle Ω enclosed within the cone described by the vector $\mathbf{t}$; the sign of Θ is determined by the direction of passage of the solid angle

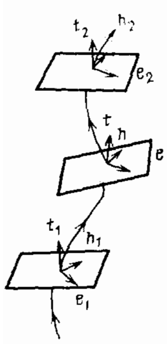

FIG. 4. The trihedron $\{t,e,h\}$ moving along a bent light ray; $\mathbf{t}$-tangent vector; $\mathbf{e}$ and $\mathbf{h}$—unit vectors of the electric and magnetic fields.

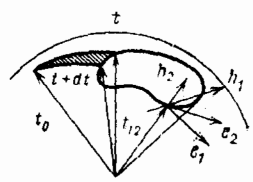

FIG. 5. The spherical segment cut out by the unit tangent vector t when its end describes the contour C on the sphere $\mathscr{S}_2$.

$$\Theta=\Omega. \tag{3.8$'$}$$

Thus, when the tangent to the ray returns to its original direction, the plane of polarization rotates by an angle equal to the solid angle described by the tangent. If the ray amounts to a plane curve, then $\Theta=0$, and upon return of the ray to its original direction, the field vectors acquire their former position. Then angle between the electric field and the tangent plane in this case will be conserved along the entire ray, since the curve here coincides with a geodesic line–an arc of a great circle lying in the plane of the ray..."

"If the tangents to the ray at the initial and final points have different directions, then the vectors $\mathbf{e}$ and $\mathbf{h}$ for these points lie in different planes, and the polarization state cannot be compared directly. It is expedient to agree to bring the ray to its former direction for comparison of its polarization by using a plane curve or by reflection from an ideal mirror that does not alter the polarization. Here the curve is closed by an arc of a great circle, and the angle of rotation of the plane of polarization can again be obtained by the formula $\Theta=\Omega$."

Thus, the cited study of Vladimirskiĭ[30] contains not only a correct expression for the angle of rotation of the plane of polarization of a bent ray, but also a rule for closure of contours corresponding to noncyclic evolutions. Rotation of the plane of polarization is equivalent to a differing phase increment for the two circular components of the field rotating against one another. Therefore it is valid to call the additional phase acquired by circularly polarized light in propagating along a nonplanar curve the *Rytov-Vladimirskiĭ phase.*

To demonstrate the analogy between the Rytov-Vladimirskiĭ phase in the optics of curved rays and the Berry phase in quantum mechanics, it is convenient to reproduce the discussions of S. M Rytov starting with Eqs. (3.3), which describe circularly polarized waves. We shall seek a solution of these equations in the form

$$\boldsymbol{\psi}_\pm=\mathbf{e}_\pm\exp(-i\omega\tau+iR). \tag{3.14}$$

Substitution of (3.14) into (3.3) yields

$$(\varepsilon\mu)^{1/2}\omega\mathbf{e}_\pm=\pm i\,[\operatorname{grad}R,\ \mathbf{e}_\pm]+\operatorname{rot}\mathbf{e}_\pm. \tag{3.15}$$

Let us introduce the notation

$$\mathbf{t}=\frac{\operatorname{grad}R}{\omega(\varepsilon\mu)^{1/2}} \tag{3.16}$$

and rewrite (3.15) in the form

$$\mathbf{e}_\pm\mp i\,[\mathbf{te}_\pm]=\frac{1}{\omega(\varepsilon\mu)^{1/2}}\operatorname{rot}\mathbf{e}_\pm. \tag{3.17}$$

In the first approximation we can neglect the right-hand side of Eq. (3.17), which yields

$$\mathbf{e}_{\pm} \mp i\,[\mathbf{t}\mathbf{e}_{\pm}] = 0. \tag{3.18}$$

Equation (3.18) implies that

$$\mathbf{t}^2 = 1. \tag{3.19}$$

Hence, according to (3.16), the eikonal equation holds

$$(\mathrm{grad}\, R)^2 - \varepsilon\mu\omega^2 = 0. \tag{3.20}$$

Taking account of the next approximation treated by Rytov in the expansion in $1/\omega$ leads to the relationship $(\hat{\mathbf{e}}_{\pm} \equiv \mathbf{e}_{\pm}/(\mathbf{e}^*_{\pm}\,\mathbf{e}_{\pm})^{1/2})$

$$(\mathbf{t}\nabla)\,\hat{\mathbf{e}}_{\pm} = -\,\mathbf{t}\,(\hat{\mathbf{e}}_{\pm}, (\mathbf{t}, \nabla)\,\mathbf{t}). \tag{3.21}$$

Upon scalar multiplication of both parts of Eq. (3.21) by $\hat{\mathbf{e}}^*_{\pm}$, we obtain

$$(\hat{\mathbf{e}}^*_{\pm}\dot{\hat{\mathbf{e}}}_{\pm}) = 0, \tag{3.22}$$

where the dot denotes differentiation along the direction

$$\dot{\hat{\mathbf{e}}}_{\pm} = (\mathbf{t}\nabla)\,\hat{\mathbf{e}}_{\pm},$$

i.e., along the arc length measured along the ray.

The relationship (3.18) is nothing other than the condition (2.34) for spin 1:

$$(\mathbf{s}\mathbf{t})\,\hat{\mathbf{e}}_{\pm} = \pm\,\hat{\mathbf{e}}_{\pm}. \tag{3.23}$$

Here the spin operator $\mathbf{s}$ is defined by Eq. (3.1). At the same time Eq. (3.22) can be considered as the Born-Fock-Simon condition (2.3). Thus we have arrived at the problem of motion of spin 1 in a magnetic field as presented in Sec. 2.4.

3.3. Experimental observations of the Rytov-Vladimirskiĭ phase

The studies of S. M. Rytov[40] and V. V. Vladimirskiĭ[30] antedated the development of the experimental technique, and as often happens, they proved to be practically forgotten. Interest in the polarization properties of bent light beams was revived with the appearance of fiber optics. A number of recent studies (see, e.g., Refs. 42 and 43) have reported observation of a rotation of the plane of polarization in bent light guides.

Thus the discovery of the Berry phase in quantum mechanics was an impetus toward a repeated "discovery" of the Rytov-Vladimirskiĭ phase in polarization optics. The basis here was the analogy between the adiabatic variation of the wave vector of a photon in a bent light beam and the adiabatic variation of the direction of the spin of a particle in a slowly varying magnetic field. Having noted this analogy, Chiao and Wu[44] proposed an extremely simple experiment to test the relationship $\theta = \Omega(C)$, which was soon realized in the study of Tomita and Chiao.[45]

Thereby the phenomenon of rotation of the plane of polarization in a bent ray was converted from a special problem of the optics of inhomogeneous media into an example of a graphic interpretation of the fundamental geometrical aspects of field theory. A number of publications[46-48] appeared popularizing the experiment of Ref. 45 and its connection with general problems. A discussion arose[49] on the need for a quantum (or photon) treatment of the phenomenon. In a number of studies[50-52] individual variants appeared of the classical theory of the experiments of Tomita and Chiao. In particular, Segert[50] noted that a rotation of

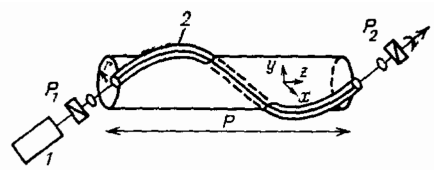

FIG. 6. Diagram of the apparatus of Tomita and Chiao.[45] 1—He-Ne laser, $P_{1,2}$—polarizers; 2—single-mode fiber light guide wound on a cylinder of length p.

the plane of polarization occurs for any transverse wave under the condition that its direction of propagation varies adiabatically, e.g., for the transverse vibrations of a bent metal rod.

The generalization of the Berry phase to the nonadiabatic evolutions of quantum systems provided by Aharonov and Anandan[19] stimulated experiments in which nonplanar trajectories of rays are formed with a series of successive reflections.[53,54]

Quantitative measurements of the corresponding additional phases were performed in a nonplanar Mach-Zehnder interferometer.[53] Here we shall take up a more detailed treatment of the experiment of Ref. 45, which corresponds to conditions of adiabatic evolution.

3.4. Rotation of the plane of polarization of radiation in a helical fiber light guide

The fundamental scheme of the experiment of Tomita and Chiao[45] includes two polarizers between which a bent fiber light guide exists, which plays the role of an optically active medium (Fig. 6). The light guide lies on the surface of a circular cylinder of length p and radius r in the form of helices of different forms. Here the ends of the light guide remain parallel. By using polarizers one measures the angle of rotation of the plane of polarization of the radiation. On the other hand, for each complex helix one can calculate the solid angle described by the tangent vector $\hat{k}$. Let us study the plane developed surface of a turn of the nonuniform helix shown in Fig. 7a. If we denote by ϕ the angle of rotation about the axis of the cylinder, and by $\theta(\phi)$ the angle between the tangent to the helix and the generator of the cylinder, then we can find the solid angle by the formula

$$\Omega(C) = \int_0^{2\pi} (1 - \cos\theta(\phi))\, d\phi. \tag{3.24}$$

Here, as we see from Fig. 7a, we have

$$\theta(\phi) = \mathrm{arctg}\left(r\frac{d\phi}{dz}\right).$$

The function $z(\phi)$, which determines the form of the devel-

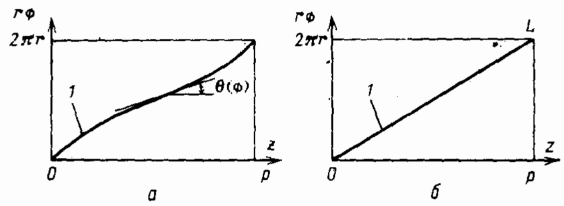

FIG. 7. 1—Development of the cylinder in a plane, which allows one to calculate the solid angle described by the vector $\hat{\mathbf{k}}$, with nonuniform winding (a) and with uniform winding for a light guide of length L (b).

opment of the helix in Fig. 7a, was defined in the form of sums of varying numbers of harmonics, after which the solid angle of (3.24) was easily calculated. In the special case of a uniform helix in Fig. 7b, we have $z(\phi) = (p/2\pi r)\phi$ and evidently $\Omega(C) = 2\pi[1-(p/L)]$, where $L = [p^2 + (2\pi r)^2]^{1/2}$ is the length of the helix. Along with the expected effect, a rotation of the plane of polarization can be caused by torsional stresses and intrinsic optical activity of the fiber. To diminish the torsional stresses, the fiber was placed in a Teflon tube so that its output end could freely rotate when wound on the cylinder. The intrinsic optical activity of the fiber in the experiment of Tomita and Chiao was rather large ($\Theta_0/L = (\omega/2c)(n_+ - n_-) = 0.436$ rad/m, where $n_\pm$ are the refractive indices for the right and left circular components). It was taken into account in processing the results, so that

$$\Omega(C) = \Theta_{н} - \Theta_0, \tag{3.25}$$

where Θ_{obs} is the observed angle of rotation.

Each form of the helical light guide corresponds to a closed curve C described on the unit sphere by the vector $\hat{k}$ subtended by the solid angle $\Omega(C)$ (Fig. 8). A uniform helix corresponds to a horizontal circle. Plane contours of the light guide correspond to motion along the equator; the solid angle $\Omega(C)$ here equals 2π, so that the added phase is trivial.

We note that the fundamental purpose of Tomita and Chiao's experiment was not to verify Eq. (3.25) upon varying the pitch of a uniform helix (this had already been done in Refs. 42 and 43), but to demonstrate the invariance of this relationship with respect to deformations of the helix that do not alter the solid angle. The results of the measurements (Fig. 9) confirmed the prediction of the theory. Actually the dots, squares, and triangles corresponding to helices of different shapes fit well the straight line $\Theta_{\text{obs}} - \Theta_0 = \Omega$.

The experiments of Tomita and Chiao employed a single-mode light guide with a step profile of the refractive index. As is known, geometrical optics is inapplicable for describing the propagation of an electromagnetic wave in such a light guide. Moreover, the Rytov-Vladimirskiĭ theory presented above was constructed in the limit of geometrical optics. Intuitively it seems clear that the nature of the geometrical phase involves only the change in direction of propagation of the transverse wave, so that taking the limit $\lambda \to 0$, which is inherent to geometrical optics, does not alter the essence of the problem. This also follows from a purely

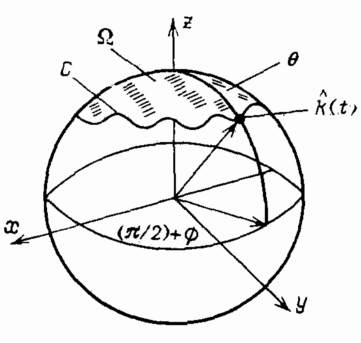

FIG. 8. Motion of the vector $\hat{\mathbf{k}} = \mathbf{k}/k$ on the unit sphere $\mathcal{S}^2(\theta,\Phi)$ in the propagation of radiation along the filament of a light guide of complex form.

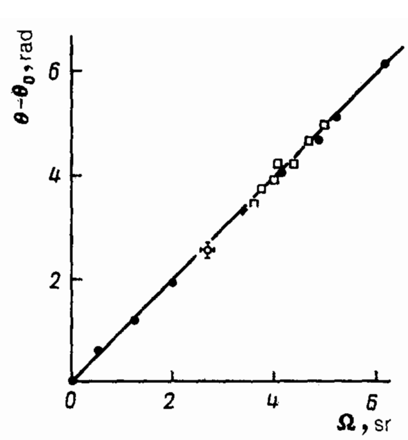

FIG. 9. Results of measuring the angle of rotation of the plane of polarization $\Theta - \Theta_0$ for different solid angles Ω.[45] The symbols correspond to helices of different forms, and the straight line to the predictions of theory.

geometrical treatment (see Ref. 50). In this case the law $\Theta = \Omega(C)$ must be derived also directly from Maxwell's equations, which describe the propagation of the light beam in the light guide. Berry[52] showed this to be actually so.

Upon using the ideas of bound local modes and the approximation of a weakly directing light guide,[55] Berry writes the electric field of the light wave in the form

$$\mathbf{E}(\rho, s) = e^{i\beta s} f(\rho)\,(C_n(s)\,\mathbf{n}(s) + C_b(s)\,\mathbf{b}(s)). \tag{3.26}$$

Here ρ is the radial coordinate (distance from the axis), s is the longitudinal coordinate measured along the axis of the light guide; β is the propagation constant; $f(\mathbf{p})$ is the radial distribution function of the field amplitude; $\mathbf{n}(s)$ and $\mathbf{b}(s)$ are the vectors of the normal and the binormal to the axis of the light guide (Fig. 10).

The smoothness of the bending of the light guide enables one to choose the quantities β and $f(\rho)$ to be the same as for a straight light guide. Upon substituting (3.26) into Maxwell's equations, Berry arrives at the fundamental equation of evolution for the coefficients $c_n(s)$ and $c_b(s)$ analogous to the Schrödinger equation:

$$i\frac{\partial}{\partial s}\begin{pmatrix} c_n(s) \\ c_b(s) \end{pmatrix} = \begin{pmatrix} \frac{k^2(s)}{2\beta} & i\tau(s) \\ -i\tau(s) & 0 \end{pmatrix}\begin{pmatrix} c_n(s) \\ c_b(s) \end{pmatrix}. \tag{3.27}$$

Here $\tau(s)$ and $k(s)$ are the torsion and the curvature of the light guide. In the derivation of (3.27) the radiation loss, the coupling with reflected modes, and elastooptic effects were not taken into account.

Since we have $\beta = 2\pi/\lambda$, while k^{-1} and τ^{-1} are of the order of the bending radius, the term $k^2/2\beta$ is small in com-

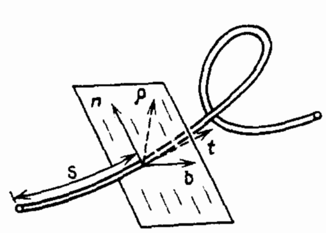

FIG. 10. The accompanying Frenet trihedron for a helically bent optical light guide.

parison with the torsion τ. Neglect of the term $k^2/2\beta$ corresponds to an adiabatic character of the evolution (conservation of the type of circular polarization). If the original light had a linear polarization and the vector E formed the angle φ_0 with the normal $\mathbf{n}$, then we have

$$\frac{c_b(s)}{c_n(s)} = \operatorname{tg}\varphi(s) = \operatorname{tg}\left(\varphi_0 - \int_0^s \tau(s')\,ds'\right). \qquad (3.28)$$

Such a law of parallel transport for **E** leads, after a complete turn of the helix, to rotation of the plane of polarization by the angle Θ, which equals the solid angle

$$\Omega = \int_0^L \frac{ds}{T(s)}. \qquad (3.29)$$

Here $T = \tau^{-1}$ is the radius of torsion of the light guide. As we see, this same result was derived by V. V. Vladimirskiĭ (3.8′) with the aid of Rytov's law (3.9).

Taking account of the term $k^2/2\beta$ in Eq. (3.27) enables us to describe nonadiabatic transformations.[7)] In the first order of smallness it leads to the following expression for the probability $P(+\to-)$ of change of the character of the circular polarization:

$$P(+\to-) = \left|\int_0^L ds k^2(s)\exp\left(2i\int_0^s \tau(s')\,ds'\right)\right|^2 (16\beta^2)^{-1}. \qquad (3.30)$$

For a helix uniformly wound on a cylinder of radius r we have

$$P(+\to-) = \left\{\frac{\Omega}{2\pi[2-(\Omega/2\pi)]}\right\}^3 \sin^2\frac{\Omega}{1-(\Omega/2\pi)}(16\beta^2 r^2)^{-1};$$

even when $r \sim 1$ mm, the magnitude of $P(+\to-)$ does not exceed $\sim 10^{-3}$. We note that for planar bends of the fiber, $\tau = 0$, a rotation of the plane of polarization does not arise.

3.5. Features of the description of geometrical phases upon rejecting the requirement of adiabaticity

When we reject the adiabatic restrictions, we must treat the cyclic evolution of the system, not in parameter space, but in projective Hilbert space (see Secs. 2.7 and 2.11). A convenient model is obtained in the language of the density matrix $\rho_\psi = |\psi\rangle\langle\psi|$ of the pure states ψ.[56] Then cyclicity implies return to the initial values of all the mean values of the physical quantities

$$\langle A\rangle_\psi = \mathrm{Sp}(\rho_\psi A),$$

that characterize the system being treated. Each element ρ_ψ of the "projective space" corresponds to a set of pure states ψ differing in phase—a ray in Hilbert space.

Just as in the adiabatic case, the geometrical phase proves to be equal to the integral of the Gaussian curvature over the surface bounded by the closed contour, but not in parameter space, but in the space of rays. Naturally the practical calculation of this integral includes a certain concrete parametrization of the abstract space $\{\rho_\psi\}$.

3.6. The geometrical phase in multiple reflection of light by a nonplanar system of ideal mirrors

In the examples of the Rytov-Vladimirskiĭ phase discussed up to now (bent light guide, sequence of glancing reflections), the helicity of the photons amounts to an adiabatic invariant. A nonplanar path of a light ray can be realized by a series of reflections, which seems far simpler, both in the sense of practical realization and in the sense of explicit theoretical description. However, for demonstrating the geometrical phase, this case is more complex. As is well known,[38] in reflection, either from a metal or from a dielectric a change generally occurs in the polarization that depends on the complex refractive index of the reflecting medium, the angle of incidence, and the state of polarization of the incident ray. This means that, e.g., a photon that had a definite helicity, transforms after reflection into a state describable by a linear combination of the eigenfunctions of the projection of the spin with coefficients depending on the cited factors. The helicity of such a photon no longer has a definite value.

What we have said implies that, as a result of a series of reflections of general type, complicated changes in the polarization can occur (in particular, rotations of the plane or the ellipse of polarization), the contribution to which yields both a change in the direction of the ray and a polarizing action of the reflectors, which has no relation to the geometrical phase being discussed.

A considerably simpler case–reflection from the surface of ideally conductive mirrors–was studied by a number of authors.[39,52,53,57] In each ideal reflection the helicity of the photon simply changes sign, while remaining fully definite. Berry[52] calls such an evolution "antiadiabatic". In classical language this implies that the ellipse of polarization conserves its shape, while changing only the direction of rotation of the electric vector with respect to the direction of the ray.

The principle result of the experimental and theoretical studies conducted by Chiao and associates[53,57] consists in the following: the fundamental relationship for the Berry-Rytov-Vladimirskiĭ phase

$$\Theta^m = -m\Omega(C)\,(m = \pm 1 \text{ is the helicity}), \qquad (3.31)$$

remains valid in the presence of ideal reflections if the following rules are obeyed: 1) the contour C and the solid angle $\Omega(C)$ are not constructed on the sphere of directions of propagation $\hat{k}$, but on the unit sphere of spin directions $\langle\hat{s}\rangle$ (see Sec. 2.11); 2) the points mapping the discrete sequence of positions of the spin on the $\langle\hat{s}\rangle$ sphere are joined with geodesics, i.e., arcs of great circles. In adiabatic evolution the direction of the spin is rigidly fixed by the direction of $\hat{\mathbf{k}}$. Hence the treatments on the $\hat{k}$ and the $\langle\hat{s}\rangle$ spheres in this case are equivalent. The sphere of spin directions, according to Refs. 53 and 57, is the surface on which one must treat the cyclic evolution according to Aharonov and Anandan[19] for photons that change their direction of propagation.

The experiment[53,57] was performed in a nonplanar Mach-Zehnder interferometer (Fig. 11). The bold dots show the vertical regions of the rays passing along the arms $\alpha = \langle\!\langle 1\rangle\!\rangle$ and $\beta = \langle\!\langle 2\rangle\!\rangle$ in the form of nonplanar broken lines that are symmetric to one another about the center O. In an arm the unit vector of the spin direction acquires the

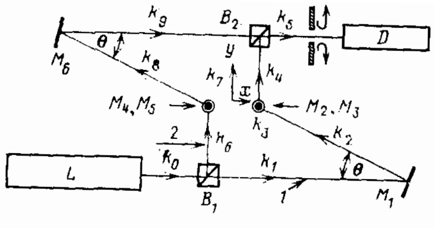

FIG. 11. Diagram of the nonplanar Mach-Zehnder interferometer used in the experiment of Ref. 53 (to view). L—laser, D—receiver for radiation.

following sequence of values:

$$
\begin{aligned}
&\langle \hat{S}_1\rangle = (1,\,0,\,0),\\
&\langle \hat{S}_2\rangle = (\cos\theta,\,-\sin\theta,\,0),\\
&\langle \hat{S}_3\rangle = (0,\,0,\,1),\\
&\langle \hat{S}_4\rangle = (0,\,-1,\,0),\\
&\langle \hat{S}_5\rangle = (1,\,0,\,0).
\end{aligned}
\tag{3.32}
$$

This result can be seen directly from Fig. 11 if we take account of the rotation of the spin in each of the four reflections from the total-reflection mirrors $M_1 - M_3$ and the semitransparent mirror B_2. The corresponding contour on the sphere of spin directions (Fig. 12) has the form of the broken line $ABCDA$. The solid angle subtended by this contour is shown by the hatching in Fig. 12 and equals $\Omega = (\pi/2) - \theta$.

In the β arm the photon passes through the same path on the spin-direction sphere, but in the opposite direction. The phases acquired by photons with right (r) and left (l) polarizations in passing through the arms α and β are respectively equal to

$$
\begin{aligned}
&\Phi_{\alpha r} = \Omega + \delta_\alpha, \qquad \Phi_{\alpha l} = -\Omega + \delta_\alpha,\\
&\Phi_{\beta r} = -\Omega + \delta_\beta, \quad \Phi_{\beta l} = \Omega + \delta_\beta.
\end{aligned}
\tag{3.33}
$$

Here $\delta_{\alpha,\beta}$ are the dynamic phases and we have $\Omega = (\pi/2) - \theta$. The interference pattern created at the output by right-polarized photons will be determined by the phase difference

$$\Delta\Phi_r = \Phi_{\alpha r} - \Phi_{\beta r} = 2\Omega + (\delta_\alpha - \delta_\beta), \tag{3.34}$$

and that by left-polarized photons by the phase difference

$$\Delta\Phi_l = \Phi_{\alpha l} - \Phi_{\beta l} = -2\Omega + (\delta_\alpha - \delta_\beta). \tag{3.35}$$

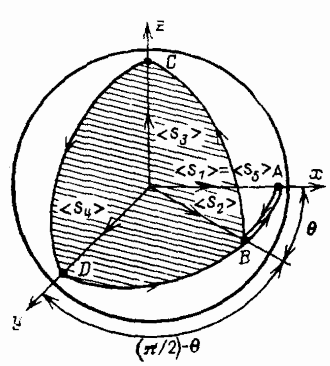

FIG. 12. Unit sphere of directions of the spin vector $\langle \mathbf{s}\rangle$ of a photon.

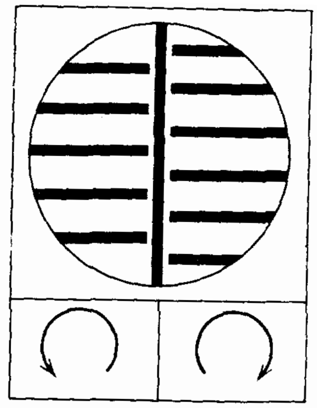

FIG. 13. Interferogram for left and right circular polarization of radiation for $\theta = 45°$.

To eliminate the dynamical phases, it suffices to measure the shift of the interference pattern upon replacing right-polarized by left-polarized radiation, as determined by the phase difference

$$\Delta\Phi = \Delta\Phi_r - \Delta\Phi_l = 4\Omega. \tag{3.36}$$

In the real experiment[53,57] *unpolarized* radiation was applied at the input of the interferometer, while the right and left circular components were isolated at the output by using two filters consisting of a $\lambda/4$ plate and a polaroid. Both interference patterns were observed simultaneously in the two alves of the field of vision, and the shift of the fringes was determined directly (Fig. 13). This method of recording was chosen to diminish the systematic errors involving the nonideality of the system. As was noted above, in real reflections the resulting phase increment depends on the initial state of polarization. The use of natural light implies averaging over the initial polarization states. Consequently the manifestation of nonideality of the reflections is reduced in considerable measure to reduction of the contrast of the interference pattern, while weakly affecting the position of the fringes.

In closing this section we note some important points of the theory of this experiment. If the direction of the incident ray and its polarization are fixed, as well as the material and the arrangement of all the mirrors in space, then evidently the direction and polarization of the output ray are determined unambiguously and can be found directly from the law of reflection and the Fresnel formulas. Such a calculation, which in this situation plays the same role as the explicit calculations of Rytov and Vladimirskiĭ in the situation with bent rays, was performed by the authors.[57] A convenient apparatus for this purpose is offered by the Jones matrices,[58] which transform the two-component vectors of the polarization states

$$\begin{pmatrix} E_p \\ E_s \end{pmatrix}.$$

In ideal reflections the component E_p, which lies in the plane of incidence, is not altered, while the component E_s perpendicular to it changes sign. One also takes account of the rotation of the local s and p axes associated with the ray upon passing from one mirror to another by an angle equal to the angle between the planes of incidence. Multiplication of the corresponding Jones matrices, whose explicit form is taken from Ref. 58, leads to authors of Ref. 57 to an expression for the geometrical phase which agrees exactly with (3.31).

3.7. The geometrical phase in a cyclic variation of the polarization state of rectilinear light beams (the Pancharatnam phase)

Let us study a plane light wave whose polarization state is varied cyclically by a series of optical elements ($\lambda/4$ and $\lambda/2$ plates, polaroids, optically active media, etc.). Here let the direction of propagation of the wave remain constant. Numerous theoretical and experimental studies[54,56,59-63] show that here the wave acquires an additional phase analogous to the Berry phase. This phase equals half the solid angle subtended by the closed contour that depicts the cyclic evolution of the polarization state of the wave on the Poincaré sphere. The topologically invariant character of this phase is manifested in the fact that it, just like the Berry phase, is not altered by deformations of the evolution contour that leave the magnitude of the solid angle unchanged.

References 59 and 64 first noted the close connection between the Berry phase in quantum mechanics and the extra phase found in the 50s by Pancharatnam[29] in studying the interference of polarized light waves. At present the term "Pancharatnam phase" is used widely in the literature to denote the geometrical phase that arises when the polarization state of a wave varies cyclically, while the direction of propagation is invariant.[27]

3.8. Representation of polarization states on the Poincaré sphere and calculation of Pancharatnam phases

Following Aharonov and Anandan,[19] we shall study the cyclic evolution in projective Hilbert space. The pure polarization states of a plane electromagnetic wave are described by the two-component Jones vectors

$$|\mathbf{E}\rangle=\begin{pmatrix}E_1\\E_2\end{pmatrix},\quad \langle\mathbf{E}|=(E_1^*,E_2^*). \tag{3.37}$$

Here E_1^* and E_2^* are the complex components of the electric-field vector along two mutually orthogonal transverse directions $x_{1,2}$. In this case

$$\rho_E=|\mathbf{E}\rangle\langle\mathbf{E}|=\begin{pmatrix}E_1E_1^* & E_1E_2^*\\ E_1^*E_2 & E_2E_2^*\end{pmatrix} \tag{3.38}$$

amounts to the well known polarization density matrix or polarization tensor[34] of the pure state $|E\rangle$. The matrix ρ_E is customarily normalized so that the relationship $Sp\,\rho_E=1$ is fulfilled. This normalization is conserved if the intensity of the wave is constant.

Let us expand ρ_E in the basis of the Pauli matrices

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}0&-i\\i&0\end{pmatrix}. \tag{3.39}$$

Then we obtain the following expression for ρ_E[34,56]

$$\rho_E=\frac{1}{2}(I+(\hat{n}\sigma)). \tag{3.40}$$

Here the vector $\hat{n}$, whose components are called the Stokes parameters[34,38] and are calculated by the formula

$$\hat{n}_i=\mathrm{Sp}\,(\rho_E\sigma_i), \tag{3.41}$$

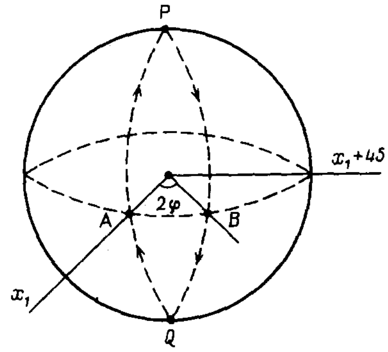

FIG. 14. A view of the polarization states on the Poincaré sphere. The points A and B on the equator depict linear polarizations. The north and south poles P and Q correspond to right and left circular polarization.

must be real, owing to the Hermitian character of ρ_E. For pure normalized states it must also be unitary, as follows from the relationships $\det\rho_E=0$, $\mathrm{Sp}\,\rho_E=1$. Thus each element ρ_E of projective Hilbert space in this case corresponds one-to-one to a point on a sphere of unit radius in ordinary three-dimensional space, which is called the Poincaré sphere.[38,56] Let a and b be the major and minor semiaxes of the polarization ellipse, $0\leqslant\xi\leqslant\pi$ be the angle between the major semiaxis and the x_1 axis, and $-\pi/4\leqslant\eta\leqslant\pi/4$ be the ellipticity, which is determined by the relationship $\mathrm{tg}\,\eta=\pm b/a$. Here the signs "+" and "−" correspond to right and left polarizations. In this notation we have[56]

$$|E\rangle=Ae^{i\delta}\begin{pmatrix}\cos\xi\cdot\cos\eta-i\sin\xi\cdot\sin\eta\\ \sin\xi\cdot\cos\eta+i\cos\xi\cdot\sin\eta\end{pmatrix}. \tag{3.42}$$

Here A and δ are real constants. Then Eq. (3.41) yields the expression for $\hat{n}$

$$\hat{n}=(\cos 2\xi\cdot\cos 2\eta,\ \sin 2\xi\cdot\cos 2\eta,\ \sin 2\eta). \tag{3.43}$$

We see from this that the quantity 2η plays the role of the latitude, and 2ξ the longitude on the Poincaré sphere. The equator corresponds to states of linear polarization, the north and south poles to right and left circular polarization, and the rest of the points to elliptical polarization. The point of intersection of the Poincaré sphere with the x_1 axis corresponds to linear polarization along the x_1 axis, and that with the x_2 axis—at a 45° angle to the x_1 axis. Upon complete passage around the sphere along a parallel, the polarization ellipse rotates by the angle π; diametrically opposite points correspond to mutually orthogonal polarizations (Fig. 14).

Upon rotation of the Poincaré sphere around some direction $\hat{n}_0$ by the angle α, the density matrix $\rho_E(\hat{n})$ naturally is not altered, yet the state vector itself $|E(n_0)\rangle$ acquires an additional phase equal to $\alpha/2$. We can convince ourselves of this most easily by studying a rotation about the vertical axis. In this case we have $\hat{n}_0=(0,0,1)$, $\eta=\pi/4$, and (3.43) leads to the expression

$$|E\rangle=A\exp\left[i(\delta-\xi)\right]\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}. \tag{3.44}$$

A change of the longitude 2ξ by α (rotation by α) changes the phase of the field components by $\alpha/2$.

The description of the polarization states by two-component Jones vectors in (3.37) is fully analogous to the description of the motion of a particle of spin 1/2 in quantum mechanics (see Sec. 2.11). The matrices σ_i of (3.39) coincide apart from the factor 1/2 with the components of the spin operator. Therefore Eq. (3.41) determines the unit vector of the spin direction. Thus, for a particle of spin 1/2 the sphere of directions of the spin vector[8)] plays the role of the Poincaré sphere. Continuing the analogy, we note that an adiabatic variation of the polarization state of the light wave corresponds to an adiabatic rotation of the spin, which can be realized, for example, in a magnetic field that adiabatically changes direction. Then the Berry formula (2.33), in which we must set $m = 1/2$, yields the result

$$\Theta = -\frac{1}{2}\Omega(C), \tag{3.45}$$

which must hold also for the Pancharatnam phase if we take the solid angle on the Poincaré sphere.

For an arbitrary evolution of polarization states (not necessarily adiabatic), the relationship (3.45) can be derived by various methods.[56,59,66] Thus, for example, the authors of Ref. 56 constructed an explicit form of the Hamiltonian and explicitly integrated the corresponding equations of evolution for the operators that transform the state vector $|E\rangle$ for displacement of the vector $\hat{n}$ on the Poincaré sphere along a given trajectory. Apparently this derivation is the most direct realization of the approach of Ref. 19 in the problem being discussed. An elegant, yet purely geometrical derivation of Eq. (3.45) stems from the Jordan theorem[66] on rotations. Let the unit vector $\hat{N}_0$ be subjected to a sequence of rotations $R_{\hat{N}_1}(\theta_1), R_{\hat{N}_2}(\theta_2), \ldots, R_{\hat{N}_k}(\theta_k)$ that transport it to the positions $\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_k = \hat{n}_0$, where $R_{\hat{N}_i}(\theta)$ denotes the matrix of rotation by the angle θ_i, which equals the angle between $\hat{n}_{i-1}$ and $\hat{n}_i$, about the axis N_i perpendicular to the plane in which $\hat{n}_{i-1}$ and $\hat{n}_i$ lie. In such a series of rotations, the end of the vector $\hat{n}$ describes the contour C on the unit sphere, composed of arcs of great circles. Jordan's theorem states that the product of the stated rotations is equivalent to *one* rotation about the original direction $\hat{n}_0$ by an angle equal to the solid angle subtended by the contour C. If we apply the Jordan theorem to the Poincarê sphere and take account of the fact that the corresponding states acquire the phase $-\alpha/2$ upon rotation by the angle α about the direction $\hat{n}$, then Eq. (3.45) has been proved.

3.9. Experiments to measure the Pancharatnam phase in laser interferometers

Two general problems that arise in various experiments to observe and measure the Pancharatnam phase[54,57,60,61,62] consist in the following. First, one must organize the cyclic variation of polarization state and convince oneself that here the dynamical phase is not altered, as determined by the optical path length. Second, one must measure the geometrical phase, having isolated it from the dynamical phase.

To organize the cyclic variation of the polarization, various combinations of half-wave and quarter-wave plates and polaroids have been used. Figure 15 shows simplified diagrams of the interferometers used in Refs. 54, 57, 60 and 61 and the arrangement of the cited elements in them. Figure 14 shows the corresponding trajectories on the Poincaré sphere. In the scheme of Bhandari and Samuel[54] shown in Fig. 15a, the initial state of the linear polarization A (see Fig. 14) is converted into a state of right circular polarization P by the plate $(\lambda/4)_1^{\rm fix}$. The second plate $(\lambda/4)_2^{\rm rot}$, which is rotated by the angle φ with respect to the first one about the direction of the ray, returns the mapping point to the equator at position B. By using the polaroid LP one performs a return to the initial state A.[9)] The solid angle subtended by the contour $APBA$ equals 2φ. The difference of Ref. 60 from Ref. 54 consists in replacing the second (movable) plate $(\lambda/4)_2^{\rm rot}$, with the half-wave plate $(\lambda/2)_2^{\rm rot}$, which converts the state P into the state Q along the path PBQ (see Fig. 14). The resulting contour $ABPQA$ subtends a solid angle of magnitude 4φ.

The authors of Ref. 61 used a Michelson interferometer (see Fig. 15b) in which the test ray was reflected from an ideal mirror M and passed through a system of two quarter-wave plates *twice*. Ideal reflection corresponds to *inversion* of the ordinary Poincaré sphere, in which the x_3 axis coincides with the direction of the wave vector. Here the conditions of construction of the closed contour of evolution and applicability of the relationship (3.45) break down. This difficulty can be easily overcome if the x_3 of the Poincaré sphere is not directed along the wave vector, but along the spin direction of the photon. The authors of Ref. 57 call this sphere the generalized Poincaré sphere. It is precisely what

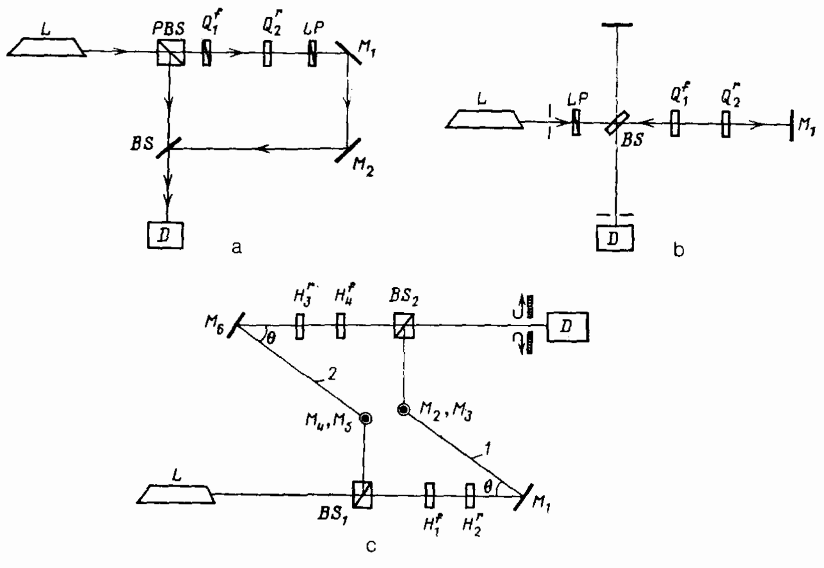

FIG. 15. Simplified diagram of laser interferometers used to measure the Pancharatnam phase in Ref. 54 (a), Ref. 61 (b), and Ref. 57 (c). L—laser, M_i—mirrors, BS_i—beam splitters, PBS—linearly polarizing splitter, LP—linear polarizers, $Q_i^{f,r}$ and $H_i^{f,r}$—movable and immovable $\lambda/4$ and $\lambda/2$ phase plates, respectively, D—radiation receiver.

the authors of Ref. 61 have used when they speak of defining the polarization in a spatially fixed system of axes. On the generalized Poincaré sphere the scheme of Ref. 61 displaces the mapping point along the same contour *ABQA* as in Ref. 60, but in four stages. An advantage of the reflection scheme is the absence of polaroids, which ensures unitarity of the evolution and corresponds to the conditions of the theory being tested.

Jiao and his associates[57] used the same Mach-Zehnder interferometer that has already been discussed in detail above. When $\theta = \pi/2$ (see Fig. 15c), the Rytov-Vladimirskiĭ phase equals zero. Two $\lambda/2$ plates were placed in the arm α, rotated with respect to one another by the angle φ, while in the arm β—similar plates, but in the reverse order. An original photon circularly right polarized in the arm α passes through the path *PBOAP* on the Poincare sphere, but the path *PAQBP* in the arm β, which differs in the direction of passage.

Thus in all cases the solid angle on the Poincaré sphere that figures in (3.45) is proportional to the angle between the axes of the corresponding phase plates, and is easily controlled in practice.

The assortment of evolution contours realizable on the Poincaré sphere in the experiments being discussed is rather meager. Strictly speaking, it does not allow us to convince ourselves of the topological invariance of the Pancharatnam phase in practice. Recent studies[56,67] have developed methods of synthesis of simple optical devices that consist of a set of half-wave and quarter-wave plates, and which enable transformation of any given initial polarization state into any previously defined final state. By using these polarization converters one can organize on the Poincaré sphere "journeys" along a given route of complex form. By altering this form so that the solid angle subtended by the contour remains constant, it will be possible to prove experimentally the topological invariance of the Pancharatnam phase.

While Refs. 54, 57, 60–62 hardly differ in the methods of transforming the polarization, the methods of measuring the phase increment are more varied. Bhandari and Samuel[54,60] use a commercially produced interferometric system designed to determine changes in the optical path length over a broad range (from $\lambda/40$ to $10^7\lambda$). In this system a laser is used. The beat frequency of two waves that have passed through the arms of the interferometer is compared with the beat frequency of a reference signal. A change in phase of the test wave caused by rotating a phase plate leads to a shift in the beat frequency, which is detected by the system at each instant of time. The phase itself is determined by integrating the instantaneous value of the frequency shift over time in a computer. This method of measurement enables one, first, to distinguish the geometrical phase against the background of the large constant dynamical phase caused by the considerable difference in course of the rays in the arms of the interferometer. Second, one can continuously follow the phase change, without restricting the measurement to the interval 2π. It was demonstrated[60] that the geometrical phase Θ linearly "tracks" the rotation of the plate by the angle φ from 0 to 4π and back in accordance with the law $\Theta = -2\varphi$.

The method of measuring the phase in Ref. 61 is far simpler. Actually the shift of the pattern of interference fringes was measured upon rotating the corresponding phase plate. The constant component of the phase that does not undergo an increment upon rotating the plate was simply rejected.

The scheme for measuring the phase shift in Ref. 57 has already been discussed in connection with the measurements of the Rytov-Vladimirskiĭ phase. The dynamical phases were compensated here by direct comparison of the interference patterns generated by right- and left-polarized photons. The use of natural light implies averaging over the initial polarization states, which diminishes the systematic error of the measurements.

In all the cited experiments the equation $\Theta = -\Omega/2$, that was predicted by the theory was confirmed to rather high accuracy.

3.10. The Pancharatnam phase in an optically active medium

In an optically active medium[38] a rotation of the ellipse of polarization occurs upon passage of the wave through the medium, which constitutes an example of the cyclic evolution of the polarization state. The mapping point executes a passage around the Poincaré sphere along a parallel; in a rotation of the polarization ellipse by 2π the Poincaré sphere is circled twice, and the corresponding solid angle amounts to $\Omega = 4\pi(1 - \sin 2\eta)$. A feature of this formulation is that Ω is determined unambiguously by the parameter η, which characterizes the ellipticity of the initial state and which is conserved in the process of evolution. Another feature consists in the impossibility of separating the geometrical from the dynamical phase, having made the latter constant, as in the experiments with phase plates.

The physical nature of the rotation of the polarization ellipse here proves to be inessential. Let us demonstrate this with the example that Garrison and Chiao[28] used to illustrate the general theory in their study.[10] This example is the known (see, e.g., Ref. 68) phenomenon of self-induced rotation of the polarization ellipse in a medium having Kerr nonlinearity. The starting equation for the complex amplitude of the electric field in the plane-wave approximation (without diffraction) has the form:[69]

$$i\frac{\partial \mathbf{E}}{\partial z} = -G_0\left((\mathbf{E}\mathbf{E})\,\mathbf{E}^* + \frac{1}{3}(\mathbf{E}^*\mathbf{E})\,\mathbf{E}\right). \tag{3.46}$$

Here $G_0 = 3kn_2/8n_0$ is the nonlinearity constant; $n = n_0 + n_2|E|^2$ is the nonlinear refractive index. If we define the field at the entrance to the medium ($z = 0$) in the form

$$\mathbf{E}(0) = E_0\begin{pmatrix}1\\ i\,\epsilon\end{pmatrix}, \tag{3.47}$$

then at an arbitrary point, according to Ref. 69, the solution of Eq. (3.46) is expressed by the formula

$$\mathbf{E}(z) = e^{i\varkappa z}R(\Gamma z)\,\mathbf{E}(0). \tag{3.48}$$

Here $R(\alpha)$ is the matrix for rotation about the z axis by the angle α.

$$\varkappa = \frac{4}{3}G_0E_0^2(1+\epsilon^2), \qquad \Gamma = -2G_0E_0^2\epsilon. \tag{3.49}$$

Thus, with increasing z the field rotates with constant angular velocity Γ and acquires the uniformly increasing phase shift $\varkappa z$. Complete rotation occurs as z varies by the amount

$\Lambda = 2\pi/|\Gamma|$. In this same interval the total phase shift, according to (3.48), amounts to $\Phi = 2\pi\varkappa/|\Gamma|$.

On the other hand, one can calculate the dynamical phase directly by the general definition[28] (see also Sec. 2.12):

$$\delta = \frac{G_0}{Q}\int_0^{\Lambda}[|(\mathbf{E},\mathbf{E})|^2 + \frac{1}{3}(\mathbf{E}^*,\mathbf{E})^2]\,dz. \tag{3.50}$$

Use of the properties of the solution of (3.48)

$$|(\mathbf{E}(z),\mathbf{E}(z))|^2 = |(\mathbf{E}(0),\mathbf{E}(0))|^2 = E_0^4(1-\epsilon^2)^2,$$
$$(\mathbf{E}^*(z),\mathbf{E}(z))^2 = Q^2 = E_0^4(1+\epsilon^2)^2$$

enables us to calculate the integral of (3.50) explicitly:

$$\delta = \frac{4\pi(1-\epsilon^2+\epsilon^4)}{3|\epsilon|(1+\epsilon^2)}. \tag{3.51}$$

The geometrical phase is determined by the solid angle Ω discussed above, which can easily be expressed in terms of the parameter ϵ. Consequently we obtain the following expression for the geometrical phase:

$$\Theta = \frac{4\pi|\epsilon|}{1+\epsilon^2}. \tag{3.52}$$

Upon adding (3.51) and (3.52), we can easily convince ourselves that the sum of the quantities δ and Θ yields the total phase $\Phi = 2\pi\varkappa/|\Gamma|$ acquired by the solution of (3.48) in one complete revolution.

We can easily see that the result obtained by Garrison and Chiao does not depend on the nature of the rotation of the polarization ellipse, and is completely general for optical activity of any type, since neither δ nor Θ depends on anything but the ellipticity parameter ϵ fixed at the entrance. Only the rate of rotation depends on the nature of the effect (in this case, on the nonlinear interaction $G_0E_0{}^2$), or as is the same, the length Λ over which a complete rotation occurs. The experimental verification of Eqs. (3.51) and (3.52) is rather complex, and primarily, it is not very interesting, since the geometrical phase is not manifested in any specifically observable effects that would not depend on the dynamical phase. The dynamical and the geometrical phases in the case of optical activity amount simply to two contributions to the total phase whose isolation is rather arbitrary.

3.11. Simultaneous observation of the Rytov-Vladimirskiĭ and Pancharatnam phases in a single experiment

If one makes the angle θ differ from $\pi/2$ in a Mach-Zehnder interferometer with a nonplanar ray contour and with phase plates (see Fig. 15c), then the additional phase will be determined both by the change in direction of the spin of the photons and by the transformation of the polarization state. Such an experiment has been performed by Jiao and his associates.[57] The scheme of the interferometer differed from that shown in Fig. 15c only in that in each arm one $\lambda/2$ plate was placed prior to the reflection by the angle θ, and the other behind it. The experiment showed that in this scheme the Pancharatnam and Rytov-Vladimirskiĭ phases amount to additive contributions to the overall geometrical phase. This fact was confirmed by explicit theoretical calculations[57] in which the Jones matrices that describe the successive conversion of the states $|E\rangle$ upon passing through the elements of the optical system were multiplied. The same result follows from analysis of the evolution on the sphere of spin directions and the generalized Poincaré sphere.

More complex experiments can include optical systems in which the light wave changes both direction and polarization state owing to one single physical process. An obvious example of a discrete variant of such a system is a sequence of nonideal reflections. Bhandari[63] proposed for a theoretical analysis of the general case of an evolution of this type to use a projective Hilbert space constructed as follows. In this space of three-component vectors corresponding to spin unity, one isolates a subspace having the elements

$$\psi = \begin{pmatrix} c_1 \\ 0 \\ c_{-1} \end{pmatrix}.$$

Here the $c_{\pm 1}$ are complex numbers, $|c_{+1}|^2 + |c_{-1}|^2 = 1$. Vanishing of the second component takes account of the transverse nature of the wave. States of the field with an arbitrary direction of propagation (polar angles θ and φ) are constructed as

$$R_z(\varphi)R_y(\theta)|\psi\rangle = |\psi(\theta,\varphi)\rangle.$$

Here $R_z(\varphi)$ and $R_y(\theta)$ are matrices for transformation of the spinors in the rotations about the z and y axes by the angles φ and θ. One further constructs from such "rotated" states a projective space of the density matrices $|\psi(\theta,\varphi)\rangle\langle\psi(\theta,\varphi)|$, as was done in Sec. 2.11.

4. CONCLUSION

The geometrical phases as the simplest topological characteristics of state spaces or phase spaces of physical systems are necessary primarily for a consistent formulation of quantum theory itself. In quantum mechanics the eigenvectors of states of the Hamiltonian fix the basis in the intrinsic subspaces apart from a unitary arbitrary factor. Thus, for example for nondegenerate states this arbitrary factor has the form of an indefinite phase factor, while for n-fold degenerate states—a certain unitary $n\times n$ matrix. In many physical problems the arbitrary phase factor is removed in universal fashion by transforming to a ray representation. However, in describing experiments with interfering beams or quantum-mechanical systems in an external varying field, when the information on the phases becomes essential, one naturally uses the language of differential geometry. In this language the ordinary state space is treated as a vector fiber space, while the missing phase factor is fixed by the element of the fiber. The very existence of the ray representation implies that the corresponding fiber space admits a global section. In the general case the phase arbitrary factor is globally not removable; the problem of classifying it is reduced to the problem of enumerating the vector fiber spaces having different bases. For linear fiber spaces this problem was solved by B. Kostant in the context of geometrical quantization.

For open systems the Hamiltonian of the quantum problem depends on the external parameters. In particular, this can be an external field that depends on the time, effective coordinates describing the medium in which the quantum system being discussed is embedded, etc. In these cases the dependence of the instantaneous Hamiltonian on the parameters is determined by the mapping f of the parameter space in the basis of the fiber space. Then the question of global determination of the phase of the wave function is reduced to studying the possibilities of extending the image f to the entire fiber space. This construction finds application in studying the adiabatic embedding of a "fast" quantum system into a slowly varying medium.

Rich geometrical structures arise in adiabatic systems in which a hierarchy of evolutions exists, i.e., separation into "fast" and "slow" subsystems. Such a separation is substantiated by the content of the adiabatic theorem, as rigorously proved by T. Kato under rather general assumptions on the form of the Hamiltonian. The analysis of adiabatic systems is reduced to studying the effective equation for the slow subsystem in which a covariant derivation is induced upon cyclic variation of the instantaneous Hamiltonian. Its form of connectivity determines the topological phase of the total wave function in the adiabatic (diagonal) approximation. In the adiabatic approach, in each intrinsic subspace of the Hamiltonian a nontrivial structure of the fiber space and a parallel transport in it are induced in different ways, depending on the topology of the parameter space. Recently E. Kiritis was able to classify all the topological obstacles to the global determination of the wave function. In essence the classification of these obstacles answers the question of what the topologically nonequivalent adiabatic systems and induced phases can be. This formulation of the problem enables finding the topological phases of the wave function geometrically.

Another mechanism of manifestation of topological phases arises in the algebraic approach in which the set of mutually commuting operators defining the quantum system depends in a coordinated way on the time (i.e., the Hamiltonian evolution and the Poisson brackets do not violate the algebraic closed character of the set). Here the evolution of the eigenvectors, as before, is defined apart from the phase, which depends on the time, and the direct use of the ray description becomes incomplete, since upon a periodic variation of the operators in time the wave function can acquire a topological phase. One can conveniently study it with the examples of quantum-mechanical systems having dynamical symmetry. We can expect that a classification of this type of the phases in the quasiclassical limit involves studying the symplectic structures within the framework of the methods developed by V. I. Arnol'd.

In closing we briefly note certain unsolved problems of the theory and experimentation to study topological phases within the framework of the presented material. The existing classification theorems are actually formulated only in the adiabatic approach. It is necessary to extend the study that has begun of topological phases in the symplectic approach with account taken of the Aharonov-Anandan phase. The problem of cyclic evolution of mixed states in the case of degenerate levels and non-Abelian Aharonov-Anandan and Pancharatnam phases requires further theoretical treatment with account taken of planning experiments to discover them.

5. APPENDIX

Historic excursion and perspectives. The authors have tried to give a pictorial view of the topological phases, while using the simplest examples from quantum mechanics and polarization optics. In this Appendix we would like to expand somewhat the scope of the material and present to the reader the possibility of tracing the history of the ideas that have led to an understanding of geometrical phases and their applications in various physical problems.

The problems involving the appearance and properties of topological phases in quantum mechanics have a long history. Actually it begins with the 1st Solvay Congress in 1911, where the participants included A. Poincaré, M. Planck, A. Sommerfeld, E. Rutherford, and A. Einstein. Under the chairmanship of G. Lorentz the question was discussed of the essence of the quantization of the energy of an oscillator having the frequency ν, i.e., $E = nh\nu$ ($n = 1,2,3,...$). Then Planck's hypothesis was still considered to be simply a formal mathematical trick. Actually, it was known that Planck introduced it in a fit of despair, having lost any hope of any other way to explain the experimentally observed spectrum of black-body emission. Therefore the physicists sought mechanical analogs, while trying to understand the essence of the new phenomena. Thus, at the end of the 1st Solvay Congress Lorentz spoke of the question, which he discussed with Einstein: "...We were speaking of a simple pendulum that could be shortened by squeezing the thread with two fingers and shifting them along it." Lorentz did not understand what will occur with a pendulum. On the one hand, it will change the frequency of oscillation, and on the other hand, the energy will somehow change. It was found that, by changing the length of the thread, one could break down the Planck relationship. Einstein partially resolved Lorentz's bewilderment with the following words: "... When the length of the pendulum is changed infinitely slowly and continuously, the energy of oscillation remains equal to $h\nu$, if initially it was equal to $h\nu$ the energy of the oscillations changes in proportion to ν." Lorentz answered: "This extremely strange result removes the stated difficulty." The strange result formulated by Einstein is intuitively obvious: in an infinitely slow and continuous change in the length of the thread, the integer (the quantum number) cannot change jumpwise, i.e., the ratio of the energy of oscillation of the pendulum to its frequency remains constant. In other words, one can say that *Lorentz and Einstein had encountered a special case of an "adiabatic invariant"*.

In the general case adiabatic invariants were introduced and studied by L. Boltzmann in 1866 in an attempt to derive the second law of thermodynamics from the principles of mechanics. He treated a mechanical system whose state was determined by a set of parameters λ. He assumed that the system for fixed values of the parameters underwent a periodic motion with the frequency ν. If heat is not absorbed nor released upon changing the parameters λ (such processes are called adiabatic), then the magnitude of I which equals the ratio of the mean kinetic energy to the frequency of the oscillations, is conserved and amounts to an adiabatic invariant.

The term "adiabatic invariant" was introduced by P. Ehrenfest. Toward the end of 1912 he proved the adiabatic invariance of the quantity I and thus gave Boltzmann's characteristic a new life. Ehrenfest proposed that Planck's hypothesis on quanta of energy must be generalized as follows: $\overline{T} = (n/2)\nu h$.

At the beginning of 1913 such a suggestion was a very bold step. Thus Ehrenfest introduced this general rule of quantization, even before N. Bohr studied the quantization of circular orbits in the hydrogen atom. Before Ehrenfest one quantized only harmonic oscillations, whereas by his formula one could quantize any uniform periodic motion. In going from the quantum representations of Planck and Einstein to the quantum theory in its present form, a great role was played by the adiabatic hypothesis of Ehrenfest: any state defined from the standpoint of quantum theory transforms upon an adiabatic change in the parameters of the system again into a defined state characterized by the same quantum numbers.

Subsequently the question of the exactness of fulfillment of the adiabatic hypothesis upon a slow variation of the external conditions was studied at the end of the 20s after the appearance of quantum mechanics. In 1928 M. Born and V. A. Fock showed that the adiabatic hypothesis of Ehrenfest is a consequence of the postulates of quantum theory. Finally, in 1949 T. Kato[70] gave a mathematically rigorous proof of the adiabatic theorem. We can consider the question to be exhausted. Yet it turned out that one simple, but important feature of the evolution of quantum-mechanical states in studying the adiabatic hypothesis in quantum theory actually has not been studied up to the present.

In 1983 the English scientist M. V. Berry found that, if the evolution of the Hamiltonian is determined by a set of time-dependent parameters $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N$, then the change in phase of the wave function $\psi(\lambda(\tau))$ in the cyclic evolution of the system, i.e., under the condition $\psi(\lambda(0)) = \psi(\lambda)(\tau_c))$, possesses remarkable geometrical properties, namely, the change in phase does not depend on the duration of the evolution (which is assumed to be long enough that the evolution is adiabatic), but is directly determined by the geometrical (topological) properties of the space of parameters $\{\lambda_i\}_1^N$.

This is how the history of the discovery itself is presented. During a lecture in the USA in 1983, Berry was not able to answer one tricky question on the behavior of the wave function of a quantum system when the symmetry of the function is broken with respect to time reversal.[11] On returning home he thought about this question two weeks and discovered what was called the quantum adiabatic phase. Of course, this is apocryphal.

In 1983 B. Simon, who was familiar with Berry's work even before its publication, showed that the change in the phase of the function $\psi(\lambda(\tau))$ is completely characterized by the Born-Fock condition, which can be interpreted as the condition of parallel transport of quantum-mechanical state vectors in Hilbert space.

In 1984 F. Wilczek and A. Zee turned their attention to the fact that, in a cyclic evolution of degenerate systems, Berry's phase transformation is generalized by a certain unitary transformation of the wave functions belonging to a certain state degenerate in energy. Ehrenfest's adiabatic hypothesis states that in the adiabatic evolution of a quantum-mechanical system the quantum numbers do not change. However, here the system does not necessarily return to its initial state. Thus, for example, if the state is degenerate with respect to the angular momentum, then the projection of the angular momentum during adiabatic evolution does not change, but the axis of quantization can change in direction.

We note in starting that the ideas involving the adiabatic phase of Berry within the framework of quantum mechanics are contained already in Refs. 72 and 73 within the framework of the more general problem of the quasi-crossing of the levels of a fast subsystem. The needed attention was not paid to them in the context of the practical problems of atomic physics. In more contemporary studies,[74] the phases were studied already as in model problems with a finite number of quasilevels, just as in the Coulomb problem of two centers.

For reference purposes let us list a number of the methodologically interesting studies. A postadiabatic generalization was proposed in Ref. 21, while Refs. 20 and 22 treated the general structure of the evolution operator and its factorization into geometrical and dynamical parts. Examples have also been studied of the evolution of states using representations of groups[12,25,35,75,76] to calculate the geometrical phases. In Refs. 77–80 coherent states were used for this same purpose. The role of the adiabatic invariants was discussed in Refs. 81-83. One-dimensional evolution equations with a periodic potential and their geometrical properties involving the Berry phase were studied, e.g., in Refs. 84 and 85. The effective action for adiabatic systems with account taken of the topological phase was carried out in Refs. 86 and 87. The effective quantum action and the quasiclassical approximation were studied in Ref. 88 for many-particle systems. An account taken of the topological phase in the three-particle problem is actually contained in Ref. 89 (see also Refs. 90 and 91). The adiabatic approach to the topological phase in the symplectic formulation was treated in Refs. 92–95. A general topological classification of nontrivial adiabatic phases was performed in Ref. 5. Individual results are contained in Refs. 96–98.

Recently the boundaries of the definition of topological phases were expanded. In 1987 Aharonov and Anandan introduced a phase without resorting to the concept of adiabaticity. It arises in a cyclic evolution in which the initial and final quantum-mechanical rays coincide: $\psi(\tau_c) = e^{i\varphi}\psi(0)$. In particular, this phase is manifested in the Aharonov-Bohm effect, which has become classical, and in a thought experiment with the system of a charge plus a Dirac monopole.

It was shown further that, if one uses as the complete basis the eigenfunctions of the invariants associated with the dynamics of a given quantum system,[19,26,32,33] then one can obtain the topological phases of the wave functions without requiring adiabaticity nor a parametric dependence of the Hamiltonian on the parameters. Study of the invariants served as an impetus for revealing the topological phase in nonlinear evolution equations.[28]

Generalizations have been made to the case of mixed states by using the density matrix. The anholonomy of the density matrix in adiabatic changes of the state vectors has been studied[99–102] in the presence of stochastic noise.

The next generalization, which does not require unitarity of the evolution operator, was performed in Refs. 27 and 56. Here mechanisms are possible that include stochastic forces, and which lead to the appearance of dissipative terms in the Hamiltonian.[103]

It was found that noncyclic evolution in the space of solutions of the quantum problem supplemented by motion along a geodesic with respect to the appropriate metric leads effectively to a cycle and to a phase increment.[27,29]

Now let us trace the history of the appearance and development of the concept of the geometrical phase in optics.

The reason for appearance of topological phases in optics involves the fact that the propagation of polarized light, according to classical electrodynamics, admits a representation in the form of an evolution equation for the complex quantity $\varepsilon^{1/2}E \pm i\mu^{1/2}H$. As early as 1938 S. M. Rytov turned attention to the rotation of the polarization vector upon displacement along the ray in an optically active medium. Then in 1941 V. V. Vladimirskiĭ wrote the angle of this rotation and Rytov's law in contemporary form—in the language of the Gaussian curvature associated with the ray, and defined the rule of parallel transport, which coincided with that introduced by Simon 50 years later. Further, in the 50s Pancharatnam studied the questions of phase change of polarized light upon passing through active elements (media) and introduced the relative phase between two rays. At present all these attainments of optics, combined with the contemporary attainments of fiber optics in interferometry, have served as a basis for planning and performing a number of experiments to measure topological and geometrical phases. The description of the most important among them has found a place in the main text of this review.

Almost simultaneously with the Berry phase, another scientist from Bristol, J. H. Hannay,[104] pointed out the mechanism of appearance of such phases within the framework of classical mechanics. These phases can be manifested in integrable mechanical systems dependent on external parameters. If such a system is adiabatically transported along a closed contour in the space of even two parameters, then in the action-angle representation a topological increment $\Delta\varphi$ to the dynamical angle φ can appear. This increment is now called the Hannay phase (see the reviews).[105,106] Curiously, although the method of averaging, which has a direct relation to the appearance of the Hannay phase, has been known for a long time (see, e.g., Ref. 107 and the review),[108] this phenomenon ($\Delta\varphi \neq 0$) remained unknown.

The Hannay angle arises in the quasiclassical limit from the Berry phase for systems with a finite number of degrees of freedom.[109] Examples are known with an infinite numbers of degrees of freedom, when the WKB approximation does not lead to a nontrivial Hannay angle.[110] We note that taking a coordinated account of the influence of adiabatic external parameters alters the symmetry structure of the phase space of the system,[111] which involves the noncanonical character of the adiabatic imaging (see also Refs. 94 and 95). Moreover, a nonadiabatic variant of the Hannay angle is introduced, which is manifested in the cyclic evolution of tori in the action-angle representation.[112]

Geometrical phases as the simplest topological invariants have found application in quantum field theory in explaining different types of anomalies and in studying the properties of vacuum quantum field theory (see, e.g., Refs. 113-118) in the Schrödinger representation.[119]

The manifestations of topological phases in supersymmetric quantum mechanics[92,93] and supersymmetric quantum scattering theory[120-123] are numerous.

Let us complete the list of the fields of physics where topological phases have been applied with the question with which I. J. R. Aitchison begins his review:[124] "What in common is manifested in the behavior of particles with spin 1/2 in a slowly rotating magnetic field,[7] in experiments on spin resonance in a weakly modulated magnetic field,[13] in nuclear quadrupole resonance spectra of slowly rotating specimens,[125] in the passage of photons through helically bent optical light guides,[44] in the classification of the spectra of diatoms and molecules in terms of half-integral angular momenta[13-15] and rotational bands,[126] in the sequences of lowest states of Jahn-Teller systems,[127] in atoms with an odd number of electrons situated in a slowly rotating electric field,[128] in the quantum Hall effect,[129] in the fractional Hall effect,[130] in the fractional statistics and the quantum Hall effect,[131] in the statistics of vortices for two-dimensional superfields[132] ...? What in common do all these effects have with the anomalies in chiral gauge fields[113-118] and with skyrmions[133]?"

Familiarity with the cited articles convinces the reader that in all these concrete, varied studies the theme was the manifestation of effects of topological phases. As a whole the impression takes shape that future theories that could claim generality in the description of physical phenomena in the great range of energies must have a topological character (see Ref. 134).

[1)] That is, the conditions of the adiabatic theorem are fulfilled.
[2)] We recall that in this context adiabaticity implies the conservation of the quantum numbers of the initial state throughout the evolution.
[3)] We recall that the scalar product $\langle ww'\rangle$ is induced from the trival $\langle\psi\psi'\rangle$ by projection on the horizontal subspace.
[4)] We note that exactly the same expression would be obtained from Eqs. (2.81) and (2.87), which shows the agreement of the definitions.
[5)] Each of these two statements is equivalent to the relativistic wave equations of the corresponding mass-free fields.[37]
[6)] Henceforth in this section the most important parts of V. V. Vladimirskiĭ's study[30] are cited in quotation marks.
[7)] The propagation constant of an intrinsic mode depends on the difference in refractive indices of the core and the cladding of the light guide. Therefore the term $k^2/2\beta$ corresponds to the terms containing the gradients of ε and μ; the dropping of them in the adiabatic limit was mentioned in discussing the equations (3.3).
[8)] Parametrization on the Poincaré sphere proves useful also in other quantum-mechanical problems, e.g., problems of scattering in a system of several particles within the framework of the adiabatic approach; see, e.g., Ref. 65.
[9)] Here the unitarity of evolution breaks down and, strictly speaking, the theory discussed above is inapplicable. While running ahead, we note that experiments with unitary[57,61] and nonunitary[54,60] evolutions yielded the very same results, so that the requirement of unitarity is apparently not essential.
[10)] The general theory developed in Ref. 28 is one of the few attempts to generalize the Berry phase to nonlinear evolution equations.
[11)] A contemporary presentation of this problem is given in Ref. 71 with the example of Fermi systems.

[1] M. Born and V. Fock, Z. Phys. **51**, 165 (1928).
[2] M. V. Berry, Proc. R. Soc. London Ser. A **392**, 45 (1984); Sci. American **259**, No. 6, 26 (1988).
[3] B. Kostant, Lect. Not. Math. **170**, 87 (1970).
[4] B. Simon, Phys. Rev. Lett. **51**, 2167 (1983).
[5] E. Kiritsis, Commun. Math. Phys. **111**, 417 (1988).
[6] B. A. Dubrovin, S. P. Novikov, and L. T. Fomenko, Contemporary Geometry [Russ original, Nauka, M., 1979, 1986] Springer-Verlag, N. Y.; 1984, 1985).
[7] T. Bitter and D. Dubbers, Phys. Rev. Lett. **59**, 251 (1987).
[8] Yu. P. Stepanovsky and L. G. Zazunov, in: Topological Phases in Quantum Physics, eds. B. Markovski and S. I. Vinitsky, World Scientific, Singapore, 1989, p. 61.
[9] F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111 (1984).
[10] V. Fock, Z. Phys. **49**, 323 (1928).
[11] H. Arodz and A. Babiuch, Examples of Non-Abelian Connections Induced in Adiabatic Processes, Preprint of the Jagellonian Univ. TPJU-18/88, 1988.
[12] J. Anandan and L. Stodolsky, Phys. Rev. D **35**, 2597 (1987).
[13] J. Moody, A. Shapere, and F. Wilczek, Phys. Rev. Lett. **56**, 893 (1986).
[14] J. H. Van Vleck, Phys. Rev. **33**, 467 (1929).
[15] A. Mead and D. G. Truhlar, J. Chem. Phys. **70**, 2284 (1979).
[16] S. N. Biswas, A Realization of Berry's Non-Abelian Gauge Field, Preprint of the Inst. Theor. Phys. BUTP-89/14, Bern, 1989.

[17] A. Zee, Non-Abelian Gauge Structure in Nuclear Quadrupole Resonance, Preprint NSF-ITP-88-07, 1988.
[18] A. I. Monastyrskiĭ and A. G. Sergeev, eds., Monopoles: Topological Variational Methods (in Russian), Mir, M., 1989.
[19] Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593 (1987).
[20] B. L. Markovski, V. M. Dubovik, and S. I. Vinitsky, see Ref. 8, p. 93.
[21] S. P. Sun. J. Phys. A **21**, 1595 (1988).
[22] A. Herdegen, Phys. Lett. A **139**, 109 (1989).
[23] D. N. Page, Phys. Rev. A **36**, 3479 (1987).
[24] R. Wells, Differential Analysis in Complex Manifolds, Prentice Hall, 4th Edn., 1973.
[25] C. Bouchiat and G. W. Gibbons, Non-integrable Quantum Phase in the Evolution of a Spin 1-System: A Physical Consequence of the Nontrivial Topology of the Quantum State-space, Preprint LPTENS 87/25, 1987.
[26] Y. Aharonov and J. Anandan, Phys. Rev. D **38**, 1863 (1988).
[27] J. Samuel and R. Bhandari, Phys. Rev. Lett. **60**, 2339 (1988).
[28] J. C. Garrison and R. Y. Chiao, ibid., p. 165.
[29] S., Pancharatnam, Proc. Ind. Acad. Sci. Ser. A **44**, 247 (1956); also: Collected Work of S. Pancharatnam, ed. G. W. Series, Oxford University Press, 1975.
[30] V. V. Vladimirskiĭ, Dokl. Akad. Nauk SSSR **31**, 222 (1941).
[31] P. A. M. Dirac, The Principles of Quantum Mechanics, 3rd ed., Clarendon Press, Oxford (1947) [Russ. transl. Mir, M., 1960, 1985].
[32] H. R. Lewis, Jr., J. Math. Phys. **9**, 1976 (1968).
[33] S. Mizrahi, Phys. Lett. A **183**, 465 (1989).
[34] A. I. Akhiezer and V. B. Berestetskiĭ, Quantum Electrodynamics [Russ. original, Nauka, M., 1964, 4th ed. 1981, p. 432] Interscience, 1965.
[35] I. Bialynicki-Birula and S. Bialynicki-Birula, Phys. Rev. D **35**, 2383 (1987).
[36] T. F. Jordan, J. Math. Phys. **28**, 1759 (1987).
[37] Yu. P. Stepanovskiĭ, Ukr. Fiz. Zh. **9**, 1165 (1964); Teor. Mat. Fiz. **47**, 343 (1981). [Theor. Math. Phys. (USSR) **47**, 501 (1981)].
[38] M. Born and E. Wolf, Principles of Optics, 4th ed. Pergamon Press, Oxford, 1970 [Russ. transl., Nauka, M., 1973].
[39] M. Kitano, T. Yabuzaki, and T. Ogawa, Phys. Rev. Lett. **58**, 523 (1987).
[40] S. M. Rytov, Dokl. Akad. Nauk SSSR **28**, 263 (1938).
[41] Yu. A. Kravtsov and Yu. I. Orlov, *Geometrical Optics of Inhomogeneous Media* (in Russian), Nauka, M., 1980, p. 304.
[42] M. P. Varnham, R. D. Birch, and D. N. Payne, Techn. Dig. of Intern. Conference on Integrated Optics: Optical Fiber Communications European Conference on Optical Communs., Geneva, 1985, p. 135.
[43] M. P. Varnham, R. D. Birch, D. N. Payne, and J. D. Love, Techn. Digest of Conference on Optical Fiber Communication, Atlanta, USA: OSA, 1986, p. 68.
[44] R. Y. Chiao and Y. S. Wu, Phys. Rev. Lett. **57**, 933 (1986).
[45] A. Tomita and R. Y. Chiao, *ibid.*, p. 937.
[46] A. L. Robinson, Science **234**, 424 (1986).
[47] M. McCallum, New Sci., Nov. 20, 1986, p. 22.
[48] J. Maddox, Nature (London) **323**, 199 (1986).
[49] F. D. Haldane, Phys. Rev. Lett. **59**, 1788 (1987).
[50] J. Segert, Phys. Rev. A **36**, 10 (1987).
[51] M. Kugler and S. Shtrikman, Berry's Phase, Locally Inertial Frames and Classical Analogues, Preprint WIS-87/49/Juhne-PH, 1987, p. 33.
[52] M. V. Berry, Nature (London) **326**, 277 (1987).
[53] R. Y. Chiao, A. Antaramian, K. M. Ganga, H. Jiao, S. R. Wilkinson, and H. Nathel, Phys. Rev. Lett. **60**, 1214 (1988).
[54] R. Bhandari and J. Samuel, *ibid.*, p. 1211.
[55] A. W. Snyder and J. D. Love, Optical Waveguide Theory, Chapman and Hall, London, 1983 [Russ. transl. Mir, M., 1987.]
[56] R. Simon, N. Mukunda, and E. C. G. Sudarshan, Pramana--J. Phys. **32**, 769 (1989).
[57] H. Jiao, S. R. Wilkinson, R. Y. Chiao, and H. Nathel, Phys. Rev. A **39**, 3475 (1989); D. N. Klyshko, Phys. Lett. **140**, 19 (1989).
[58] W. W. Chow, J. Gea-Banacloche, L. M. Pedrotti, V. Sanders, W. Scheich, and M. O. Scully, Rev. Mod. Phys. **57**, 61 (1985).
[59] M. V. Berry, J. Mod. Opt. **34**, 1401 (1987).
[60] R. Bhandari, Phys. Lett. A **133**, 1 (1988).
[61] T. H. Chyba, L. J. Wang, L. Mandel, and R. Simon, Opt. Lett. **13**, 562 (1988).
[62] R. Simon, H. J. Kimble, and E. C. G. Sudarshan, Phys. Rev. Lett. **61**, 19 (1988).
[63] R. Bhandari, Phys. Lett. A **135**, 240 (1989).
[64] S. Ramaseshan and R. Nityananda, Curr. Sci. **55**, 1225 (1986).
[65] R. T. Pack and G. A. Parker, J. Chem. Phys. **87**, 3888 (1987).
[66] T. F. Jordan, J. Math. Phys. **29**, 2042 (1988).
[67] R. Bhandari, Phys. Lett. A **138**, 469 (1989).
[68] P. -D. Marker, R. W. Terhene, and C. M. Savage, Phys. Rev. Lett. **12**, 507 (1964).
[69] R. Y. Chiao and J. Godine, Phys. Rev. **185**, 430 (1969).
[70] T. Kato, J. Phys. Soc. Jpn. **5**, 435 (1950).
[71] J. E. Avron, L. Sadun, J. Segert, and B. Simon, Commun. Math. Phys. **124**, 595 (1989).
[72] Yu. N. Demkov, Zh. Eksp. Teor. Fiz. **45**, 195 (1963) [Sov. Phys. JETP **18**, 138 (1964)]; Dokl. Akad. Nauk SSSR **49**, 1157.
[73] Yu. N. Demkov and V. I. Osherov, Zh. Eksp. Teor. Fiz. **53**, 1589 (1967) [Sov. Phys. JETP **26**, 916 (1968)].
[74] E. A. Solov'ev, Teor. Mat. Fiz. **28**, 115 (1976) [Theor. Math. Phys. (USSR) **28**, 669 (1976)] Usp. Fiz. Nauk **157**, 437 (1989) [Sov. Phys. Usp. **32**, 228 (1989)].
[75] D. J. Rowe, P. Rochford, and J. Repka, J. Math. Phys. **29**, 572 (1988); E. A. Tolkachev and A. Ya. Tregubovich, Dokl. Akad. Nauk SSSR **34**, 119 (1990).
[76] G. John, J. Phys. A **21**, 1805 (1988).
[77] G. Giavarini and E. Onofri, *Generalized Coherent States and Berry's Phase*, Preprint UTF 165, Povo, Italy, 1988.
[78] H. Yi. Fan and A. R. Zaidi, Can. J. Phys. **66**, 978 (1988).
[79] G. Giavarini and E. Onofri, *Vector Coherent States and Non-Abelian Gauge Structures in Quantum Mechanics*, Preprint UTNF-198, 1989.
[80] S. Abe, Phys. Rev. D **39**, 2327 (1989).
[81] A. S. Bakai and Yu. P. Stepanovskiĭ, *Adaibatic Invariants* (in Russian), Naukova dumka, Kiev, 1981.
[82] V. V. Dodonov and V. I. Man'ko, see Ref. 8, p. 74.
[83] I. A. Malkin, V. J. Man'ko, and D. A. Trifonov, J. Math. Phys. **14**, 576 (1973).
[84] V. S. Buslaev and L. A. Dmitrieva, see Ref. 8, p. 218.
[85] A. P. Breuer, K. Dietz, and M. Holthaus, *Adiabatic Motion and the Structure of Quasienergy Surfaces of Periodically Driven Quantum Systems*, Preprint BONN-AM 89-01, 1989.
[86] D. Dusedau, Phys. Lett. B, **205**, 312 (1977).
[87] H. Kuratsuji and S. Iida, Phys. Lett. A **111**, 220 (1985); Phys. Lett. B **184**, 212 (1987); Prog. Theor. Phys. **77**, 739 (1985); Phys. Rev. Lett. **56**, 1003 (1986); O. K. Vorov, see Ref. 8, p. 112.
[88] K. Yabana and H. Horiuchi, Prog. Theor. Phys. **75**, 592 (1986).
[89] I. Iwai, J. Math. Phys. **28**, 964, 1315 (1987); **29**, 1325 (1988).
[90] V. M. Dubovik, M. B. Kadomtsev, B. L. Markovski, A. A. Susko, and S. I. Vinitsky, *Proceedings of the International Few-body Workshop*, Kalinin, August 15-21, 1988, Vol. 1, p. 1.
[91] S. I. Vinitsky, A. A. Susko, B. L. Markovski, M. B. Kadomtsev, and V. M. Dubovik, see Ref. 8, p. 173.
[92] E. Gozzi and W. D. Thacker, Phys. Rev. D **35**, 2398 (1987).
[93] H. Kuratsuji and S. Iida, *ibid.* **37**, 441 (1988).
[94] J. P. Ralson, Berry's Phase and the Symplectic Character of Quantum Time Evolution, Preprint, Univ. of Kansas, 1989.
[95] J. Szczesny, M. Biesiada, and M. Szydlowski, Acta Phys. Pol. B **20**, 789 (1989).
[96] T. D. Imbo and E. C. G. Sudarshan, *Inequivalent Quantizations and Fundamentally Perfect Spaces*, Preprint DOE-ER40200-102, 1987.
[97] E. C. G. Sudarshan, T. D. Imbo, and T. R. Govindarajan, *Configuration Space Topology and Quantum Internal Symmetries*, Preprint DOE-ER40200-137, 1988.
[98] B. F. Bednarz, *Geometrical Methods and the Gauge Structure of Berry's Phase*, Preprint DOE-ER40200-127, 1987.
[99] L. Dabrowski and A. Jadczyk, *Quantum Statistical Holonomy*, Preprint ITCP: Report 155/88/FM, 1988.
[100] A. Jadczyk, *7th Summer Workshop on Math. Physics: Group Theoretical Methods and Physical Applications: Proceedings of 7th Summer Workshop on Mathematical Physics*, Claustal, FRG, ed. H. Doebner, Springer-Verlag, Berlin a.o., 1988.
[101] L. Dabrowski and H. Grosse, *On Quantum Holonomy for Mixed States*, Preprint UWThPh-36, 1988.
[102] A. Uhlmann, Rep. Math. Phys. **24**, 229 (1986).
[103] D. Camlied and J. A. Freed, Phys. Rev. A **39**, 3238 (1989).
[104] J. H. Hannay, J. Phys. A **18**, 221 (1986).
[105] R. Jackiw, Comments At. Mol. Phys. **21**, 71 (1988); Intern. Mod. Phys. Ser. A **3**, 285 (1988).
[106] E. Gozzi and W. O. Thacker, Proceedings of the Workshop on Skyrmions and Anomalies, Krakow, Poland, ed. M. Jezabek, World Scientific, Singapore, 1987.
[107] N. N. Bogolyubov and Yu. A. Mitropol'skiĭ, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Gordon and Breach, N. Y., 1964. [Russ. original, Nauka, M., 1963, 2nd ed. 1977].
[108] V. I. Arnol'd, V. V. Kozlov, and A. I. Neĭshtadt, *Mathematical Aspects of Classical Celestial Mechanics* (in Russian), VINITI AN SSSR, M., 1983 (Advances of Science and Technology. Ser. "Contemporary Problems of Mathematics. Fundamental Directions", Vol. 3).
[109] M. V. Berry, J. Phys. A **18**, 15 (1985).
[110] G. Giavarini, E. Gozzi, D. Rohrlich, and W. D. Thacker, Phys. Rev. D **39**, 3007 (1989).
[111] M. V. Berry and J. H. Hannay, J. Phys. A **21**, L325 (1988).

[112] H. Kuratsuji, Phys. Rev. Lett. **61**, 1687 (1988).
[113] E. Witten, Commun. Math. Phys. **100**, 187 (1985).
[114] J. M. Bismut and D. S. Freed, ibid., **107**, 109 (1986).
[115] T. Itoh and K. Odaha, Lett. Math. Phys. **15**, 297 (1988).
[116] M. Stone and F. Gaitan, Ann. Phys. (N.Y.) **178**, 89 (1987).
[117] A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **55**, 327 (1985).
[118] A. Sonoda, Nucl. Phys. B **266**, 410 (1986).
[119] R. Floreanini and R. Jackiw, Phys. Lett. B **175**, 428 (1986); Ann. Phys. (N.Y.) **178**, 227 (1987); Phys. Rev. Lett. **59**, 1873 (1987); Phys. Rev. D **37**, 2206 (1988).
[120] D. Boyanovsky and R. Blankenbecler, Phys Rev. D **30**, 1821 (1984).
[121] A. A. Andrianov, N. V. Borisov, and M. V. Ioffe, Phys. Lett. B **181**, 141 (1986).
[122] N. V. Borisov, W. Muller, and R. Schrader, Commun. Math. Phys. **114**, 475 (1988).
[123] N. V. Borisov, see Ref. 8, p. 130.
[124] I. J. R. Aiktchison, Phys. Scripta **23**, 12 (1988).
[125] R. Tycko, Phys. Rev. Lett. **58**, 2281 (1987).
[126] G. Delacretaz, E. R. Grant, R. L. Whetten, L. Woste, and J. W. Zwanziger, ibid. **56**, 2598 (1986).
[127] F. S. Ham, *ibid.* **58**, 725 (1987).
[128] C. A. Mead, *ibid.* **59**, 161 (1987).
[129] D. Thouless, M. Kohomoto, M. Nightingale, and M. van den Nijs, *ibid.* **49**, 405 (1982).
[130] G. Semenoff and P. Sodano, *ibid.* **58**, 1195 (1986).
[131] D. Arovas, J. R. Schrieffer, and F. Wilczek, *ibid.* **53**, 722 (1984).
[132] F. D. M. Haldane and Y. -S. Wu, *ibid.* **55**, 2887 (1985).
[133] I. J. R. Aitchison, Acta Phys. Pol. B **18**, 207 (1987).
[134] E. Witten, Commun. Math. Phys. **117**, 353 (1988); **118**, 601 (1988).

Translated by M. V. King