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REVIEWS OF TOPICAL PROBLEMS

Supersymmetry in quantum mechanics

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This review is an elementary introduction to supersymmetry. The example of supersymmetric quantum mechanics is used to discuss the basic concepts of supersymmetry and its characteristic features: anticommuting variables, supercharges, the cancellation of divergences, the vanishing of the vacuum energy, the degeneracy of energy spectra, and the spontaneous breaking of supersymmetry. The form taken by supersymmetry in problems in quantum mechanics and nuclear physics is discussed. The use of a supersymmetric formalism in statistical physics and field theory is also discussed.

TABLE OF CONTENTS

1. INTRODUCTION

Our intention in this review is to provide a first introduction to supersymmetry, primarily as a new type of symmetry. We feel that this purpose is served best by choosing for discussion the simplest possible models which can be used to identify and analyze the characteristic properties of supersymmetry, which are retained also in the more complex models.

The most important property of supersymmetry is that it combines *continuous* transformations (e.g., translations) with *discrete* transformations of a special kind (of the reflection type) in an extremely nontrivial way. The formal analogy between these two types of transformations, which are very different in nature, is preserved. This analogy is the heat of supersymmetry.

This analogy was noted a long time ago in quantum field theory, where it is an analogy between boson and fermion operators. Boson operators correspond to continuous transformations, and fermion operators to discrete transformations. The formal analogy is that for boson fields we have *commutation* relations, while for fermion fields we have *anticommutation* relations. In view of this distinction, many of the equations in the boson and fermion field theories are surprisingly similar. This similarity was noted at the very

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birth of quantum mechanics (by, for example, Dirac in his monumental book), but nearly half a century was to pass before Gol'fand and Likhtman,[1] Volkov and Akulov,[2] and Wess and Zumino[3] pointed out that this property makes it possible to *combine* in a single group (the "supergroup") the transformations corresponding to boson and fermion operators. This approach led to the first field theories in which bosons and fermions finally had an equal footing.

Before the appearance of the supersymmetry theories, bosons (e.g., photons) and fermions (e.g., electrons) were regarded as particles fundamentally different in nature: Bosons were regarded as the carriers of "interactions," while fermions were regarded as carriers of "matter." This distinction was particularly reinforced by the advent of the gauge theories, because in these theories the boson fields are gauge fields, directly related to the symmetry group of the theory, while the fermion fields are introduced "manually." Because of this property, the boson fields were *unambiguously* determined by the symmetry of the theory, while the fermion fields could belong to *arbitrary* representations of the symmetry group.

Only in the supersymmetry theories did it first become possible to combine "matter" and "interaction" or, more precisely, to remove the distinction between them. In these theories the bosons and fermions are combined in common (super-) multiplets. This property of the supersymmetry theories attracted considerable interest, of course.

A second extremely important property of the supersymmetry theories turned out to be the dramatic reduction of the divergences which until now are one of the unresolved fundamental problems in quantum field theory. Furthermore, the first field theories which were completely free of divergences in four-dimensional space-time finally appeared.

The reduction of divergences in supersymmetry theories immediately spurred attempts to construct a quantum theory of gravity (supergravity) since all previous attempts had run into an insurmountable obstacle: the nonrenormalizability of the gravitational interaction, if the quanta of this interaction are assumed to be exclusively bosons (spin-2 gravitons). Supersymmetry has accordingly become one of the central ideas in attempts to construct a unified quantum field theory which combines all interactions, including the gravitational interaction.

What can be said today about the realization of supersymmetry in nature?

In the physics of elementary particles, the situation remains indefinite. All that can be said with certainty is that the simplest versions of the supersymmetry models are unsuccessful, but there is still a chance for more complex theories, which are presently being developed actively throughout the world. Special programs have been dedicated to searching for manifestations of supersymmetry in high-energy physics.[1)]

The attractive features of supersymmetry are so obvious, however, that even the absence of experimental confirmation in the physics of elementary particles—the field where supersymmetry was invented—could not prevent the progressively increasing interest in this topic.

The apparent reason is that supersymmetry is a new type of symmetry which expands our understanding of the symmetry of physical systems. The ideas and methods of supersymmetry have found their way into statistical physics[40,41,44–46] and nuclear physics.[47,48] They have led to the development of new mathematical theories.[15,16] Supersymmetry has also found a place in problems in quantum mechanics.[33]

We can thus now confidently assert that supersymmetry *is realized* in nature. To a large extent, its manifestations are the subject of this review.

One reason why the simple and elegant ideas of supersymmetry have not been introduced to a wide number of physicists is that the existing reviews[4–8,12,13] have been aimed primarily at specialists in modern quantum field theory. While unusual itself, supersymmetry in field theories is necessarily complicated by the spinor structure of the generators which is required for the correct coupling of the spin with the statistics. Furthermore, there are specific difficulties which are inherent in quantum field theory because it is a system with an infinite number of degrees of freedom.

Becoming acquainted with supersymmetry is made much easier by the advent of "supersymmetric quantum mechanics,"[17,18] which originally arose as a laboratory for studying supersymmetry but has since proved a good representation of actual problems in quantum mechanics.

In addition to its pedagogical value, a study of supersymmetry in quantum mechanics is worthwhile for two other reasons: First, supersymmetry provides a new vantage point for looking at the "classical" problems of quantum mechanics, which are widely used in a variety of fields; second, this approach is useful also for developing supersymmetry theories themselves, since these theories include a number of concepts which arose from the extensive experience accumulated in research on the problems of quantum mechanics.

This review is organized as follows: Section 2 is a detailed examination of a very simple model: the supersymmetric oscillator, which has proved to be just as useful in a study of supersymmetry as the ordinary oscillator is in a study of quantum mechanics. Section 3 deals with supersymmetric quantum mechanics and emphasizes those of its properties which are the most characteristic and which are retained in supersymmetric field theories. Section 4 is on the Berezin formalism, which today is the basic formalism used to describe fermion degrees of freedom. Section 5 deals with a problem in which supersymmetry is a physical symmetry: the problem of an electron in a magnetic field. Section 6 deals with the relationship between supersymmetry and zero modes; this relationship has served as the basis for the application of supersymmetry methods in topology. Section 7 deals with supersymmetry in nuclear physics. Finally, Section 8 is devoted to the use of supersymmetry for functional changes in variables in the path-integral method. Supersymmetry has thus found its way into the theory of the quantization of gauge fields, statistical physics, and the theory of random processes.

2. A SIMPLE MODEL: THE SUPERSYMMETRIC OSCILLATOR

a) Two properties of supersymmetry

The term "supersymmetry" usually means two properties:

1. There exist transformations which convert bosons into fermions and vice versa. The Hamiltonian is invariant under such transformations.

2. The algebra of this symmetry includes not only the customary commutation relations for generators but also anticommutation relations. In other words, it is not a Lie algebra but a generalization of a Lie algebra: a Lie superalgebra.

These two properties are intimately related to each other. To see this, we start from the simplest case, in which there is only a single boson degree of freedom and only a single fermion degree of freedom, and we find the simplest supersymmetric Hamiltonian. This Hamiltonian will describe a "supersymmetric harmonic oscillator."

b) States and operators

The basis vectors of the states of this system can be written in a natural way as

$$|n_B, n_F\rangle, \quad n_B = 0, 1, 2, \ldots, \infty, \quad n_F = 0, 1, \tag{2.1}$$

where n_B and n_F are the boson and fermion occupation numbers. The creation and annihilation operators, which we denote for convenience by b^+ and b^- for the bosons and f^+ and f^- for the fermions, act on the state vectors $|n_B, n_F\rangle$, changing the occupation numbers n_B and n_F in a standard way. The operators $b^\pm$ and $f^\pm$ satisfy the following commutation and anticommutation relations:

$$[b^-, b^+] = 1, \quad \{f^-, f^+\} \equiv f^-f^+ + f^+f^- = 1, \tag{2.2}$$

$$f^{+2} = f^{-2} = 0, \tag{2.3}$$

$$[b, f] = 0. \tag{2.4}$$

The property of the operators $f^\pm$ expressed by the relation $f^2 = 0$ is called *nilpotency*. It will play a major role in the discussion below.

c) The first appearance of the generators of supersymmetry and of a supersymmetric Hamiltonian

How do we define operators which convert a boson into a fermion and vice versa? In the simplest case, the effect of such operators, $Q_\pm$, may be

$$Q_+ |n_B, n_F\rangle \propto |n_B - 1, n_F + 1\rangle, \tag{2.5}$$

$$Q_- |n_B, n_F\rangle \propto |n_B + 1, n_F - 1\rangle; \tag{2.6}$$

i.e., the operator Q_+ converts a boson into a fermion, while Q_- does the opposite, converting a fermion into a boson. Expressing these operators in terms of creation and annihilation operators, we find

$$Q_+ = qb^-f^+, \quad Q_- = qb^+f^-, \tag{2.7}$$

where the arbitrary constant q is the same, so that Q_+ and Q_- are adjoints of each other.

By virtue of the presence of the fermion operators $f^\pm$, the operators $Q_\pm$ are also nilpotent:

$$Q_+^2 = Q_-^2 = 0. \tag{2.8}$$

This property of these operators is retained in more general models. It is closely related to *anticommutation*: If we define Hermitian operators

$$Q_1 = Q_+ + Q_-, \quad Q_2 = -i(Q_+ - Q_-), \tag{2.9}$$

we find that they anticommute with each other by virtue of (2.8):

$$\{Q_1, Q_2\} = 0, \tag{2.10}$$

Furthermore, their squares are equal:

$$Q_1^2 = Q_2^2 = \{Q_+, Q_-\}. \tag{2.11}$$

These relations suggest a form for a very simple Hamiltonian H which is invariant under transformations which mix bosons and fermions [see (2.5) and (2.6)], i.e., which has the property of *supersymmetry*:

$$H = Q_1^2 = Q_2^2 = \{Q_+, Q_-\}. \tag{2.12}$$

The condition under which the Hamiltonian has supersymmetry,

$$[H, Q] = 0, \tag{2.13}$$

where Q is either of the operators $Q_\pm$ or $Q_{1,2}$, is satisfied by virtue of the nilpotency of the operators $Q_\pm$, as is easily seen.

d) First appearance of a superalgebra

Combining (2.10), (2.12), and (2.13) in the form

$$\{Q_i, Q_k\} = 2\delta_{ik}H, \quad i, k = 1, 2, \tag{2.14}$$

$$[Q_i, H] = 0, \tag{2.15}$$

we find a very simple *Lie superalgebra*, i.e., an algebra which incorporates *both* commutation relations *and* anticommutation relations. This superalgebra characterizes a new type of dynamic symmetry: *supersymmetry*. The dynamic nature of the symmetry is seen in the fact that the Hamiltonian H is one of the generators of the superalgebra.

e) Structure of superalgebras

We will be discussing only the simplest Lie superalgebras of the type in (2.14) and (2.15), but we must point out that this is only an extremely particular form of Lie superalgebras. We can describe the general structure of such algebras. Their basic property is Z_2 *graduation*. In other words, all the generators are classified as either even or odd. The structure of a superalgebra is

$$[\mathrm{E}, \mathrm{E}] \propto \mathrm{E}, \tag{2.16}$$

$$[\mathrm{E}, \mathrm{O}] \propto \mathrm{O}, \tag{2.17}$$

$$\{\mathrm{O}, \mathrm{O}\} \propto \mathrm{E}, \tag{2.18}$$

where E and O represent even and odd generators. The structure constants which are not written explicitly on the right sides of (2.16)–(2.18) must (first) satisfy natural symmetry conditions [for (2.18)] or antisymmetry conditions [for (2.16) and (2.17)] and (second) a relation which is a generalization of the Jacobi identity to the case of superalgebras. The reason for the last requirement is that the corresponding Lie superalgebra must be associative.

In the case of superalgebra (2.14), (2.15), the operators Q_i are odd generators, and the Hamiltonian H is the only even generator.

Let us focus on relation (2.17). It shows that the odd generators belong to some representation of a Lie algebra (an ordinary algebra, not a superalgebra), and the generators of this Lie algebra are even generators. This was one of the principal heuristic ideas in the development of supersymmetry in relativistic field theory: Odd generators—the generators of the supersymmetry—transform there under a spinor representation of the Lorentz group. By virtue of the Pauli principle, the odd nature of the generators of the supersymmetry is related in relativistic field theory to the spinor structure of these generators.

f) Nonnegativity of the spectrum of the Hamiltonian

Let us return to our simple Hamiltonian (2.12) and examine its properties, because some of them will be retained in the more general case, since these properties do not depend on the specific model and are instad determined exclusively by the presence of supersymmetry.

We note at the outset that the spectrum of Hamiltonian H is nonnegative, since H is a square of a Hermitian operator according to (2.12).

g) Twofold degeneracy of the levels of the Hamiltonian ($E\neq 0$)

The levels of Hamiltonian H with energies $E\neq 0$ are twofold degenerate. Since this is one of the most characteristic properties of supersymmetry theories, we will show how it is derived from the superalgebra relations in (2.14), (2.15).

By virtue of (2.15) we can choose a common system of eigenvectors of the Hamiltonian H and of *one* of the operators Q_1, Q_2. We choose Q_1, and we take one state of this system, for which the following hold:

$$Q_1\psi_1 = q\psi_1, \quad H\psi_1 = q^2\psi_1. \tag{2.19}$$

We will show that the operator Q_2 converts ψ_1 back into an eigenvector of the operator Q_1, but with an eigenvalue of the opposite sign. We introduce

$$\psi_2 = Q_2\psi_1. \tag{2.20}$$

We find

$$Q_1\psi_2 = Q_1Q_2\psi_1 = -Q_2Q_1\psi_1 = -qQ_2\psi_1 = -q\psi_2. \tag{2.21}$$

On the other hand, since $[H, Q_2] = 0$, we have

$$H\psi_2 = HQ_2\psi_1 = Q_2H\psi_1 = q^2Q_2\psi_1 = q^2\psi_2; \tag{2.22}$$

i.e., ψ_2 is also an eigenstate of H, and its eigenvalue is *the same* as that of ψ_1. Consequently, if $q\neq 0$, i.e., if $E=q^2>0$, this level of the Hamiltonian is twofold degenerate. States which are degenerate in energy and which are sent into each other by the application of odd generators of a supersymmetry are called *superpartners*.

h) Clifford algebra and the multiplicity of level degeneracy

We will take a look at this twofold degeneracy from a more general standpoint, and we will find the multiplicity of the level degeneracy of the Hamiltonian for superalgebras of the type in (2.14), (2.15), but which contain an arbitrary number (N) of supercharges Q_i. This is the form of the superalgebras in supersymmetric field theories in a frame of reference in which the 3-momentum vanishes, so that the multiplicity of the degeneracy determines the composition of the *supermultiplets*.

The multiplicity of the degeneracy of a level of Hamiltonian H with energy E is evidently equal to the dimensionality of the subspace which is invariant under the application of all the operators Q_i.

If $E=0$, the corresponding subspace is one-dimensional: Supersymmetry does not lead to a degeneracy of this level. We call this the "zeroth"level, since it corresponds to a zero eigenvalue of the Hamiltonian.

We now consider the multiplicity of the degeneracy of nonzero levels, for which we have $E\neq 0$.

In the subspace of state vectors which belong to a given eigenvalue E of the Hamiltonian H, this Hamiltonian is evidently a multiple of the unit operator. We modify the normalization of the operators Q_i, introducing

$$q_i = \frac{Q_i}{\sqrt{E}}. \tag{2.23}$$

This change in normalization converts Hamiltonian H into simply a unit operator on the subspace in question. In place of algebra (2.14), (2.15) we now have

$$q_iq_k + q_kq_i = 2\delta_{ik}. \tag{2.24}$$

These relations determine a *Clifford algebra* with N generators q_i ($i=1,\dots,N$). The problem of finding the multiplicity of the level degeneracy thus reduces to one of determining the dimensionality of the representations of the Clifford algebra. This problem was solved a long time ago, but we will review it at a simple level here in order to keep our discussion complete and also since Clifford algebras are crucial to supersymmetry theories. For this brief review we will use fermion operators, with which physicists are well acquainted.

We introduce n pairs of fermion creation and annihilation operators: f_α^+, f_α^- ($\alpha=1,\dots,n$). These operators satisfy the anticommutation relations

$$\{f_\alpha^+, f_\beta^-\} = \delta_{\alpha\beta}, \quad \{f_\alpha^+, f_\beta^+\} = 0,$$
$$\{f_\alpha^-, f_\beta^-\} = 0 \qquad (\alpha, \beta = 1, \dots, n). \tag{2.25}$$

By analogy with the boson case, we introduce the fermion "coordinates" x_α and "momenta" p_α:

$$x_\alpha = f_\alpha^+ + f_\alpha^-, \quad p_\alpha = \frac{1}{i}(f_\alpha^+ - f_\alpha^-). \tag{2.26}$$

From (2.25) we find

$$\{x_\alpha, x_\beta\} = 2\delta_{\alpha\beta}, \ \{p_\alpha, p_\beta\} = 2\delta_{\alpha\beta}, \ \{x_\alpha, p_\beta\} = 0. \tag{2.27}$$

It is easy to see that we have obtained none other than a Clifford algebra with N generators, and we have $N = 2n$. It is a simple matter to change the notation is such a way that we can switch from relations (2.27) to relations (2.24): $x_\alpha \to q_{2\alpha-1}$, $p_\alpha \to q_{2\alpha}$. Incidentally, in this example we see the important difference between fermion "coordinates" and "momenta" from ordinary boson coordinates and momenta. In the fermion case, the coordinates x_α and momenta p_α in anticommutation relations (2.27) are completely uncoupled: Each operator *is its own* canonical conjugate (in the anticommutation sense). Accordingly, we can completely drop the distinction between "coordinates" and "momenta," designating them in a common way; in transforming to the operators q_i, this is just what we have done.[2)]

It is now a simple matter to answer the question of the dimensionality of a representation of a Clifford algebra (2.24) with an even number of generators, $N = 2n$. The dimensionality of a representation is evidently equal to the number of different states in a system with n fermion degrees of freedom. Each of the n single-particle fermion states may be either vacant or filled, so that the total number of states and thus the dimensionality (ν) of the representation of a Clifford algebra with $N = 2n$ generators are given by

$$\nu = 2^n = 2^{N/2}. \tag{2.28}$$

This is the unique, irreducible representation of a Clifford algebra for even N. For odd $N = 2n + 1$ there are two irreducible representations, each of which has a dimensionality

$$\nu = 2^n = 2^{[N/2]}, \tag{2.29}$$

where [. . .] means the greatest integer. We will not prove these assertions here (see Ref. 65).

Expression (2.29) covers the cases of both even and odd values of N, and it gives the multiplicity of the degeneracy of the levels of the supersymmetric Hamiltonian H with energy $E \neq 0$, i.e., the number of states in one supermultiplet.

I) Charges and supercharges

We wish to express Hamiltonian H in terms of creation and annihilation operators. For the case in which $Q_\pm$ are given in the form in (2.7) we find

$$\begin{aligned} H &= \{Q_+, Q_-\} = q^2 (b^+b^- + f^+f^-) \\ &= q^2\left(b^+b^- + \frac{1}{2}\right) + q^2\left(f^+f^- - \frac{1}{2}\right) \equiv H_B + H_F. \end{aligned} \tag{2.30}$$

The latter equations are written to show that Hamiltonian H is the sum of the Hamiltonians of boson and fermion oscillators H_B and H_F, which do not interact with each other, with respective energies F_B and E_F:

$$H_B = q^2\left(b^+b^- + \frac{1}{2}\right), \quad E_B = q^2\left(n_B + \frac{1}{2}\right), \quad n_B = 0,\ 1,\ \ldots,\ \infty, \tag{2.31}$$

$$H_F = q^2\left(f^+f^- - \frac{1}{2}\right), \quad E_F = q^2\left(n_F - \frac{1}{2}\right), \quad n_F = 0,\ 1. \tag{2.32}$$

The frequencies $\omega = q^2$ of these oscillators are identical—it is this property which gives Hamiltonian H its supersymmetry. It is pertinent to recall here the relationship between an agreement of frequencies and the appearance of an additional symmetry for the ordinary two-dimensional oscillator with the Hamiltonian

$$H_{2B} = \frac{1}{2}(p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2). \tag{2.33}$$

At $\omega_1 = \omega_2$ this Hamiltonian becomes invariant under rotations generated by the angular-momentum operator $L = q_1 p_2 - q_2 p_1$:

$$[H_{2B}, L] = 0. \tag{2.34}$$

If we replace the two real coordinates q_1 and q_2 by the single complex coordinate $q = q_1 + iq_2$, the rotation transformation takes the familiar form of a phase transformation:

$$q \to e^{i\varphi} q. \tag{2.35}$$

In field theory, invariance under this transformation implies the conservation of charge (e.g., electric charge). In this case, the angular momentum L has served as a charge.

The analogs of the "charge" L in the case of *super*symmetry are the *super*charges Q. To see this analogy more clearly, we express L in terms of creation and annihilation operators:

$$L = -i(b_1^+ b_2^- - b_2^+ b_1^-). \tag{2.36}$$

It is not difficult to see that each term in this expression is also an integral of motion:

$$[H_{2B},\ b_1^+ b_2^-] = 0, \quad [H_{2B},\ b_2^+ b_1^-] = 0. \tag{2.37}$$

These integrals of motion,

$$L_+ = b_1^+ b_2^-, \quad L_- = b_2^+ b_1^- \tag{2.38}$$

are explicit analogs of the operators $Q_\pm$ [see (2.7)] and also transform a quantum of the excitation of one degree of freedom into a quantum of the excitation of another degree of freedom, but in the present case this transformation is not boson $\leftrightarrow$ fermion but boson $\leftrightarrow$ boson. This mutual conversion of quanta does not change the energies (i.e., the generators $L_\pm$ commute with the Hamiltonian H_{2B}) in the particular case in which the frequencies ω_1 and ω_2 are equal.

By analogy with the generation by L of rotations in the space of two boson degrees of freedom, q_1 and q_2, the operators Q generate "rotations" in a "boson-fermion" space. In the former case, the parameter of the transformation is the rotation angle φ: an ordinary c-number. What should play the role of transformation parameter in the case of supersymmetry?

J) Unusual properties of the transformation parameters

The change in some operator A caused by a transformation generated by another operator B is known to be

$$\delta A \backsim [\varepsilon B, A], \tag{2.39}$$

where ε is the transformation parameter.[3)]

In the case of a rotation in the space q_1, q_2, since the generator of the transformation is the operator L, we find

$$\delta q_1 \backsim [\varphi L,\, q_1] \backsim \varphi q_2,\quad \delta q_2 \backsim [\varphi L,\, q_2] \backsim \varphi q_1. \quad (2.40)$$

Let us attempt to follow an analogous path in the case of supersymmetry, in which the transformation generators are the operators Q. We initially assume that the transformation parameter ε is also a c-number.

For a variation of boson operators, e.g., b^+, we find

$$\delta b^+ \backsim [\varepsilon Q_+,\, b^+] = [\varepsilon b^- f^+,\, b^+] = \varepsilon f^+ [b^-,\, b^+] = \varepsilon f^+. \quad (2.41)$$

This result is the same as the expected structure, $\delta b \propto \varepsilon f$, similar to (2.40). For a variation of fermion operators, in contrast, we do not find the structure we want, $\delta f \propto \varepsilon b$:

$$\delta f^- \backsim [\varepsilon Q_+,\, f^-] = [\varepsilon b^- f^+,\, f^-] = \varepsilon b^- [f^+,\, f^-] \neq \varepsilon b. \quad (2.42)$$

If the commutator $[f^+, f^-]$ in (2.42) had been replaced by the *anti*commutator $\{f^+, f^-\}$, we would have found the transformation structure that we wanted, $\delta f \propto \varepsilon b$. How can we kill two birds with one stone? Retain the commutator $[b^-, b^+]$ in relation (2.41) but get an anticommutator $\{f^+, f^-\}$ in (2.42)? We can do this by requiring that the transformation parameter ε have some slightly unusual properties: that it commute with boson operators but *anti*commute with fermion operators. In this case we have

$$\begin{aligned}\delta b &\backsim [\varepsilon Q,\, b] = \varepsilon Q b - b \varepsilon Q \\ &= \varepsilon Q b - \varepsilon b Q = \varepsilon\, [Q,\, b] \backsim \varepsilon f, \qquad (2.43)\\ \delta f &\backsim [\varepsilon Q,\, f] = \varepsilon Q f - f \varepsilon Q \\ &= \varepsilon Q f + \varepsilon f Q = \varepsilon\, \{Q,\, f\} \backsim \varepsilon b. \qquad (2.44)\end{aligned}$$

k) Even-odd

We have arrived at yet another important property of supersymmetry theories: Not only the operators but also the transformation parameters can be classified as either *even* or *odd*. The even parameters are the ordinary c- numbers, and they commute with everything (operators and parameters), while the odd parameters, such as ε, commute with all even quantities but anticommute with odd quantities (with both operators and parameters). This property of transformation parameters in supersymmetry theories is certainly one of the most unusual, because—while in the case of operators we have "agreed" to allow arbitrary commutation relations—we have always before regarded transformation parameters as ordinary numbers. In supersymmetry, odd parameters are unusual in that, while formally having the status of numbers, rather than operators, they acquire a new property—noncommutativity— which we have accustomed to associating with operators alone. For this reason, in "supermathematics," a new branch of mathematics[15,16] which has developed in a close relationship with research on supersymmetry in physics, the very concept of commutation is generalized in such a way that for even quantities a commutator remains a commutator, but for odd quantities it converts into an anticommutator. It might be said that it is this generalization of the concept of commutation which embodies the formal analogy between boson and fermion operators which we were discussing in the Introduction. This entire matter is closely related to the Z_2 graduation of Lie superalgebras which we mentioned above [see (2.16)–(2.18)]. For even and odd quantities of all types (operators and parameters) the following natural "multiplication rules" hold:

$$\mathrm{E{\cdot}E} = \mathrm{E},\ \mathrm{E{\cdot}O} = \mathrm{O},\ \mathrm{O{\cdot}O} = \mathrm{E}\,, \quad (2.45)$$

where E and O stand for even and odd quantities. In particular, relations (2.43) and (2.44) show that a variation of an even operator b is also even, and a variation of an odd operator f is odd.

l) Grassmann algebra

Odd parameters anticommute not only with other odd quantities but also with themselves; i.e., their squares are zero. That this is true can be seen from the following arguments. Transformed fermion operators $\tilde{f} = f + \delta f = f + \varepsilon b$, while remaining fermion operators, must conserve the property $\tilde{f}^2 = 0$. We thus find

$$0 = \tilde{f}^2 = (f + \varepsilon b)\,(f + \varepsilon b) = f^2 + \{f,\ \varepsilon b\} + \varepsilon^2 b^2. \quad (2.46)$$

The first term on the right side is zero, since f is a fermion operator; the second term is zero because of the anticommutation of ε and f and thus $\varepsilon^2 b^2 = 0$, which gives us

$$\varepsilon^2 = 0, \quad \text{or} \quad \{\varepsilon,\, \varepsilon\} = 0. \quad (2.47)$$

The set of quantities ε_i, which have the property

$$\{\varepsilon_i,\ \varepsilon_j\} = 0 \quad (2.48)$$

for *all i, j*, is called the set of generators of a *Grassmann algebra*. We can see that these algebras are similar to Clifford algebras: In Clifford algebras, the generators also anticommute, but only *with each other*, and their squares are not zero. In fact, each Grassmann algebra can be associated with a Clifford algebra with twice the number of generators.[65] Those quantities which are generators of Grassmann and Clifford algebras play an extremely important role not only in supersymmetry theories but also generally in models which contain fermion degrees of freedom—and it is sometimes extremely convenient to introduce such degrees of freedom even if they are absent in the original formulation of the problem. This question will be discussed in more detail in the sections which follow.

m) Zero energy of the vacuum

We now examine the energy spectrum of a supersymmetric oscillator. It might appear at first glance that the interesting features of this spectrum are a consequence of the particular simple model chosen, but actually they are characteristic of all supersymmetric models.

Furthermore, the spectrum of a supersymmetric oscillator will remind the reader of a standard problem in quantum mechanics.

It follows from (2.30)–(2.32) that each energy level is characterized by two occupation numbers, n_{B} and n_{F}, with

$$E_{n_{\mathrm{B}},\, n_{\mathrm{F}}} = \omega\,(n_{\mathrm{B}} + n_{\mathrm{F}}). \quad (2.49)$$

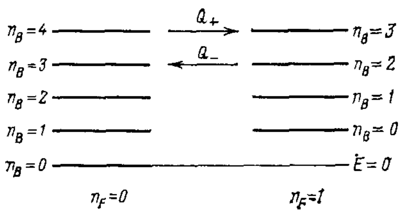

FIG. 1. Energy spectrum of a supersymmetric oscillator. The effect of the operators $Q_{\pm}$ is shown by the arrows.

This spectrum is shown in Fig. 1. We first note that the ground state (the vacuum) has a zero energy. The energy of the boson zero-point vibrations is cancelled exactly by the negative energy of the fermion "zero-point vibrations!" This cancellation is a manifestation of a monumental reduction of the infinite energy of the zero-point vibrations in supersymmetry theories, by virtue of which the energy of the vacuum becomes zero. Before the advent of supersymmetry theories the vacuum energy was forced to zero by an artificial approach: through a normal ordering of the creation and annihilation operators. The requirement of normal ordering follows in no way from the basic principles of quantum field theory, so it has given the impression that there is an internal contradiction in the theory. On the other hand, the zero-point vibrations cannot simply be thrown out: They are completely real. For example, they are responsible for the radiative corrections to the energy levels of atoms (the Lamb shift). From the standpoint of supersymmetry theories, on the other hand, infinite energies of boson and fermion vacuums (positive and negative energies, respectively) are simply a consequence of the artificial breaking up of the zero energy of the vacuum of the "unified" theory (including both bosons and fermions) into positive and negative (both infinite) terms. This natural vanishing of the vacuum energy has done much to attract interest in supersymmetry. A circumstance which has proved particularly important is that this property is a property of not only the "free" theories of such entities as our harmonic oscillator but also a property in problems incorporating an interaction, and this is true *outside the framework of perturbation theory*, if the interaction satisfies certain requirements. We will discuss this matter in the following section; at this point we simply note that it would be possible to take the opposite approach: to reduce the nonzero energy of the zero-point boson vibrations we could introduce in a purely boson theory some fictitious additional degrees of freedom, treating this step as simply following a recipe whose "legitimacy" ranked at the same level as that of normal ordering. These additional degrees of freedom would then be of a fermion nature (for otherwise the energies of the zero-point vibrations would not cancel out); furthermore, the new effective Hamiltonian would have supersymmetry, i.e., a symmetry with respect to the mutual conversions of "real" bosons and "fictitious" fermions. This phenomenon has also been studied in quantum field theory.

Another distinguishing feature of the spectrum of a supersymmetric oscillator (Fig. 1) is the twofold degeneracy of all energy levels other than the zeroth. It was shown above that this degeneracy is a consequence of supersymmetry, so that it is retained in other supersymmetric models. We will discuss some models of this type in the following sections.

n) Fermions without a Dirac sea

The supersymmetry theories suggest an interpretation of the fermion vacuum and of fermion states which is completely analogous to that in the case of bosons. In this case it is not necessary to introduce a Dirac "sea" of filled states with a negative energy and an infinite charge.

First we need to refine the terminology. The word "state" is frequently used in two distinct meanings:

1) as a state of a *system* (generally a multiparticle system);

2) as a *one-particle* state.

We will need to draw a distinction between these two meanings, so we will use the word "state" only in the first of these meanings; we will call one-particle states "cells."

In the ordinary theory of fermions one introduces cells with positive and negative energies, in accordance with the two signs of the energy for the solutions of the Dirac equation. For fixed values of the momentum and of the spin projection there is the following set of four states:

$E>0$, $E<0$
(0, 1) for the vacuum,
(1, 1) for a one-electron state,
(0, 0) for a one-positron state (a hole),
(1, 0) for an electron-positron pair.

The first number in a set of parentheses here gives the occupation number of a cell with positive energy, while the second is that of a cell with a negative energy. This description of states is obviously asymmetric with respect to a charge transformation which sends electrons into positrons and vice versa. Furthermore, it is necessary to introduce an artificial redefinition of charge in order that the vacuum will have a zero charge. Nevertheless, this description appeared satisfactory at one time; furthermore, it was believed that antiparticles corresponding to holes in a sea of states with a negative energy arise in a natural way. Today, in contrast, we know that the presence of antiparticles is not peculiar to fermions but instead a consequence of the charge symmetry of the theory and is just as applicable to bosons as it is to fermions.

Furthermore, when the theory of fermions which has now become conventional was being founded it appeared natural to assign a zero energy to an empty cell. In the case of bosons, on the other hand, a nonzero energy corresponds to an empty cell ($n_B=0$) by virtue of the relation

$$E_B=n_B+\frac{1}{2}\,. \tag{2.50}$$

For this reason, there was an important difference between the way in which the energies of one-particle states were compared with each other and the way in which the energies corresponding to the solutions of the Dirac equation in the case of fermions and the Klein-Gordon equation in the case of bosons were compared.

The energy of a fermion oscillator is determined by the following expression, according to (2.32):

$$E_F = n_F - \frac{1}{2}. \quad (2.51)$$

As in the case of bosons, an empty cell ($n_F = 0$) corresponds to a nonzero energy [see (2.50)], but in this case negative. Consequently, both states, with positive and negative energies, can be realized by means of *a single* cell—not the two cells in the conventional interpretation. In field theory, this cell is characterized not by the sign of the energy but by other characteristics of fermion excitations, e.g., a charge. To describe, say, electrons and positrons we introduce two cells (which are related by a charge-conjugation transformation). The set of four physical states written above now takes the form

(0, 0) for the vacuum,
(1, 0) for a one-electron state,
(0, 1) for a one-positron state,
(1, 1) for an electron-positron pair.

The first number in a set of parentheses here is the occupation number of an "electron" cell, and the second is that of a "positron" cell.

This description is completely symmetric under charge conjugation ($e^- \leftrightarrow e^+$), and the charge of the vacuum is zero without any artificial redefinition of the charge. The energy of a fermion vacuum, on the other hand, is negative according to expression (2.51). By analogy with the case of bosons, in which the vacuum energy is also nonzero, this appears to be a natural situation—indeed necessary if the energy of the vacuum is to be zero in a "unified" theory incorporating both bosons and fermions.

The analogy between boson and fermion oscillators becomes even clearer if we write the expressions for the Hamiltonians

$$H_B = b^+b^- + \frac{1}{2}, \quad H_F = f^+f^- - \frac{1}{2} \quad (2.52)$$

(here we are assuming $\omega = 1$) in the form

$$H_B = \frac{1}{2}\{b^+, b^-\}, \quad H_F = \frac{1}{2}[f^+, f^-], \quad (2.53)$$

which directly reflects the difference between boson statistics and fermion statistics.

c) First example of supersymmetry in the real world

The extremely simple supersymmetric model which we will discuss in this section corresponds to a real physical problem. The spectrum in Fig. 1 is exactly the same as the picture of Landau levels for an electron in a uniform magnetic field. The energy levels in this problem are given by[56]

$$E = \frac{eB}{m}\left(n + \frac{1}{2}\right) \pm \frac{1}{2}\frac{eB}{m}$$
$$= \frac{eB}{m}\left[\left(n_B + \frac{1}{2}\right) + \left(n_F - \frac{1}{2}\right)\right], \quad (2.54)$$

where we have set the component of the electron momentum along the field equal to zero, and B is the magnetic induction.

The second equality in (2.54) is written so that the reader will immediately recognize the energy of the "boson oscillator" in the first term and that of the "fermion oscillator" in the second. The boson oscillator arises from the quantization of the orbital motion of an electron in a magnetic field, while the fermion oscillator arises from the interaction of the magnetic moment of the electron with the magnetic field. The frequencies of these oscillators are *identical*, so that all the electron levels except the zeroth are doubly degenerate—this is the degeneracy which stems from the presence of supersymmetry in this problem.

Interestingly, the agreement of the frequencies of the boson and fermion oscillators, and also the presence of supersymmetry, arises only if the magnetic moment of the electron is exactly equal to the Bohr magneton $\mu_B = e\hbar/2mc$, i.e., a value predicted only by a relativistic theory. This question is analyzed in detail in Section 5 below, where it is shown that this agreement is not fortuitous. In the same place, an infinite degeneracy of levels (2.54) with respect to the "center of the orbit" is also taken into account. This degeneracy is lifted in a nonuniform magnetic field, but the supersymmetry nevertheless remains for a wide range of fields.

In this example of a real problem we can clearly see what supersymmetry transformations are and how supersymmetry combines continuous and discrete transformations.

The role of a fermion degree of freedom in the problem of an electron in a magnetic field is played by the spin; as the boson degree of freedom we could choose one of the coordinates perpendicular to the magnetic field (it is the presence of a second boson degree of freedom which is responsible for the infinite degeneracy with respect to the center of the orbit). In a supersymmetry transformation, the spin of an electron flips, and the electron *simultaneously* undergoes a transition from one orbit to another—with no change in its energy. The spin flip is a discrete transformation, while the transition from one orbit to another, although it may appear also to be discrete, is actually caused by boson annihilation and creation operators constructed from the (ordinary) coordinate and momentum. The momentum, on the other hand, is a generator of translations—continuous transformations—and the coordinate (the coordinate operator) is a generator of translations in momentum space. Supersymmetry transformations generated by the operators Q combine the properties of these transformations. The effect of the generators $Q_\pm$ is shown by the arrows in Fig. 1. The energy-degenerate states or "superpartners" correspond to the motion of an electron in different orbits and with different spin directions.

3. SUPERSYMMETRIC QUANTUM MECHANICS

a) Supersymmetry can be preserved when an interaction is incorporated.

The example of a supersymmetric harmonic oscillator which we discussed in the preceding section corresponds to a free theory. How can we incorporate an interaction while retaining the supersymmetric structure of the Hamiltonian?

The easiest way to see that an interaction can be incorporated in this way is to recall that the invariance of the Hamiltonian

$$H = Q_+Q_- + Q_-Q_+ \quad (3.1)$$

under supersymmetry transformations generated by the operators $Q_\pm$ is a consequence of their nilpotent nature, i.e., a consequence of the relation $Q_\pm^2 = 0$:

$$HQ_+ = Q_+Q_-Q_+ + 0, \tag{3.2}$$

$$Q_+H = 0 + Q_+Q_-Q_+, \tag{3.3}$$

i.e.,

$$[H, Q_+] = 0. \tag{3.4}$$

Analogously, $[H, Q_-] = 0$.

We now note that the operators $Q_\pm$ remain nilpotent if we generalize the simple model of the preceding section in the following way:

$$Q_+ = B^-(b^-, b^+)f^+, \tag{3.5}$$

$$Q_- = B^+(b^-, b^+)f^-, \tag{3.6}$$

where $B^\pm$ are *arbitrary functions* of the boson operators (which are adjoints of each other if Q_+ and Q_- remain adjoints). For this reason, Hamiltonian H given by (3.1), with the operators $Q_\pm$ given in the form in (3.5) and (3.6), remains supersymmetric. In the present case, however, it is no longer quadratic in the boson operators; i.e., it incorporates an interaction between "bosons." Furthermore, it incorporates an interaction between "bosons" and "fermions."

For a supersymmetric Hamiltonian which incorporates an interaction, we need to generalize our original picture of supersymmetry transformations as transformations which send a boson into a fermion and vice versa. When an interaction is incorporated, the Hamiltonian ceases to be diagonal in the representation of boson occupation numbers, so that the eigenstates of the Hamiltonian are no longer characterized by a definite "number of bosons." It is thus more correct to state that the supersymmetry transformations send a boson *state* into a fermion *state* and vice versa. In the case of supersymmetric quantum mechanics, this separation of states into "boson" and "fermion" states is arbitrary, but in supersymmetric field theories it acquires a physical meaning. The generators of the supersymmetry transform there under a spinor representation of the Lorentz group and thus change the spin by $\pm 1/2$; these changes correspond to a transition from a boson state to a fermion state and vice versa.

b) Matrix realization of fermion operators and "abnormality" of boson operators

Since the fermion occupation number n_F can taken on only the two values $n_F = 0, 1$, it is convenient to choose a representation for the state vectors in which the wave functions are two-component functions:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}, \tag{3.7}$$

where the upper component ψ_1 corresponds to $n_F = 1$, and the lower, ψ_0, to $n_F = 0$. The fermion creation and annihilation operators are realized in this case by 2×2 matrices:

$$f^+ = \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f^- = \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.8}$$

The operators B^+ and B^-, which are adjoints of each other, and which act in the space of boson variables are written in the form

$$B^+ = B_1 + iB_2, \quad B^- = B_1 - iB_2, \tag{3.9}$$

where B_1 and B_2 are Hermitian operators. For the operators Q_1 and Q_2, defined as in the preceding subsection, we find

$$Q_1 = Q_+ + Q_- = B_1\sigma_1 + B_2\sigma_2, \tag{3.10}$$

$$Q_2 = -i(Q_+ - Q_-) = B_1\sigma_2 - B_2\sigma_1. \tag{3.11}$$

These operators anticommute with each other and have different squares [which agree with Hamiltonian H in (3.1)] *regardless* of the commutation relations between the operators B_1 and B_2. In our representation, the Hamiltonian H becomes

$$H = \frac{1}{2}\{B^-, B^+\} + \frac{1}{2}[B^-, B^+]\sigma_3. \tag{3.12}$$

This Hamiltonian includes a fermion degree of freedom only if $[B^-, B^+] \neq 0$. This circumstance means that the operator B^- (and B^+) must not be "normal" in the terminology of the theory of operators.[59] The "abnormality" of the operators B^- and B^+ is seen in the circumstance that they may have *different* eigenvectors; in particular, one may have a kernel (such as $\psi \neq 0$, for which $B\psi = 0$), while the other may not. For the supersymmetric operator of the preceding subsection, for example, we have $B^\pm = b^\pm$, and b^- has a kernel (a vacuum state), while b^+ does not. The difference in the eigenvectors of the adjoint operators B^- and B^+ turns out to be important (as we will see below) in a study of the spontaneous breaking of supersymmetry.

c) Hamiltonian and superpotential

If Hamiltonian H in (3.12) is quadratic in the momenta p, the operators B are

$$B^\pm = \frac{1}{\sqrt{2}}[\mp ip + W(x)], \tag{3.13}$$

where $W(x)$ is an arbitrary function of the coordinate x. Corresponding to the supersymmetric oscillator of Section 2 is $W(x) = x$; in this case we have $B^\pm = b^\pm$.

Substituting (3.13) into (3.12), we find the following expression for the Hamiltonian H:

$$H = \frac{1}{2}[p^2 + W^2(x) + \sigma_3 W'(x)]. \tag{3.14}$$

This is the Hamiltonian of "Witten's supersymmetric quantum mechanics."[17] It has the important property, which is retained in all the supersymmetric models, that the interaction of bosons with bosons [in this case the term $W^2(x)$] and that of fermions with bosons [in this case, $\sigma_3 W'(x)$] are determined by *the same* function $W(x)$. Although we will call this function the "superpotential" below, that term is usually applied to another function, $V(x)$, which is related to $W(x)$ by $V'(x) = W(x)$. The choice of the function $V(x)$ is determined by the formalism (of superfields), which we will not discuss here.

d) Common interaction constant and cancellation of divergences

Consequently, as can be seen from the example of Hamiltonian (3.14), supersymmetry establishes the relationship between the interactions of bosons with bosons and the interactions of bosons with fermions. The nature of this relationship is such that in the transition to a "free" theory both the interactions of bosons with bosons and those of bosons with fermions are turned off simultaneously. Corresponding to a "free" theory, i.e., the harmonic oscillator, is $W(x) = x$. Here we have $W^2 = x^2$, corresponding to the absence of an interaction of bosons with bosons, and $W' = \text{const}$, which corresponds to the absence of an interaction of bosons with fermions. In the weak-coupling limit, on the other hand, where we can introduce a small interaction constant, the two interactions are determined by *the same* constant. It is this property of the supersymmetric theories which leads to the cancellation of the divergences when perturbation theory is used. The boson and fermion loops in the perturbation-theory diagrams are known to correspond to contributions with opposite signs, so that the equality of the constants at the vertices of the diagrams may lead to a cancellation of infinite contributions from boson and fermion loops. Unfortunately, however, supersymmetry by itself is not enough to eliminate the divergences completely, although the number of types of divergences does decrease in supersymmetric theories, and the divergences which remain become less strong (e.g., a power-law divergence changes into a logarithmic divergence). It nevertheless turns out that if a supersymmetric theory has an additional, sufficiently high symmetry, there may be no divergences in it at all. The first well-known example of a quantum field theory in a four-dimensional space-time which is free of divergences is the supersymmetric theory of gauge fields with the O(4) symmetry group. The cancellation of divergences in supersymmetric theories is important in the derivation of a quantum theory of gravity (supergravity), in which the superpartner of the spin-2 graviton is a new hypothetical particle: a spin-3/2 gravitino.

e) Energy of the vacuum and topology of the superpotential

Let us return to Hamiltonian (3.14) and determine which properties of a supersymmetric oscillator remain in force in this more general case.

In the first place, the twofold degeneracy of the energy levels with energies $E > 0$ persists for an *arbitrary* function $W(x)$, since this degeneracy is due exclusively to the supersymmetry.

Furthermore, for a supersymmetric oscillator the ground-state energy is exactly zero, and this zero state is the only nondegenerate state. Is this property retained in the more general case with an interaction?

This question was studied by Witten,[17] who discovered an elegant property: The presence or absence of a nondegenerate level with an energy $E = 0$ is determined exclusively by the *global properties of the superpotential* $W(x)$ and *does not depend* on its particular form. A generalization of this observation led to the discovery of criteria for spontaneous breaking of supersymmetry in field theories.

To study the question of the ground state of Hamiltonian (3.14), we write it in matrix form [see (3.12)]:

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \tag{3.15}$$

where $H_+ = B^-B^+$, $H_- = B^+B^-$.

The Hamiltonians H_+ and H_- act in the space of *one-component* wave functions, each of which is factorized, i.e., has the form of a product of two *first*-order differential operators which are adjoints of each other. For this reason, the problem of determining the state with the energy $E = 0$ reduces to one of finding the solutions of the equation $B^-\psi = 0$ or $B^+\psi = 0$.

These equations are equivalent to the equations $H_-\psi = 0$ or $H_+\psi = 0$. Let us prove this equivalence. From $B^-\psi = 0$ we find $H_-\psi = 0$ in the obvious manner. On the other hand, from $H_-\psi = 0$ we find $B^-\psi = 0$, since $H_- = B^+B^-$:

$$0 = \langle\psi \mid B^+B^- \mid \psi\rangle = \mid B^- \mid \psi\rangle \mid^2. \tag{3.16}$$

We denote the solutions of the equations $B^\pm \psi = 0$ by $\psi_\pm$. The indices $\pm$ correspond to the sign of the eigenvalue σ_3 in (3.14).

Using expression (3.13) for the operators $B^\pm$, we can write the equations $B^\pm \psi = 0$ in the form

$$\left(\frac{d}{dx} \mp W\right)\psi_\pm = 0. \tag{3.17}$$

The solutions of these equations are

$$\psi_\pm = C\exp\left[\pm \int_0^x W(x')\,dx'\right]. \tag{3.18}$$

For the functions $\psi_\pm$ actually to be eigenfunctions of the Hamiltonian H, however, we must require that they be quadratically-integrable. We will not discuss here the case in which the functions $\psi_\pm$ oscillate as $x \to \pm\infty$. We then require for quadratic integrability of ψ_-, according to (3.18), the following:

$$\int_0^x W(x')\,dx' \to \infty \quad \text{as} \quad x \to \pm\infty. \tag{3.19}$$

For the normalizability of ψ_+, on the other hand, we must have

$$\int_0^x W(x')\,dx' \to -\infty \quad \text{as} \quad x \to \pm\infty. \tag{3.20}$$

Conditions (3.19) and (3.20) are obviously incompatible, so that only *one* of the functions $\psi_\pm$ *can* be normalized. It may turn out, however, that *neither* of these functions can be normalized.

Consequently, if a state with an energy $E = 0$ *does exist*, then it is nondegenerate.[4)] This state corresponds to that of the functions $\psi_\pm$ which is normalizable.

If neither condition (3.19) nor (3.20) holds, then no normalizable function corresponding to a state with an energy $E = 0$ exists, and the nonnegativity of the spectrum of the

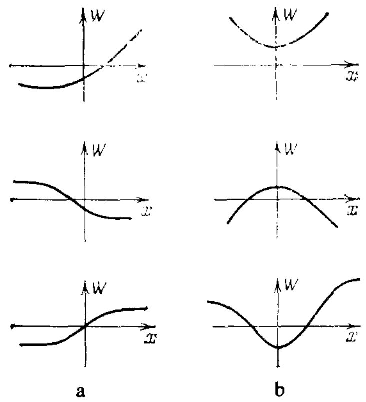

FIG. 2. Some superpotentials $W(x)$. a—Exact supersymmetry; b—spontaneously broken supersymmetry.

supersymmetric Hamiltonian means that the ground state has an energy $E_0 > 0$.

Conditions (3.19) and (3.20) are those global conditions on the superpotential which we mentioned above: A slight deformation of the superpotential $W(x)$ in a finite region cannot give rise to or eradicate a level with a zero energy. The existence of such a level, as implied by the mathematics, has a homotopic invariance under deformations of the superpotential.

Conditions (3.19) and (3.20) take a particularly simple form when $W(x)$ has definite signs as $x \to \pm\infty$.

If the signs of $W(x)$ in the limits $x \to \pm\infty$ are identical, then neither condition (3.19) nor (3.20) can hold, so we have $E_0 > 0$. This is the situation, e.g., in the case $W(x) = x^2$.

If, on the other hand, $W(x)$ has different signs as $x \to \pm\infty$, and $W(x)$ does not vanish, then one of conditions (3.19), (3.20) holds. In this case the ground-state energy is $E_0 = 0$, regardless of the particular form of $W(x)$. The simplest example is the supersymmetric oscillator discussed in Section 2, which corresponds to $W(x) = x$.

Figure 2 shows examples of superpotentials $W(x)$ for which the ground-state energy is $E_0 = 0$ (Fig. 2a) and for which we have $E_0 > 0$ (Fig. 2b).

f) Spontaneous breaking of supersymmetry

Like other types of symmetry, supersymmetry can be broken spontaneously, but in the case of supersymmetry there are some distinctive features in this breaking.

We recall that the "spontaneous breaking" of a symmetry has the following meaning: The Hamiltonian H is invariant under certain transformations $[H, R] = 0$, where R represents the generators of the transformations, but the ground state $|0\rangle$ (the vacuum) is not invariant; i.e., $R|0\rangle \neq 0$.

In the case of an exact symmetry, without spontaneous breaking, the generators of the transformations annihilate the vacuum when they act on it: $R|0\rangle = 0$. This relation means that the vacuum remains invariant under finite transformations of the form $\exp(i\alpha R)$, where α is a transformation parameter.

The generators of the supersymmetry are the operators Q. Consequently, if the supersymmetry is exact, then we have $Q|0\rangle = 0$ and thus $H|0\rangle = Q^2|0\rangle = 0$; i.e., *there exists* a state with a zero energy. The converse is also true: If there is a state with a zero energy, the supersymmetry is exact; i.e., there is no spontaneous breaking. The proof is analogous to that in Subsection 2e.

The question of the spontaneous breaking of supersymmetry is an extremely important one. If supersymmetry is realized in the realm of elementary particles, then it is necessarily spontaneously broken, since in the case of exact supersymmetry the bosons and fermions would have to be degenerate in mass, in contradiction of reality.

In this example of supersymmetric quantum mechanics we see that the question of spontaneous breaking of the symmetry is related to the topology of this potential: The signs of $W(x)$ in the limits $x \to \pm\infty$ are topological characteristics in the sense that they are not changed by small deformations of the superpotential.

An important distinction between the spontaneous breaking of supersymmetry and the spontaneous breaking of an internal symmetry is that a spontaneous breaking is possible in a system with a *finite* number of degrees of freedom. In supersymmetric quantum mechanics, for example, this is true even in the case of a single boson degree of freedom and a single fermion degree of freedom [for the superpotentials $W(x)$ in Fig. 2b].

g) Using supersymmetry to derive the spectra of the Schrödinger equation

The Hamiltonian of supersymmetric quantum mechanics in (3.14) may be thought of as a set of two ordinary one-dimensional Hamiltonians

$$H_\pm = \frac{1}{2}[p^2 + W^2(x) \pm W'(x)], \tag{3.21}$$

which, by virtue of the supersymmetry, have an *identical* spectrum for an *arbitrary* function $W(x)$. The only possible exception is the lower level of one of $H_\pm$, and in this case its energy would be *exactly* zero. These two properties of supersymmetric theories—the twofold degeneracy of all levels with energies $E > 0$ and the vanishing of the energy of the ground state if the latter is nondegenerate—can be used to find the exact spectrum.[5)]

We turn now to a characteristic example of this determination of the spectrum.

We choose the superpotential W to be $W(\alpha, x) = \alpha \tanh x$.

In the limits $x \to \pm\infty$ the superpotential W has different signs. If $\alpha > 0$, there is a zero level for

$$H_-(a) = \frac{1}{2}[p^2 + W^2(x) - W'(x)]$$
$$= \frac{1}{2}\left[p^2 - \frac{a(a+1)}{\mathrm{ch}^2 x}\right] + \frac{a^2}{2}. \tag{3.22}$$

On the other hand, for H_+ we find

$$H_+(a) = \frac{1}{2}\left[p^2 - \frac{a(a-1)}{\mathrm{ch}^2 x}\right] + \frac{a^2}{2}. \tag{3.23}$$

Introducing $a_1 = a - 1$, we can then write the following by virtue of (3.22) and (3.23):

$$H_+(a) = H_-(a_1) + \frac{a^2}{2} - \frac{a_1^2}{2}. \qquad (3.24)$$

If $a_1 > 0$, the lower level for the Hamiltonian $H_-(a_1)$ is again zero, so that Eq. (3.24) can be used to determine the lower level of the Hamiltonian $H_+(a)$. It is

$$E_1 = \frac{a^2 - a_1^2}{2}. \qquad (3.25)$$

The levels of $H_+(a)$ and $H_-(a)$ are, however, identical except for the lower level of $H_-(a)$, so that the E_1 is the energy of the level of Hamiltonian $H_-(a)$ just above its ground (zero) level. We thus now have *two* levels of Hamiltonian $H_-(a)$: $E_0 = 0$ and E_1.

The procedure described here can be repeated, if we use the substitution $a_n = a_{n-1} - 1 = a - n$ in each step, as long as the condition $a_n > 0$ holds. As a result we find the complete discrete spectrum of $H_-(a)$: The energy of the nth level is given by the expression

$$E_n = \frac{1}{2}[(a^2 - a_1^2) + (a_1^2 - a_2^2) + \ldots + (a_{n-1}^2 - a_n^2)]$$
$$= \frac{a^2}{2} - \frac{a_n^2}{2}. \qquad (3.26)$$

For the familiar potential

$$U(a, x) = -\frac{a(a+1)}{\mathrm{ch}^2 x}, \qquad (3.27)$$

which differs from $H_-(a)$ by the additive constant $-a^2/2$, we thus find the energy spectrum

$$E_n = -\frac{a_n^2}{2} = -\frac{(a-n)^2}{2}. \qquad (3.28)$$

To generalize this example we note that the Hamiltonian H_+ and H_- differ in this case only in the values of their *parameters* (including the additive constant); it is this property which made it possible to devise an iterative procedure for finding the spectrum. It is natural to pose the problem of finding all potentials which satisfy this "form-invariance" condition. This problem was studied in Ref. 28. It was found that the class of potentials for which spectra can be found by elementary calculations by the technique described above includes all the problems of one-dimensional quantum mechanics which have been solved exactly over the entire x axis.

We have used supersymmetry to solve a problem which did not contain this symmetry in its original formulation (the problem of finding the spectra of a one-dimensional Schrödinger equation). It turned out to be useful to extend the problem, moving to supersymmetric quantum mechanics, by analogy with the switch to a complex plane which is sometimes useful in the evaluation of integrals along the real axis.

4. BEREZIN INTEGRALS

A major role is played in the construction and development of supersymmetric models by the Berezin formalism, in which bosons and fermions are described in a surprisingly similar way.[14,15] Since the rules for working with fermion variables may appear rather unusual, we will take a detailed look at the reasons behind these rules, so that they will arise necessarily, in a "natural way." This approach also clarifies the correspondence between boson and fermion variables: a correspondence which is the central idea of supersymmetry.

Even without reference to supersymmetry, however, the Berezin formalism is extremely convenient and useful in studying systems with fermion degrees of freedom. Furthermore, it makes it possible to introduce such degrees of freedom formally in problems which do not originally contain them and thereby to simplify the study of these problems.

The basic goal in the development of this formalism was to construct a path integration in fermion variables. This problem was taken up some time ago by Feynman,[60] but he was not able to extend his path-integration method to fermion variables. He perceived the problem as being the absence of a classical limit for the anticommuting variables describing the fermion degrees of freedom.

a) Why there is no "coordinate representation" for fermion degrees of freedom

In searching for a common description of bosons and fermions, Berezin pointed out that the ordinary representation in terms of coordinates and momenta, used for boson degrees of freedom, cannot be directly generalized to the fermion case. Why is it not possible to draw a direct analogy between boson and fermion variables in a "coordinate" representation?

In the coordinate representation, a complex function of *real* variables (a wave function) is associated with each state of the system:

$$|\psi\rangle \leftrightarrow f(q_1, \ldots, q_N), \qquad (4.1)$$

where N is the number of degrees of freedom. In this representation all the coordinate operators can be diagonalized *simultaneously*, but this is possible only if the coordinate operators *commute* with each other.

This condition naturally holds in the case of boson degrees of freedom. However, an attempt to construct an analogous representation for a system with fermion degrees of freedom immediately runs into a fundamental difficulty: The fermion operators *anticommute* with each other, so that they cannot be diagonalized simultaneously. Consequently, in this case it is not possible to associate with a state a function of real variables or "coordinates."

b) Holomorphic representation for bosons

Is there a representation, common to the description of boson and fermion degrees of freedom, in which it is still possible to associate a function with each state? If so, what properties should arguments of this function have?

We begin with bosons. For a system with a single degree of freedom we choose as a basis the states

$$(b^+)^n|0\rangle, \qquad (4.2)$$

where $|0\rangle$ is the vacuum. Any state vector can be written as a linear combination of vectors (4.2):

$$|\psi\rangle=\sum_{n=0}^{\infty} c_n (b^+)^n|0\rangle. \qquad (4.3)$$

where c_n are complex coefficients. Expansion (4.3) is unique by virtue of the orthogonality of vectors (4.2).

We now note that the sum in (4.3) may be thought of as a Taylor-series expansion of some *analytic function* of the operator b^+:

$$|\psi\rangle = F(b^+)\,|0\rangle. \qquad (4.4)$$

This relation shows that it is possible to associate an analytic function of a complex variable with each state $|\psi\rangle$. This variable is usually chosen to be not z but z^*, in view of the correspondence with a representation in terms of coherent states.[53–55]

The association

$$\text{state} \leftrightarrow \text{function:} \quad |\psi\rangle \leftrightarrow F(z^*) \qquad (4.5)$$

is a representation which can be generalized to the fermion case. A representation of this sort is called "holomorphic" since the function $F(z^*)$ is holomorphic.[6)]

The generalization to the case of several boson degrees of freedom is obvious:

$$|\psi\rangle=\sum_{\{n_i\}} a_{\{n_i\}}(b_1^+)^{n_1}(b_2^+)^{n_2}\ldots(b_N^+)^{n_N}|0\rangle, \qquad (4.6)$$

where $\{n_i\}$ are sets of integers (a multiple index), and the $a_{\{n_i\}}$ are coefficients which may be assumed to be *symmetric* with respect to the interchange of the indices i since the boson operators commute with each other.

The scalar product in a holomorphic representation is determined from the condition

$$\langle m\mid n\rangle=\delta_{mn}, \qquad (4.7)$$

where $|n\rangle$ is a normalized "n-particle" state.

We write the scalar product in integral form:

$$\langle\psi_1\mid\psi_2\rangle=\int F_1(z)\,F_2(z^*)\,\rho(z^*,z)\,dz^*\,dz, \qquad (4.8)$$

where the weight function $\rho(z^*, z)$ must be determined from condition (4.7), and where we are using the circumstance that, in accordance with (4.2)–(4.5), the function $(z^*)^n\sqrt{n!}$ is associated with the state $|n\rangle$. Condition (4.7) thus takes the form

$$\frac{1}{n!}\int (z^*)^n z^m \rho(z^*, z)\,dz^*\,dz=\delta_{mn}. \qquad (4.9)$$

This relation is satisfied by one and only one function:

$$\rho(z^*, z)=\frac{1}{2\pi i}e^{-z^*z}. \qquad (4.10)$$

c) Holomorphic representation for fermions

We turn now to the case of fermion degrees of freedom. By analogy with the boson case, we choose as a basis wave functions of the type

$$(f_1^+)^{n_1}(f_2^+)^{n_2}\ldots(f_N^+)^{n_N}|0\rangle, \qquad (4.11)$$

where N is the number of fermion degrees of freedom, and $n_i=0, 1$ in accordance with $(f_i^+)^2=0$. Any state vector is a linear combination of vectors (4.11):

$$|\psi\rangle=\sum_{\{n_i\}} c_{\{n_i\}}(f_1^+)^{n_1}(f_2^+)^{n_2}\ldots(f_N^+)^{n_N}|0\rangle. \qquad (4.12)$$

but the coefficients $c_{\{n_i\}}$, are now *antisymmetric* with respect to permutation of the indices i, by virtue of the anticommutation of the operators f_i^+.

Pursuing the analogy with the case of bosons, we may also understand (4.12) as an expansion of a *function* in a Taylor series. The arguments of this function, however, must be thought of as anticommuting (*Grassmann*) quantities, so that the property of antisymmetry of the coefficients $c_{\{n_i\}}$, can be taken into account. Consequently, and as in the case of bosons [see (4.5)], we can associate a function with each state, but in the fermion case this is a function of Grassmann variables ξ_i^+:

$$|\psi\rangle \longleftrightarrow F(\xi_1^*, \xi_2^*, \ldots, \xi_N^*). \qquad (4.13)$$

These Grassmann variables are sometimes called "fermion coordinates." That term is not completely successful since there is the implicit assumption of a correspondence between these variables ξ and the boson coordinates q, while actually there is no such correspondence. The correspondence which does exist is between the *complex* quantities z, which characterize the *coherent* states of bosons, and the Grassmann variables ξ. Incidentally, this correspondence makes it possible to determine coordinate states for fermions also, as was essentially done by Berezin.

d) Integration in Grassmann variables

How do we define a scalar product in a holomorphic representation in the fermion case?

Let us take one fermion degree of freedom. By analogy with the boson case, we write the scalar product as the integral

$$\langle\psi_1\mid\psi_2\rangle=\int F_1(\xi)\,F_2(\xi^*)\,\rho(\xi^*,\xi)\,d\xi^*\,d\xi, \qquad (4.14)$$

and we require that the states $|n\rangle$ be orthonormal (in this case, $n=0, 1$). Here, however, we immediately run into *two* problems: How do we define the weight function $\rho(\xi^*, \xi)$? What are we to understand by "integration in Grassmann variables" in (4.14)? We have to make a choice: either *define* integration rules and then *find* a weight function, or take the opposite approach of defining a weight function and then finding integration rules from the requirement that the basis be orthonormal.

The customary approach is the first of these approaches, but we will take the second here; we will attempt to defend this choice below.

By analogy with bosons [see (4.10)], we define the weight function

$$\rho(\xi^*, \xi)=e^{-\xi^*\xi}, \qquad (4.15)$$

and we find rules for integrating in Grassmann variables from the conditions for orthonormality of the basis:

$$\langle 0\mid 0\rangle=\langle 1\mid 1\rangle=1,\quad \langle 0\mid 1\rangle=\langle 1\mid 0\rangle=0. \qquad (4.16)$$

Since the states $|0\rangle$ and $|1\rangle$ are represented by the functions

$$|0\rangle \leftrightarrow f(\xi^*) = 1, \quad |1\rangle \leftrightarrow f(\xi^*) = \xi^*, \tag{4.17}$$

the orthonormality conditions (4.16) become

$$\begin{aligned} &\int \rho(\xi^*, \xi)\, d\xi^* d\xi = \int \xi\xi^* \rho(\xi^*, \xi)\, d\xi^* d\xi = 1, \\ &\int \xi^* \rho(\xi^*, \xi)\, d\xi^*\, d\xi = \int \xi \rho(\xi^*, \xi)\, d\xi^*\, d\xi = 0. \end{aligned} \tag{4.18}$$

In accordance with the Grassmann nature of ξ and ξ^* we have

$$\rho(\xi^*, \xi) = \exp(-\xi^*\xi) = 1 - \xi^*\xi, \tag{4.19}$$

so from (4.18) we find

$$\begin{aligned} &\int d\xi^*\, d\xi - \int \xi^*\xi\, d\xi^*\, d\xi = 1, \\ &\int \xi\, d\xi^*\, d\xi = \int \xi^*\, d\xi^*\, d\xi = 0, \\ &\int \xi\xi^*\, d\xi^*\, d\xi = 1. \end{aligned} \tag{4.20}$$

If we understand the multiple integrals in (4.20) as repeated integrals, we find Berezin's famous integration rules:

$$\int d\xi = 0, \quad \int \xi\, d\xi = 1. \tag{4.21}$$

If we had instead taken the path mentioned above, i.e., that of *defining* integration rules (4.21) first, then we believe that it would not be clear how it would be possible to guess such unusual integration rules. On the other hand, the introduction of the weight function $\rho(\xi^*, \xi)$ for the fermions is completely analogous to the boson case and should cause no "internal protest."

In general, in dealing with Grassmann variables we should bear in mind that they *do not take on numerical values of any sort*. A description of states as *functions* by means of these variables corresponds, in the figurative expression of one mathematician, to the specification of functions "in the eighteenth-century sense," i.e., not as mappings but as a way of writing—in this case, a way of specifying sets of coefficients for an expansion in some basis. Incidentally, this is precisely the situation which we have in the boson case, where a state is represented by a function of a complex variable (a holomorphic representation). This function is understood not as a mapping of the complex plane onto itself but as the specification of an infinite set of coefficients of a Taylor-series expansion. In the case of a finite number of fermion degrees of freedom, on the other hand, the basis is finite-dimensional.

e) Gauss integrals

The integration in Grassmann variables was introduced for, and is basically used for, path integration. Basic features of the path-integration method are Gauss integrals.

For the simplest such integral we find

$$\int e^{-a\xi\xi^*}\, d\xi^*\, d\xi = \int (1 - a\xi\xi^*)\, d\xi^*\, d\xi = -a, \tag{4.22}$$

where a is an ordinary number. We wish to call attention to the fact that a would be in the denominator in the case of an "ordinary" integration.

In the case of several Grassmann variables we find

$$\int e^{-a_{ik}\xi_i\xi_k^*} \prod_{i=1}^{n} d\xi_i^* \prod_{i=1}^{n} d\xi_i = (-1)^n \det |a_{ik}|. \tag{4.23}$$

Here again, the determinant would be in the denominator in the case of an "ordinary" integration.

This distinction between Gauss integrals in Grassmann variables and ordinary Gauss integrals allows us to use these integrals in functional changes in variables. This question is examined in detail in Section 7.

5. SUPERSYMMETRY IN THE REAL WORLD

Let us examine some problems in which supersymmetry is realized as a physical symmetry.

a) Electron in a magnetic field

The Dirac equation for an electron in a magnetic field which is constant in time can be written

$$\begin{aligned} \sigma\pi\psi_a &= (H_D + m)\,\psi_b, \\ \sigma\pi\psi_b &= (H_D - m)\,\psi_a, \end{aligned} \tag{5.1}$$

where ψ_a and ψ_b are two-component spinors, $\boldsymbol{\pi} = -i\partial/\partial\mathbf{x} - e\mathbf{A}(\mathbf{x})$, and $H_D = i\partial/\partial t$. Both spinors, ψ_a and ψ_b, satisfy the equation

$$(\sigma\pi)^2\,\psi = (H_D^2 - m^2)\,\psi, \tag{5.2}$$

so that in the nonrelativistic limit we have a Pauli equation for an electron with a magnetic moment equal to the Bohr magneton:

$$(\sigma\pi)^2\psi = 2mH_P\psi. \tag{5.3}$$

An important point for the discussion below is that Eqs. (5.2) and (5.3), for relativistic and nonrelativistic electrons, have a common structure:

$$Q^2\psi = H\psi, \tag{5.4}$$

where $Q = \boldsymbol{\sigma\pi}$, and the operator H is defined by $H = H_D^2 - m^2$ in the relativistic case or by $H = 2mH_P$ in the nonrelativistic case. Accordingly, after the sign of the energy is chosen in the relativistic case, the symmetry of the "Hamiltonian" H determines both the symmetry of the relativistic Hamiltonian H_D and that of the nonrelativistic Hamiltonian H_P. Below we will accordingly treat these cases simultaneously, calling the operator H in Eq. (5.4) the "Hamiltonian."

Equation (5.4) is of the form of *one* of the relations of supersymmetry algebra [see (2.12), (2.14)]; in particular, the supercharge Q is an integral of motion. This integral of motion,

$$Q = \sigma(\mathrm{p} - e\mathrm{A}), \tag{5.5}$$

has been familiar to experimentalists for a long time now, especially to those who have been measuring the radiative corrections to magnetic moments. Conservation of Q means that the spin and the velocity precess in a magnetic field at an identical frequency, so that the angle between them remains constant. A change in this angle with time, on the other hand, results from a small deviation of the magnetic moment of the electron from the Bohr magneton; this deviation is caused by the radiative corrections.

As we have already seen (Section 2), the operator Q by itself is not sufficient to bring about the degeneracy of the energy spectrum which is characteristic of supersymmetry: The multiplicity of the degeneracy is $\nu = 2^{[N/2]}$, where N is the number of supercharges. With $N=1$ we find $\nu = 1$; i.e., there is no degeneracy at all.

If there exists an operator T with the properties

$$\{T, Q\} = 0, \quad T^2 = 1. \tag{5.6}$$

there will be two supercharges: $Q_1 = Q$ and $Q_2 = iQT$.

It is easy to see that the operator Q has the form of a Dirac operator in a three-dimensional Euclidean space, so that the operator T has an effect analogous to a γ_5 transformation. In the three-dimensional case, however, because of the odd dimensionality, there exists no matrix which is an analog of γ_5, i.e., which anticommutes with all three Pauli matrices σ. Consequently, in the general case, if no further restrictions of any sort are imposed on the magnetic field, there are no supercharges other than Q, and there is no spectral degeneracy.

There are, however, two wide classes of fields for which it is possible to find an operator T with properties (5.6):

1. A "two-dimensional field," i.e., a field which is directed along one of the coordinate axes and which depends in an arbitrary way on the two others, with $B_z = B_z(x, y)$, $B_x = B_y = 0$.

2. A three-dimensional field with a definite parity: $\mathbf{B}(-x) = \pm \mathbf{B}(x)$.

We first consider the fields of the first class, and we choose the vector potential in the form $A_x = A_x(x, y)$; $A_y = A_y(x, y)$; $A_z = 0$.

Equation (5.4) remains the same in form if we take Q to be the two-dimensional Dirac operator

$$Q = \sigma_1 \pi_1 + \sigma_2 \pi_2 \tag{5.7}$$

and if we transform Hamiltonian H: $H \to H - p_z^2$.

In this case the role of the operator T is played by σ_3. The existence of an operator with these properties is a consequence of the even dimensionality.

For our purposes below we also introduce operators $Q_\pm$ (Section 2), given by

$$Q_\pm = \frac{1}{2}(Q_1 \pm iQ_2) = Q\,\frac{1 \mp \sigma_3}{2}. \tag{5.8}$$

The analogy with supersymmetric quantum mechanics, discussed in Section 3, immediately becomes clear if we write the generators $Q_\pm$ as

$$Q_+ = \pi^- f^+, \quad Q_- = \pi^+ f^-, \tag{5.9}$$

where $\pi^\pm = \pi_x \pm i\pi_y$, and $f^\pm = \sigma^\pm = (\sigma_1 \pm i\sigma_2)/2$.

The operators $f^\pm$ satisfy the algebra of fermion annihilation and creation operators: $f^{\pm 2} = 0$, $\{f^+, f^-\} = 1$.

The operators $\pi^\pm$ satisfy the commutation relation

$$[\pi^-, \pi^+] = 2\,|\,e\,|\,B_z(x, y), \tag{5.10}$$

where we have used $e = -|e|$ for an electron. In a *uniform* field $B_z = B_0 > 0$, the operators $b^\pm = \pi^\pm/\sqrt{2|e|B_0}$ may be thought of as boson creation and annihilation operators (they interchange roles if $B_0 < 0$).

The problem of an electron in a uniform magnetic field thus reduces to the problem of a supersymmetric harmonic oscillator (discussed in Section 2), as we have already mentioned.

The supersymmetric structure is preserved, however, even in a *nonuniform* field, and in this case we are dealing not with a "free" theory of noninteracting oscillators but with a theory which incorporates an interaction, analogous to the supersymmetric quantum mechanics which we discussed in Section 3.

The presence of supersymmetry in this problem turns out to be related to the gauge nature of the electromagnetic field. The replacement $\mathbf{p} \to \boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}(\mathbf{x})$, in accordance with the principle of a minimal incorporation of the interaction, leads to a "normal" value for the magnetic moment, specifically, the Bohr magneton, $\mu_B = e\hbar/2mc$; only in the case $\mu = \mu_B$ does Hamiltonian H have a supersymmetric structure. Feynman[60] mentioned the relationship between the circumstance that the Hamiltonian is an "exact square" and the relation $\mu = \mu_B$.

In the basis of eigenfunctions of the operator σ_3, Hamiltonian H splits up into two factorized Hamiltonians (Section 3):

$$H = H_+ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + H_- \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.11}$$

where

$$H_+ = \pi^- \pi^+, \quad H_- = \pi^+ \pi^-. \tag{5.12}$$

The Hamiltonians $H_\pm$ act on the single-component functions corresponding to spin projections $\pm 1/2$. The factorization of these Hamiltonians substantially simplifies the problem of finding the ground-state wave function. Precisely as in supersymmetric quantum mechanics (Section 3), the ground-state wave function must satisfy one of the *first*-order differential equations

$$\pi^+ \psi = 0 \quad \text{or} \quad \pi^- \psi = 0, \tag{5.13}$$

depending on the sign of the total magnetic flux. It can be shown that Eqs. (5.13) are incompatible with each other, so that only one of them can be satisfied.

A factorization of the Pauli and Dirac Hamiltonians in a two-dimensional magnetic field was discovered (in a manner unrelated to supersymmetry) by Aharonov and Casher,[30] who also found the degree of degeneracy of the ground state in the case of a finite field, in which the total magnetic flux Φ is finite. This degree of degeneracy is given by $N_0 = [\Phi/\Phi_0]$, where $\Phi_0 = 2\pi/|e|$ is the quantum of magnetic flux. If the flux is equal to an integer number of quanta, then the degree of degeneracy is one unit lower, as was mentioned in Ref. 31. In the same paper there was a study of the case of a doubly periodic two-dimensional magnetic field, where it was shown that the vacuum remains degenerate with an infinite multiplicity despite the loss of translational invariance. The supersymmetric structure of the problem was studied in Ref. 33, and it was shown that in terms of the ground-state wave functions there is an infinite-dimensional irreducible representation of the Heisenberg-Weyl algebra (the algebra of boson creation and annihilation operators) even if the magnetic field has no regularity of any sort, but the total flux is infinite.

The problem of an electron in a two-dimensional magnetic field is an example of supersymmetric quantum mechanics with *two* boson degrees of freedom and *one* fermion degree of freedom. We thus see that supersymmetry does not require that the numbers of boson and fermion degrees of freedom be equal, despite a widespread opinion to the contrary. The properties of "two-dimensional" supersymmetric quantum mechanics are quite different from those of Witten's one-dimensional mechanics (discussed in Section 3), primarily because the ground state (the vacuum) may be degenerate even if there is an exact supersymmetry (not spontaneously broken). However, as in the one-dimensional case, the degree of degeneracy of the vacuum is determined by the global properties of the superpotential (by the total magnetic flux Φ in the case of a two-dimensional field), not by the particular form of this superpotential. In the case of an infinite flux, the degree of degeneracy is infinite, and it is this circumstance which is related to the well-known degeneracy of infinite degree with respect to the "center of the orbit" in a uniform magnetic field. In a nonuniform field with an infinite total flux, the degeneracy of all levels except the ground level can be lifted (except for a twofold degeneracy stemming from the supersymmetric structure of the problem), but the ground level remains infinitely degenerate. We might say that even after the interaction is incorporated the vacuum "remembers" the symmetry of the free Hamiltonian corresponding to the case of a uniform field, when *all* the levels, including the ground level, are infinitely degenerate. A distinction between a nonuniform field and a uniform field is that in a uniform field the wave packet representing the ground state can be shifted without a change in shape: The energy of the state vanishes. In a nonuniform field,the wave packet changes shape when shifted, "adapting" to the field irregularities in such a way that its energy remains zero.

Going over to the case of a three-dimensional field, we note that the "momentum" operator $\mathbf{p} = -i\partial/\partial\mathbf{x}$ anticommutes with the parity operator P. Consequently, for fields which have a definite parity, $\mathbf{A}(-\mathbf{x}) = \pm\mathbf{A}(\mathbf{x})$, the Hamiltonian has a supersymmetry with two supercharges. The role of the operator T, which we discussed above [see (5.6)], is played by the parity operator P in the case of an odd field, $\mathbf{A}(-\mathbf{x}) = -\mathbf{A}(\mathbf{x})$, or by the operator R, which has the effect[33] $R\psi(\mathbf{x}) = i\sigma_2\psi^*(-\mathbf{x})$, in the case of an even field, $\mathbf{A}(-\mathbf{x}) = \mathbf{A}(\mathbf{x})$. We mean here the parity of the vector potential $\mathbf{A}(\mathbf{x})$; the magnetic field $\mathbf{B}(x)$ itself has a parity opposite that of the field $\mathbf{A}(\mathbf{x})$ (by virtue of the relation $\mathbf{B} = \mathrm{curl}\mathbf{A}$).

It is interesting to see the role played by the "fermion charge" in our problem of an electron in a magnetic field.

In two dimensions, this "charge" is the spin projection (along or opposite the field). In three dimensions, the role of the "fermion charge" is played by the parity. In both cases, the introduction of a "fermion degree of freedom" is simply an approach which is related in a natural way to the partitioning of the total space of states into two orthogonal subspaces. In the former case, the subspaces are determined by the sign of s_z, while in the latter they are determined by the parity of the wave function. This example shows that the fermion degrees of freedom can also be used to advantage in problems in which they are not present at the outset.

Some other examples will be discussed below.

A twofold degeneracy of the levels of an electron in a *uniform* magnetic field has also been linked with the existence of a dynamic symmetry group.[54] In states with identical energy in this case we have a spinor representation of the SU(2) group, whose generators X_i satisfy, along with commutation relations, the *anti*commutation relation $\{X_i, X_j\} = 2\delta_{ij}$. After a change in the normalization of the generators, this anticommutation relation becomes one of the relations of the superalgebra (Section 2). Supersymmetry, on the other hand, explains the twofold degeneracy of the energy levels even in a *nonuniform* two-dimensional field, in which the operators X_i cannot be represented in terms of creation and annihilation operators. This representation is the fundamental representation in the use of a dynamic symmetry group (not supergroup).

b) Nuclear physics

The concept of a nucleon-nucleon potential is used frequently in research on nuclear interactions. At short range ($r \lesssim 2$ fm) this potential is repulsive (the "core" of the nucleons), while at large range it is attractive. Whereas the core is formed as a result of complicated one- and many-particle processes (the exchange of vector mesons ρ, ω, ...; two-pion exchange; etc.), the long-range part of the nucleon-nucleon potential will be formed exclusively by one-pion exchange. The pion-nucleon interaction thus plays an important role in nuclear physics. As we will show below, this interaction has a supersymmetric structure in certain cases.

We first consider the one-particle problem of a nucleon in a classical (external) π-meson field. This problem is interesting from the methodological standpoint, but in addition it is pertinent to a study of the properties of nuclear matter with a π condensate,[62] since the π condensate is a classical pion field.

In the case of a pseudoscalar coupling, the Lagrangian of the πN interaction is

$$\mathscr{L}_{\mathrm{PS}} = g_{\pi\mathrm{N}}\bar{\psi}\gamma_5\boldsymbol{\tau}\boldsymbol{\pi}(\mathbf{x})\psi, \tag{5.14}$$

and in the case of a pseudovector coupling it is

$$\mathscr{L}_{\mathrm{PV}} = f_{\pi\mathrm{N}}\bar{\psi}\gamma_\mu\gamma_5\partial_\mu\boldsymbol{\tau}\boldsymbol{\pi}(\mathbf{x})\psi; \tag{5.15}$$

where $\boldsymbol{\pi}(\mathbf{x})$ is the isovector pion field, $\boldsymbol{\tau}$ are the isotopic Pauli matrices, and ψ is the isodoublet of nucleons. The pseudoscalar interaction is characterized by a dimensionless coupling constant $g_{\pi N}$ and is renormalizable; the coupling constant of the pseudovector interaction, $f_{\pi N}$, has the dimensionality of a reciprocal length, and this interaction is not renormalizable. We do know, however, that the two theories are equivalent in an expansion in which no terms beyond the terms linear in the coupling constant are retained (see Ref. 61, for example). Here we have $f_{\pi N} = g_{\pi N}/2M$, where M is the mass of a nucleon.

The Hamiltonian which appears in the Dirac equation for a nucleon in an external pion field is

$$H_{PS}^{D} = \alpha\hat{\mathbf{p}} + \beta M + ig\gamma_5 \tau\pi(\mathbf{x}) \tag{5.16}$$

in the case of a pseudoscalar coupling or

$$H_{PV}^{D} = \alpha\hat{\mathbf{p}} + \beta M - f\alpha\gamma_5 \frac{\partial}{\partial \mathbf{x}} \tau\pi(\mathbf{x}) \tag{5.17}$$

in the case of a pseudovector coupling. In the nonrelativistic approximation we have

$$H_{PS} = \frac{\hat{p}^2}{2M} + \frac{g^2}{2M}\pi^2(\mathbf{x}) + \frac{g}{2M}\sigma\frac{\partial}{\partial \mathbf{x}}\tau\pi(\mathbf{x}), \tag{5.18}$$

$$H_{PV} = \frac{\hat{p}^2}{2M} + f\sigma\frac{\partial}{\partial \mathbf{x}}\tau\pi(\mathbf{x}), \tag{5.19}$$

where σ are spin matrices.

The terms in (5.18) and (5.19) which are linear in the coupling constant describe p-wave attraction and are the same if $f = g/2M$. For pseudoscalar coupling, however, an additional term appears in the nonrelativistic Hamiltonian and contributes to an s-wave repulsion.

In a field of neutral pions, with $\boldsymbol{\pi}(\mathbf{x}) = (0, 0, \pi(\mathbf{x}))$, the Hamiltonian H_{PS} can be written

$$H_{PS} = \frac{1}{4M}\{Q_+, Q_-\}, \tag{5.20}$$

where $Q_{\pm}$ are nilpotent operators which are adjoints of each other:

$$Q_{\pm} = (\sigma\hat{\mathbf{p}} \pm ig\pi(\mathbf{x}))\tau^{\pm}. \tag{5.21}$$

These operators serve as supercharges. The Hamiltonian H_{PS} thus has a supersymmetry with two supercharges, and for this reason the energy levels of nucleons are doubly degenerate in an arbitrary three-dimensional field of a π^0 condensate. The superpartner states are different functions of the spatial coordinates and have opposite isospin projections.

The pseudovector coupling does not give the Hamiltonian a supersymmetric structure. It is pertinent here to compare the pion-nucleon interaction with the electromagnetic interaction (a gauge interaction). Supersymmetry for the Dirac equation in a magnetic field (Subsection 5a) is related to the principle of minimal incorporation of the interaction by the replacement

$$\partial_\mu \to \partial_\mu - ieA_\mu(x). \tag{5.22}$$

The phenomenological incorporation of an anomalous magnetic momentum $\mu'\sigma_{\mu\nu}F_{\mu\nu}/2m$ disrupts the supersymmetry, at the same time making the theory unrenormalizable because of the presence of a dimensional constant. A completely analogous situation is found in the case of the πN interaction, since the pseudoscalar coupling satisfies the principle of minimal incorporation $[\mathbf{p} \to \mathbf{p} - g\boldsymbol{\pi}(\mathbf{x})]$, is renormalizable, and leads to a supersymmetric Hamiltonian. It should be noted, however, that the description of the πN interaction is phenomenological, in contrast with that in electrodynamics.

A supersymmetric structure has also been discovered in the nucleon-nucleon potential itself—more precisely, in that part of this potential which is caused by one-pion exchange and which depends on the spins of the nucleons.[48]

6. ZERO MODES, TOPOLOGY, AND SUPERSYMMETRY

We have already seen, in our discussion of supersymmetric quantum mechanics (Section 3) and in our study of the problem of an electron in a magnetic field (Section 5), that the presence or absence of levels with zero energy is related to the global characteristics of the superpotentials which play the role of topological characteristics. Witten[17,18] has offered a general formulation of the problem of this relationship and has analysed it in detail.

In field theory, the role of levels with zero energy is played by so-called zero modes. These modes, first discovered for Fermi systems by Jackiw and Rebbi,[49] have played a significant role in the subsequent development of field theory.

As the simplest example we consider a field theory in a two-dimensional (1 + 1) space-time with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - U(\varphi) + \bar{\psi}(i\gamma_\mu\partial_\mu - g\varphi)\psi, \tag{6.1}$$

where φ is a real scalar field, ψ is a fermion field, and g is the constant of the coupling of fermions with bosons.

To determine the spectrum of particles we should first determine the ground state (the vacuum), since one-particle excitations are the quanta of "small vibrations" above the vacuum. The vacuum corresponds to a minimum of the potential energy. In order to determine it, we solve the classical equations of motion for the boson part of the Lagrangian in the limit $g \to 0$ and then linearize the problem, examining small deviations from the classical solution which we have found.

We denote by $\varphi_c(x)$ the solution of the classical equations of motion corresponding to the minimum energy. It satisfies the equation

$$\frac{d^2\varphi(x)}{dx^2} = \frac{dU(\varphi)}{d\varphi}. \tag{6.2}$$

What is the spectrum of fermion excitations in this classical boson field $\varphi_c(x)$? For such excitations we find

$$(i\gamma_\mu\partial_\mu - m(x))\psi(x, t) = 0, \tag{6.3}$$

where $m(x) = g\varphi_c(x)$. Since $\varphi_c(x)$ depends only on x, we can seek a solution of (6.3) in the form $\psi(x, t) = \exp(-i\omega t)\psi(x)$. Substituting this solution into (6.3), we find the equation

$$H_D\psi(x) = \omega\psi(x). \tag{6.4}$$

For the choice $\gamma_0 = -\sigma_1$, $\gamma_1 = -i\sigma_3$ of the two-dimensional γ matrices (this choice is frequently used), the Hamiltonian H_D is

$$H_D = \sigma_2 p - \sigma_1 m(x). \tag{6.5}$$

This Hamiltonian has the properties of a supercharge in supersymmetric quantum mechanics (Section 3) with the superpotential $m(x)$.

We can thus use the criterion given in Section 3 for the existence of a level with zero energy to determine whether there exists a solution of Dirac equation (6.4) with zero frequency ω, i.e., with zero energy. With regard to the case at hand, we find that there is such a solution if the signs of $m(x)$ in the limits $x \to \pm\infty$ are different, i.e., if the boson field $\varphi_c(x)$ has different asymptotes in the limits $x \to \pm\infty$. The very existence of fermion "excitations" with zero energy in field theory was an unexpected result, and it attracted much interest. These excitations are called "zero modes." They have recently been discovered experimentally in a study of the conductivity properties of a linear polymer: polyacetylene. The effective Lagrangian for this system is analogous to the model discussed here.

Steady-state classical solutions for a boson field can have different asymptotes in the limits $x \to \pm\infty$ only in models in which soliton solutions are possible; such solutions exist in the example with Lagrangian (6.1) because the potential $U(\varphi)$ has two wells. There is thus a relationship between the existence of fermion zero modes and the existence of solitons in the boson sector. These soliton solutions are topologically nontrivial ground states (vacuums).

The relationship between zero modes and topology was exploited by Witten[18] to find criteria for spontaneous breaking of supersymmetry. He introduced the quantity

$$\Delta_W = n_b^0 - n_f^0, \tag{6.6}$$

where n_b^0 and n_f^0 are the numbers of boson and fermion states with zero energy. The condition $\Delta_W = 0$ is a necessary (but not sufficient) condition for spontaneous breaking of the supersymmetry. If $\Delta_W \neq 0$, then at least one of the numbers n_b^0, n_f^0 is nonzero; i.e., there exist states with zero energy, so that the supersymmetry is exact (Section 3). The quantity Δ_W has been called the "Witten index."

We partition the total space of states into two subspaces—boson and fermion states—in such a manner that a state vector has the form

$$\begin{pmatrix} B \\ F \end{pmatrix}. \tag{6.7}$$

In this representation, a supersymmetry generator which sends boson states into fermion states and vice versa has the structure

$$Q = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}. \tag{6.8}$$

We choose Q to be Hermitian; then M and M^* are adjoints of each other.

States with zero energy are annihilated by the operator Q since $H = Q^2$. According to (6.7) and (6.8), boson and fermion states with zero energy satisfy the equations $M\psi = 0$ and $M^*\psi = 0$, respectively. Boson states with zero energy are therefore a kernel of the operator M, while fermion states are the same for the operator M^*. The dimensionality of the kernels is precisely equal to the number of states with zero energy:

$$n_b^0 = \dim \operatorname{Ker} M, \quad n_f^0 = \dim \operatorname{Ker} M^*. \tag{6.9}$$

The quantity

$$\operatorname{Ind} M = \dim \operatorname{Ker} M - \dim \operatorname{Ker} M^* \tag{6.10}$$

is, by definition, called the "index of the operator M." It follows from (6.6) and (6.9) that the Witten index Δ_W is the same as the operator index M:

$$\operatorname{Ind} M = \Delta_W. \tag{6.11}$$

Relation (6.11) has proved useful "in both directions": On the one hand, since the operator index is a topological characteristic, it does not change upon a small deformation of the operator, so that (6.11) can be used to determine the Witten index and thus the possibility of a spontaneous breaking of the supersymmetry in some field-theory model. Here one makes use of the constancy of the Witten index upon variations of the parameters of the theory, and the theory is "continuously" varied to a form in which the Witten index can be calculated explicitly. On the other hand, the use of the supersymmetry formalism and an integration in Grassmann variables has made it possible to derive a simple proof of the Atiyah-Singer theorem regarding the index of an operator.[37,38] For a given operator M (e.g., the Dirac operator in an external gauge field) one constructs a corresponding supersymmetric theory whose Witten index agrees with the operator index M and can be calculated by a path integration.

7. SUPERSYMMETRY AND FUNCTIONAL CHANGES IN VARIABLES

In field-theory problems, functional determinants arise as transformation Jacobians upon a change in variables in the path integrals. These determinants can be expressed as integrals in Grassmann variables (Section 4). Consequently, in the path integral

$$Z = \int D\{\varphi\}\, e^{-iS\{\varphi\}} \tag{7.1}$$

we can change variables while *preserving* the form, which is extremely convenient for the derivation of a perturbation theory, for taking various averages, etc. The action is modified: In terms of the new variables, it also includes fictitious Fermi fields and has an additional symmetry of the nature of a supersymmetry.

a) Stochastic differential equations

We begin with the simplest example, in which the path integrals reduce to ordinary integrals. We denote by ρ a random quantity with a distribution function $P(\rho)$ which satisfies the normalization condition $\int d\rho P(\rho) = 1$.

We denote by x another random quantity, which is related to ρ by

$$W(x) = \rho. \tag{7.2}$$

We are interested in the expectation value of some function of x:

$$\langle g(x)\rangle = \int g[x(\rho)]\,P(\rho)\,d\rho. \tag{7.3}$$

If the function $x(\rho)$ which is the inversion of (7.2) can be found explicitly, then it is sufficient to substitute this function into (7.3) in order to obtain $\langle g(x)\rangle$. It may happen, however, that the inversion of (7.2) is either impossible technically or undesirable because this equation has symmetry properties of some sort, and we wish to preserve this symmetry explicitly. In such a case we need to change variables in (7.3) and transform from an integration over ρ to one over x. The Jacobian of the transformation, $d\rho/dx = dW/dx$, can be written as an integral in Grassmann variables with the help of (4.22). As a result we find

$$\langle g(x)\rangle = \int dx\,g(x)\,e^{-F(x)}\frac{d\rho}{dx}$$
$$= \int dx\,d\eta\,d\bar{\eta}\,g(x)\exp\left[-F(x)+\bar{\eta}\frac{dW}{dx}\eta\right], \tag{7.4}$$

where $F = -\ln P$.

It is not difficult to see that the "action"

$$I(x,\ \eta,\ \bar{\eta}) = F(x) - \bar{\eta}\frac{dW}{dx}\eta,$$

in the exponential function in the last integral in (7.4), is invariant under transformations which mix the "boson" variable x and the "fermion" variables η, $\bar{\eta}$:

$$x \to x' = x + \bar{\varepsilon}\eta + \bar{\eta}\varepsilon,$$
$$\eta \to \eta' = \eta + \varepsilon\frac{dF}{dW},\qquad \bar{\eta} \to \bar{\eta}' = \bar{\eta} + \bar{\varepsilon}\frac{dF}{dW}, \tag{7.5}$$

where ε and $\bar{\varepsilon}$ are the Grassmann parameters of the transformation. In the case of a Gaussian distribution, $F \propto W^2$, the "action" $I(x, \eta, \bar{\eta})$ takes a form typical of supersymmetric theories with a superpotential $W(x)$ (Section 3).

The transformation to the one-dimensional case corresponds to a transformation from a random quantity ρ to a random function $\rho(t)$. The integrals become path integrals (functional integrals).

Following Refs. 40 and 41, we can show that the problem of the dissipative dynamics of a system which is subject to a random force $\rho(t)$ is analogous to the supersymmetric quantum mechanics which we discussed in Section 3.

We consider the Langevin equation

$$\frac{dx}{dt} = -W(x) + \rho(t), \tag{7.6}$$

where $\rho(t)$ is a random force with a Gaussian distribution function

$$P\{\rho(t)\} \backsim \int D\rho\exp\left[-\frac{1}{2\alpha}\int dt\rho^2(t)\right],$$
$$\langle\rho\rangle = 0,\quad \langle\rho(t)\rho(t')\rangle = \alpha\delta(t-t') \tag{7.7}$$

(a "white noise"). We are interested in the correlation function

$$\langle x(t)\,x(t')\rangle \backsim \int D\rho\,x(t)\,x(t')\,P\{\rho\}. \tag{7.8}$$

Here the functional integral is over all realizations of the random process $\rho(t)$. We transform to a path integration $x(t)$, and we write the Jacobian of the transformation, $\det|\delta\rho/\delta x|$, as a Gauss integral of the Fermi fields ψ and $\bar{\psi}$ [see (4.23)]. As a result we find

$$\langle x(\tau)\,x(\tau')\rangle \backsim \int Dx\,D\bar{\psi}\,D\psi\,x(\tau)\,x(\tau')$$
$$\exp\left\{-\frac{1}{2\alpha}\int d\tau\,[\dot{x}_\tau + W(x)]^2 + \int d\tau\bar{\psi}\,[\partial_\tau + W'(x)]\,\psi\right\}. \tag{7.9}$$

We have used the change of notation $t\to\tau$, because this correlation function is the same as the Green's function in imaginary time for Witten's supersymmetric quantum mechanics (Section 3). The parameter α, which determines the fluctuation intensity in (7.7), serves as a Planck's constant, if (7.9) is thought as a Green's function. Setting $\alpha = 1$, and discarding the total time derivative $\dot{x}_\tau W(x)$ from the argument of the exponential function, we find the Lagrangian of supersymmetric quantum mechanics:

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - \frac{1}{2}W^2(x) + \bar{\psi}\,[i\partial_t + W'(x)]\,\psi. \tag{7.10}$$

Interestingly, the presence or absence of a steady-state regime for random process $x(t)$ turns out to be related to the presence or absence of a spontaneous breaking of the supersymmetry, as we will now demonstrate.

The probability density $\mathscr{P}\{x(t);\ t\}$ for the random function $x(t)$ satisfies the Fokker-Planck equation[57]

$$\frac{\partial\mathscr{P}}{\partial t} = \frac{\partial\{W(x)\mathscr{P}\}}{\partial x} + \frac{1}{2}\alpha\frac{\partial^2\mathscr{P}}{\partial x^2}. \tag{7.11}$$

This equation has a steady-state (time-independent) solution

$$\mathscr{P}(x) \sim \exp\left[-\frac{2}{\alpha}\int_{-\infty}^{x} dy\,W(y)\right] \tag{7.12}$$

only if the following normalization integral converges:

$$\int_{-\infty}^{\infty} dx\,\mathscr{P}(x) = 1. \tag{7.13}$$

Equations (7.12) and (7.13) are compatible only if the "superpotential" $W(x)$ which serves as the determinate part of the random force in Langevin equation (7.6) has different signs in the limits $x\to\pm\infty$. Physically, this requirement is obvious: Otherwise, the regular component of the force would carry the particle off to infinity. On the other hand, the condition that $W(x)$ must have different signs in the limits $x\to\pm\infty$ is exactly the same as the condition that the supersymmetry in the corresponding problem is exact—that there is no spontaneous breaking (Section 3).

b) Faddeev-Popov "ghosts"

Historically the first introduction of fictitious Fermi fields for a functional change in variables is apparently represented by the well-known Faddeev-Popov ghosts,[63] which have proved very useful in the derivation of a quantum theory of gauge fields. The presence of a supersymmetric structure in the effective action incorporating the Fermi fields of ghosts was discovered by Bakki, Rouer, and Storoy[64] and Tyutin and used by them to simplify the proofs of the Slavnov-Taylor identities,[63,64] which play the role of Ward identities in non-Abelian gauge theories.

Let us recall the distinctive features of a path integration in a gauge theory. The classical Lagrangian is invariant under gauge transformations which contain arbitrary *functions*, so that it actually depends on fewer fields than it formally contains. On the fields we must impose an additional condition, e.g., the Lorentz gauge condition, which reduces the number of independent field components. In the path integral we would integrate over only the physically different field configurations (e.g., over only the transverse fields in a Lorentz gauge). In order to derive a perturbation theory, however, it is more convenient to carry out the integration over *all* configurations of the gauge fields, introducing a δ-function in the integral to eliminate the integration over nonphysical variables. The functional change in variables is then used to avoid inverting the gauge condition explicitly.

The simplest example in which path integrals reduce to ordinary integrals may appear slightly contrived in this case, but we believe that this example helps clarify the basic idea. We denote by x a "physical" variable, and by y a "nonphysical" variable in the sense that the action $S(x, y)$, while formally a function of x and y, is actually a function of only one variable. This circumstance is seen in the fact that a relationship is imposed on x and y (a "gauge condition"):

$$F(x, y)=\rho, \tag{7.14}$$

where ρ is some arbitrary number. We are interested in an integral over the "physical" variable x,

$$Z=\int \mathrm{d}x e^{-S}, \tag{7.15}$$

and we wish to write it in a form in which x and y appear as if they were equivalent variables; this approach would correspond to an integration over all configurations of the gauge fields. Furthermore, we wish to retain the form of integral (7.15) as an integral of an exponential function, since we will then be able to construct a perturbation theory. Finally, our third wish is to avoid going through an explicit solution of coupling condition (7.14).

We first introduce an integration over y:

$$Z=\int \mathrm{d}x\, \mathrm{d}y e^{-S} \delta\left(y-f_{\rho}(x)\right), \tag{7.16}$$

where $y=f_{\rho}(x)$ is a solution of coupling equation (7.14) (we do not need to find it explicitly). The δ-function which appears in (7.16) is eliminated through an additional integration over ρ, which can be carried out with any convenient weight, since integral (7.15) does not depend on ρ. For a reason which we will see below, it is more convenient to integrate over ρ with a Gaussian weight. A subsequent transformation of integral (7.16) then gives us the following result, where we are using (4.22):

$$\begin{aligned}
Z &= \int \mathrm{d}x\, \mathrm{d}y e^{-S} \delta[F(x, y)-\rho] \frac{\partial F}{\partial y} \\
&\backsim \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\rho e^{-S-\rho^{2}} \delta(F-\rho) \frac{\partial F}{\partial y} \\
&= \int \mathrm{d}x\, \mathrm{d}y e^{-S-F^{2}(x, y)} \frac{\partial F}{\partial y} \\
&= \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\bar{\eta}\, \mathrm{d}\eta \exp\left[-S-F^{2}(x, y)-\bar{\eta} \frac{\partial F}{\partial y} \eta\right].
\end{aligned} \tag{7.17}$$

It is easy to see that this expression is a realization of all three of the properties that we were hoping to find. An integration over ρ with a Gaussian weight is convenient, as can be seen from (7.17), in cases in which coupling equation (7.14) is linear in x and y: In this case, the increment in the "action" turns out to be quadratic.

We turn now to the problem of the quantization of gauge fields. The coupling condition takes the following form in the Lorentz gauge [an analog of (7.14)]:

$$\partial_{\mu} A_{\mu}^{a}=\rho^{a}(x), \tag{7.18}$$

where a is an isotopic index, and μ is a Lorentz index.

The integration over $\rho^a(x)$ is carried out with a Gaussian fuctional distribution in the imaginary time:

$$P\{\rho^{a}\} \backsim \exp\left[-\frac{1}{2\alpha} \int \mathrm{d}^{4}x (\rho^{a})^{2}\right]. \tag{7.19}$$

As a result we find, in accordance with the example above, the following expression for the path integral:

$$Z \backsim \int \mathrm{D}A_{\mu}^{a}\, \mathrm{D}\bar{\eta}^{a}\, \mathrm{D}\eta^{a} \exp\left[i \int \mathrm{d}^{4}x\left(\mathscr{L}_{\mathrm{cl}}+\mathscr{L}_{\mathrm{g.f}}+\mathscr{L}_{\mathrm{gh}}\right)\right]. \tag{7.20}$$

Let us examine the analogy with (7.14). First, the integration in (7.20) is carried out over all configurations of the gauge fields $A_{\mu}^{a}(x)$; this circumstance corresponds to the integration over x and y in (7.14). Next, fictitious Fermi fields $\eta^a(x)$, $\bar{\eta}^a(x)$ arise; they are analogous to the Grassmann variables η, $\bar{\eta}$ in (7.17). These Fermi fields are the Faddeev-Popov ghosts. Next, as in (7.17), there are three terms in the exponential function (in the integral): first, the original Lagrangian for the gauge fields, which formally depends on all the field components [it corresponds to S in (7.17)]; second, a gauge-fixing term

$$\mathscr{L}_{\mathrm{g.f}}=-(1/2\alpha)\left(\partial_{\mu} A_{\mu}^{a}\right)^{2} \tag{7.21}$$

(α is a gauge parameter), which corresponds to $F^2(x, y)$ in (7.17); and, finally, the Lagrangian of the interaction with Faddeev-Popov ghosts,

$$\mathscr{L}_{\mathrm{gh}}=\bar{\eta}^{a} \frac{\delta\left(\partial_{\mu} A_{\mu}^{a}\right)}{\delta \omega^{b}} \eta^{b}, \tag{7.22}$$

which corresponds to the last term, $\bar{\eta}\frac{\partial F}{\partial y}\eta$, in the exponential function in (7.17). The quantities ω^b in (7.22) are the parameters of gauge transformations: for the infinitesimal transformations

$$\delta A_{\mu}^{a}=D_{\mu}^{ab} \omega^{b}, \quad D_{\mu}^{ab}=\partial_{\mu} \delta^{ab}+f^{abc} A_{\mu}^{c}, \tag{7.23}$$

where f^{abc} are structure constants of the gauge symmetry group (the coupling constant of the gauge fields has been set equal to unity here). In the Lorentz gauge, the Lagrangian of the interaction with ghosts, $\mathscr{L}_{\text{gh}}$, takes its standard form:

$$\mathscr{L}_{\text{gh}} = \partial_\mu \eta^a D_\mu^{ab} \eta^b. \tag{7.24}$$

The Feynman integral in the field configurations is thus represented in (7.20) as an integral of an exponential function with a modified action and with a relativistically invariant measure.

Supersymmetry is found quite unexpectedly in a study of the behavior of the various terms in the exponential function in (7.20) under gauge transformations. Both the total action (the sum of the first three terms) and the classical action for the gauge fields (the first term) are unchanged by the gauge transformations. Consequently, the last two terms (the gauge-fixing term and the Lagrangian of the interaction with ghosts) are as a *sum* also invariant. However, neither of these terms is *separately* gauge-invariant. The gauge invariance of the sum of these two terms is a consequence of the particular supersymmetry of the modified action. Supersymmetry transformations which mix fermion fields of ghosts and longitudinal components of gauge fields are of the form

$$\delta A_\mu^a = \varepsilon D_\mu^{ab} \eta^b, \quad \delta \bar{\eta}^a = -\frac{1}{\alpha} \varepsilon \partial_\mu A_\mu^a,$$
$$\delta \eta^a = -\frac{1}{\alpha} \varepsilon f^{abc} \eta^b \eta^c, \tag{7.25}$$

where ε is a Grassmann parameter.

8. CONCLUSION

It may happen that nature does not wish to make use of the attractive properties of supersymmetry in order to resolve the problems of the physics of fundamental interactions, such as the problem of divergences and the nonrenormalizability of quantum gravitation. Even in such a case, however, supersymmetry will apparently remain with us, primarily because it is a new and elegant symmetry which combines continuous and discrete transformations.

Supersymmetry has already manifested itself in problems in quantum mechanics, and it has proved to be an extremely convenient formalism for studying problems even if they do not contain fermion degrees of freedom in their original formulation. The number of studies which use the ideas and methods of supersymmetry continues to grow, and supersymmetry is finding new fields of application.

[1)] See also some other papers in this Issue (editor's note).

[2)] For this reason, "the number of fermion degrees of freedom" is understood in some papers as meaning the number of operators q_i. In this definition, we would have not n but $N = 2n$ fermion degrees of freedom in the case at hand.

[3)] We are writing this and the following relations in a schematic form to call the reader's attention primarily to the commutation and anticommutation relations.

[4)] This assertion holds only in the case of *one* boson degree of freedom. As we will show in Section 5, as soon as there are even two boson degrees of freedom the multiplicity of the degeneracy of the ground state can be arbitrary (even infinite).

[5)] This method of determining the spectra is related to the factorization method of Refs. 26 and 27 and reveals the symmetry underlying that method.

[6)] Some other names are "Fok-Bargmann representation" and "representation by coherent states." It is closely related to the second-quantization representation.

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