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Homotopy classification of elliptic operators on stratified manifolds

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Abstract. We give a homotopy classification of elliptic operators on a stratified manifold. Namely, we establish an isomorphism between the set of elliptic operators modulo stable homotopy and the K-homology group of the manifold. By way of application, we obtain an explicit formula for the obstruction of Atiyah–Bott type to the existence of Fredholm problems in the case of stratified manifolds.

§ 1. Introduction

In the classical paper [1], Atiyah observed that *abstract elliptic operators* on a compact space X (that is, Fredholm operators acting on $C(X)$-modules and commuting modulo compact operators with the operators of multiplication by functions) determine elements of the K-homology group of X. Moreover, Kasparov [2] and Brown, Douglas, and Fillmore [3] showed that one can not only obtain elements of K-homology groups but also realize K-homology as a generalized homology theory if one takes the quotient of the set of abstract elliptic operators by the equivalence relation given by stable homotopy.

However, to obtain the K-homology group of a smooth manifold, one need not consider abstract elliptic operators. It suffices to restrict oneself to *differential* or *pseudo-differential* operators, which arise naturally in the theory of partial differential equations. Moreover, if the manifold is also equipped with a spin$^{\mathrm{c}}$-structure, then it suffices to consider only (twisted) Dirac operators. This example suggests the study of the natural problem of comparing the K-homology group with the group generated by the elliptic pseudo-differential operators (ψDO) for non-smooth spaces (see Singer's problem in [4]), in particular, for stratified manifolds.

This problem has been solved in some special cases. The classification of general elliptic ψDO in terms of K-homology was established in [5] for manifolds with isolated singularities. The same result was proved independently in [6] and [7] using groupoids and KK-theory. A classification of edge-degenerate elliptic operators for stratified manifolds with two strata was obtained in [8] and [9].

Pseudo-differential calculus on general stratified manifolds is nothing new. It was constructed, for example, in [10] and, within the framework of the general approach [13] to the construction of ψDO associated with a given Lie algebra of vector fields using the techniques of groupoid theory, in [11] and [12]. However,

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no results have yet been obtained on the homotopy classification of elliptic operators on such manifolds. We establish such results in this paper.

Our main theorem states that if X is a compact stratified manifold with arbitrarily (but finitely) many strata, then there is a group isomorphism

$$\mathrm{Ell}(X) \simeq K_0(X), \tag{1.1}$$

where $\mathrm{Ell}(X)$ is the group generated by the elliptic pseudo-differential operators on X modulo stable homotopy and $K_0(X)$ is the even K-homology group of X. Special cases of this isomorphism were obtained in the papers [5]–[9].

The isomorphism (1.1) enables us to apply the topological methods of K-homology theory in the elliptic case. To give examples of such applications, we compute the obstruction of Atiyah–Bott type to the existence of Fredholm problems on stratified manifolds and generalize the theorem on the cobordism invariance of the index (see § 8).

In addition to these applications to elliptic operators, the isomorphism (1.1) has an interesting interpretation within the framework of non-commutative geometry. Namely, the algebra of ψDO on a stratified manifold is associated with a certain groupoid (see [14] and [15]). Moreover, the group $\mathrm{Ell}(X)$ is related to the K-group of the C^*-algebra of the groupoid [12]. The well-known Baum–Connes conjecture [16] asserts that this K-group is isomorphic to the topological K-group of the classifying space of the groupoid (see [17]). Explicit computations for the simplest stratified manifolds show that the K-group of the classifying space is isomorphic to $K_0(X)$, that is, to the right-hand side of (1.1). It would be interesting to investigate further comparisons of (1.1) with the Baum–Connes map.

To conclude the introduction, we wish to express our gratitude to Professor T. Fack (Lyon) for indicating the possible relationship between the isomorphism (1.1) and the Baum–Connes isomorphism for groupoids.

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§ 2. Stratified manifolds and ψDO

In this section we describe the class of manifolds where elliptic theory will be studied and the class of pseudo-differential operators to be considered. These topics are well known (for example, see [10], [18], [19]), and so our exposition is rather concise.

2.1. Stratified manifolds. We use the following terminology. A *manifold with singularities* is a triple

$$\pi\colon M \to \mathfrak{M},$$

where $\mathfrak{M}$ is a Hausdorff space, M is a manifold with corners[1] and π is a continuous projection. The manifold M is called the *blow-up* of $\mathfrak{M}$. We do not discuss the uniqueness of the blow-up and, when speaking about manifolds with singularities, we always assume that the triple $\pi\colon M \to \mathfrak{M}$ is given.

[1] We recall that a *manifold of dimension n with corners* is a Hausdorff space locally homeomorphic to the product $\overline{\mathbb{R}}^k_+ \times \mathbb{R}^{n-k}$, $0 \leqslant k \leqslant n$, with smooth transition functions between such domains.

A *diffeomorphism* of such manifolds is a pair $(f\colon \mathfrak{M}_1 \to \mathfrak{M}_2,\ \widetilde{f}\colon M_1 \to M_2)$ such that f is a homeomorphism, $\widetilde{f}$ is a diffeomorphism, and the following diagram commutes:

$$\begin{tikzcd} M_1 \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_1"'] & M_2 \arrow[d, "\pi_2"] \\ \mathfrak{M}_1 \arrow[r, "f"'] & \mathfrak{M}_2, \end{tikzcd}$$

where π_1 and π_2 are the natural projections. A special class of manifolds with singularities is obtained as follows. On a manifold M with corners, consider a smooth Riemannian metric that is non-degenerate in the interior and possibly degenerate on the boundary, and define $\mathfrak{M}$ as the quotient of M by the following equivalence relation determined by the metric: two points are equivalent if the metric distance between them is zero. (Of course, one must also assume that $\mathfrak{M}$ is Hausdorff.)

Now let us describe the class of manifolds with singularities to be studied in this paper, namely, stratified manifolds. The description is by induction.

Definition 2.1. A *filtration of length k* on a topological space $\mathfrak{M}$ is a sequence

$$\mathfrak{M} = \mathfrak{M}_k \supset \mathfrak{M}_{k-1} \supset \cdots \supset \mathfrak{M}_0 \tag{2.1}$$

of closed subspaces of $\mathfrak{M}$ such that each $\mathfrak{M}_j$ is contained in the closure of the set $\mathfrak{M}^\circ_{j+1} = \mathfrak{M}_{j+1} \setminus \mathfrak{M}_j$, $j = 0, 1, \dots, k-1$.

Definition 2.2 (inductive base). A *stratified manifold of length zero* is an arbitrary smooth manifold. In this case, $M = \mathfrak{M}$, $\pi = \mathrm{id}$ and the blow-up M is a smooth manifold without boundary.

Definition 2.3 (inductive step). A *stratified manifold of length $k > 0$* is a Hausdorff space $\mathfrak{M}$ equipped with a filtration (2.1) such that the following additional conditions hold.

1. The set $\mathfrak{M}_0$ has the structure of a smooth manifold.

2. The set $\mathfrak{M} \setminus \mathfrak{M}_0$ is equipped with the structure of a stratified manifold of length $k-1$ with respect to the filtration

$$\mathfrak{M} \setminus \mathfrak{M}_0 = \mathfrak{M}_k \setminus \mathfrak{M}_0 \supset \mathfrak{M}_{k-1} \setminus \mathfrak{M}_0 \supset \cdots \supset \mathfrak{M}_1 \setminus \mathfrak{M}_0.$$

3. We have a bundle with fibre K_Ω over $\mathfrak{M}_0$, where Ω is a compact stratified manifold of length at most $k-1$ and K_Ω is the cone with base Ω. We also have a homeomorphism from a neighbourhood $U \subset \mathfrak{M}$ of $\mathfrak{M}_0$ onto a neighbourhood of the subbundle formed by the vertices of the cones, and the restriction of this homeomorphism to $\mathfrak{M}_0$ is the identity map.

4. The structure in Condition 3 is compatible with that in Condition 2 on $\mathfrak{M} \setminus \mathfrak{M}_0$ in the sense described below.

It follows by induction that

(i) the sets $\mathfrak{M}_j \setminus \mathfrak{M}_{j-1} \simeq M^\circ_j$ (*open strata*) are smooth manifolds[2] for all $j = 0, 1, \dots, k$,

[2] We put $\mathfrak{M}_{-1} = \varnothing$ for convenience. These sets are the interiors of the corresponding blow-ups, which are smooth manifolds with corners.

(ii) each open stratum M_j°, $j<k$, has a neighbourhood $U_j \subset \mathfrak{M}$ homeomorphic to a bundle with fibre K_{Λ_j}, where the base Λ_j of the cone is a stratified manifold of length at most $k-j-1$.

Let us give precise statements of Conditions 3 and 4.

The cone in Condition 3 is defined by the formula

$$K_\Omega = \{\overline{\mathbb{R}}_+ \times \Omega\} \;/\; \bigl\{\{0\} \times \Omega\bigr\}.$$

We require that the transition functions of the bundle with fibre K_Ω preserve the variable $r \in \overline{\mathbb{R}}_+$ and are induced by *diffeomorphisms of the stratified manifold* Ω of length $\leqslant k-1$, that is, by maps that are diffeomorphisms of manifolds with singularities and preserve the stratification and the fibrations of neighbourhoods of the strata into cones. Hence we actually require that our bundle is obtained by fibrewise conification from some bundle with fibre Ω over $\mathfrak{M}_0$.

The complement $U \setminus \mathfrak{M}_0$ is an open subset of the bundle with fibre

$$K_\Omega^\circ \simeq \mathbb{R}_+ \times \Omega$$

(the cone with vertex deleted) over $\mathfrak{M}_0$ and hence possesses the natural structure of a stratified manifold of length at most $k-1$. Indeed, local trivializations of this bundle are given by $V \times \mathbb{R}_+ \times \Omega$, where $V \subset \mathbb{R}^l$ ($l = \dim \mathfrak{M}_0$) is a coordinate patch on $\mathfrak{M}_0$. Hence they are stratified manifolds of the same length as Ω, that is, of length at most $k-1$. For the strata in $V \times \mathbb{R}_+ \times \Omega$, one can take $V \times \mathbb{R}_+ \times \Omega_j$, where the Ω_j are the corresponding strata in Ω and the fibrations of neighbourhoods of these strata into cones are obtained from those of neighbourhoods of the corresponding strata in Ω by taking their products by $V \times \mathbb{R}_+$.

Now let us clarify Condition 4. We have seen that $U \setminus \mathfrak{M}_0$ is equipped with two structures of a stratified manifold: one is the restriction of the corresponding structure on $\mathfrak{M} \setminus \mathfrak{M}_0$ and the other comes from the bundle. The compatibility condition requires that these two structures coincide (that is, that the identity map be a diffeomorphism).

Finally, let us define the blow-up of $\mathfrak{M}$. Let $\tilde\pi\colon \widetilde{M} \to \mathfrak{M} \setminus \mathfrak{M}_0$ be the blow-up of $\mathfrak{M} \setminus \mathfrak{M}_0$ (known by the inductive hypothesis). The blow-up M of $\mathfrak{M}$ is obtained as the union of $\widetilde{M}$ and some set 'over $\mathfrak{M}_0$'. Hence, to describe M and the projection $\pi\colon M \to \mathfrak{M}$, it suffices to study what happens over U. We can assume that U is fibred over $\mathfrak{M}_0$. Then $\tilde\pi^{-1}(U \setminus \mathfrak{M}_0)$ is fibred over $\mathfrak{M}_0$ by the composite map

$$\tilde\pi^{-1}(U \setminus \mathfrak{M}_0) \xrightarrow{\tilde\pi} U \setminus \mathfrak{M}_0 \to \mathfrak{M}_0.$$

The structure of this bundle is easily described in local trivializations: it is given by

$$V \times \mathbb{R}_+ \times \widetilde{\Omega} \xrightarrow{\mathrm{id} \times \mathrm{id} \times p} V \times \mathbb{R}_+ \times \Omega \to V,$$

where $p\colon \widetilde{\Omega} \to \Omega$ is the blow-up of the base of the cone. In this local trivialization, we define the blow-up M of $\mathfrak{M}$ by adding the point $r=0$ to the second factor, that is, by passing from $\mathbb{R}_+$ to $\overline{\mathbb{R}}_+$. The projection takes each point $(v,0,\omega)$ to v.

Now the meaning of Definition 2.3 is completely clear.

Each stratum $\mathfrak{M}_j$ is a stratified manifold (of length j). It has a blow-up M_j, which will be called a *closed stratum* in what follows. The corresponding projection will be denoted by $p_j\colon M_j \to \mathfrak{M}_j$.

Remark 2.1. By Condition 3, the cone bundles are defined over the open strata M_j° in $\mathfrak{M}$. However, Condition 4 readily implies that these bundles can be extended canonically to the closed strata M_j.

This remark is used in what follows because the operator-valued symbols of our pseudo-differential operators are defined over the closed strata.

Metrics, measures, and L^2-spaces. Let us introduce some natural metrics and measures on stratified manifolds. They will be used in the definition of spaces of square-integrable functions.

First, let us give an inductive description of metrics. We take an arbitrary Riemannian metric on a stratified manifold of length zero. To describe the inductive step, we define a metric locally on $V \times \mathbb{R}_+ \times \widetilde{\Omega}$ by the formula

$$ds^2 = dv^2 + dr^2 + r^2\, d\widetilde{\omega}^2, \tag{2.2}$$

where $d\widetilde{\omega}^2$ is the metric defined on $\widetilde{\Omega}$ by the inductive hypothesis. Globally, the metric on $\mathfrak{M}$ is obtained by patching together these local expressions (defined in a neighbourhood of $\mathfrak{M}_0$) and the metric $d\tilde{\rho}^2$ (defined on $\mathfrak{M} \setminus \mathfrak{M}_0$ by the inductive hypothesis) outside a slightly smaller neighbourhood of $\mathfrak{M}_0$ using a partition of unity. Metrics of this form are called *edge-degenerate metrics.*

The measure naturally corresponding to a metric is defined as the volume element equal to unity on an orthonormal frame. In terms of the inductive formula (2.2), the corresponding formula for the measure is

$$d\,\mathrm{vol} = r^n\, dv\, dr\, d\,\mathrm{vol}_\Omega,$$

where $d\,\mathrm{vol}_\Omega$ is the volume form on Ω (known by the inductive hypothesis) and $n = \dim \Omega$ is the dimension of Ω.

From now on, all operators on $\mathfrak{M}$ are considered in the space

$$L^2(\mathfrak{M}) \equiv L^2(\mathfrak{M}, d\,\mathrm{vol}),$$

and the operators on the cone K_Ω are considered in the space

$$L^2(K_\Omega) \equiv L^2(K_\Omega,\, r^n\, dr\, d\,\mathrm{vol}_\Omega).$$

The cotangent bundle. Let us define a space $\mathrm{Vect}_{\mathfrak{M}}$ of vector fields on $\mathfrak{M}$. If $\mathfrak{M}$ is a smooth manifold, then $\mathrm{Vect}_{\mathfrak{M}}$ is the space of all vector fields on $\mathfrak{M}$. Next, we define $\mathrm{Vect}_{\mathfrak{M}}$ by induction locally, assuming that it is a $C^\infty(M)$-module. On the product $V \times \mathbb{R}_+ \times \widetilde{\Omega}$, the space $\mathrm{Vect}_{\mathfrak{M}}$ consists of vector fields of the form

$$\theta = a\frac{\partial}{\partial v} + b\frac{\partial}{\partial r} + \frac{1}{r}\theta_1,$$

where a and b are smooth functions and $\theta_1 \in \mathrm{Vect}_\Omega$.

The metric ds^2 defines a $C^\infty(M)$-valued pairing on $\mathrm{Vect}_{\mathfrak{M}}$, and the formula

$$\big\langle \varphi(\theta), \mu \big\rangle = ds^2(\theta, \mu)$$

defines a bijection φ between the space $\mathrm{Vect}_{\mathfrak{M}}$ and some $C^\infty(M)$-module $\Lambda^1(\mathfrak{M}) \subset \Lambda^1(M)$ of differential forms on the blow-up M. (To see this, it suffices to note that $\mathrm{Vect}_M \subset \mathrm{Vect}_{\mathfrak{M}}$ and the embedding is epimorphic on the dense main stratum.)

Definition 2.4. The *cotangent bundle* $T^*\mathfrak{M}$ of $\mathfrak{M}$ is the vector bundle over M (which exists by Swan's theorem) whose sections are elements of $\Lambda^1(\mathfrak{M})$.

Remark 2.2. The elements of $\Lambda^1(\mathfrak{M})$ are precisely the forms vanishing on the fibres of the projection $\pi\colon M \to \mathfrak{M}$.

Remark 2.3. Since differential operators on a stratified manifold $\mathfrak{M}$ are polynomials with smooth coefficients (belonging to $C^\infty(M)$) in vector fields in $\mathrm{Vect}_{\mathfrak{M}}$, one can readily show that their interior symbols are smooth functions on $T^*\mathfrak{M}$.

The cotangent bundles of the strata $\mathfrak{M}_j$ are defined in a similar way.

The space $C^\infty(\mathfrak{M})$. We define the elements of $C^\infty(\mathfrak{M})$ as smooth functions on the blow-up M of $\mathfrak{M}$ satisfying the following additional condition: in the coordinates (v, r, ω) in a neighbourhood of any stratum of non-maximal dimension, these functions depend only on v for sufficiently small r.

2.2. Pseudo-differential operators and symbols. Let us describe the algebra of zero-order ψDO on a stratified manifold $\mathfrak{M}$. As noted in the introduction, there are several different expositions of the theory of ψDO on stratified manifolds. We use the construction in [21] and recall the corresponding definitions and facts for the readers' convenience. The proofs of the assertions in this section (except for Proposition 2.2) are omitted because of their awkwardness (they can be found in [21]).

We shall define families of ψDO with a parameter v ranging over a finite-dimensional vector space V. By successive trivial generalizations at each inductive step, one can first treat the case of ψDO depending smoothly on some additional parameter x and then the case of ψDO parametrized by points of a finite-dimensional vector bundle over a smooth manifold.

Negligible families. Let us introduce the space of operator families modulo which ψDO will be defined below. Let $\mathfrak{M}$ be a stratified manifold (possibly non-compact). We denote by $J_\infty(V, \mathfrak{M}) \equiv J_\infty(\mathfrak{M}) \equiv J_\infty$ the space of smooth operator families

$$D(v)\colon L^2(\mathfrak{M}) \to L^2(\mathfrak{M}) \tag{2.3}$$

such that all the operators $D(v)$ are compact in $L^2(\mathfrak{M})$, we have

$$\left\| \frac{\partial^\beta D(v)}{\partial v^\beta} \right\| \leqslant C_{\beta N}\bigl(1 + |v|\bigr)^{-N}, \qquad |\beta|, N = 0, 1, 2, \dots, \tag{2.4}$$

and these conditions still hold if $D(v)$ is replaced by a product

$$V_1 \cdots V_p D(v) V_{p+1} \cdots V_{p+q}$$

of arbitrary length $p + q \geqslant 0$. Here $V_1, \dots, V_{p+q}$ are smooth vector fields on M representable near each stratum in the form $V = \bigl(V(x), 0, \widetilde{V}(x, \omega)\bigr)$ in the local coordinates $(x, r, \omega) \in \mathbb{R}^m \times \overline{\mathbb{R}}_+ \times \Omega$, where $V(x)$ is a smooth vector field on the stratum, $\widetilde{V}(x, \omega)$ is a vector field with the same properties on the manifold Ω of shorter length, and $\widetilde{V}(x, \omega)$ depends smoothly on $x \in X$.

ψDO with parameters on smooth manifolds. We are now in a position to define ψDO. Our definition is by induction, and we start by describing the class of ψDO with parameters on smooth manifolds.

Definition 2.5. A *pseudo-differential operator with parameter $v \in V$* on a smooth manifold $\mathfrak{M}$ is an operator family

$$D(v)\colon L^2(\mathfrak{M}) \to L^2(\mathfrak{M})$$

that is a zero-order ψDO on $\mathfrak{M}$ with parameter $v \in V$ in the sense of Agranovich–Vishik. The *symbol* of D (corresponding to the unique stratum of $\mathfrak{M}$) is, by definition, the symbol $\sigma(D)(x,\xi,v)$ in the sense of Agranovich–Vishik. It is defined on the total space of the vector bundle $T^*\mathfrak{M} \times V$ over $\mathfrak{M}$ outside the zero section.

The space of pseudo-differential operators with parameter $v \in V$ on $\mathfrak{M}$ is denoted by $\Psi(V,\mathfrak{M})$. (In what follows we omit V if it is clear from the context or trivial.)

ψDO with parameters on stratified manifolds. Here we define ψDO by induction on the length k of the stratified manifold. We simultaneously define ψDO and their symbols. The inductive base is already available. It is provided by Definition 2.5 for $k = 0$.

Definition 2.6. Let $\mathfrak{M}$ be a stratified manifold of length $k > 0$. A smooth family

$$D(v)\colon L^2(\mathfrak{M}) \to L^2(\mathfrak{M})$$

of linear operators is called a *pseudo-differential operator on $\mathfrak{M}$* (*with parameter $v \in V$ in the sense of Agranovich–Vishik*) if the following conditions hold.

1. If $\varphi, \psi \in C^\infty(\mathfrak{M})$ and $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \varnothing$, then $\psi A \varphi \in J_\infty$.

2. The operator D is a ψDO with parameter on $\mathfrak{M} \setminus \mathfrak{M}_0$.[3] In a neighbourhood U of $\mathfrak{M}_0$, the operator D is representable modulo elements of $J_\infty(V,\mathfrak{M})$ as

$$D = P\biggl(x,\, r,\, rv,\, -ir\frac{\partial}{\partial x},\, ir\frac{\partial}{\partial r} + i\,\frac{n+1}{2}\biggr), \qquad n = \dim\Omega, \tag{2.5}$$

where $P(x,r,v,\eta,p) \in \Psi(V \times T^*_x\mathfrak{M}_0 \times \mathbb{R},\, \Omega)$ is a ψDO with parameters on Ω depending smoothly on the additional parameters $x \in \mathfrak{M}_0$ and $r \in \overline{\mathbb{R}}_+$, and $P = 0$ for $r > r_0$, where r_0 is sufficiently small.

The *symbols $\sigma_j(D)$ of D corresponding to the strata $\mathfrak{M}_j \setminus \mathfrak{M}_{j-1}$ $(j > 0)$ of $\mathfrak{M}$* are defined as the symbols of the operator D regarded as an element of $\Psi(V, \mathfrak{M} \setminus \mathfrak{M}_0)$. The symbol $\sigma_k(D)$ is called the *interior symbol* of D. The *symbol of D corresponding to the stratum* $\mathfrak{M}_0$ is the operator family

$$\sigma_0(D) = P\biggl(x,\, 0,\, rv,\, r\xi,\, ir\frac{\partial}{\partial r} + i\,\frac{n+1}{2}\biggr)\colon L^2(K_\Omega) \to L^2(K_\Omega) \tag{2.6}$$

parametrized by points of the bundle $V \times T^*\mathfrak{M}_0$ over $\mathfrak{M}_0$ minus the zero section. The *symbols $\sigma_j\bigl(\sigma_0(D)\bigr)$ of the symbol* $\sigma_0(D)$ are defined as the symbols $\sigma_j\bigl(P(x,0,v,\eta,p)\bigr)$ of the ψDO $P(x,0,v,\eta,p)$ with parameters on Ω, $j = 1,\dots,k$.

[3] The stratum $\mathfrak{M}_0$ has measure zero in $\mathfrak{M}$. Hence D is automatically interpreted as an operator on $L^2(\mathfrak{M} \setminus \mathfrak{M}_0)$.

Let us clarify formulae (2.5) and (2.6). One can prove by induction on the length of the manifold that a pseudo-differential operator $A \in \Psi(V, \mathfrak{M})$ satisfies the estimates

$$\left\| \frac{\partial^\alpha A(v)}{\partial v^\alpha} \right\| \leqslant C_\alpha \bigl(1+|v|\bigr)^{-|\alpha|}, \qquad |\alpha| = 0,1,2,\dots, \tag{2.7}$$

and all the derivatives of order at least one are compact-valued. Applying this assertion with $\mathfrak{M} = \Omega$, we see that the operator family

$$F(x,t,v,\xi,p) = P(x, e^{-t}, ve^{-t}, \xi e^{-t}, p)$$

satisfies the estimates

$$\begin{gathered} \left\| \frac{\partial^{\alpha+l+\beta+\gamma+k} F(x,t,v,\xi,p)}{\partial x^\alpha\, \partial t^l\, \partial v^\beta \partial \xi^\gamma\, \partial p^k} \right\| \leqslant C_{\alpha l\beta\gamma k} \bigl(1+|v|+|\xi|\bigr)^{-|\beta|-|\gamma|} \bigl(1+|p|\bigr)^{-k}, \\ |\alpha|+l+|\beta|+|\gamma|+k = 0,1,2,\dots, \end{gathered} \tag{2.8}$$

and the operator family $\widetilde{F}(x,t,v,\xi,p) = P(x,0,ve^{-t},\xi e^{-t},p)$ satisfies the estimates

$$\begin{aligned} \left\| \frac{\partial^{\alpha+l+\beta+\gamma+k} \widetilde{F}(x,t,v,\xi,p)}{\partial x^\alpha\, \partial t^l\, \partial v^\beta\, \partial \xi^\gamma\, \partial p^k} \right\| &\leqslant C_{\alpha l\beta\gamma k} \bigl(e^t+|v|+|\xi|\bigr)^{-|\beta|-|\gamma|} \bigl(1+|p|\bigr)^{-k} \\ &\leqslant C_{\alpha l\beta\gamma k} \bigl(|v|+|\xi|\bigr)^{-|\beta|-|\gamma|} \bigl(1+|p|\bigr)^{-k}, \quad |\alpha|+l+|\beta|+|\gamma|+k = 0,1,2,\dots. \end{aligned} \tag{2.9}$$

Moreover, both families have compact variation in the parameters (v,ξ,p). Now one can readily show that the operators on the right-hand sides of (2.5) and (2.6) are well defined as ψDO in the sense of Luke [20]. Indeed, the change of variable $r = e^{-t}$ takes the cone K_Ω to the cylinder $\Omega \times \mathbb{R}$ and the operator $ir\partial/\partial r$ to $-i\partial/\partial t$. It remains to note that the operator $-i\partial/\partial t + i(n+1)/2$ is self-adjoint in the L^2 space with weight $e^{-(n+1)t}$ on the cylinder. This space is just the image of $L^2(K_\Omega)$ under our change of variable. Therefore, the substitution of this operator as an operator argument is well defined. Moreover, the resulting ψDO satisfies the estimates (2.7), and the inductive step is complete.

Remark 2.4. Since the cone is non-compact (in the variable r), it follows that the operator-valued symbol (2.6) has only *almost compact* variation in (ξ, v) in the general case. (That is, the variation becomes compact if we multiply it by a cut-off function compactly supported in r.) However, as we shall see shortly, the fibre variation of the symbol (2.6) is compact if all its symbols $\sigma_j\bigl(\sigma_0(D)\bigr)$ are zero, $j = 1,\dots,k$.

Definition 2.7. The *conormal symbol* $\sigma_c\bigl(\sigma_0(D)\bigr) \in \Psi(\mathfrak{M}_0^\circ \times V \times \mathbb{R}, \Omega)$ of the family (2.6) is the family $\sigma_c\bigl(\sigma_0(D)\bigr) = P(x,0,v,0,p)$.

Compatibility conditions. We have defined the notion of ψDO with parameters on a stratified manifold $\mathfrak{M}$. Such ψDO have symbols defined a priori on the cotangent bundles of the *open strata* times the parameter space without the zero section. However, if we compare the representation (2.5) with the representation valid in $U \setminus \mathfrak{M}_0$

by the inductive hypothesis, then we can see that these symbols can actually be extended continuously (and smoothly) up to the boundary of the cotangent bundle and hence are defined on the cotangent bundles of the corresponding *closed strata*. Moreover, the following compatibility conditions hold at the points where a stratum $\mathfrak{M}_j$ meets a stratum $\mathfrak{M}_i$, $j > i$:

$$\sigma_l\big(\sigma_j(D)\big)\big|_{\mathfrak{M}_i} = \sigma_l\big(\sigma_i(D)\big), \qquad l = j, \dots, k. \tag{2.10}$$

Naturally we write $\sigma_j\big(\sigma_j(D)\big) = \sigma_j(D)$.

Main properties of the calculus of ψDO. Let $\Psi(T^*X \times V, K_\Omega)$ be the set of all symbols of the form (2.6) over the manifold $X = \mathfrak{M}_0$.

Proposition 2.1. *The set $\Psi(T^*X \times V, K_\Omega)$ is a local C^*-algebra.*

The norm is given by the supremum of the operator norm over all parameter values.

Theorem 2.1 (main properties of ψDO). *Pseudo-differential operators have the following properties.*

1. *The set $\Psi(V, \mathfrak{M})$ of pseudo-differential operators on a stratified manifold $\mathfrak{M}$ is an algebra with respect to the usual composition of operators and is a local C^*-algebra with respect to the supremum of the operator norm over the parameter. Pseudo-differential operators compactly commute with the operators of multiplication by continuous functions on $\mathfrak{M}$.*

2. *The symbol map*

$$\begin{aligned} \sigma\colon \Psi(V, \mathfrak{M}) &\to \bigoplus_{j=0}^{k} \Psi(T^*\mathfrak{M}_j \times V, K_{\Omega_j}), \\ D &\mapsto \big(\sigma_0(D), \dots, \sigma_k(D)\big), \end{aligned} \tag{2.11}$$

is a local C^*-algebra homomorphism and induces an isomorphism

$$\sigma\colon \Psi(V, \mathfrak{M}) \,/\, J(V, \mathfrak{M}) \to \Sigma(V, \mathfrak{M}) \subset \bigoplus_{j=0}^{k} \Psi(T^*\mathfrak{M}_j \times V, K_{\Omega_j})$$

onto the local C^*-algebra of symbols satisfying the compatibility conditions (2.10). *Here $J(V, \mathfrak{M}) \subset \Psi(V, \mathfrak{M})$ is the ideal of compact-valued operator families vanishing at infinity.*

3. *The algebra $\Psi(V, \mathfrak{M})$ is invariant under diffeomorphisms of $\mathfrak{M}$.*

Definition 2.8. An operator $D \in \Psi(V, \mathfrak{M})$ is said to be *elliptic* if all its symbols $\sigma_j(D)$, $j = 0, \dots, k$, are invertible outside the zero sections of the corresponding bundles.

The following assertion is a corollary of Theorem 2.1.

Theorem 2.2. 1. *Elliptic operators on a compact stratified manifold $\mathfrak{M}$ are Fredholm* (*for all values of the parameter*).

2. *If $V \neq \{0\}$, then an elliptic operator with parameter is invertible for large $|v|$.*

We shall also use the following proposition.

Proposition 2.2. *Let $\Sigma_0 \subset \Psi(V, K_\Omega)$ be the set of symbols whose symbols corresponding to the strata of $\Omega \times \mathbb{R}_+$ are zero. Then each symbol $\sigma \in \Sigma_0$ has compact variation in v.*

Proof. By definition (see (2.6)), σ can be written as

$$\sigma(v) = P\biggl(rv,\, ir\frac{\partial}{\partial r} + i\,\frac{n+1}{2}\biggr)\colon L^2(K_\Omega) \to L^2(K_\Omega)$$

for some operator function $P(w,p) \in \Psi(V \times \mathbb{R}, \Omega)$. Theorem 2.1 implies that $P(w,p) \in J(V \times \mathbb{R}, \Omega)$ and satisfies the estimates (2.7) with respect to (w,p).

We must show that the derivative $\partial\sigma/\partial v$ is compact for $v \neq 0$. We have

$$\frac{\partial\sigma}{\partial v} = r\frac{\partial P}{\partial w}\biggl(rv,\, ir\frac{\partial}{\partial r} + i\,\frac{n+1}{2}\biggr).$$

The symbol $r\partial P/\partial w(rv,p)$ satisfies all the estimates needed for it to define a bounded operator on $L^2(K_\Omega)$ (see (2.9)). Furthermore, it is compact-valued. It remains to show that it tends to zero as $r \to 0$ and as $r \to \infty$. For $r \to 0$, this is obvious (because of the factor r), and for $r \to \infty$, one must consider the representation

$$r\,\frac{\partial P}{\partial w}(rv,p) = \frac{1}{|v|}\,|rv|\,\frac{\partial P}{\partial w}(rv,p) = \frac{1}{|v|}\biggl[|w|\,\frac{\partial P}{\partial w}\biggr]_{w=rv}$$

and apply the following lemma to $P(w,p)$.

Lemma 2.1. *Let $f \in C^2(\mathbb{R}^n)$ be an operator-valued function such that $f(\xi) \to 0$ as $\xi \to \infty$ and $\bigl|f''(\xi)\bigr| \leqslant C|\xi|^{-2}$ for large ξ. Then*

$$|\xi|\,\bigl|f'(\xi)\bigr| \to 0 \quad \text{as} \quad \xi \to \infty.$$

Semiclassical quantization. One can consider the quantization (2.6) with a 'semiclassical' parameter h:

$$T_h\colon \Psi(V \times \mathbb{R}, \Omega) \to \Psi(V, K_\Omega) \subset \mathcal{B}\bigl(L^2(K_\Omega)\bigr).$$

This map takes a family of operators with parameter on the base of the cone to a family of operators on the infinite cone by the rule

$$(T_h D)(v) := D\biggl(\overset{2}{r}v,\, ihr\overset{1}{\frac{\partial}{\partial r}} + ih\frac{n+1}{2}\biggr). \tag{2.12}$$

By the same argument as in the appendix of [8], one can show that this quantization is asymptotic in L^2, that is, the following estimates hold in the operator norm:

$$T_h(a)T_h(b) = T_h(ab) + o(1), \qquad h \to 0. \tag{2.13}$$

We shall use this quantization to compute the boundary map in the K-theory of algebras of ψDO.

§ 3. Ell-groups

Let $\mathfrak{M}$ be a stratified manifold, which we assume throughout to be compact.

Definition 3.1. Two elliptic operators

$$D\colon L^2(\mathfrak{M},E)\to L^2(\mathfrak{M},F),\qquad D'\colon L^2(\mathfrak{M},E')\to L^2(\mathfrak{M},F')$$

acting between sections of vector bundles over the blow-up M of $\mathfrak{M}$ are said to be *stably homotopic* if there is a continuous homotopy[4]

$$D\oplus 1_{E_0}\sim f^*(D'\oplus 1_{F_0})e^*$$

of elliptic operators, where $E_0,F_0\in \mathrm{Vect}(M)$ are vector bundles and

$$e\colon E\oplus E_0\to E'\oplus F_0,\qquad f\colon F'\oplus F_0\to F\oplus E_0$$

are vector bundle isomorphisms.

Here, ellipticity is understood as the invertibility of the components of a symbol on all the strata (see Definition 2.8), and we consider only homotopies in the class of elliptic ψDO.

3.1. Even groups $\mathbf{Ell_0(\mathfrak{M})}$. Stable homotopy is an equivalence relation on the set of all elliptic pseudo-differential operators acting between sections of vector bundles. Let $\mathrm{Ell}_0(\mathfrak{M})$ be the quotient by this equivalence relation. This quotient is a group with respect to the direct sum of elliptic operators. The inverse is defined as an almost inverse operator (that is, an inverse modulo compact operators).

3.2. Odd groups $\mathbf{Ell_1(\mathfrak{M})}$. In a similar way, one defines the odd elliptic theory $\mathrm{Ell}_1(\mathfrak{M})$ as the group of stable homotopy classes of elliptic self-adjoint operators. In this case, stabilization is in terms of the operators $\pm\,\mathrm{Id}$.

Remark 3.1. An equivalent definition of the odd Ell-group can be given in terms of elliptic families on $\mathfrak{M}$ parametrized by the circle $\mathbb{S}^1$, modulo constant families.

The *homotopy classification problem for elliptic operators* is the problem of computing the groups $\mathrm{Ell}_*(\mathfrak{M})$.

§ 4. Main theorem

4.1. A map into *K*-homology. Let

$$D\colon L^2(\mathfrak{M},E)\to L^2(\mathfrak{M},F)$$

be an elliptic operator, as in the preceding section. By Theorems 2.2 and 2.1, part 1, this operator can be treated as an abstract elliptic operator in the sense of Atiyah [1] on $\mathfrak{M}$. Hence it determines an element in the K-homology of $\mathfrak{M}$. The corresponding Fredholm module is defined in the standard way. Namely (see [22]), if D is self-adjoint (and $E=F$), then we consider the normalization

$$\mathcal{D}=(P_{\ker D}+D^2)^{-1/2}D\colon L^2(\mathfrak{M},E)\to L^2(\mathfrak{M},E), \tag{4.1}$$

where $P_{\ker D}$ is the projection onto the null-space of D.

[4] We can and shall always assume that the homotopies of all symbols and their derivatives are continuous in the corresponding norms.

In the general case, we consider the self-adjoint operator

$$\mathcal{D}=\begin{pmatrix} 0 & (P_{\ker D}+D^*D)^{-1/2}D^* \\ D(P_{\ker D}+D^*D)^{-1/2} & 0 \end{pmatrix}:$$
$$L^2(\mathfrak{M},E\oplus F)\to L^2(\mathfrak{M},E\oplus F), \tag{4.2}$$

which is odd with respect to the $\mathbb{Z}_2$-grading of the space $L^2(\mathfrak{M},E)\oplus L^2(\mathfrak{M},F)$.

Proposition 4.1. 1. *The operators* (4.1) *and* (4.2) *determine elements in K-homology denoted by $[D]\in K_*(\mathfrak{M})$, where $*=1$ for self-adjoint operators and $*=0$ in the general case.*

2. *There is a well-defined group homomorphism*

$$\mathrm{Ell}_*(\mathfrak{M})\xrightarrow{\varphi} K_*(\mathfrak{M}),\qquad D\mapsto [D].$$

Proof. The operators $\mathcal{D}$ in (4.1) and (4.2) are self-adjoint and act on $*$-modules over the C^*-algebra $C(\mathfrak{M})$. To complete the proof of part 1, it suffices to verify that

$$[\mathcal{D},f]\in\mathcal{K},\qquad (\mathcal{D}^2-1)f\in\mathcal{K} \tag{4.3}$$

for each $f\in C(\mathfrak{M})$, where $\mathcal{K}$ is the ideal of compact operators. These relations easily follow from the composition formula for pseudo-differential operators since $\mathcal{D}$ is a pseudo-differential operator. The map is well defined because homotopies of elliptic operators induce continuous homotopies of the corresponding Fredholm modules and, therefore, result in the same element in K-homology. Bundle isomorphisms give degenerate Fredholm modules. (We recall [22] that a module is *degenerate* if both the left-hand sides in (4.3) are zero.)

4.2. The classification theorem. The following theorem solves the classification problem for stratified manifolds.

Theorem 4.1. *The map*

$$\mathrm{Ell}_*(\mathfrak{M})\overset{\varphi}{\simeq} K_*(\mathfrak{M})$$

that takes each elliptic operator D to the element defined in Proposition 4.1 is an isomorphism.

The non-degeneracy of the index pairing $K_0(\mathfrak{M})\times K^0(\mathfrak{M})\to\mathbb{Z}$ (on the torsion-free parts of the groups) yields the following assertion.

Corollary 4.1. *Two elliptic operators D_1 and D_2 are stably rationally homotopic if and only if their indices with coefficients in any vector bundle over $\mathfrak{M}$ coincide.*

We shall obtain Theorem 4.1 as a special case of the more general theorem stated below.

4.3. Classification of partially elliptic operators. An operator D on $\mathfrak{M}$ is said to be *elliptic on the set* $\mathfrak{M}\setminus\mathfrak{M}_j$ if the components $\sigma_k(D),\dots,\sigma_{j+1}(D)$ of its symbol are invertible on their domains, that is, everywhere outside the zero sections of the respective bundles $T^*\mathfrak{M}_k,\dots,T^*\mathfrak{M}_{j+1}$.

Let $\mathrm{Ell}_*(\mathfrak{M},\mathfrak{M}_j)$ be the group of stable homotopy classes of pseudo-differential operators that are elliptic on $\mathfrak{M}\setminus\mathfrak{M}_j$. Here we mean that the homotopies are also taken in this class.

By analogy with the map φ constructed in Proposition 4.1, we define a map

$$\mathrm{Ell}_*(\mathfrak{M},\mathfrak{M}_j)\xrightarrow{\varphi}K_*(\mathfrak{M}\setminus\mathfrak{M}_j).$$

Here the operators $(P_{\ker D}+D^*D)^{-1/2}$ must be replaced by self-adjoint pseudo-differential operators with $k-j$ leading components of the symbol equal to

$$\bigl(\sigma_k(D)^*\sigma_k(D)\bigr)^{-1/2},\ \dots,\ \bigl(\sigma_{j+1}(D)^*\sigma_{j+1}(D)\bigr)^{-1/2}.$$

We note that both constructions give the same K-homology element for an operator elliptic on the whole of $\mathfrak{M}$.

Theorem 4.2. *For each j, $-1\leqslant j\leqslant k-1$, there is an isomorphism*

$$\mathrm{Ell}_*(\mathfrak{M},\mathfrak{M}_j)\overset{\varphi}{\simeq}K_*(\mathfrak{M}\setminus\mathfrak{M}_j).$$

The proof of Theorem 4.2 is given in §§ 5–7. First, in § 5, we represent the Ell-groups as the K-groups of certain algebras. (This is a non-commutative analogue of the Atiyah–Singer difference construction.) This enables us to define exact sequences for Ell-groups. Then we prove the theorem by induction on the strata (§§ 6 and 7).

§ 5. Connection between the Ell-groups and K-theory

5.1. The Ell-groups as K-groups of C^*-algebras. Pseudo-differential operators acting on sections of vector bundles can be described in terms of the embedding

$$C^\infty(M)\subset\Psi(\mathfrak{M})$$

of algebras of scalar operators. (The embedding corresponds to the usual action of functions as operators of multiplication.) Namely, an arbitrary zero-order ψDO acting between spaces of sections of vector bundles can be represented in the form

$$D'\colon\ \mathrm{Im}\,P\to\mathrm{Im}\,Q,$$

where P and Q are matrix projections ($P^2=P$ and $Q^2=Q$) with entries in $C^\infty(M)$ and D' is a matrix operator with entries in $\Psi(\mathfrak{M})$.

Let

$$\Sigma(\mathfrak{M}\setminus\mathfrak{M}_j)\overset{\mathrm{def}}{=}\mathrm{Im}(\sigma_k,\dots,\sigma_{j+1})\subset\bigoplus_{l\geqslant j+1}C^\infty\bigl(S^*M_l,\mathcal{B}\bigl(L^2(K_{\Omega_l})\bigr)\bigr)$$

be the algebra generated by the $k-j$ leading components of the symbol.

Theorem 4 in [5] gives isomorphisms

$$\mathrm{Ell}_*(\mathfrak{M},\mathfrak{M}_j)\overset{\chi}{\simeq}K_*\bigl(\mathrm{Con}\bigl(C^\infty(M)\xrightarrow{f}\Sigma(\mathfrak{M}\setminus\mathfrak{M}_j)\bigr)\bigr) \tag{5.1}$$

between the Ell-groups and the K-groups of a special local C^*-algebra. Here

$$f\colon C^\infty(M) \to \Sigma(\mathfrak{M}\setminus\mathfrak{M}_j)$$

is the embedding taking a smooth function on the blow-up M to the symbol of the operator of multiplication by this function, and

$$\operatorname{Con}(A \xrightarrow{f} B) = \bigl\{\bigl(a, b(t)\bigr) \in A \oplus C_0\bigl([0,1), B\bigr) \mid f(a) = b(0)\bigr\}$$

is the mapping cone of the algebra homomorphism $f\colon A \to B$.

In many interesting cases, the right-hand side of (5.1) can be represented in an equivalent form that does not contain the mapping cone.

Lemma 5.1. *There is an isomorphism*

$$K_{*+1}\bigl(\operatorname{Con}\bigl(C^\infty(M) \to \Sigma(\mathfrak{M}\setminus\mathfrak{M}_j)\bigr)\bigr) \simeq K_*\bigl(\Sigma(\mathfrak{M}\setminus\mathfrak{M}_j)\bigr)/K_*\bigl(C^\infty(M)\bigr)$$

provided that there is a non-vanishing vector field on M. (This condition holds if, for example, M has no components with empty boundary.)

Proof. A vector field $M \to S^*M$ defines a section $\Sigma(\mathfrak{M}\setminus\mathfrak{M}_j) \to C^\infty(M)$. Thus the exact sequence of the mapping cone for the embedding of algebras splits. This gives the desired isomorphism.

Remark 5.1. In the odd case, the composite of this isomorphism with χ shows that, modulo stable homotopy, elliptic self-adjoint operators are isomorphic to symbols-projections modulo projections determining sections of vector bundles (compare with [23]).

5.2. Exact sequence of the pair in Ell-theory. Let us construct an exact sequence corresponding to the pair of spaces

$$\mathfrak{M}_j\setminus\mathfrak{M}_{j-1} \subset \mathfrak{M}\setminus\mathfrak{M}_{j-1}$$

in the elliptic theory. We define the maps in the desired sequence in terms of maps in K-theory. To this end, we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \ker(\sigma_k,\dots,\sigma_{j+1}) & \to & \Sigma(\mathfrak{M}\setminus\mathfrak{M}_{j-1}) & \xrightarrow{(\sigma_k,\dots,\sigma_{j+1})} & \Sigma(\mathfrak{M}\setminus\mathfrak{M}_j) & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \to & 0 & \to & C^\infty(M) & = & C^\infty(M) & \to 0 \end{array} \tag{5.2}$$

with exact rows.

The ideal $\ker(\sigma_k,\dots,\sigma_{j+1})$ of symbols with the leading $k-j$ components equal to zero will be denoted by Σ_0 for brevity. The diagram induces the exact sequence

$$0 \to S\Sigma_0 \to \operatorname{Con}\bigl(C^\infty(M) \to \Sigma(\mathfrak{M}\setminus\mathfrak{M}_{j-1})\bigr) \to \operatorname{Con}\bigl(C^\infty(M) \to \Sigma(\mathfrak{M}\setminus\mathfrak{M}_j)\bigr) \to 0 \tag{5.3}$$

of the mapping cones of the vertical embeddings. Here $S\Sigma_0 = C_0\bigl((0,1), \Sigma_0\bigr)$ stands for the suspension.

According to (5.1), the K-groups of the mapping cones in (5.3) classify elliptic operators on $\mathfrak{M}\setminus\mathfrak{M}_{j-1}$ and $\mathfrak{M}\setminus\mathfrak{M}_j$ respectively. The K-groups of the ideal also classify elliptic operators.

Lemma 5.2. *The K-groups of $S\Sigma_0$ classify elliptic operators on $\mathfrak{M} \setminus \mathfrak{M}_{j-1}$ whose symbols $(\sigma_k, \dots, \sigma_{j+1})$ are induced by multiplication by constant functions in $C^\infty(M)$.*

Proof. To be definite, we consider the even group $K_0(S\Sigma_0)$. Then $K_0(S\Sigma_0) = K_1(\Sigma_0) = K_1(\Sigma_0^+)$, where Σ_0^+ stands for the algebra with the identity adjoined. It now follows from the definitions of K_1 and Σ_0 that $K_1(\Sigma_0^+)$ classifies the elliptic operators described in the lemma.

By Lemma 5.2, we can define the map (compare with Proposition 4.1)

$$\varphi \colon K_*(S\Sigma_0) \to K_*(\mathfrak{M}_j \setminus \mathfrak{M}_{j-1}) \tag{5.4}$$

that restricts elliptic operators to a neighbourhood U of $\mathfrak{M}_j \setminus \mathfrak{M}_{j-1}$, where U possesses the structure of a bundle $\pi \colon U \to \mathfrak{M}_j \setminus \mathfrak{M}_{j-1}$ with conical fibre. We note that the structure

$$C_0(\mathfrak{M}_j \setminus \mathfrak{M}_{j-1}) \xrightarrow{\pi^*} C(U) \to \mathcal{B}\bigl(L^2(U)\bigr)$$

of a $C_0(\mathfrak{M}_j \setminus \mathfrak{M}_{j-1})$-module on the corresponding L^2-spaces (where the operator acts) is obtained by the pullback from the base of the bundle. The operator has a well-defined restriction since, by construction, outside an arbitrarily small neighbourhood of $\mathfrak{M}_j \setminus \mathfrak{M}_{j-1}$ it is the operator of multiplication by some function.

In what follows (see Lemma 7.1) we shall show that there is an isomorphism

$$K_*(S\Sigma_0) \simeq K_c^*(T^*M_j) = \operatorname{Ell}_*(\mathfrak{M}_j, \mathfrak{M}_{j-1}). \tag{5.5}$$

Therefore we can replace all K-groups by Ell-groups in the exact sequence induced in K-theory by the sequence (5.3). As a result, we obtain the following periodic six-term exact sequence relating the Ell-groups.

$$\begin{tikzcd}
& \operatorname{Ell}_0(\mathfrak{M}_j, \mathfrak{M}_{j-1}) \arrow[r] & \operatorname{Ell}_0(\mathfrak{M}, \mathfrak{M}_{j-1}) \arrow[dr] & \\
\operatorname{Ell}_1(\mathfrak{M}, \mathfrak{M}_j) \arrow[ur, "\partial"] & & & \operatorname{Ell}_0(\mathfrak{M}, \mathfrak{M}_j) \arrow[dl, "\partial"] \\
& \operatorname{Ell}_1(\mathfrak{M}, \mathfrak{M}_{j-1}) \arrow[ul] & \operatorname{Ell}_1(\mathfrak{M}_j, \mathfrak{M}_{j-1}) \arrow[l] &
\end{tikzcd} \tag{5.6}$$

We do not dwell upon how to define all the maps in this sequence directly in terms of elliptic operators.

§ 6. Induction

For j ranging from k down to -1, we shall prove by induction that the map φ in Theorem 4.2 is an isomorphism on the set $\mathfrak{M} \setminus \mathfrak{M}_j$.

For $j = k$, this is obvious. Let us prove the inductive step: the isomorphism for j implies an isomorphism for $j-1$.

The pair $\mathfrak{M}_j \setminus \mathfrak{M}_{j-1} \subset \mathfrak{M} \setminus \mathfrak{M}_{j-1}$ gives the diagram

$$\begin{array}{ccccccccc} \cdots \to \mathrm{Ell}_{*+1}(\mathfrak{M}, \mathfrak{M}_j) & \xrightarrow{\partial} & \mathrm{Ell}_*(\mathfrak{M}_j, \mathfrak{M}_{j-1}) & \to & \mathrm{Ell}_*(\mathfrak{M}, \mathfrak{M}_{j-1}) & \to & \mathrm{Ell}_*(\mathfrak{M}, \mathfrak{M}_j) & \xrightarrow{\partial} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots \to K_{*+1}(\mathfrak{M} \setminus \mathfrak{M}_j) & \xrightarrow{\partial} & K_*(\mathfrak{M}_j \setminus \mathfrak{M}_{j-1}) & \to & K_*(\mathfrak{M} \setminus \mathfrak{M}_{j-1}) & \to & K_*(\mathfrak{M} \setminus \mathfrak{M}_j) & \xrightarrow{\partial} & \cdots \end{array} \tag{6.1}$$

We note that the vertical maps in the diagram are defined by Proposition 4.1 and formulae (5.4) and (5.5). Once we have proved that the diagram commutes, the inductive hypothesis can be combined with the five lemma to obtain that the vertical map with values in $K_*(\mathfrak{M} \setminus \mathfrak{M}_{j-1})$ is an isomorphism, as desired. Let us establish the commutativity.

The commutativity of the square

$$\begin{array}{ccc} \mathrm{Ell}_*(\mathfrak{M}, \mathfrak{M}_{j-1}) & \longrightarrow & \mathrm{Ell}_*(\mathfrak{M}, \mathfrak{M}_j) \\ \Big\downarrow & & \Big\downarrow \\ K_*(\mathfrak{M} \setminus \mathfrak{M}_{j-1}) & \longrightarrow & K_*(\mathfrak{M} \setminus \mathfrak{M}_j) \end{array}$$

is obvious since the horizontal arrows are just the forgetful maps.

It is also easy to prove the commutativity of the square

$$\begin{array}{ccc} \mathrm{Ell}_*(\mathfrak{M}_j, \mathfrak{M}_{j-1}) & \longrightarrow & \mathrm{Ell}_*(\mathfrak{M}, \mathfrak{M}_{j-1}) \\ \Big\downarrow & & \Big\downarrow \\ K_*(\mathfrak{M}_j \setminus \mathfrak{M}_{j-1}) & \longrightarrow & K_*(\mathfrak{M} \setminus \mathfrak{M}_{j-1}) \end{array} \tag{6.2}$$

which corresponds to the embedding $i\colon \mathfrak{M}_j \setminus \mathfrak{M}_{j-1} \to \mathfrak{M} \setminus \mathfrak{M}_{j-1}$. Indeed, consider the composite of the arrows passing through the top right corner of the square (6.2). It takes each elliptic operator on $\mathfrak{M} \setminus \mathfrak{M}_{j-1}$ equal to a multiplication operator outside a neighbourhood U of the stratum $\mathfrak{M}_j \setminus \mathfrak{M}_{j-1}$ to the same operator with the natural $C_0(\mathfrak{M} \setminus \mathfrak{M}_{j-1})$-module structure on the spaces between which the operator acts. If we now restrict the operator to a neighbourhood of $\mathfrak{M}_j \setminus \mathfrak{M}_{j-1}$ (which does not change the K-homology element since the corresponding Fredholm module varies by a degenerate module) and make a homotopy of the module structure to the composite

$$C_0(\mathfrak{M} \setminus \mathfrak{M}_{j-1}) \xrightarrow{i^*} C_0(\mathfrak{M}_j \setminus \mathfrak{M}_{j-1}) \xrightarrow{\pi^*} C(U) \longrightarrow \mathcal{B}\big(L^2(U)\big),$$

where $\pi\colon U \to \mathfrak{M}_j \setminus \mathfrak{M}_{j-1}$ is the projection, then we obtain the element to which the operator is taken by the composite of the arrows passing through the bottom left corner of the square. This proves the commutativity of the square (6.2).

To justify the inductive step, it remains to verify the commutativity of the squares containing the boundary maps in (6.1). This is the most technically involved part of the proof, and it occupies the next section.

§ 7. Comparison of the boundary maps

In this section we prove the commutativity of the squares containing the boundary maps in (6.1). The computations are carried out in terms of the K-groups of symbol algebras. First we outline the scheme of the proof, which is rather lengthy.

(i) We show that the K-homology boundary map is the composite of the restriction to the boundary of the stratum and the direct image map (see § 7.1).

(ii) The K-theory boundary map is also the composite of the restriction map and some boundary map ∂' corresponding to the algebra $\Psi(T^*M_j, K_\Omega)$ of symbols on the stratum M_j (§ 7.2). Hence the comparison of the boundary maps in the K-theory and K-homology is reduced to a computation on M_j.

(iii) The boundary map ∂' is not so easy to work with directly. Therefore we replace $\Psi(T^*M_j, K_\Omega)$ by the simpler algebra $\Psi(T^*M_j \times \mathbb{R}, \Omega)$ of families with parameters, for which the boundary map ∂'' is easier to describe. More precisely, we define an asymptotic homomorphism of one algebra to the other (§ 7.3) and show that the asymptotic homomorphism induces an isomorphism of K-groups (§ 7.4).

(iv) The boundary map ∂'' corresponding to the algebra $\Psi(T^*M_j \times \mathbb{R}, \Omega)$ is expressed in § 7.5 in terms of the index of elliptic families with parameters. Then the compatibility of the boundary map ∂'' with the direct image in K-homology, and hence the commutativity of the diagram (6.1), are obtained in § 7.5.

Let us proceed to a detailed proof of the commutativity of the diagram (6.1).

7.1. The boundary map in the lower row of the diagram (6.1). We start with some notation. Let U be a neighbourhood of the open stratum $\mathfrak{M}_j \setminus \mathfrak{M}_{j-1} = \mathfrak{M}_j^\circ$, fibred with a conical fibre over the stratum. Let $\pi\colon U \to \mathfrak{M}_j \setminus \mathfrak{M}_{j-1}$ be the projection. The corresponding bundle of cone bases will be denoted by

$$\pi'\colon \Xi \to \mathfrak{M}_j \setminus \mathfrak{M}_{j-1}.$$

A typical fibre of the bundle will be denoted by Ω. One obviously has $U \setminus \mathfrak{M}_j^\circ \simeq \mathbb{R}_+ \times \Xi$. Then the boundary map $\partial\colon K_*(\mathfrak{M} \setminus \mathfrak{M}_j) \to K_{*+1}(\mathfrak{M}_j \setminus \mathfrak{M}_{j-1})$ is equal to the composite

$$K_*(\mathfrak{M} \setminus \mathfrak{M}_j) \longrightarrow K_*(U \setminus \mathfrak{M}_j^\circ) = K_{*+1}(\Xi) \xrightarrow{\pi'_*} K_{*+1}(\mathfrak{M}_j \setminus \mathfrak{M}_{j-1}) \tag{7.1}$$

of the restriction to $U \setminus \mathfrak{M}_j^\circ$, the periodicity isomorphism and the direct image map. One can readily obtain this decomposition from the fact that the boundary map is natural.

7.2. Reduction to the boundary. Let us compute the boundary map in the K-theory of algebras. Since the boundary map

$$\partial\colon K_*\bigl(\operatorname{Con}\bigl(C^\infty(M) \to \Sigma(\mathfrak{M} \setminus \mathfrak{M}_j)\bigr)\bigr) \to K_{*+1}(S\Sigma_0)$$

is natural, it is equal to the composite

$$\begin{aligned} &K_*\bigl(\operatorname{Con}\bigl(C^\infty(M) \to \Sigma(\mathfrak{M} \setminus \mathfrak{M}_j)\bigr)\bigr) \\ &\qquad\longrightarrow K_*\bigl(\operatorname{Con}\bigl(C^\infty(\partial_j M) \to \Sigma_{M_j}\bigr)\bigr) \xrightarrow{\partial'} K_{*+1}(S\Sigma_0) \end{aligned} \tag{7.2}$$

with the restriction of the symbols to M_j, where $\Sigma_{M_j} \equiv \Sigma(T^*M_j, \Omega)$ is the algebra of symbols of ψDO on Ω with parameters in T^*M_j (from now on, we fix some isomorphism $T^*\mathfrak{M}_j \simeq T^*M_j$) and $\partial_j M \subset \partial M$ is the closure of $\pi^{-1}\mathfrak{M}_j^\circ$. We note that $\partial_j M$ is a manifold fibred over M_j with fibre isomorphic to the blow-up of the stratified manifold Ω.

7.3. The asymptotic homomorphism. We recall that $\Psi(T^*M_j, K_\Omega)$ is the algebra of jth symbols on M_j. To compute the boundary map ∂', we replace the algebra $\Psi(T^*M_j, K_\Omega)$ (without changing its K-group) by an algebra of pseudodifferential operators with parameters, for which the boundary map is simpler.

To this end, we consider the map (see (2.12))

$$T_h\colon \Psi(T^*M_j \times \mathbb{R}, \Omega) \longrightarrow \Psi(T^*M_j, K_\Omega), \qquad h \in (0,1],$$

$$(T_h u)(\xi) = u\biggl(\overset{2}{r}\xi, ihr\overset{1}{\frac{\partial}{\partial r}} + ih\frac{n+1}{2}\biggr), \qquad (\xi, p) \in T^*M_j \times \mathbb{R},$$

where $\Psi(T^*M_j \times \mathbb{R}, \Omega)$ is the algebra of smooth families of ψDO on the fibres Ω with parameters in $T^*M_j \times \mathbb{R}$. As $h \to 0$, we have

$$T_h(ab) = T_h(a)T_h(b) + o(1), \qquad \bigl(T_h(a)\bigr)^* = T_h(a^*) + o(1), \tag{7.3}$$

where $a, b \in \Psi(T^*M_j \times \mathbb{R}, \Omega)$ are arbitrary and $o(1)$ refers to the uniform operator norm.

This semiclassical quantization is a special case of the so-called asymptotic homomorphisms, which play an important role in the theory of C^*-algebras [24]–[26]. In particular, it follows from (7.3) that T_h induces a K-group homomorphism

$$T\colon K_*\bigl(\Psi(T^*M_j \times \mathbb{R}, \Omega)\bigr) \to K_*\bigl(\Psi(T^*M_j, K_\Omega)\bigr).$$

Now let us consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J(T^*M_j \times \mathbb{R}, \Omega) \arrow[r] \arrow[d, "T_h"] & \Psi(T^*M_j \times \mathbb{R}, \Omega) \arrow[r] \arrow[d, "T_h"] & \Sigma_{M_j} \arrow[r] \arrow[d, "t_h"] & 0 \\
0 \arrow[r] & \Sigma_0 \arrow[r] & \Psi(T^*M_j, K_\Omega) \arrow[r] & \Sigma_{M_j} \arrow[r] & 0
\end{tikzcd} \tag{7.4}$$

where t_h is the induced map on the symbols of families. (It is an algebra isomorphism.)

The algebra $C^\infty(\partial_j M)$ is embedded in each of the algebras $\Psi(T^*M_j \times \mathbb{R}, \Omega)$, $\Psi(T^*M_j, K_\Omega)$ and Σ_{M_j}. The diagram of the mapping cones of these embeddings gives the square

$$\begin{tikzcd}
K_*\bigl(\operatorname{Con}\bigl(C^\infty(\partial_j M) \to \Sigma_{M_j}\bigr)\bigr) \arrow[r, "\partial''"] \arrow[d, equal] & K_*\bigl(C_0(T^*M_j \times \mathbb{R})\bigr) \arrow[d, "T"] \\
K_*\bigl(\operatorname{Con}\bigl(C^\infty(\partial_j M) \to \Sigma_{M_j}\bigr)\bigr) \arrow[r, "\partial'"] & K_{*+1}(S\Sigma_0)
\end{tikzcd} \tag{7.5}$$

of K-groups. (The horizontal arrows are just the boundary maps in the corresponding sequences.) Here we have used the isomorphism

$$K_*\bigl(J(T^*M_j \times \mathbb{R}, \Omega)\bigr) \simeq K_*\bigl(C_0(T^*M_j \times \mathbb{R})\bigr)$$

induced by the embedding $J(T^*M_j \times \mathbb{R}, \Omega) \subset C_0(T^*M_j \times \mathbb{R}, \mathcal{K})$ of the local C^*-algebra in its closure.

The square (7.5) commutes since the boundary map in K-theory is natural with respect to asymptotic homomorphisms (for example, see [27]).

7.4. The map $T\colon K_*\bigl(C_0(T^*M_j \times \mathbb{R})\bigr) \to K_*(\Sigma_0)$. First let us compute the groups $K_*(S\Sigma_0)$. To this end, we consider the short exact sequence

$$0 \longrightarrow \ker \sigma_c \longrightarrow \Sigma_0 \xrightarrow{\sigma_c} J(M_j \times \mathbb{R}, \Omega) \longrightarrow 0 \tag{7.6}$$

of local C^*-algebras. (Here σ_c is the conormal symbol map; see Definition 2.7.) The algebra $\ker \sigma_c$ is formed by families of compact operators.[5] Since the embeddings

$$\ker \sigma_c \subset C\bigl(S^*M_j, \mathcal{K}(K_\Omega)\bigr), \qquad J(M_j \times \mathbb{R}, \Omega) \subset C_0\bigl(M_j \times \mathbb{R}, \mathcal{K}(\Omega)\bigr)$$

induce isomorphisms in K-theory, it follows that the K-theory long exact sequence corresponding to (7.6) can be written as the upper row of the diagram

$$\begin{array}{ccccccc}
\longrightarrow K^*(S^*M_j) & \longrightarrow & K_*(\Sigma_0) & \longrightarrow & K_c^*(M_j \times \mathbb{R}) \longrightarrow \\
\Big\| & & \Big\downarrow L & & \Big\| \\
\longrightarrow K^*(S^*M_j) & \longrightarrow & K_c^{*+1}(T^*M_j) & \longrightarrow & K_c^*(M_j \times \mathbb{R}) \longrightarrow
\end{array} \tag{7.7}$$

where the lower row is the sequence of topological K-groups of the pair $S^*M_j \subset B^*M_j$ formed by the unit sphere and ball bundles in T^*M_j, and the map L is the difference construction for ψDO with operator-valued symbols in the sense of Luke (see [20] and [28]). We recall that this map is defined as follows. An element σ_j of the unitalized algebra Σ_0^+ is an operator-valued function on S^*M_j with compact variation in the fibres of $S^*_xM_j$ (see Proposition 2.2). If this function is invertible, then L takes each element $[\sigma_j] \in K_1(\Sigma_0)$ to the index

$$L[\sigma_j] := \operatorname{ind} \tilde{\sigma}_j \in K_c(T^*M_j) \tag{7.8}$$

of an extension $\tilde{\sigma_j}$ of σ_j to the unit ball bundle in T^*M_j preserving the compact variation property in the fibres. (The extension is a family that is Fredholm in B^*M_j and invertible on S^*M_j. Thus its index is an element of the K-group with compact supports mentioned above.) The index in (7.8) is independent of the choice of the extension. For even K-groups, the map L is defined in a similar way.

Lemma 7.1. *The diagram (7.7) commutes, and hence L is an isomorphism.*

---

[5] Just as in the theory of operators on manifolds with isolated singularities, a family in Σ_0 is compact if and only if its conormal symbol is zero.

Proof. 1. The commutativity of the squares

$$\begin{array}{ccc} K^*(S^*M_j) & \longrightarrow & K_*(\Sigma_0) \\ \Big\| & & \Big\downarrow L \\ K^*(S^*M_j) & \longrightarrow & K^{*+1}(T^*M_j) \end{array}$$

follows from the fact that L coincides with the Atiyah–Singer difference construction in the case of finite-dimensional symbols.

2. Now let us consider the square

$$\begin{array}{ccc} K_*(\Sigma_0) & \xrightarrow{\sigma_c} & K_c^*(M_j \times \mathbb{R}) \\ \Big\downarrow L & & \Big\| \\ K_c^{*+1}(T^*M_j) & \xrightarrow{j^*} & K^{*+1}(M_j) \end{array}$$

where $j\colon M_j \to T^*M_j$ is the embedding of the zero section. Its commutativity follows from the index formula (for example, see [29])

$$\beta \operatorname{ind} D_y = \operatorname{ind} \sigma_c(D_y) \in K_c^1(Y \times \mathbb{R}) \tag{7.9}$$

for a family D_y, $y \in Y$, of elliptic operators with unit interior symbol on the infinite cone. Here Y is a compact parameter space and β is the periodicity isomorphism $K(Y) \simeq K_c^1(Y \times \mathbb{R})$.

Indeed, if $a \in K_1(\Sigma_0)$, then the element $j^*L(a)$ (resp. $\sigma_c(a)$) is the left-hand side (resp. the right-hand side) of the index formula (7.9). If $a \in K_0(\Sigma_0)$, then one must first pass to the suspension and then apply formula (7.9).

3. We now consider the square

$$\begin{array}{ccc} K_c^{*+1}(M_j \times \mathbb{R}) & \xrightarrow{\partial} & K^*(S^*M_j) \\ \Big\| & & \Big\| \\ K^*(M_j) & \xrightarrow{p^*} & K^*(S^*M_j) \end{array}$$

where $p\colon S^*M_j \to M_j$ is the natural projection. Its commutativity also follows from the index formula (7.9). Indeed, for an element $a \in K_c^1(M_j \times \mathbb{R})$, which can be represented by a family of invertible conormal symbols with unit symbol, we obtain the element $p^* \operatorname{ind} a \in K^0(S^*M_j)$ by passing through the left bottom corner of the square. On the other hand, passing through the upper right corner, we obtain

$$\partial a = p^* \operatorname{ind} \hat{a},$$

where $\hat{a}$ is an operator family on K_Ω with unit interior symbol and conormal symbol a. This equality holds because the boundary map ∂ takes an invertible symbol to the index of the corresponding operator. Applying (7.9), we obtain the desired relation $p^* \operatorname{ind} a = p^* \operatorname{ind} \hat{a} \in K^0(S^*M_j)$. For $a \in K_c^0(M_j \times \mathbb{R})$, one must first pass to the suspension.

Lemma 7.2. *The map*

$$T\colon K_*\bigl(C_0(T^*M_j\times\mathbb{R})\bigr)\to K_*(\Sigma_0)$$

is the inverse of the isomorphism L (see (7.7)).

Proof. For example, consider the map

$$T\colon K_1\bigl(C_0(T^*M_j\times\mathbb{R})\bigr)\to K_1(\Sigma_0).$$

Let us prove that LT is the identity map on the K-group.

1. Indeed, if $u(\xi,p)\in\bigl(1+J(T^*M_j\times\mathbb{R},\Omega)\bigr)$ is a symbol invertible for $(\xi,p)\in T^*M_j\times\mathbb{R}$ and equal to the identity on the complement of a compact set, then

$$LT[u]=\operatorname{ind}u\biggl(\overset{2}{r}\xi,\,ihr\overset{1}{\frac{\partial}{\partial r}}+ih\frac{n+1}{2}\biggr)\in K^0_c(T^*M_j)$$

(for sufficiently small h), where $[u]\in K^1_c(T^*M_j\times\mathbb{R})$. The index element is well defined since the operator-valued function $(T_hu)(\xi)$ has compact variation on the fibres of $T^*M_j\setminus\mathbf{0}$ and is invertible for $\xi\neq0$ (see Proposition 2.2 and formula (2.13), respectively).

2. Let $\overline{T_hu}$ be the family of Fredholm operators parametrized by $T^*M_j\setminus\mathbf{0}$ and equal to T_hu for $|\xi|<1$ and to

$$u\biggl((\overset{2}{r}+|\xi|-1)\xi,\,ihr\overset{1}{\frac{\partial}{\partial r}}+ih\frac{n+1}{2}\biggr)$$

for $|\xi|\geqslant1$. For sufficiently small h, this family is invertible for all ξ. (This follows since the support of $1-u$ is bounded and the estimates (2.13) hold uniformly with respect to the parameter λ, const $>\lambda\geqslant0$, if we replace $r\xi$ by $(r+\lambda)\xi$ in (2.12).) By construction,

$$\operatorname{ind}T_hu=\operatorname{ind}\overline{T_hu}.$$

However, the family $\overline{T_hu}$ consists of identity operators for ξ lying outside a compact set. Hence its index can be calculated by (7.9) and is equal to the index of the family of conormal symbols $u(\xi,p)$ (modulo the Bott periodicity isomorphism), that is, it does give the original element $LT[u]=[u]$ of the K-group.

7.5. Comparison of the boundary maps. Let us compare the expressions obtained above for the boundary maps in K-homology and K-theory of algebras (see (7.1) and (7.2), (7.5)). We ignore the restriction maps in §§ 7.1 and 7.2. Consider the diagram (from now on, $I=(0,1)$)

$$\begin{array}{ccc}
K_*(\Xi\times I) & \xrightarrow{\pi'_*} & K_*(M_j^\circ\times I)\\
\big\uparrow{\scriptstyle\varphi} & & \big\uparrow{\scriptstyle\varphi}\\
K_*\bigl(\mathrm{Con}\bigl(C^\infty(\partial_jM)\to\Sigma_{M_j}\bigr)\bigr) & \xrightarrow{\partial''} & K_c^*(T^*M_j\times\mathbb{R})\\
\Big\| & & \big\uparrow{\scriptstyle L}\\
K_*\bigl(\mathrm{Con}\bigl(C^\infty(\partial_jM)\to\Sigma_{M_j}\bigr)\bigr) & \xrightarrow{\partial'} & K_*(\Sigma_0)
\end{array}\qquad(7.10)$$

containing the boundary maps in the middle and bottom rows. Only the vertical arrow φ in the left column has not yet been defined. To define it, we note that the groups $K_*\bigl(\mathrm{Con}\bigl(C^\infty(\partial_j M) \to \Sigma_{M_j}\bigr)\bigr)$ classify the elliptic families $\sigma(x,\xi,p)$ of operators on Ω with parameters $(x,\xi,p) \in T^*M_j \times \mathbb{R}$. Such a family defines an operator on the product $\Xi \times I$ by the formula

$$\sigma\left(x,\, -i\,\frac{\partial}{\partial x},\, -i\,\frac{\partial}{\partial t}\right).$$

This operator specifies an element of $K_*(\Xi \times I)$ provided that the family $\sigma(x,\xi,p)$ is elliptic.

The bottom square in (7.10) is isomorphic to (7.5) by Lemma 7.2. Hence it commutes. We claim that the top square also commutes. Let $z \in K_*\bigl(\mathrm{Con}\bigl(C^\infty(\partial_j M) \to \Sigma_{M_j}\bigr)\bigr)$ be the element defined by an elliptic family $\sigma(x,\xi,p)$. Since the family is elliptic, it is Fredholm and the boundary map ∂'' applied to z is just the index of the family with parameters in $T^*M_j \times \mathbb{R}$. On the other hand, the element $\pi'_* z$ corresponds to the elliptic operator

$$\sigma\left(x,\, -i\,\frac{\partial}{\partial x},\, -i\,\frac{\partial}{\partial t}\right)$$

regarded as an operator on $C_0(M_j^\circ \times I)$-modules. That these two elements actually coincide is a consequence of the following general result.

Theorem 7.1. *Let $p(x,\xi)$ be an operator-valued symbol, elliptic in the sense of Luke* [20], *on a compact manifold X with corners. Then*

$$\left[p\left(x, -i\,\frac{\partial}{\partial x}\right)\right] = \varphi\bigl(\mathrm{ind}\, p(x,\xi)\bigr) \in K_*(X^\circ), \tag{7.11}$$

where X° is the interior of the manifold, square brackets stand for an element in K-homology and $\varphi\colon K^*_c(T^*X) \to K_*(X^\circ)$ is the Poincaré isomorphism on manifolds with corners (*for example, see* [30]).

The proof of Theorem 7.1 is given in the appendix (see § 9).

Thus, Theorem 7.1 ensures that diagram (7.10) commutes. Hence the squares containing the boundary maps in the diagram (6.1) also commute.

The proof of Theorem 4.2 is complete.

§ 8. Applications

8.1. Topological obstruction to the Fredholm property. Let $\mathfrak{M} \supset X$ be a stratified pair. It is of interest to find conditions under which a given elliptic operator on $\mathfrak{M} \setminus X$ can be transformed into an elliptic operator on $\mathfrak{M}$ without changing the components of the symbol over $\mathfrak{M} \setminus X$. This question is similar to the Atiyah–Bott problem of determining topological conditions on the symbol on a smooth manifold with boundary under which there is a Fredholm boundary condition for the corresponding operator.

We shall answer a similar question for elements of Ell-groups in the case of an arbitrary stratification. To this end, let us consider the diagram

$$\begin{tikzcd} \mathrm{Ell}(\mathfrak{M}) \arrow[r] \arrow[d, "\varphi", "\simeq"'] & \mathrm{Ell}(\mathfrak{M}, X) \arrow[d, "\varphi"', "\simeq"] & \\ K_0(\mathfrak{M}) \arrow[r] & K_0(\mathfrak{M} \setminus X) \arrow[r, "\partial"] & K_1(X) \end{tikzcd}$$

It obviously commutes since we are dealing with forgetful maps. Hence the non-vanishing of $\partial\varphi(a)$ is a necessary and sufficient condition for the existence of a lift of $a \in \mathrm{Ell}(\mathfrak{M}, X)$ to the group $\mathrm{Ell}(\mathfrak{M})$.

The boundary map in K-homology plays a similar role to the obstruction in other problems (see [31]–[33]).

We note that the vanishing of $\partial\varphi(a)$ is a condition on the interior symbol of the operator (a finite-dimensional condition) if $X = \mathfrak{M}_{k-1}$ is the set of all singularities of the manifold $\mathfrak{M} = \mathfrak{M}_k$.

8.2. Cobordism invariance of the index. Let us consider a generalization of the usual cobordism invariance of the index of Dirac operators on a smooth manifold. Suppose that X is a smooth stratum. Then we have a commutative diagram

$$\begin{tikzcd} \mathrm{Ell}_1(\mathfrak{M}, X) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Ell}(X) \arrow[d, "\simeq"] & \\ K_1(\mathfrak{M} \setminus X) \arrow[r] & K_0(X) \arrow[r] & K_0(\mathfrak{M}) \end{tikzcd}$$

Since the map $K_0(X) \to K_0(\mathfrak{M})$ preserves the index ($\in \mathbb{Z}$), we see that the index of an elliptic operator D on X is zero provided that the element $[D] \in \mathrm{Ell}(X)$ is the image of some element of $\mathrm{Ell}_1(\mathfrak{M}, X)$.

Remark 8.1. For non-smooth X, the construction of such a commutative diagram is an open problem (not even the exact sequence in Ell-theory is known).

§ 9. Appendix. Proof of Theorem 7.1

We note that Theorem 7.1 is a refinement of Luke's theorem [20] on the index of operators with operator-valued symbols. (The latter is obtained from equation (7.11) if one assumes that $\partial X = \varnothing$ and applies the index map to the K-homology elements.)

If the symbols are homogeneous for large $|\xi|$, then the proof is a verbatim repetition of [20]. The general case (non-homogeneous symbols) can be reduced to the case of homogeneous symbols by a method suggested in [28], where it was adapted to the computation of the index. The reduction used in the present paper is based on the following standard fact of Kasparov's KK-theory.

Proposition 9.1. *Let P_t be a $*$-strongly continuous homotopy of bounded operators such that the families*

$$f[P_tP_t^* - 1], \qquad f[P_t^*P_t - 1], \qquad [f, P_t], \qquad [f, P_t^*] \qquad \forall f \in C_0(X^\circ)$$

are norm-continuous families of compact operators. Then the corresponding element in K-homology is independent of the parameter:

$$[P_0] = [P_1] \in K_*(X^\circ).$$

Proof. The hypotheses mean that the family $\{P_t\}$ defines an operator on the $\big(C_0(X^\circ), C\big([0,1]\big)\big)$-bimodule $C_0\big(X^\circ \times [0,1], L^2(H)\big)$. This operator defines a homotopy (in the sense of KK-theory) between P_0 and P_1 (for example, see [34]). The fact that the K-homology element $[P_t]$ is independent of the parameter t follows from the equivalence of the definitions of K-homology in terms of homotopies and in terms of operator homotopies.

Let us apply Proposition 9.1 in our case. We can assume without loss of generality that the symbol $p(x,\xi)$ is smooth up to the zero section in T^*X and normalized: $p^*(x,\xi)p(x,\xi) = 1$ for large $|\xi|$. Let $\psi(t)$, $t \geqslant 0$, be a smooth positive function such that

$$\psi(t) = \begin{cases} 1 & \text{for } t < 1, \\ 1/t & \text{for } t > 2. \end{cases}$$

We consider the symbol

$$p_\varepsilon(x,\xi) = p\big(x, \xi\psi\big(\varepsilon|\xi|\big)\big).$$

A straightforward computation (compare with [35], Theorem 19.2.3) establishes the following properties.

1. We have $p_0 = p$. The symbol p_ε with $\varepsilon > 0$ is homogeneous for large $|\xi|$ and elliptic for sufficiently small ε.

2. The symbols p_ε and p_ε^* are uniformly bounded in the class of symbols of compact variation in the fibres of T^*X for $\varepsilon \in [0,1]$.

3. The compactly supported compact-valued symbols $p_\varepsilon p_\varepsilon^* - 1$ and $p_\varepsilon^* p_\varepsilon - 1$ are independent of ε for sufficiently small $\varepsilon > 0$.

We define the operator

$$P_\varepsilon = p_\varepsilon\left(x, -i\frac{\partial}{\partial x}\right). \tag{9.1}$$

(We fix coordinate neighbourhoods and a subordinate partition of unity independent of ε on X.) We claim that, for sufficiently small $\varepsilon \geqslant 0$,

(a) $\operatorname{ind} p_\varepsilon \in K_c(T^*X)$ is independent of ε,

(b) the family P_ε satisfies the hypotheses of Proposition 9.1.

Theorem 7.1 now follows since, on the one hand, the symbol p_ε is homogeneous at infinity for $\varepsilon > 0$ and hence $[P_\varepsilon] = \varphi(\operatorname{ind} p_\varepsilon)$ by the first part of the proof and, on the other hand, passage to the limit as $\varepsilon \to 0$ is possible by Proposition 9.1. Thus it remains to prove (a) and (b).

Assertion (a) follows from the homotopy invariance of the index since variations of ε change the symbol p_ε only outside a sufficiently large ball $\big\{|\xi| > R\big\}$, where $R \simeq 1/\varepsilon$, and leave it invertible outside this ball.

Assertion (b) can be proved as follows.

1. The families P_ε and P_ε^* are strongly continuous since they are uniformly bounded and each summand in their definitions in coordinate patches on X is strongly continuous on the set of functions whose Fourier transform is compactly supported.

2. If $f \in C_0(X^\circ)$, then the operators $f(P_\varepsilon P_\varepsilon^* - 1)$ and $f(P_\varepsilon^* P_\varepsilon - 1)$ are compact and depend continuously on ε. Indeed, their compactness is obvious and their continuity follows from the fact that their complete symbols and their derivatives in local coordinates are uniformly continuous in ε on compact subsets in ξ, are uniformly bounded and decay as $\xi \to \infty$, and hence they are uniformly continuous in ε for all ξ.

The compactness and continuity of the commutators $[f, P_\varepsilon]$ and $[f, P_\varepsilon^*]$ can be proved along the same lines. The proof of Theorem 7.1 is complete.

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