Lawrence P Horwitz and Rafael I Arshansky

Chapter 6

Quantum relativistic statistical mechanics, spin statistics and quantum field theory

In chapter 5, we discussed the classical relativistic statistical mechanics of a many-body system. In this chapter, we discuss the construction of quantum statistical mechanics. The development of this theory for the special choice of $\kappa = -\frac{1}{2}Mc^2$ was discussed in [1]. Here we work in the more general framework discussed in section 5.2. We show that much of the analysis given there is applicable to the quantum case as well.

Quantum mechanics brings with it the notion of many-body systems of indistinguishable particles that appear in nature to occur either symmetrically or antisymmetrically (Bose–Einstein or Fermi–Dirac statistics) in their quantum descriptions. The nonrelativistic quantum theory admits the construction of many-body states with these properties in a very simple way in the construction of tensor product states [2]. We show in this chapter how a similar method can be followed for the Stueckelberg, Horwitz and Piron (SHP) relativistic quantum theory, starting from a generalization of Wigner's method [3] of describing relativistic spin which can be applied to many-body quantum theory. We show, in our discussion of spin and angular momentum, that it is necessary to introduce a covariant time-like unit vector [4, 5], universal for the system, which foliates spacetime in such a way that the space-like surfaces orthogonal to this vector admit representations of $O(3)$ for which one can add angular momenta [6] in a many-body state with the usual Clebsch–Gordan coefficients and prove a spin-statistics theorem; one can also construct in a natural way a Pauli–Lubanski operator (in an appendix to this chapter) which provides a covariant signature of the state of angular momentum.

6.1 Relativistic quantum statistical mechanics

In this section we show how the results of the previous chapter on classical statistical ensembles can be extended to the quantum case using density matrix methods.

doi:10.1088/978-1-6817-4948-8ch6

In the quantum theory the density operator for a quantum state [1] corresponds to a microcanonical ensemble represented by

$$\rho(\kappa, E) = \psi_{\kappa,E}\psi^{*}_{\kappa,E}, \tag{6.1}$$

where $\psi_{\kappa,E}$ are (generalized) eigenfunctions of the total K operator, and E is the value of the total energy operator (well-defined by translation invariance of the whole system), as discussed for the classical microcanonical ensemble. The theory, assuming that the microcanonical shell is specified by a nonrelativistic limit for which κ is chosen to be precisely $\kappa = -Mc^2/2$, is treated in detail in [1] and [4]. Here, we follow the more general approach of section 5.2, where we admit in principle all values of κ (permissible by the conservation law) in the construction of the quantum canonical ensemble. We see from this that one can obtain a mass 'temperature' and achieve mass stability of a 'particle' constructed from an ensemble of events in the same way as in the classical case. A mass chemical potential is obtained in the grand canonical ensemble. We show that there is a high temperature phase transition for which this mass chemical potential controls the mass of the particle in the resulting transition.

The total number of states in the quantum microcanonical ensemble is given by

$$\Gamma(\kappa, E) = \mathrm{Tr}\,\rho = \left\|\psi_{\kappa,E}\right\|^2, \tag{6.2}$$

where Tr is the trace over a complete set (one obtains (6.2) immediately by using the set $\{\psi_{\kappa,E}\}$); we have suppressed the additional quantum numbers needed in case of degeneracy. The entropy is then defined as

$$S(\kappa, E) = k_B \ln \Gamma(\kappa, E). \tag{6.3}$$

The canonical ensemble is defined as for the classical case discussed in chapter 5, and is given in terms of a partition of the total ensemble into a bath and subsystem, now assuming that the complete set of states of the subsystem is orthogonal to those of the bath, forming orthogonal subspaces. As for the classical case, this corresponds to neglecting interactions between the bath and subsystem particles, so effectively the Hilbert space for the system becomes a direct product $\mathcal{H}_b \otimes \mathcal{H}_s$. The density of states can then be represented as

$$\begin{aligned}\rho(\kappa, E) &= \sum\nolimits_s \psi_{\kappa-\kappa',\, E-E'}\psi_{\kappa_s,E'}\psi^{*}_{\kappa-\kappa',E-E'}\psi^{*}_{\kappa',E'} \\ &= \sum\nolimits_s \rho_{\kappa-\kappa',E-E'} \otimes \rho_{\kappa_s,E'},\end{aligned} \tag{6.4}$$

where we use primed quantities to represent subsystem properties. The total number of states is then

$$\Gamma(\kappa, E) = \sum_s \Gamma_b(\kappa - \kappa', E - E')\Gamma_s(\kappa', E'). \tag{6.5}$$

This result is exactly of the same form as (5.26) for the classical case, up to (5.41). Assuming a maximum contribution on both variables to the sum, on this maximum value,

$$\Gamma(\kappa, E) = \Gamma_b(\kappa - \kappa', E - E')|_{\max} \Gamma_s(\kappa', E')|_{\max}, \tag{6.6}$$

and as in (5.38) the entropy defined as $S(\kappa, E) = \ln \Gamma(\kappa, E)$ is additive, i.e.

$$\begin{aligned} S(\kappa, E) &= \ln \Gamma_b(\kappa - \kappa', E - E') + \ln \Gamma_s(\kappa', E') \\ &= S_b(\kappa - \kappa') + S_s(\kappa', E'). \end{aligned} \tag{6.7}$$

We may use here the definitions (5.36) and (5.37) as well, i.e.

$$\begin{aligned} S_b(\kappa, E) &= \ln \Gamma_b(\kappa, E) \\ S_s(\kappa, E) &= \ln \Gamma_s(\kappa, E), \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} \frac{\partial S_b}{\partial E} &= \frac{\partial S_s}{\partial E} = \frac{1}{T} \\ \frac{\partial S_b}{\partial \kappa} &= \frac{\partial S_s}{\partial \kappa} = \frac{1}{T_\kappa}. \end{aligned} \tag{6.9}$$

Since, as for the classical case, we have

$$S(\kappa, E) \cong \ln \Gamma_b(\kappa - \kappa', E - E') + \ln \Gamma_s(\kappa', E') \tag{6.10}$$

so that

$$\Gamma_b(\kappa - \kappa', E - E') = \mathrm{e}^{S_b(\kappa - \kappa', E - E')}. \tag{6.11}$$

For κ' and E' small compared to κ and E,

$$\begin{aligned} \Gamma_b(\kappa - \kappa', E - E') &= \mathrm{e}^{S_b(\kappa - \kappa', E - E')} \\ &\cong \mathrm{e}^{S_b(\kappa, E) - \kappa' \frac{\partial S_b}{\partial \kappa} - E' \frac{\partial S_b}{\partial E}} \\ &= \mathrm{e}^{S_b(\kappa, E)} \mathrm{e}^{-\frac{\kappa'}{T_\kappa}} \mathrm{e}^{-\frac{E'}{T}}. \end{aligned} \tag{6.12}$$

Integrating over κ' and E' (essentially restricted to the neighborhood of the maximum), we then have

$$\Gamma(\kappa, E) = \int \mathrm{d}\kappa' \, \mathrm{d}E' \mathrm{e}^{S_b(\kappa, E)} \mathrm{e}^{-\kappa' \frac{\partial S_b}{\partial \kappa}} \mathrm{e}^{-E' \frac{\partial S_b}{\partial E}} \Gamma_s(\kappa', E'). \tag{6.13}$$

Recall that, for the quantum case (using the spectral representation of K_s, E_s),

$$\Gamma_s(\kappa', E') = \mathrm{Tr}_s \delta(K_s - \kappa') \delta(E_s - E') \tag{6.14}$$

and therefore we may write

$$\begin{aligned} \Gamma(\kappa, E) &= \mathrm{Tr}_s \, \delta(K_s - \kappa') \delta(E_s - E') \\ &\quad \times \mathrm{e}^{S_b(\kappa, E)} \mathrm{e}^{-\frac{\kappa'}{T_\kappa}} \, \mathrm{e}^{-\frac{E'}{T}} \\ &= \mathrm{e}^{S_b(\kappa, E)} Tr_s \, \mathrm{e}^{-\frac{K_s}{T_\kappa}} \, \mathrm{e}^{-\frac{E_s}{T}}. \end{aligned} \tag{6.15}$$

The distribution density operator for the N-body canonical ensemble is then (we drop the subscript s)

$$\rho_N = \mathrm{e}^{-\frac{K}{T_\kappa}} \mathrm{e}^{-\frac{E}{T}} \tag{6.16}$$

and define the partition function as

$$Q_N(T_\kappa, T) = \mathrm{Tr} \mathrm{e}^{-\frac{K}{T_\kappa}} \mathrm{e}^{-\frac{E}{T}}. \tag{6.17}$$

Let us now define the quantum Helmholtz free energy A by

$$Q_N(T_\kappa, T) = \mathrm{e}^{-A(T_\kappa, T)\beta}, \tag{6.18}$$

so that

$$\mathrm{Tr} \mathrm{e}^{-\beta_\kappa K} \mathrm{e}^{(A-E)\beta} = 1, \tag{6.19}$$

where $\beta = 1/T$ and $\beta_\kappa = 1/T_\kappa$. Differentiating with respect to β one obtains, as for the classical case,

$$\begin{aligned} A &= < E > -\beta \frac{\partial A}{\partial \beta} \\ &= < E > +T \frac{\partial A}{\partial T}. \end{aligned} \tag{6.20}$$

It then follows that if we define

$$S = -\frac{\partial A}{\partial T}, \tag{6.21}$$

we have

$$A = < E > -TS, \tag{6.22}$$

which can, as before, be derived from the grand canonical ensemble.

There is another relation, however, that we can obtain by taking the derivative of (3.12) with respect to β_κ; one obtains in this way

$$\begin{aligned} 0 &= \mathrm{Tr} \left\{ -K \mathrm{e}^{-\beta_\kappa K} + \mathrm{e}^{-\beta_\kappa K} \beta \frac{\partial A}{\partial \beta_\kappa} \right\} \mathrm{e}^{(A-E)\beta} \\ &= - < K > +\beta \frac{\partial A}{\partial \beta_\kappa}, \end{aligned} \tag{6.23}$$

so that

$$< K > = \frac{1}{T} \frac{\partial A}{\partial \beta_\kappa} = -\frac{T_\kappa^2}{T} \frac{\partial A}{\partial T_\kappa}. \tag{6.24}$$

We therefore obtain a mean value for $< K >$, the effective center-of-mass mass of the subensemble, which is determined by T_κ and T under the quantum canonical distribution, corresponding to an equilibrium of both heat and mass, without exchange of particles with the bath.

The partition function is then, for the $N_s \equiv N$-particle subspace,

$$Q_N(T_\kappa, T) = Tr_N \mathrm{e}^{-\frac{K}{T_\kappa}} \mathrm{e}^{-\frac{E}{T}}, \tag{6.25}$$

where the trace is now taken over a complete set in $\mathcal{H}_s$[1]. The expectation value of an operator $\mathcal{O}$ (on $\mathcal{H}_s$) is then

$$< \mathcal{O} >_N = \frac{\mathrm{Tr}_N(\rho\mathcal{O})}{Q_N}. \tag{6.26}$$

We now turn to the grand canonical ensemble.

The partition function for the grand canonical ensemble is defined as

$$\begin{aligned} Q_N(V, T, T_\kappa) = \sum\nolimits_{N_s=0}^{N} &\mathrm{Tr}_s\, \mathrm{e}^{-\beta_\kappa K_s} \mathrm{e}^{-\beta E_s} \\ &\times Q_{N-N_s}(V - V_s, T, T_\kappa), \end{aligned} \tag{6.27}$$

where s refers to the N_s-body subsystem (T and T_κ are equilibrium parameters for the whole system), and for the bath at each N_s. The normalized density operator is then

$$\begin{aligned} \rho(\Omega_s, N_s) = &\frac{1}{Q_N(V, T, T_\kappa)} \mathrm{e}^{-\beta_\kappa K_s}\, \mathrm{e}^{-\beta E_s} \\ &\times Q_{N-N_s}(V - V_s, T, T_\kappa), \end{aligned} \tag{6.28}$$

We can now write, as for the classical case (for N_s small compared to N, which can be arbitrarily large),

$$\begin{aligned} Q_{N-N_s}(V - V_s, T, T_\kappa) &= \mathrm{e}^{-\beta A(V-V_s, T, T_\kappa, K-K_s, N-N_s)} \\ &\cong Q_N(V, T, T_\kappa) \mathrm{e}^{\beta V_s \frac{\partial A}{\partial V} + \frac{\partial A}{\partial K} K_s + \frac{\partial A}{\partial N} N_s}. \end{aligned} \tag{6.29}$$

It then follows that

$$\begin{aligned} \frac{\partial A}{\partial V} &= - P \\ \frac{\partial A}{\partial N} &= \mu \end{aligned} \tag{6.30}$$

with the *quantum mass chemical potential*

$$\frac{\partial A}{\partial K} = -\mu_\kappa. \tag{6.31}$$

[1] As we discuss below, the N-body states are constructed on the basis of N-fold tensor products of wave functions in $\mathcal{H}_s$.

The remainder of the results given in chapter 5 (section 5.5 and 5.6) are exactly the same for the quantum case, where the integrals over $d\Omega_s$ are replaced by the trace of the corresponding operator valued integrands. The conclusions therefore remain the same for the quantum theory.

6.2 The ideal free quantum gas

In this section we discuss the relativistic version of Boltzmann counting to obtain the corresponding Bose–Einstein, Boltzmann, and Fermi–Dirac statistics [1]. We shall explain in our discussion of the spin states in a later section of this chapter how the spin-statistics theorem emerges in the relativistic quantum theory, and assume its validity in this section.

For the ideal free quantum gas in a spacetime box of dimension

$$-L/2 \leqslant x, y, z \leqslant L/2, \ \Delta t/2 \leqslant t \leqslant \Delta t/2,$$

the microcanonical distribution is characterized by the spectrum

$$2MK = \hbar^2(k_1^2 + k_2^2 + k_3^2 - k_0^2), \tag{6.32}$$

where

$$k_0 = \frac{2\pi}{\Delta t}\nu_0, \quad k_j = \frac{2\pi\nu_j}{L} \quad j = 1, 2, 3, \tag{6.33}$$

and $\nu_0, \nu_j = 0, \pm 1, \pm 2 \ldots$. Then, $\mathbf{p} = (2\pi\hbar/L)\boldsymbol{\nu}$ and $\varepsilon = (2\pi\hbar/\Delta t)\nu_0$. The integral measure is given by

$$d^3p d\varepsilon \sim \frac{(2\pi\hbar)^4}{V^{(4)}}, \quad V^{(4)} = L^3\Delta t. \tag{6.34}$$

We now compute the Bose–Einstein, Fermi–Dirac, and Boltzmann distributions in terms of the discrete sums characteristic of kinetic theory. Let

$$\begin{aligned} &i = \text{cell around } \mathbf{p}, \varepsilon, m \in \mu, \\ &g_i = \text{number of mass and momentum states in each cell} \\ &n_i = \textstyle\sum_{\mathbf{p},\varepsilon} n_{\mathbf{p},\varepsilon}, \end{aligned} \tag{6.35}$$

where $n_{\mathbf{p},\varepsilon}$ is the number of particles with energy momentum $\mathbf{p}$, ε. Let $W(\{n_i\})$ be the number of states associated with the distribution $\{n_i\}$. Then, the total number of states in phase space is

$$\Gamma(E, K_0) = \sum_{\{n_i\}} W(\{n_i\}) \tag{6.36}$$

with the constraints

$$E = \textstyle\sum_i \varepsilon_i n_i \quad K_0 = \sum_i K_i n_i \quad N = \sum_i n_i, \tag{6.37}$$

where K_i is the average value of $K(\mathbf{p}, \varepsilon)$ in the ith cell. Taking into account the constraints (6.37), we wish to find $\{n_i\}$ such that

$$\delta\left\{\ln W(\{n_i\}) - \alpha\sum_i n_i - \beta\varepsilon_i n_i + \gamma\sum_i K_i n_i\right\} = 0, \tag{6.38}$$

where α, β, γ are Lagrange parameters implementing the constraints. Permitting up to g_i states in each cell for Fermi–Dirac statistics, and all integer values for Bose–Einstein statistics, we find the distributions

$$\begin{aligned} W(\{n_i\}) &= \Pi_i \frac{(n_i + g_i - 1)!}{N_i!(g_i - 1)!} \quad \text{(Bose–Einstein)} \\ &= \Pi_i \frac{g_i!}{n_i!(g_i - n_i)!} \quad \text{(Fermi–Dirac)} \\ &= \Pi_i \frac{g_i!}{n_i!} \quad \text{(Boltzmann)} \end{aligned} \tag{6.39}$$

and obtain the average occupation number distributions (the sign of K_i is important in establishing the sign of the second variation)

$$\begin{aligned} \bar{n}_i &= \frac{g_i}{z^{-1}\zeta^{-K_i}\mathrm{e}^{\beta\varepsilon_i} - 1} \quad \text{(Bose–Einstein)} \\ &= \frac{g_i}{z^{-1}\zeta^{-K_i}\mathrm{e}^{\beta\varepsilon_i} + 1} \quad \text{(Fermi–Dirac)} \\ &= g_i z\zeta^{K_i}\mathrm{e}^{-\beta\varepsilon_i} \quad \text{(Boltzmann)}, \end{aligned} \tag{6.40}$$

where $z = \mathrm{e}^{\alpha}$ and $\zeta = \mathrm{e}^{\gamma}$. Using the maximal distributions in (6.38), the entropy is given by

$$S = k_B \ln W(\{\bar{n}_i\}) \tag{6.41}$$

Pinching down the size of the cells to obtain continuum distributions, we can write (taking $g_i = 1$)

$$\begin{aligned} \bar{n}_{\mathbf{p},\varepsilon} &= \frac{1}{z^{-1}\zeta^{-K(\mathbf{p},\varepsilon)}\mathrm{e}^{\beta\varepsilon} - 1} \quad \text{(Bose–Einstein)} \\ &= \bar{n}_{\mathbf{p},\varepsilon} \\ &= \frac{1}{z^{-1}\zeta^{-K(\mathbf{p},\varepsilon)}\mathrm{e}^{\beta\varepsilon} + 1} \quad \text{(Fermi–Dirac)} \\ &= z\zeta^{K(\mathbf{p},\varepsilon)}\mathrm{e}^{-\beta\varepsilon} \quad \text{(Boltzmann)}. \end{aligned} \tag{6.42}$$

The parameters z, β, ζ are to be determined from

$$\begin{aligned} \sum\nolimits_{\mathbf{p},\varepsilon} \varepsilon\bar{n}_{p,\varepsilon} &= E, \\ \sum\nolimits_{\mathbf{p},\varepsilon} \bar{n}_{\mathbf{p},\varepsilon} &= N \\ \sum\nolimits_{\mathbf{p},\varepsilon} K(\mathbf{p}, \varepsilon)\bar{n}_{p,\varepsilon} &= K_0, \end{aligned} \tag{6.43}$$

where the sums are to be taken over a narrow range of masses Δm. Comparing the Boltzmann case with the classical grand canonical distributions, we identify $\beta = \frac{1}{k_B T}$, $z = e^{\mu\beta}$, $\zeta = e^{-\mu_K \beta}$.

Note that in the Fermi–Dirac distribution we have counted as distinct states the several values of ε for each p which lie within the admissible widths of the particles. Although the distributions we have obtained are formally very similar to the usual one [7] (except for the factor ζ^{-K}), the usual notion of Fermi–Dirac statistics treats all of these states the same. The additional multiplicity would cancel out in the expectation values of observables, so that the results should be very close to those of the usual distribution functions.

Using Stirling's approximation for the factorials, one finds that for the Boltzmann gas,

$$S/k_B = \beta E - K \ln \zeta - N \ln z. \tag{6.44}$$

We now turn to a study of the ideal gas from the point of view of the grand canonical ensemble.

For Boltzmann statistics, the canonical partition function can be written, in the quantum version of (5.72) and (5.75) (taking into account *a priori* normalizations), as

$$\hat{Q}_N(V^{(4)}, \zeta, T) = \sum_{n_{\mathbf{p},\varepsilon}} \frac{1}{N!} \left(\frac{N!}{\Pi_{\mathbf{p},\varepsilon} n_{\mathbf{p},\varepsilon}} \right) e^{-\beta E(\mathbf{p},\varepsilon)} \zeta^{K(p,\varepsilon)}, \tag{6.45}$$

where

$$\begin{aligned} E(\{n_{\mathbf{p},\varepsilon}\}) &= \sum\nolimits_{\mathbf{p},\varepsilon} \varepsilon n_{p,\varepsilon}, \\ K(\{n_{\mathbf{p},\varepsilon}\}) &= \sum\nolimits_{\mathbf{p},\varepsilon} K(\mathbf{p}, \varepsilon) \end{aligned} \tag{6.46}$$

and

$$N = \sum\nolimits_{\mathbf{p},\varepsilon} n_{\mathbf{p},\varepsilon} \tag{6.47}$$

as a constraint.

With the constraint (6.47), the sum in (6.45) becomes

$$\hat{Q}_N(V^{(4)}, \zeta, T) = \frac{1}{N!} \left(\sum\nolimits_{\mathbf{p},\varepsilon} e^{-\beta\varepsilon} \zeta^{K(\mathbf{p},\varepsilon)} \right)^N, \tag{6.48}$$

and therefore

$$\mathcal{Q}(V^{(4)}, \zeta, z, T) = \exp \left\{ z \sum\nolimits_{\mathbf{p},\varepsilon} e^{-\beta\varepsilon} \zeta^{K(\mathbf{p},\varepsilon)} \right\}. \tag{6.49}$$

The equation of state can then be obtained explicitly by noting that, as in the classical case,

$$< N > = z\frac{\partial}{\partial z} \ln \mathcal{Q} = \ln \mathcal{Q} = \frac{PV}{k_B T}. \tag{6.50}$$

Evaluating the distributions for Bose–Einstein and Fermi–Dirac statistics one obtains [1]

$$\begin{aligned} \mathcal{Q}(V^{(4)}, \zeta, z, T) &= \Pi_{\mathbf{p},\varepsilon}\frac{1}{1 - z\mathrm{e}^{-\beta\varepsilon}\zeta^{K(\mathbf{p},\varepsilon)}} \quad \text{(BE)} \\ &= \Pi_{\mathbf{p},\varepsilon}(1 + z\mathrm{e}^{-\beta\varepsilon}\zeta^{K(\mathbf{p},\varepsilon)}) \quad \text{(FD)}. \end{aligned} \tag{6.51}$$

The equations of state for the relativistic free quantum gas are

$$\begin{aligned} \frac{PV}{k_B T} = \ln \mathcal{Q} &= -\sum\nolimits_{p,\varepsilon} \ln(1 - z\mathrm{e}^{-\beta\varepsilon}\zeta^{K(\mathbf{p},\varepsilon)}) \quad \text{(BE)} \\ &= \sum\nolimits_{\mathbf{p},\varepsilon} \ln(1 + z\mathrm{e}^{-\beta\varepsilon}\zeta^{K(\mathbf{p},\varepsilon)}) \quad \text{(FD)}. \end{aligned} \tag{6.52}$$

The total number of particles is

$$\begin{aligned} N = z\frac{\partial}{\partial z} \ln \mathcal{Q} &= \sum\nolimits_{\mathbf{p},\varepsilon} \frac{z\mathrm{e}^{-\beta\varepsilon}\zeta^{K(p,\varepsilon)}}{1 - z\mathrm{e}^{-\beta\varepsilon}\zeta^{K(\mathbf{p},\varepsilon)}} \quad \text{(BE)} \\ &= \sum\nolimits_{\mathbf{p},\varepsilon} \frac{z\mathrm{e}^{-\beta\varepsilon}\zeta^{K(p,\varepsilon)}}{1 + z\mathrm{e}^{-\beta\varepsilon}\zeta^{K(\mathbf{p},\varepsilon)}} \quad \text{(FD)}. \end{aligned} \tag{6.53}$$

Similarly, by differentiating with respect to $\beta\varepsilon - K(\mathbf{p}, \varepsilon) \ln \zeta$, we find that the average occupation numbers are given by

$$< n_{\mathbf{p},\varepsilon} > = \frac{z\mathrm{e}^{-\beta\varepsilon}\zeta^{K(p,\varepsilon)}}{1 \mp z\mathrm{e}^{-\beta\varepsilon}\zeta^{K(\mathbf{p},\varepsilon)}}. \tag{6.54}$$

Equations (6.53) then correspond to

$$N = \sum_{\mathbf{p},\varepsilon} < n_{\mathbf{p},\varepsilon} > . \tag{6.55}$$

6.3 Relativistic high temperature Boson phase transition

The existence of the grand canonical ensemble with this chemical potential enables us to define a particle *mass* as a result of a high temperature phase transition. Since this determination is in a statistical framework, fluctuations nevertheless admit the development of the off-shell theory for individual particles which provides the framework for this result.

Haber and Weldon [8] showed, in the usual (mass shell) form of relativistic quantum mechanics, that taking into account both the particle and antiparticle distribution functions, a system of bosons can undergo a high temperature phase transition. The introduction of antiparticles in the theory, by application of the arguments of Haber and Weldon, implies the addition of another term in the total number expectation with a negative sign, carrying an opposite sign for the energy chemical potential, i.e. formula (6.53) (for the boson case) is written as [9] (dividing numerator and denominator by the numerator factor)[2]

$$N = V^{(4)} \sum_{k^\mu} \left[\frac{1}{e^{\left(E-\mu-\mu_K \frac{m^2}{2M}\right)/T} - 1} - \frac{1}{e^{\left(E+\mu-\mu_K \frac{m^2}{2M}\right)/T} - 1} \right]. \tag{6.56}$$

As assumed by Haber and Weldon, the total particle number remains unchanged in the equilibrium state, but the presence of antiparticles implies annihilation and creation processes. Thus, in counting the total number of particles, the antiparticle distribution must carry a negative sign, consistent with the interpretation of Stueckelberg as given in the early chapters of the book. On the other hand, both the terms in the sum in equation (6.56) must separately be positive, implying the

$$\begin{aligned} m - \mu - \mu_K \frac{m^2}{2M} &\geqslant 0, \\ m + \mu - \mu_K \frac{m^2}{2M} &\geqslant 0, \end{aligned} \tag{6.57}$$

resulting in the inequalities representing the nonnegativeness of the discriminants in the mass quadratic formulas,

$$-\frac{M}{2\mu_K} \leqslant \mu \leqslant \frac{M}{2\mu_K}. \tag{6.58}$$

The bounds of the intersection of the regions satisfying the inequalities (6.57) are given by

$$\frac{M}{\mu_K}\left(1 - \sqrt{1 - \frac{2|\mu|\mu_K}{M}}\right) \leqslant m \leqslant \frac{M}{\mu_K}\left(1 + \sqrt{1 - \frac{2|\mu|\mu_K}{M}}\right), \tag{6.59}$$

which for small $\frac{|\mu|\,\mu_K}{M}$ reduces, as in the no-antiparticle case, to

$$|\mu| \leqslant m \leqslant \frac{2M}{\mu_K}. \tag{6.60}$$

[2] Since the sign of the energy of the antiparticle is opposite to that of the particle, the chemical potential μ must change sign for the antiparticle, but the mass squared of both particle and antiparticle are positive, and therefore the sign of μ_K does not change.

Replacing the summation in (6.56) by integration, one obtains the formula for the number density

$$n = \frac{1}{4\pi^3}\int_{m_1}^{m_2} m^3 \mathrm{d}m \int_{-\infty}^{\infty} \sinh^2\beta \mathrm{d}\beta$$
$$\times\left[\frac{1}{\mathrm{e}^{\left(m\cosh\beta-\mu-\mu_K m^2/2M\right)/T}-1},\right.$$
$$\left.-\frac{1}{\mathrm{e}^{\left(m\cosh\beta+\mu-\mu_K m^2/2M\right)/T}-1}\right] \tag{6.61}$$

where m_1 and m_2 are defined by the bounds (6.59). Integrating out the β variable, one finds for high temperature $\mu/\mathrm{T} \ll 1$,

$$n \cong \frac{1}{\pi^3}\left(\frac{M}{\mu_K}\right)^2 \mu T\sqrt{1-\frac{2|\mu|\mu_K}{M}}. \tag{6.62}$$

For T above a critical value, the range of admissible masses is pinched down to zero, corresponding to a phase transition where the dispersion

$$\delta m = \sqrt{<m^2> - <m>^2}$$

vanishes as $\sqrt{T-T_c}$; a second-order transition corresponding to a ground state with $p_\mu p^\mu = -(M/\mu_k)^2$. States with temperature $T > T_c$ correspond to off-shell excitations of such a ground state.

The phase transition that we have described selects a definite mass for the particles, but this result is statistical. Although the mean fluctuations vanish, there is nevertheless sufficient freedom in the phase space for each particle to fulfil the off-shell requirements for the formulation of the Stueckelberg theory.

This mechanism provides an insight into a possibly more general formulation which would explain the stability of the asymptotic mass of a particle in the Stueckelberg theory in the presence of arbitrary number of collisions; the existence of several solutions could give rise to what appears phenomenologically as mass spectra of observed particles [10].

6.4 Quantization of the electromagnetic field and black-body radiation

The existence of quanta of the electromagnetic field and their role in black-body radiation, through the work of Planck [11] lies at the foundation of the quantum theory. It is therefore important to understand how this phenomenon can be understood in the framework of the generalized Maxwell electromagnetism discussed in chapter 2, which appears to be a necessary consequence of the SHP theory.

As we have explained in chapter 2, the Stueckelberg–Schrödinger equation implies that the electromagnetic gauge fields are five-dimensional, including both a Lorentz four-vector field a_μ, which compensates the action of the four-derivative on the wave function, and a Lorentz scalar a_5 field which compensates for the τ derivative of the evolving wave function. The usual argument for two polarization states of the four-dimensional Maxwell field is that, of the four degrees of freedom, there is a gauge condition and the constraint of the Gauss law, leaving two polarization states. The factor of two on the Bose–Einstein distribution is essential for the computation of the specific heat of a black body, but the argument of the existence of two constraints leaves the possibility of three polarization states. In the following, we show that the observable radiation field of a black body indeed carries just two polarization states [12].

The canonical quantization of the five-dimensional radiation field was carried out by Shnerb and Horwitz [13] following the basic ideas of Teitelboim and Henneaux [14] and Haller [15] using algebraic methods. Taking for this discussion, as in [16], the signature of the five-dimensional manifold to be $[\sigma, +,-,-,-]$, we write the action for the interacting fields (in this section we work in the framework of both quantized gauge fields and quantized wave functions ψ) as

$$\begin{aligned} S = \int_{-\infty}^{\infty} \mathrm{d}^5 x \Bigg\{ & -\frac{\lambda}{4} f^{\alpha\beta} f_{\alpha\beta} - G(x)[\partial_\alpha a^\alpha(x)] + \frac{1}{2\lambda} G^2(x) \\ & i\psi^\dagger(x) \frac{\partial \psi(x)}{\partial \tau} \\ & - \frac{1}{2M} \psi^\dagger(x) \frac{\partial \psi(x)}{\partial \tau} - \frac{1}{2M} \psi^\dagger [\partial^\mu - \mathrm{i}e' a^\mu(x)] \big[\partial_\mu - \mathrm{i}e' a_\mu(x)\big] \psi(x) \\ & + e' \psi^\dagger(x) a_\tau(x) \psi(x) \Bigg\}, \end{aligned} \tag{6.63}$$

where λ is a quantity with dimensions of length (it will play the role of the τ correlation length of the wave function in the Maxwell limit). As discussed in chapter 2, e' is the coupling constant of the covariant theory, which also has dimension of length, and G plays the role of an auxiliary field [15] (somewhat analogous to the Fadeev–Popov ghosts [17] of the path integral approach). The canonically conjugate momenta are given by

$$\begin{aligned} \pi^\mu &= \frac{\delta \mathcal{L}}{\delta\big(\partial_\tau a_\mu\big)} = -\lambda f^{\tau\mu}, \\ \pi^\tau &= \frac{\delta \mathcal{L}}{\delta(\partial_\tau a_\tau)} = -\sigma G, \\ \pi_\psi &= \frac{\delta \mathcal{L}}{\delta(\partial_\tau \psi)} = \mathrm{i}\psi^\dagger. \end{aligned} \tag{6.64}$$

We now impose equal time (τ) commutation relations

$$\big[\pi^\alpha(x), a_\beta(y)\big] = -\mathrm{i}\delta^\alpha_\beta \delta(x - y) \tag{6.65}$$

and (we are assuming ψ a boson field for our present purposes)

$$[\mathrm{i}\psi^\dagger(x), \psi(y)] = -\mathrm{i}\delta(x - y). \tag{6.66}$$

The Hamiltonian (generating unitary evolution in ψ and a^α) then takes the form

$$\begin{aligned} K &= \sigma \int d^4x\big[\pi^\mu(\partial_\tau a_\mu) \ + \pi^\tau(\partial_\tau a_\tau) + i\psi^\dagger\partial_\tau\psi - \mathcal{L}\big] \\ &= K_\gamma + K_m + K_{\gamma m}, \end{aligned} \tag{6.67}$$

where

$$\begin{aligned} K_\gamma = \int d^4x \bigg\{ &-\frac{1}{2\lambda}\pi^\mu\pi_\mu - \frac{\lambda\sigma}{4} f^{\mu\nu} f_{\mu\nu} \\ &+ \pi^\mu\big(\partial_\mu a^\tau\big) - \pi^\tau\big(\partial_\mu a^\mu\big) - \frac{1}{2\lambda}\pi^\tau\pi_\tau \bigg\} \end{aligned} \tag{6.68}$$

and

$$\begin{aligned} K_m &= \frac{\sigma}{2M}\int d^4x\psi^\dagger\partial_\mu\partial^\mu\psi, \\ K_{\tau m} &= d^4x\bigg\{ -e'\psi^\dagger a_\tau\psi - \frac{ie'}{2M}\psi^\dagger\big[a^\mu\partial_\mu + \big(\partial_\mu a^\mu\big)\big] \\ &\quad - \frac{e'^2}{2M}\psi^\dagger\psi a^\mu a_\mu \bigg\}. \end{aligned} \tag{6.69}$$

The stability condition on the states for the restriction to the Gauss law

$$< \partial_\mu\pi^\mu + j^\tau > = 0 \tag{6.70}$$

implies that $< \pi^\tau > = 0$; one can then eliminate the longitudinal part of the field a^μ.

In case the four vector k^μ in the Fourier decomposition of the a^μ field is time-like, for which the $O(4, 1)$ theory is stable, one can eliminate by a unitary transformation (as in the Maxwell case) the time component of a^μ. There remain, except for the Coulomb term, three space-like polarization components a^i, and the Hilbert space has positive norm.

We will show in our discussion of spin and angular momentum that vector bosons must lie in a representation of angular momentum with spin 1; as discussed in Jauch and Rohrlich [18], page 41), these components with canonical commutation relations provide a representation in any choice of gauge that meets this requirement. For the asymptotic photons of the black-body radiation, the components for k^μ space-like, for which the stable solutions are representations of $O(2, 1)$ do not meet this requirement. Furthermore, in the case that k^μ is light-like, the elimination of longitudinal modes corresponds exactly to the removal of both a_0 and $a_\parallel$, leaving just two polarization states. This limiting case is realized for the asymptotic photons of the black body when $\tau \to \infty$, leaving, by application of the Riemann–Lebesgue lemma, the 'massless' zero mode[3]. We make this argument explicit in the following.

[3] It was suggested by Andrew Bennett (private communication) that the concatenated field equations, corresponding to an integral over τ, would equivalently lead to this result.

The analog of the radiation gauge (e.g. [19]) for the five-dimensional fields would correspond to setting the a_5 field equal to zero; this corresponds to subtracting the 5-gradient of the indefinite integral of the a_5 field from the a_α fields, i.e. for

$$a^{5\prime} = a^5 + \partial^5\Lambda, \tag{6.71}$$

we can take

$$\Lambda(x, \tau) = -\int^\tau a^5(x, \tau') + \tilde{\Lambda}(x). \tag{6.72}$$

Then, since the second term is independent of τ, $a^{5\prime} = 0$. Furthermore, since

$$a^{\mu\prime} = a^\mu + \partial^\mu\Lambda(x), \tag{6.73}$$

it follows that

$$a^{0\prime} = a^0(x, \tau) - \partial^0\int^\tau a^5(x, \tau')\mathrm{d}\tau' + \partial^0\tilde{\Lambda}(x). \tag{6.74}$$

Under the assumption that the asymptotic fields are independent of τ, assuming convergence of the indefinite integral in (6.74) for large τ, we can make $a^{0\prime} = 0$ asymptotically with the choice

$$\tilde{\Lambda}(x) = -\int^t a^0(\mathbf{x}, t', \tau)\mathrm{d}t' + \int^t\int^\tau a^5(\mathbf{x}, t', \tau')\mathrm{d}\tau'\mathrm{d}t'. \tag{6.75}$$

The remaining term of the generalized Lorentz gauge $\partial_\alpha a^\alpha = 0$ is just the condition $\nabla \cdot \mathbf{a} = 0$, exhibiting the required rotational invariance on the orbit of the induced representation for the a^μ field. The longitudinal component along the $\mathbf{k}$ vector must therefore vanish, and we are left with two effective polarization states.

Therefore, with the Gauss law and the additional gauge condition on the five-dimensional fields, there are three constraints on the fields, leaving two degrees of freedom.

The remaining degrees of freedom correspond, in the induced representation, to two polarization states that are directly interpretable as angular momentum states of the photon in $SU(2)$ on the orbit.

The boson distribution function obtained above with the remaining two degrees of freedom then gives the usual result for the specific heat for black-body radiation [4].

We remark that for the relativistic Gibbs ensembles worked out above (section 6.2) we assumed for simplicity that there were no antiparticles (the Boltzmann counting construction did not make this assumption). The existence of the a_5 field makes possible, as we have seen in chapter 2, the (classical) particle–antiparticle transition on particle world lines. The analog of the radiation gauge requirement that we have imposed above as a second gauge condition, resulting in residually two degrees of freedom for the radiation field, would not admit this mechanism in the detectors. The presence of pair production [20] (expected to be very small) in the detector would therefore suggest that there may be this additional degree of freedom in the boson gas, with a concomitantly small correction in the black-body radiation formula.

6.5 Manifestly covariant relativistic Boltzmann equation

In this section, we shall derive a covariant Boltzmann equation with collision terms obtained from the binary scattering of events as described by relativistic scattering theory. We give here the basic ideas, and refer the reader to [21] for further details.

We study the case of N identical particles, and for convenience use the formalism of second quantization. We shall discuss the general construction of the many-body Fock space and second quantization in detail in the next section, but it will suffice for our treatment of scalar fields to assume here (6.66) as well, to define annihilation and creation operators for the boson fields.

The field which annihilates an event at the point $q = (\mathbf{q}, t)$ is related to the operator which annihilates an event of energy momentum $p = (\mathbf{p}, E/c)$ by the Fourier transform ($\hbar = 1$)

$$\psi(q) = \frac{1}{(2\pi)^2} \int \mathrm{d}d^4 p \psi(p) \mathrm{e}^{\mathrm{i}p\cdot q}. \tag{6.76}$$

An arbitrary operator A on the Hilbert space of events can be represented as

$$A = \sum_{s=1}^{N} \frac{1}{s!} \int \mathrm{d}^4 q_1 \dots \mathrm{d}^4 q_s \psi^\dagger(q_1) \cdots \psi^\dagger(q_s) \hat{A}_s \psi(q_1) \cdots \psi(q_s), \tag{6.77}$$

where $\hat{A}$ are operators acting on the space associated with every s event subspace of the N event system. The expectation value of such an operator can be expressed in terms of the trace with the density matrix ρ as

$$< A > = \mathrm{Tr}(\rho A). \tag{6.78}$$

The Weyl correspondence applies, as in the nonrelativistic theory, to every s event operator represented as [22]

$$\hat{A}_s = \int \mathrm{d}^4 k_1 \mathrm{d}^4 j_1 \dots \mathrm{d}^4 k_s \mathrm{d}^4 j_s A_s(k_1 j_1 \cdots k_s j_s) \exp\left\{ \mathrm{i} \sum_{n=1}^{s} k_n \cdot \hat{q}_n + j_n \cdot \hat{p}_n \right\}, \tag{6.79}$$

where the operators $\hat{q}_n$, $\hat{p}_n$ satisfy the canonical commutation relations

$$\left[\hat{q}_n^\mu, \hat{p}_{n'}^\nu \right] = \mathrm{i} g^{\mu\nu} \delta_{n,n'}. \tag{6.80}$$

There is a corresponding function $A_s(q_1, p_1, \dots q_s p_s)$ of the classical variables containing the same coefficients $A_s(k_1 j_1 \cdots k_s j_s)$ which is its classical limit. Consider, in particular, the case $s = 1$. Then, the quantity $< A_1 >$ is given by

$$< A_1 > = \int \mathrm{d}^4 q \int \mathrm{d}^4 k \mathrm{d}^4 j A_1(k, j) \mathrm{Tr}(\rho \psi^\dagger(q) \mathrm{e}^{\mathrm{i}(k\cdot\hat{q} + j\cdot\hat{p})} \psi(q)). \tag{6.81}$$

The exponential can be factorized to

$$\exp(ik \cdot q + j\partial) = \exp(ik \cdot q)\exp(j \cdot \partial)\exp(ik \cdot j/2). \tag{6.82}$$

Then (6.81) becomes

$$< A_1 > = \int d^4q d^4p A_1(q, p) f_1^W(q, p), \tag{6.83}$$

where $A_1(q, p)$ is the classical function corresponding to the operator $\hat{A}_1$ through the Weyl correspondence, and we have defined the *one particle relativistic Wigner function*

$$\begin{aligned} f_1^W(q, p) &= \frac{1}{(2\pi)^4} \int d^4 j e^{-ij\cdot p} Tr\left(\rho\psi^\dagger\left(q - \frac{j}{2}\right)\psi\left(q + \frac{j}{2}\right)\right) \\ &= \frac{1}{(2\pi)^4} \int d^4 k e^{ik\cdot q} Tr\left(\rho\psi^\dagger\left(p - \frac{k}{2}\right)\psi\left(p + \frac{k}{2}\right)\right). \end{aligned} \tag{6.84}$$

As for the nonrelativistic analog of this procedure, $f_1^W(q, p)$ is not necessarily positive, and cannot be interpreted as a pointwise probability density. It has the advantage, as we shall see, that the equations of motion are very analogous to the classical equations in phase space, and the results are immediately applicable to classical transport theory. Furthermore, note that

$$\int d^4q f_1^W(q, p) = \mathrm{Tr}(\rho\psi^\dagger(p)\psi(p)) \geqslant 0, \tag{6.85}$$

and that

$$\int d^4q d^4p f_1^W(q, p) = \int d^4q \, \mathrm{Tr}(\rho\psi^\dagger(q)\psi(q)). \tag{6.86}$$

Since

$$\int d^4q \psi^\dagger(q)\psi(q) = \int d^4q \psi^\dagger(p)\psi(p) = N, \tag{6.87}$$

the number operator for the total absolutely conserved number of the set of events is a superselection rule for this system, and therefore just a simple classical number,

$$\int d^4q d^4p f_1^W(q, p) = N \, \mathrm{Tr}\, \rho = N, \tag{6.88}$$

i.e. a 'normalization' for the Wigner function.

We now consider the τ evolution of the one particle distribution function. To do this in a convenient way, we study the Fourier transform

$$f_1^W(k, p) = \int d^4q e^{ik\cdot q} f_1^W(q, p) = \mathrm{Tr}\left(\rho\psi^\dagger\left(p - \frac{k}{2}\right)\psi\left(p + \frac{k}{2}\right)\right). \tag{6.89}$$

Using the cyclic properties of operators under a trace with the density matrix, it then follows from the Stueckelberg–Schrödinger evolution that

$$\partial_\tau f_1^W(k, p) = -\mathrm{i}\mathrm{Tr}\left[\left(\rho\psi^\dagger\left(p - \frac{k}{2}\right)\psi\left(p + \frac{k}{2}\right)\right), K\right]. \tag{6.90}$$

We assume that K has the form

$$K = K_0 + V, \tag{6.91}$$

where

$$K_0 = -\int \mathrm{d}^4 q \psi^\dagger(q)\frac{\partial^\mu\partial_\mu}{2M}\psi(q), \tag{6.92}$$

and

$$V = \frac{1}{2}\int \mathrm{d}^4 q' \mathrm{d}^4 q'' \psi^\dagger(q')\psi^\dagger(q'')V(q' - q'')\psi(q'')\psi(q') \tag{6.93}$$

is the two-body operator (Poincaré invariant) corresponding to a two-event interaction potential. Carrying out the commutator with this model, one finds that the time dependence of the one particle Wigner function depends on the two particle Wigner function, defined by

$$\begin{aligned} f_2^W(k_1 p_1, k_2 p_2) &= \int \mathrm{d}^4 q_1 \mathrm{d}^4 q_2 \mathrm{e}^{-\mathrm{i}k_1\cdot q_1 - \mathrm{i}k_2\cdot q_2} f_2^W(q_1 p_1, q_2 p_2) \\ &= \mathrm{Tr}(\rho\psi^\dagger(p_1 - k_1/2)\psi^\dagger(p_2 - k_2/2) \\ &\quad \times \psi(p_1 + k_1/2)\psi(p_2 + k_2/2)), \end{aligned} \tag{6.94}$$

according to

$$\begin{aligned} \partial_\tau f_1^W(k_1, p_1) &= L_0 f_1^W(k_1, p_1) \\ &\quad + \int \mathrm{d}^4 p_2 \mathrm{d}^4 k_2 \delta^4(k_2) L_{12} f_2^W(k_1 p_1, k_2 p_2), \end{aligned} \tag{6.95}$$

where L_0 and L_{12} are differential operators induced by the commutator with K_0. This procedure may be applied again to every f_s^W for $s = 1, 2 \ldots N$, and results in a set of equations of precisely the same form as the well-known Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy [22] for the nonrelativistic case. One obtains in this way a relativistically covariant generalization of the BBGKY hierarchy derived from basic dynamical principles.

The higher order relations invoke higher order correlations, and for a dilute gas of events, we may assume that truncation at the level of two-body correlations will suffice. Furthermore, the two-body correlation terms can be represented to fairly good accuracy, as in the non-relativistic case, by two-body scattering amplitudes consisting of two basic terms: one scattering events into the quasi-equilibrium ensemble, and the other scattering events out. The basic ingredients needed are derived in chapter 4 on scattering theory. The scattering, as for the nonrelativistic case, induces changes in the distribution function, i.e. the rate of change of f due to collisions is

$$D_c f(q, p) = D_c^+ f(q, p) - D_c^- f(p, q), \tag{6.96}$$

where $D_c^- f(p, q)\mathrm{d}^4q\mathrm{d}^4p\delta\tau$ is the number of collisions in the interval $\delta\tau$ in which one of the events is in $\mathrm{d}^4q\mathrm{d}^4p$, and $D_c^+ f(p, q)\mathrm{d}^4q\mathrm{d}^4p\delta\tau$ is the number of collisions in $\delta\tau$ in which one of the final events is in $\mathrm{d}^4q\mathrm{d}^4p$. Denoting by $\dot{P}$ the transition rate derived from the two-body scattering theory for this potential, we have

$$\begin{aligned} D_c^+ f(q, p) &= \int \mathrm{d}^4p_1\mathrm{d}^4p_1'\mathrm{d}^4p'\,\dot{P}(p_1'p' \to p_1 p) f(q, p') f(q, p_1'), \\ D_c^- f(q, p) &= \int \mathrm{d}^4p_1\mathrm{d}^4p_1'\mathrm{d}^4p'\,\dot{P}(p_1 p \to p_1'p') f(q, p) f(q, p_1). \end{aligned} \tag{6.97}$$

Furthermore, these results can be put into terms of the experimentally measured scattering cross sections [23] in the form (we denote $q_1 - q_2$ by q_r, and $\frac{1}{2}(p_1 - p_2)$ by p_r, $P = p_1 + p_2$, and assume a narrow distribution over the mass shifts)

$$\begin{aligned} \partial_\tau f(q, p) + \frac{p^\mu}{M}\frac{\partial}{\partial q^\mu} f(q, p) = 4\pi \int \mathrm{d}^3p_r\mathrm{d}^3p_r' \frac{|\mathbf{p}_r'|}{M}\frac{\mathrm{d}\sigma^{exp}}{\mathrm{d}^3p_r}\left(p_r' \to p_r; P\right) \\ \times \left\{f(q, p')f\left(q, p_1'\right) - f(q, p)f(q, p_1)\right\}. \end{aligned} \tag{6.98}$$

With this final form of the Boltzmann equation, we can discuss the relativistic H theorem. Defining the functional [7]

$$H(\tau) = \int \mathrm{d}^4q\mathrm{d}^4p f(q, p, \tau)\ln f(q, p, \tau) \equiv -S(\tau)/k_B, \tag{6.99}$$

where $S(\tau)$ is the entropy. Then, the derivative of $H(\tau)$ is

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}\tau} = \frac{1}{64}\int \mathrm{d}^4q\mathrm{d}^4p\mathrm{d}^4p_1\mathrm{d}^4p_1'[\ln f(q, p_1)f(q, p) - \ln f(q, p_1')f(q, p')] \\ \times \{f(q, p')f(q, p)f(q, p_1)\}\dot{P}\left(\left(\frac{p_1 - p}{2} \to \frac{p_1' - p'}{2}\right); P\right). \end{aligned} \tag{6.100}$$

Since $\dot{P}(p_r \to p_r' : P) \geqslant 0$, and the remaining factor in the integrand is non-positive, we obtain

$$\frac{\mathrm{d}H(\tau)}{\mathrm{d}\tau} \leqslant 0, \tag{6.101}$$

the relativistic H theorem.

This result implies that the entropy $S(\tau)$ is monotonically increasing as a function of τ, but *not necessarily in t*, since the directions of t and τ for the antiparticle are opposite. In the nonrelativistic limit, the standard H theorem is recovered, since t and τ become identical.

6.6 Spin, statistics and correlations

We shall discuss in this chapter the basic idea of a relativistic particle with spin, based on Wigner's seminal work [3]. The theory is adapted here to be applicable to relativistic quantum theory; in this form, Wigner's theory together with the requirements imposed by the observed correlation between spin and statistics in nature for identical particle systems make it possible to define the total spin of a state of a relativistic many-body system.

Furthermore, we shall show that a generalization of the construction of Wigner yields, in the framework we shall present here, a representation for tensor operators corresponding to an *invariant* decomposition in terms of irreducible representations of $SO(3)$; this procedure may be applied as well to spinorial valued operators, such as Rarita–Schwinger fields [24].

6.6.1 Introduction

The spin of a particle in a nonrelativistic framework corresponds to the lowest dimensional nontrivial representation of the rotation group; the generators are the Pauli matrices σ_i divided by two, the generators of the fundamental representation of the double covering of $SO(3)$. The self-adjoint operators that are the generators of this group measure intrinsic angular momentum and are associated with magnetic moments [25].

In the nonrelativistic quantum theory, the spin states of a two or more particle system are defined by combining the spins of these particles at equal time using appropriate Clebsch–Gordan coefficients [2] at each value of the time. The restriction to equal time follows from the tensor product form of the representation of the quantum states for a many-body problem [2]. This correlation at equal time in the nonrelativistic quantum theory is the source of the famous Einstein–Podolsky–Rosen discussion [26] and provides an important model for quantum information transfer. In the SHP theory, as we shall see below, these correlations persist for not precisely equal times.

The standard Pauli description of a particle with spin is not, however, relativistically covariant, but Wigner [3] has shown how to describe this dynamical property of a particle in a covariant way. The method developed by Wigner has provided the foundation for what is now known as the theory of induced representations [27], with very wide applications including a very powerful approach to finding the representations of noncompact groups.

The formulation of Wigner [3] is, however, not appropriate for application to quantum theory, since it does not preserve the covariance of the expectation value of coordinate operators. We first briefly review Wigner's method in its original form, and show how the difficulties arise. We then review the extension of Wigner's approach necessary to describe the spin of a particle in the framework of the manifestly covariant SHP theory [7]. We then show that the observed correlation of spin and statistics for identical particles necessitates a structure for which the Hilbert space of states of a many-body system of identical particles is represented as a direct integral over all values of a (normalized) time-like vector, a structure called *foliation*.

The relativistic many-body system then admits the description of total spin (in general, total angular momentum) states through computation with Clebsch–Gordan coefficients as in the nonrelativistic case, and implies correlations between the spins of the particles much in the same way (although not necessarily at precisely equal times).

We have shown here (see also [1]) that relativistically covariant canonical ensembles can be constructed in the framework of the SHP theory [7] as well as a corresponding Boltzmann transport theory [21]. The results that we achieve here admit an extension to particles with spin; the results obtained in the previous section may be embedded in the foliation implied by the accommodation of spin.

The foliation universally induced in the representation for physical many-body systems applies both to fermion and boson sectors of the full Fock space, and therefore to the quantum fields.

As Wigner [3] has shown (see also the detailed discussion in [28, 29]), constructing a representation of the Lorentz group by inducing a representation on the stability group of the (time-like) four-momentum, one obtains a representation $\psi(p, \sigma)$ with the transformation property

$$\psi'(p, \sigma) = \psi(\Lambda^{-1}p, \sigma')D_{\sigma',\sigma}(\Lambda, p), \tag{6.102}$$

under the action of the Lorentz group, taking into account the spin degrees of freedom of the wave function, where the matrix transformation factor (Wigner's 'little group' [3]) is constructed of the 2×2 matrices of $SL(2, C)$.

The presence of the p-dependent matrices representing the spin of a relativistic particle in the transformation law of the wave function, however, destroys the covariance in a relativistic quantum theory of the expectation value of the coordinate operators [30] in states transforming as in (6.102). To see this, consider the expectation value of the dynamical variable x^μ, i.e.

$$< x^\mu > = \sum\nolimits_\sigma \int \mathrm{d}^4p\psi(p, \sigma)^\dagger \mathrm{i}\frac{\partial}{\partial p_\mu}\psi(p, \sigma). \tag{6.103}$$

A Lorentz transformation would introduce the p-dependent 2×2 unitary transformation on the function $\psi(p, \sigma)$, and the derivative with respect to momentum would destroy the covariance property that we would wish to see of the expectation value $< x^\mu >$.

In this framework, it is also not possible to form wave packets of definite spin by (four-dimensional) Fourier transform over the momentum variable, since this would add functions over different parts of the orbit, with a different $SU(2)$ at each point.

These problems were solved [30] by inducing a representation of the spin on a time-like unit vector, say n^μ in place of the four-momentum.

Using a representation induced on a time-like vector n^μ, which is independent of x^μ or p^μ [4] permits the linear superposition of momentum eigenstates to form wave

[4] Note that the resulting Stueckelberg type wave functions $\psi_n(x, \sigma)$ are local [12] and do not have the non-local properties discussed by Newton and Wigner [31].

packets of definite spin, and admits the construction of definite spin states for many-body relativistic systems. In the following, we show how such a representation can be constructed, and discuss some of its dynamical implications.

6.6.2 Induced representation on a time-like vector n^μ

We briefly review here the construction given in [11] in order to make clear the nature of the resulting foliation of the Hilbert space. Let us define

$$|n, \sigma, x> \equiv U(L(n))|n_0, \sigma, x> , \tag{6.104}$$

where we may admit a dependence on x (or, through Fourier transform, on p). Here, we distinguish the action of $U(L(n))$ from the general Lorentz transformation $U(\Lambda)$; $U(L(n))$ acts only on the manifold of $\{n^\mu\}$. Its infinitesimal generators are given by

$$M_n^{\mu\nu} = -\mathrm{i}\left(n^\mu \frac{\partial}{\partial n_\nu} - n^\nu \frac{\partial}{\partial n_\mu}\right), \tag{6.105}$$

while the generators of the transformations $U(\Lambda)$ act on the full space of both n^μ and x^μ (as well as p^μ); its generators are given by

$$M^{\mu\nu} = M_n^{\mu\nu} + (x^\mu p^\nu - x^\nu p^\mu). \tag{6.106}$$

The two terms of the generator commute, and therefore the full group is a (diagonal) direct product.

We now investigate the properties of a total Lorentz transformation, i.e. as in Wigner's procedure [3],

$$U(\Lambda^{-1})|n, \sigma, x> = U(L(\Lambda^{-1}n)(U^{-1}(L(\Lambda^{-1}n))U(\Lambda^{-1})U(L(n)))|n_0, \sigma, x> . \tag{6.107}$$

Now, consider the conjugate of (6.107),

$$<n, \sigma, x|U(\Lambda) = <n_0, \sigma, x|(U(L^{-1}(n))U(\Lambda)U(L(\Lambda^{-1}n)))U^{-1}(L(\Lambda^{-1}n)). \tag{6.108}$$

The operator in the first factor (in parentheses) preserves n_0, and therefore corresponds to an element of the little group associated with n^μ which may be represented by the matrices of $SL(2, C)$. Due to the factor $U(\Lambda)$ (for which the generators are those of the Lorentz group acting both on n and x (or p)), it also takes $x \rightarrow \Lambda^{-1}x$ in the conjugate ket on the left. Taking the product on both sides with $|\psi>$, we obtain

$$<n, \sigma, x|U(\Lambda)|\psi> = <\Lambda^{-1}n, \sigma', \Lambda^{-1}x|\psi> D_{\sigma',\sigma}(\Lambda, n), \tag{6.109}$$

or [30]

$$\psi'_{n,\sigma}(x) = \psi_{\Lambda^{-1}n,\sigma'}(\Lambda^{-1}x)D_{\sigma',\sigma}(\Lambda, n), \tag{6.110}$$

where

$$D(\Lambda, n) = L^{-1}(n)\Lambda L(\Lambda^{-1}n), \tag{6.111}$$

with Λ and $L(n)$ the corresponding 2×2 matrices of $SL(2, C)$.

It is clear that, with this transformation law, one may take the Fourier transform to obtain the wave function in momentum space, and vice versa. The matrix $D(\Lambda, n)$ is an element of $SU(2)$, and therefore linear superpositions over momenta or coordinates maintain the definition of the particle spin for each n^μ, and interference phenomena for relativistic particles with spin may be studied consistently. Furthermore, if two or more particles with spin are represented in representations induced on n^μ, at the same value of n^μ on their respective orbits, and therefore in the *same* $SU(2)$ representation, their spins can be added by the standard methods with the use of Clebsch–Gordan coefficients. This method therefore admits the treatment of a many-body relativistic system with spin, as in the proposed experiment of [32] (see also [33]).

Our assertion of the unitarity of the n-dependent part of the transformation has assumed that the integral measure on the Hilbert space, to admit integration by parts, is of the form $\mathrm{d}^4n\mathrm{d}^4x$, where the support of the wave functions on n^μ is in the time-like sector. The action of the generator of Lorentz transformations on n^μ maintains the normalization of $n^\mu n_\mu$, which we shall take to be -1 in our discussion of the Dirac representation for the wave function. Although the time-like vector n^μ in many applications is degenerate, it carries a probability interpretation under the norm, and may play a dynamical role (for example, as for the space-like inducing vector for the two-body bound state problem in the covariant Zeeman formulation of [34]).

There are two fundamental representations of $SL(2, C)$ which are inequivalent [29, 30]. Multiplication of a two-dimensional spinor representing one of these by the operator $\sigma \cdot p$, expected to occur in any dynamical theory, results in an object transforming like the other representation, and therefore the state of lowest dimension in spinor indices of a physical system should contain both representations. As we shall emphasize, however, in our treatment of more than one particle system, for the rotation subgroup, both of the fundamental representations yield the same $SU(2)$ matrices up to a unitary transformation, and therefore the Clebsch–Gordan decomposition of the product state into irreducible representations may be carried out independently of which fundamental $SL(2, C)$ representation is associated with each of the particles. This is therefore true for the Dirac representation, incorporating both fundamental representations, constructed as follows [30].

As in [30], one finds the Dirac spinor [35]

$$\psi_n(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} L(n)\hat{\varphi}_n(x) \\ \underline{L}(n)\hat{\phi}_n(x) \end{pmatrix}, \tag{6.112}$$

which transforms as

$$\psi'_n(x) = S(\Lambda)\psi_{\Lambda^{-1}n}(\Lambda^{-1}x), \tag{6.113}$$

where $S(\Lambda)$ is a (nonunitary) transformation generated infinitesimally, as in the standard Dirac theory (see, for example, [19]), by $\sum^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

Following the arguments of [30], one can construct in the presence of a $U(1)$ gauge field the covariant Hamiltonian

$$K = \frac{1}{2M}(p - eA)^2 + \frac{e}{2M}\sum\nolimits_n^{\mu\nu} F_{\mu\nu}(x) - eA_5, \tag{6.114}$$

where

$$\sum\nolimits_n^{\mu\nu} = \sum\nolimits_{\mu\nu} + K^\mu n^\nu - K^\nu n^\mu, \tag{6.115}$$

and $K^\mu = \sum^{\mu\nu} n_\nu$. The A_5 field arises as a compensation field for the τ derivative in the Stueckelberg–Schrödinger equation [36]. In general, in this framework, the A^μ and A^5 fields may depend on τ, since they correspond to gauge compensation fields for the local gauge transformation $\psi_\tau(x) \to \exp \mathrm{i}\Lambda(x, \tau)\psi_\tau(x)$. The τ-independent Maxwell fields correspond to the zero mode of the A^μ fields used here [36]. The currents constructed from the Lagrangian associated with (6.114) are according to (6.112) also foliated, and therefore the fields A_μ, A_5 generated by these currents will be foliated as well.

The expression (6.114) is quite similar to that of the second-order Dirac operator; it is, however, Hermitian and has no direct electric coupling to the electromagnetic field in the special frame for which $n^\mu = (1, 0, 0, 0)$ in the minimal coupling model we have given here (note that in his calculation of the anomalous magnetic moment [19], Schwinger puts the electric field to zero; a non-zero electric field would lead to a non-Hermitian term in the standard Dirac propagator, the inverse of the Klein–Gordon square of the interacting Dirac equation). Note that in the derivation of the anomalous magnetic moment given by Bennett [20], this restriction is not necessary since the generator of the interacting motion is intrinsically Hermitian.

The matrices $\sum_n^{\mu\nu}$ are, in fact, a relativistically covariant form of the Pauli matrices.

To see this [30], we note that the quantities K^μ and $\sum_n^{\mu\nu}$ satisfy the commutation relations

$$\begin{aligned}
[K^\mu, K^\nu] &= -\mathrm{i}\sum\nolimits_n^{\mu\nu} \\
\left[\sum\nolimits_n^{\mu\nu}, K^\lambda\right] &= -\mathrm{i}[(g^{\nu\lambda} + n^\nu n^\lambda)K^\mu - (g^{\mu\lambda} + n^\mu n^\lambda)K^\nu], \\
\left[\sum\nolimits_n^{\mu\nu}, \sum\nolimits_n^{\lambda\sigma}\right] &= -\mathrm{i}\Big[(g^{\nu\lambda} + n^\nu n^\lambda)\sum\nolimits_n^{\mu\sigma} - (g^{\sigma\mu} + n^\sigma n^\mu)\sum\nolimits_n^{\lambda\nu} \\
&\quad - (g^{\mu\lambda} + n^\mu n^\lambda)\sum\nolimits_n^{\nu\sigma} + (g^{\sigma\nu} + n^\sigma n^\nu)\sum\nolimits_n^{\lambda\nu}\Big].
\end{aligned} \tag{6.116}$$

Since $K^\mu n_\mu = n_\mu \sum^{\mu\nu} = 0$, there are only three independent K^μ and three $\sum_n^{\mu\nu}$. The matrices $\sum_n^{\mu\nu}$ are a covariant form of the Pauli matrices, and the last of (6.116) is the Lie algebra of $SU(2)$ in the space-like surface orthogonal to n^μ. The three independent K^μ correspond to the non-compact part of the algebra which, along with the $\sum_n^{\mu\nu}$, provide a representation of the Lie algebra of the full Lorentz group. The covariance of this representation follows from

$$S^{-1}(\Lambda)\sum\nolimits_{\Lambda n}^{\mu\nu}S(\Lambda)\Lambda_\mu^\lambda\Lambda_\nu^\sigma = \sum\nolimits_n^{\lambda\sigma}. \tag{6.117}$$

In the special frame for which $n^\mu = (1, 0, 0, 0)$, $\sum_n^{ij}$ become the Pauli matrices $\frac{1}{2}\sigma^k$ with (i, j, k) cyclic, and $\sum_n^{0j} = 0$. In this frame there is no direct electric interaction with the spin in the minimal coupling model (6.114). We remark that there is, however, a natural spin coupling which becomes pure electric in the special frame [30], given by (in gauge covariant form)

$$\mathrm{i}[K_T, K_L] = -\mathrm{i}e\gamma^5(K^\mu n^\nu - K^\nu n^\mu)F_{\mu\nu}. \tag{6.118}$$

Note that the matrices

$$\gamma_n^\mu = \gamma_\lambda \pi^{\lambda\mu}, \tag{6.119}$$

with the projection

$$\pi^{\lambda\mu} = g^{\lambda\mu} + n^\lambda n^\mu, \tag{6.120}$$

appearing in (6.116), play an important role in the description of the dynamics in the induced representation. In (6.115), the existence of projections on each index in the spin coupling term implies that $F^{\mu\nu}$ can be replaced by $F_n{}^{\mu\nu}$, a tensor projected into the foliation subspace. As we shall see, this foliation induced by the spin has a profound effect on the tensor products (and therefore on the full Fock space) of identical particle systems, both in the boson and fermion sectors[5].

6.6.3 The many-body problem with spin, and spin statistics

As in the nonrelativistic quantum theory, one represents the state of an N-body system in terms of a basis given by the tensor product of N one-particle states, each an element of a one-particle Hilbert space. The general state of such an N-body system is given by a linear superposition over this basis [38]. Second quantization then corresponds to the construction of a Fock space, for which the set of all N-body states, for all N, are embedded in a large Hilbert space for which operators that change the number N are defined [2]. We shall discuss this structure in this section, and show with our discussion of the relativistic spin given in the previous section that the spin of a relativistic many-body system can be well-defined and, furthermore, that the quantum fields associated with the particles of the system carry the induced foliation structure.

In order to construct the tensor product space corresponding to the many-body system, we consider, as for the nonrelativistic theory, the product of wave functions which are elements of isomorphic Hilbert spaces. In the nonrelativistic theory, this

[5] Note that for the $SO(1, 1)$ covariant generalization of one-dimensional systems treated, for example, by methods utilizing the Bethe ansatz [37], the relation between spin and statistics is not so direct, and therefore this problem requires a separate discussion.

corresponds to functions at equal time; in the relativistic theory, the functions are at equal τ. Thus, in the relativistic theory, there are correlations at unequal t, within the support of the Stueckelberg wave functions. Moreover, for particles with spin we argue, as a consequence of the spin-statistics relation, that in the induced representation, these functions must be taken at *identical values of* n^μ, i.e. taken at the same point on the orbits of the induced representation of each particle:

Statement: Identical particles must be represented in tensor product states by wave functions not only at equal τ *but also at equal* n^μ.

This statement follows from the observation that the spin-statistics relation appears to be a universal fact of nature. The elementary proof of the spin-statistics theorem, for example, for a system of two spin 1/2 particles, is that a π rotation of the system introduces a phase factor of $e^{i\frac{\pi}{2}}$ for each particle, thus introducing a minus sign for the two-body state. However, the π rotation is equivalent to an interchange of the two identical particles. This argument rests on the fact that each particle is in the same representation of $SU(2)$, which can only be achieved in the induced representation with the particles at the same point on their respective orbits. We therefore see that identical particles *must carry the same value of* n^μ, and the construction of the N-body system must follow this rule[6]. It therefore follows that the two-body relativistic system can carry a spin computed by use of the usual Clebsch–Gordan coefficients, and entanglement would follow even at unequal time (within the support of the equal τ wave functions), as in the proposed experiment in [12]. This argument can be followed for arbitrary N, and therefore the Fock space of the quantum field theory carries the properties usually associated with fermion (or boson) fields with the entire Fock space foliated over the orbit of the inducing vector n^μ.

Although, due to the Newton–Wigner problem [31] noted above, the solutions of the Dirac equation are not suitable for the covariant local description of a quantum theory, the functions constructed in (6.112) can form the basis of a consistent, local (off-shell) covariant quantum theory.

To show how the many-body Fock space develops, we start by constructing a two-body Hilbert space in the framework of the relativistic quantum theory. The states of this two-body space are given by linear combinations over the product wave functions, where the wave functions are given by Dirac functions of the type described in (6.112), i.e. temporarily suppressing the indices n, τ,

$$\psi_{ij}(x_1, x_2) = \psi_i(x_1) \otimes \psi_j(x_2), \tag{6.121}$$

where $\psi_i(x_1)$ and $\psi_j(x_2)$ are elements of the one-particle Hilbert space $\mathcal{H}$. Let us introduce the notation, often used in differential geometry, that

$$\psi_{ij}(x_1, x_2) = \psi_i \otimes \psi_j(x_1, x_2), \tag{6.122}$$

[6] Note that symmetrization and antisymmetrization can, of course, be carried out with factors in the tensor product on any sequence in n, but the symmetry properties would not then correspond to the phases associated with spin.

identifying the arguments according to a standard ordering. Then, without specifying the spacetime coordinates, we can write

$$\psi_{ij} = \psi_i \otimes \psi_j, \tag{6.123}$$

formally, an element of the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$. The scalar product is carried out by pairing the elements in the two factors according to their order, since it corresponds to integrals over x_1, x_2, i.e.

$$(\psi_{ij}, \psi_{k,\ell}) = (\psi_i, \psi_k)(\psi_j, \psi_\ell). \tag{6.124}$$

For two identical particle states satisfying Bose–Einstein or Fermi–Dirac statistics, according to our argument given above we must write

$$\psi_{ijn} = \frac{1}{\sqrt{2}}[\psi_{\text{in}} \otimes \psi_{jn} \pm \psi_{jn} \otimes \psi_{\text{in}}]. \tag{6.125}$$

This expression has the required symmetry or antisymmetry only if both functions are on the same points of their respective orbits in the induced representation. Furthermore, they transform under the *same* $SU(2)$ representation of the rotation subgroup of the Lorentz group, and thus for spin 1/2 particles, under a π spatial rotation (defined by the space orthogonal to the time-like vector n^μ) they both develop a phase factor $e^{i\frac{\pi}{2}}$. The product results in an overall negative sign. As in the usual quantum theory, this rotation corresponds to an interchange of the two particles, but here with respect to a 'spatial' rotation around the vector n^μ. The spacetime coordinates in the functions are rotated in this (foliated) subspace of spacetime, and correspond to an actual exchange of the positions of the particles in spacetime, as in the formulation of the standard spin-statistics theorem. It therefore follows that the interchange of the particles occurs in the foliated space defined by n^μ. For identical bosonic particles, the π rotation produces a positive sign. These conclusions are valid for unequal times that lie in support of the SHP wave functions (at equal τ). We therefore have the following:

Statement: The antisymmetry of identical half-integer spin (fermionic) particles remains at unequal times (within the support of the wave functions). This is true for the symmetry of identical integer spin (bosonic) particles as well.

Furthermore, the construction we have given enables us to define the spin of a many-body system, even if the particles are relativistic and moving arbitrarily with respect to each other. Since all particles with representations on a common n^μ of their orbits transform in the space-like submanifold orthogonal to n^μ under the same $SU(2)$, it is also true that

Statement: The spin of an N-body system of identical particles is well-defined, independent of the state of motion of the particles of the system, by the usual laws of combining representations of $SU(2)$, i.e. with the usual Clebsch–Gordan coefficients, since the states of all the particles in the system are in induced representations at the same point of the orbit n^μ.

Thus, for example, in the quark model for hadrons, the total spin of the hadron can be computed from the spins (and orbital angular momenta projected into the foliated space) of the individual quarks using the usual Clebsch–Gordan coefficients even if they are in significant relative motion within the same $SU(2)$; a similar conclusion would be valid for nucleons in a nucleus even at high excitation. The validity of spin assignations in high energy scattering would provide an important example of such quantum mechanical correlations.

In the course of our construction, we have seen that the foliation of the spacetime follows from the arguments based on the representations of a relativistic particle with half-integer spin. However, as we have remarked, our considerations of the nature of identical particles, and their association with the spin-statistics properties observed in nature, require that the foliation persists in the bosonic sector as well, where a π rotation, exchanging two particles, must be in a definite representation of the rotation group, specified by the foliation vector n^μ, to achieve a positive sign. Since there is no extra phase (corresponding to integer representations of $SU(2)$) for the Bose–Einstein case, the boson symmetry can then be extended to a covariant symmetry with important implications; for example, for the statistical mechanics of relativistic boson systems in Bose–Einstein condensation.

We remark in this connection that the Cooper pairing [39] of superconductivity must be between electrons on the same point of their induced representation orbits, so that the superconducting state is defined on the corresponding foliation of spacetime as well. The resulting (quasi-)bosons have identical particle properties inferred from our discussion of the boson sector. As remarked above, the two electrons of the Cooper pair may not be at equal time, a result which may be accessible to experiment. A similar remark applies to the Josephson effect [40] (where a single gate may be opened at two successive times, as in [41]).

These results have important implications in atomic and molecular physics; for example, for the construction of the exchange interaction.

6.6.4 Quantum fields

We now extend our argument for the finite Fock space to the general structure of quantum field theory.

The N-body state of Fermi–Dirac particles can be written as (the N-body boson system should be treated separately since the normalization conditions are different, but we give the general result below)

$$\Psi_{nN} = \frac{1}{N!}\sum(-)^P P\psi_{nN} \otimes \psi_{nN-1} \otimes \cdots \psi_{n1}, \tag{6.126}$$

where the permutations P are taken over all possibilities, and no two functions are equal. By the arguments given above, any pair of particle wave functions in this set has the Fermi–Dirac symmetry properties. We may now think of such a function as an element of a larger Hilbert space, the *Fock space*, which contains all values of the number N. On this space, one can define an operator that adds another particle (in the tensor product), performs the necessary antisymmetrization, and changes the normalization appropriately. This operator is called a *creation operator*, which we shall denote by $a^\dagger(\psi_{nN+1})$ and has the property that

$$a^{\dagger}(\psi_{nN+1})\Psi_{nN} = \Psi_{nN+1}, \tag{6.127}$$

now to be evaluated on the manifold $(x_{N+1}, x_N, x_{N-1} \dots x_1)$. Taking the scalar product with some $N + 1$ particle state Φ_{nN+1} in the Fock space, we see that

$$(\Phi_{nN+1}, a^{\dagger}(\psi_{nN+1})\Psi_{nN}) \equiv (a(\psi_{nN+1})\Phi_{nN+1}, \Psi_{nN}), \tag{6.128}$$

thus defining the *annihilation* operator $a(\psi_{nN+1})$.

The existence of such an annihilation operator, as in the usual construction of the Fock space (see e.g. [2]) implies the existence of an additional element in the Fock space, the *vacuum*, or the state of no particles. The vacuum defined in this way lies in the foliation labeled by n^μ. The covariance of the construction, however, implies that, since all sectors labeled by n^μ are connected by the action of the Lorentz group, this vacuum is a vacuum for any n^μ, i.e. the vacuum $\{\Psi_{n0}\}$ over all n^μ is Lorentz invariant.

The commutation relations of the annihilation–creation operators can be easily deduced from a low dimensional example, following the method used in the nonrelativistic quantum theory [2]. Consider the two-body state (6.125) (we use the antisymmetric form here), and apply the creation operator $a^{\dagger}(\psi_{n3})$ to create the three-body state

$$\begin{aligned}\Psi(\psi_{n3}, \psi_{n2}, \psi_{n1}) = \frac{1}{\sqrt{3!}}\{&\psi_{n3} \otimes \psi_{n2} \otimes \psi_{n1} + \psi_{n1} \otimes \psi_{n3} \otimes \psi_{n2} \\ &+ \psi_{n2} \otimes \psi_{n1} \otimes \psi_{n3} - \psi_{n2} \otimes \psi_{n3} \otimes \psi_{n1} \\ &- \psi_{n1} \otimes \psi_{n2} \otimes \psi_{n3} - \psi_{n3} \otimes \psi_{n1} \otimes \psi_{n2}\}.\end{aligned} \tag{6.129}$$

One then takes the scalar product with the three-body state

$$\begin{aligned}\Phi(\phi_{n3}, \phi_{n2}, \phi_{n1}) = \frac{1}{\sqrt{3!}}\{&\phi_{n3} \otimes \phi_{n2} \otimes \phi_{n1} + \phi_{n1} \otimes \phi_{n3} \otimes \phi_{n2} \\ &+ \phi_{n2} \otimes \phi_{n1} \otimes \phi_{n3} - \phi_{n2} \otimes \phi_{n3} \otimes \phi_{n1} \\ &- \phi_{n1} \otimes \phi_{n2} \otimes \phi_{n3} - \phi_{n3} \otimes \phi_{n1} \otimes \phi_{n2}\}.\end{aligned} \tag{6.130}$$

Carrying out the scalar product term by term, and picking out the terms corresponding to the scalar product of some function with the two-body state

$$\frac{1}{\sqrt{2}}\{\psi_{n2} \otimes \psi_{n1} - \psi_{n1} \otimes \psi_{n2}\}, \tag{6.131}$$

one finds that the action of the operator $a(\psi_{n3})$ on the state $\Phi(\phi_{n3}, \phi_{n2}, \phi_{n1})$ is given by

$$\begin{aligned}a(\psi_{n3})\Phi(\phi_{n3}, \phi_{n2}, \phi_{n1}) = &(\psi_{n3}, \phi_{n3})\phi_{n2} \otimes \phi_{n1} \\ &- (\psi_{n3}, \phi_{n2})\phi_{n3} \otimes \phi_{n1} + (\psi_{n3}, \phi_{n1})\phi_{n3} \otimes \phi_{n2},\end{aligned} \tag{6.132}$$

i.e. the annihilation operator acts like a derivation with alternating signs due to its fermionic nature; the relation of the two- and three-body states we have analyzed has a direct extension to the N-body case. The action of boson annihilation–creation operators can be derived in a similar way.

Applying these operators to N and $N + 1$ particle states, one finds directly their commutation and anticommutation relations

$$[a(\psi_n),\, a^\dagger(\phi_n)]_\mp = (\psi_n,\, \phi_n), \tag{6.133}$$

where the $\mp$ sign, corresponds to commutator or anticommutator for the boson or fermion operators. If the functions ψ_n, ϕ_n belong to a normalized orthogonal set $\{\phi_{nj}\}$, then

$$[a(\phi_{ni}),\, a^\dagger(\phi_{nj})]_\mp = \delta_{ij}. \tag{6.134}$$

Let us now suppose that the functions ϕ_{nj} are plane waves in spacetime, i.e. in terms of functions

$$\phi_{np}(x) = \frac{1}{(2\pi)^2}\mathrm{e}^{-\mathrm{i}p^\mu x_\mu}, \tag{6.135}$$

so that

$$(\phi_{np},\, \phi_{np'}) = \delta^4(p - p'). \tag{6.136}$$

The quantum fields are then constructed as follows. Define

$$\phi_n(x) \equiv \int \mathrm{d}^4 p a(\phi_{np})\mathrm{e}^{\mathrm{i}p^\mu x_\mu}. \tag{6.137}$$

It then follows that, by the commutation (anticommutation) relations (6.133), these operators obey the relations

$$[\phi_n(x),\, \phi_n^\dagger(x')]_\mp = \delta^4(x - x'), \tag{6.138}$$

corresponding to the commutation relations of boson and fermion *fields* (we suppress the spinor indices here, arising from the spinor form which must be used for (6.135)). Under Fourier transform, one finds the commutation relations in momentum space

$$[\phi_n(p),\, \phi_n^\dagger(p')]_\mp = \delta^4(p - p'). \tag{6.139}$$

The relation of these quantized fields with those of the usual on-shell quantum field theories can be understood as follows. Let us suppose that the fourth component of the energy momentum is $E = \sqrt{\mathbf{p}^2 + m^2}$, where m^2 is close to a given number, the on-shell mass of a particle. Then, noting that $\mathrm{d}E = \frac{\mathrm{d}m^2}{2E}$, if we multiply both sides of (6.139) by $\mathrm{d}E$ and integrate over the small neighborhood of m^2 occurring in both E and E', the delta function $\delta(E - E')$ on the right-hand side integrates to unity. On the left-hand side, there is a factor of $\mathrm{d}m^2/2E$, and we may absorb $\sqrt{\mathrm{d}m^2}$ in each of the field variables, obtaining (with $\varphi_n(\mathbf{p}) \equiv \sqrt{dm^2}\,\phi_n(p)$ on-shell)

$$[\varphi_n(\mathbf{p}),\, \varphi_n^\dagger(\mathbf{p}')]_\mp = 2E\delta^3(\mathbf{p} - \mathbf{p}'), \tag{6.140}$$

the usual formula for on-shell quantum fields.

We remark that these algebraic results have been constructed in the foliation involved in the formulation of a consistent theory of relativistic spin. They therefore admit the action of the $SU(2)$ group (in the Dirac representation (6.113), $S(\Lambda)$ has the form $\mathrm{e}^{\mathrm{i}\Sigma_n^{\mu\nu}\omega_{\mu\nu}}$ for $\omega_{\mu\nu}$ parameters corresponding to the $SU(2)$ subgroup leaving n^μ invariant) for a many-body system, applicable for unequal times, within the support of the Stueckelberg wave functions at equal τ.

We have discussed here the construction of quantum fields as they emerge from the structure of a Fock space. Local observables can be formed from the Hermitian operators built with these fields. According to the methods generally attributed to Schwinger [42] and Tomanaga [43] (see the book of Jauch and Rohrlich [44], for example, for a discussion of the ideas and additional references), a quantum state is defined by assigning values to the spectra of a complete set of such local observables which necessarily commute, according to the causal nature of measurements, if they are associated with a space-like surface. The sequence of space-like surfaces then forms a parametrization of the evolution of such states (the basis of the Schwinger–Tomonaga equation); it follows from our considerations that, for states of identical particles, the set of local observables is defined on the foliation provided by the inducing parameter n^μ, and therefore the Schwinger–Tomonaga state lies on this foliation as well. Furthermore, since the local fields in the Heisenberg picture evolve unitarily in τ, and the corresponding space-like surfaces are isomorphic, a correspondence can be established between τ and an invariant parameter labeling the sequence of space-like surfaces. Moreover, it is clear from (6.116) that the action of the operators $\sum_n^{\mu\nu}$, due to the occurrence of the projections $g^{\nu\lambda} + n^\nu n^\lambda$ in the coefficients of the Lie algebra, correspond to rotations in the space-like surface orthogonal to the time-like vector n^μ (as we have remarked, in the frame for which $n^\mu = (1, 0, 0, 0)$, these operators reduce to the ordinary Pauli matrices). Together with the operators K^μ, they constitute a representation of the Lorentz group, forming the fundamental representation of a group oriented with its maximal compact subgroup, corresponding to the $SU(2)$ little group of Wigner, acting on the wave functions, and the corresponding quantum fields, as a rotation in the space-like surface orthogonal to n^μ. We may therefore identify the space-like surfaces on which the quantum fields are defined with the space-like surfaces on which the little group induces rotations (as in the nonrelativistic theory). Local variations in the space-like surfaces, contemplated by Schwinger and Tomonaga, then correspond as well to local variations in the orbit of the induced representation, clearly preserving the local commutation and anticommutation relations.

The correlations imposed by the existence of the universal time-like vector permit us to construct the spin states of many-body systems through direct product of spin $\frac{1}{2}$

states (with appropriate Clebsch–Gordan coefficients) as well as higher spin states of a particle [45].

Appendix: Pauli–Lubanski vector

In this appendix we discuss a covariant Pauli–Lubanski vector W_μ^n [6] which, in the rest frame of the particle, carries the physical internal angular momentum of the particle, and for which the invariant $W_\mu{}^n W_{\mu_n}$ serves as the second Casimir operator for the Poincaré group. The angular momentum operator embedded in this definition generates rotations in the hyperplane orthogonal to the stability vector n^μ labeling the point on the orbit of the induced representation.

The operator $M^{\mu\nu}{}_n$ acts as an $SU(2)$ rotation in a space-like plane perpendicular to the time-like vector n^μ, which is identified with the physical angular momentum of the particle [14]. To demonstrate this property we first compute

$$\begin{aligned}[M^{\mu\nu}{}_n, M^{\lambda\sigma}{}_n] = -\,\mathrm{i}\{&(g^{\nu\lambda} + n^\nu n^\lambda)M^{\mu\sigma}{}_n + (g^{\sigma\mu} + n^\sigma n^\mu)M^{\lambda\nu}{}_n \\ &- (g^{\mu\lambda} + n^\mu n^\lambda)M^{\nu\sigma}{}_n + (g^{\sigma\nu} + n^\sigma n^\nu)M^{\lambda\mu}{}_n\}\end{aligned} \tag{6A.1}$$

In the special frame for which $n^\mu = (1, 0, 0, 0)$, equation (6A.1) becomes the equation for the algebra of ordinary three-dimensional angular momentum; the operator $M^{\mu\nu}{}_n$ is therefore a covariant form of the Lie algebra of $SU(2)$, valid also for the spin 1/2 representation [21][7].

Since $M^{\mu\nu}{}_n n_\nu$ is identically zero, it is clear that $M^{\mu\nu}{}_n$ rotates the vector x^λ in a plane perpendicular to n_ν, but it is of some interest to see the action of this infinitesimal transformation. Computing $[M^{\mu\nu}{}_n, x^\lambda]$ explicitly, one finds

$$\begin{aligned}[M^{\mu\nu}{}_n, x^\lambda] = &-\mathrm{i}(g^{\nu\lambda} + n^\nu n^\lambda)x^\mu \\ &+ \mathrm{i}(g^{\mu\lambda} + n^\mu n^\lambda)x^\nu \\ &\mathrm{i}(g^{\mu\lambda}n^\nu - g^{\nu\lambda}n^\mu)(x \cdot n).\end{aligned} \tag{6A.2}$$

This form is orthogonal to the vector n_ν, so the infinitesimal shift in x^λ is in a space-like plane orthogonal to n.

We now define the Pauli–Lubanski [6] vector in our context

$$W_\mu{}^n = \epsilon_{\mu\nu\lambda\sigma} M^{\lambda\sigma}{}_n p^\nu. \tag{6A.3}$$

We may easily demonstrate that this operator is Hermitian. Moreover, since the commutator of $M^{\lambda\sigma}{}_n$ with p^μ has precisely the same form as with x^μ (as in (6A.2), with p^μ replacing x^μ), and since $\epsilon_{\mu\nu\lambda\sigma}$ is totally antisymmetric, it follows that

$$[\epsilon_{\mu\nu\lambda\sigma} M^{\lambda\sigma}{}_n, p^\nu] = 0. \tag{6A.4}$$

We can therefore define the 'Casimir' on the orbit as

$$C_n \equiv W_\mu{}^n W^{\mu n}. \tag{6A.5}$$

[7] Note that the projection $g^{\mu\nu} + n^\nu n^\mu$ brings the metric into a three-dimensional space with a metric equivalent to (+++) by the operation $(g^{\mu\nu} + n^\nu n^\mu)g_{\nu\lambda}(g^{\lambda\gamma} + n^\lambda n^\gamma) = g^{\mu\gamma} + n^\mu n^\gamma$.

This operator commutes with the first Poincaré Casimir $p_\mu p^\mu$, corresponding to the mass of the particle (not necessarily a constant of the motion in the relativistic dynamics of SHP). If the momentum of the particle takes on a value parallel to n^μ, the Pauli–Lubanski operator that we have defined then coincides with the covariant relativistic generalization of the intrinsic physical angular momentum on the orbit. In this case, a Lorentz transformation to the frame for which $n^\mu = (1, 0, 0, 0)$ (then coinciding with the rest frame of the particle) brings $M^{\mu\nu}{}_n$ explicitly to the form of a generator of $SU(2)$. Note that in an asymptotic state with well-defined wave packet, if n^μ and p^μ coincide in the sense that $n^\mu \cong p^\mu/m$ (for $m = \sqrt{-p^\mu p_\mu}$), the derivative of the little group factor due to Lorentz transformation would be proportional to

$$\frac{\partial n^\mu}{\partial p^\nu} = \frac{1}{m}\left(g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right), \tag{6A.6}$$

a vector approximately orthogonal to $x^\mu \sim p^\mu$. The state of this wave packet, on which we can expect its modulation by the action of the little group and its derivative to have only a small effect on the result, then forms in the construction of $< x^\mu >$ an (approximate) expectation value of the operator (6A.6). In such an asymptotic state, for which the momentum is fairly sharp, the expected value of this operator would be very small. In this way, the approximate alignment of p^μ and n^μ would retain the required covariance of the expectation value of x^μ[8].

Thus, the vector n^μ could be thought of as defining a frame (for example, for the Stern–Gerlach measurement of the spin of an asymptotic state; we thank Y Aharonov for a discussion of this point) in which the intrinsic angular momentum, corresponding to the physical angular momentum of the particle as it occurs explicitly in the induced representation, can be directly measured in the laboratory.

References

[1] Horwitz L P, Schieve W C and Piron C 1981 *Ann. Phys.* **137** 306–40

[2] Baym G 1969 *Lectures on Quantum Mechanics* (New York: Benjamin)

[3] Wigner E 1939 *Ann. Math.* **40** 149

[4] Horwitz L P 2015 *Relativistic Quantum Mechanics (Fundamental Theories of Physics)* vol 180 (Dordrecht: Springer)

[5] Horwitz L P, Piron C and Reuse F 1975 *Helv. Phys. Acta* **48** 546
Arshansky R and Horwitz L P 1982 *J. Phys. A: Math. Gen.* **15** L659–62

[6] Horwitz L P and Zeilig-Hess M 2015 *J. Math. Phys.* **56** 092301
Lubanski J K 1942 *Physica* **9** 310 325 1942 see also [29]

[7] Huang K 1967 *Statistical Mechanics* (New York: Wiley)

[8] Haber H E and Weldon H A 1981 *Phys. Rev. Lett.* **46** 1497

[8] This argument effectively reinstates the validity of the original procedure of Wigner, using the four momentum as the stability vector for the little group, in the framework of a manifestly covariant quantum theory, for asymptotic states with fairly well-defined (off-shell) momentum.

[9] Burokovsky L, Horwitz L P and Schieve W C 1996 *Phys. Rev.* D **54** 4029
[10] Kirsten T 1994 *Proc. Int. Conf. on Non-Accelerator Particle Physics* ed R Cowsik (Singapore: World Scientific)
[11] Planck M 1914 *The Theory of Heat Radiation* (Philadelphia, PA: Blakiston)
[12] Horwitz L P and Shnerb N 1993 *Symmetries in Science VI: From the Rotation Group to Quantum Algebras* (Berlin: Springer) p 335; see also [4]
[13] Horwitz L P and Shnerb N 1998 *Found. Phys.* **28** 1509
[14] Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton, NJ: Princeton University Press)
[15] Tomczak S P and Haller K 1972 *Nuovo Cimento* B **8** 1
Haller K and Sohn R B 1979 *Phys. Rev.* A **20** 1541
Haller K 1975 *Acta Phys. Aust.* **42** 163
Haller K 1987 *Phys. Rev.* D **36** 1830
[16] Shnerb N and Horwitz L P 1993 *Phys. Rev.* A **48** 4068
[17] Fadeev L D and Popov V N 1967 *Phys. Lett.* B **25** 29
[18] Jauch J and Rohrlich F 1955 *The Theory of Photons and Electrons* (Cambridge, MA: Addison-Wesley)
[19] Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw Hill)
[20] Schwinger J S 1951 *Phys. Rev.* **82** 664
[21] Horwitz L P, Shashoua S and Schieve W C 1989 *Physica* A **161** 300
[22] Balescu R 1975 *Equilibrium and Non-Equilibrium Statistical Mechanics* (New York: Wiley)
[23] Horwitz L P ans Lavie Y 1982 *Phys. Rev.* D **26** 819
[24] Rarita W and Schwinger J 1941 *Phys. Rev.* **60** 61
[25] Biedenharn L C and Louck J D 1981 *The Racah–Wigner Algebra in Quantum Theory* (Reading, MA: Addison-Wesley)
[26] Einstein A, Podolsky B and Rosen N 1978 *Phys. Rev.* **41** 1881
[27] Mackey G W 1968 *Induced Representations of Groups and Quantum Mechanics* (New York: Benjamin)
[28] Stueckelberg E C G 1941 *Helv. Phys. Acta* **14** 322–3 588–94
Stueckelberg E C G 1942 *Helv. Phys. Acta* **15** 23–7
Horwitz L P and Piron C 1973 *Helv. Phys. Acta* **46** 316–26; see also [4]
[29] Weinberg S 1995 *The Quantum Theory of Fields* vol 1 (Cambridge: Cambridge University Press)
[30] Horwitz L P, Piron C and Reuse F 1975 *Helv. Phys. Acta* **48** 546
Arshansky R and Horwitz L P 1982 *J. Phys. A: Math. Gen.* **15** L659
[31] Newton T D and Wigner E 1949 *Rev. Mod. Phys.* **21** 400
[32] Palacios A, Rescigno T N and McCurdy C W 2009 *Phys. Rev. Lett.* **103** 253001
[33] Horwitz L P and Arshansky R 2017 arXiv: 1707.03294
[34] Land M C and Horwitz L P 1995 *J. Phys. A: Math. Gen.* **28** 3289
[35] Dirac P A M 1930 *Quantum Mechanics* 1st edn (Oxford: Oxford University Press)
[36] Saad D, Horwitz L P and Arshansky R I 1989 *Found. Phys.* **19** 1125–49
Aharonovich I and Horwitz L P 2006 *J. Math. Phys.* **47** 122902
[37] Bethe H 1931 *Z. Phys.* **71** 205
[38] Fetter A L and Walecka J D 1971 *Quantum Theory of Many Particle Systems* (New York: McGraw Hill)
[39] Cooper L N 1956 *Phys. Rev.* **104** 1189

[40] Josephson B D 1974 *Phys. Lett.* **1** 251
[41] Lindner F, Schätzel M G, Walther H, Baltuska A, Goulielmakis E, Krausz F, Miloševi D B, Bauer D, Becker W and Paulus G G 2005 *Phys. Rev. Lett.* **95** 040401
Horwitz L P 2006 *Phys. Lett.* A **355** 1
[42] Schwinger J 1951 *Proc. Natl Acad. Sci.* **37** 452
[43] Tomonaga S 1946 *Prog. Theor. Phys.* **1** 27
[44] Jauch J M and Rohrlich F 1976 *The Theory of Photons and Electrons* 2nd edn (New York: Springer)
[45] Horwitz L P and Zeilig-Hess M 2015 *J. Math. Phys.* **56** 092301