Relativistic Many-Body Theory and Statistical Mechanics

Online at: https://doi.org/10.1088/978-1-6817-4948-8

Relativistic Many-Body Theory and Statistical Mechanics

Lawrence P Horwitz and Rafael I Arshansky
Tel Aviv University, Israel

Morgan & Claypool Publishers

ISBN 978-1-6817-4948-8 (ebook)
ISBN 978-1-6817-4945-7 (print)
ISBN 978-1-6817-4946-4 (mobi)

DOI 10.1088/978-1-6817-4948-8

Version: 20180501

IOP Concise Physics
ISSN 2053-2571 (online)
ISSN 2054-7307 (print)

A Morgan & Claypool publication as part of IOP Concise Physics
Published by Morgan & Claypool Publishers, 1210 Fifth Avenue, Suite 250, San Rafael, CA, 94901, USA

IOP Publishing, Temple Circus, Temple Way, Bristol BS1 6HG, UK

Dedicated to Yuval Ne'eman

Contents

Preface

In this book, we describe the theory of Stueckelberg, Horwitz and Piron (SHP), which provides a comprehensive classical and quantum mechanical relativistically covariant framework for the discussion of many body problems. This theory has the essential property that the *time t of Einstein*, which is the time of arrival of an event as measured on a standard universal clock in an inertial laboratory, corresponding as well to the variable t occurring in the Maxwell equations, is considered to be an *observable*. The time t of the occurrence of an event is subject, as well as the position of the event $\mathbf{x}$, to equations of motion according to a universal evolution parameter τ corresponding to Newton's postulated time. The universality of this parameter enables us to write both classical and quantum dynamical equations for relativistic many body systems. We also develop, in this framework, the corresponding relativistically manifestly covariant quantum field theory.

We first study the two body problem for both bound and scattering states. The bound state solutions are found to yield a spectrum agreeing with the known nonrelativistic Schrödinger equation, up to relativistic corrections, when the coordinates are chosen to be those of a 'reduced Minkowski space' (RMS). To describe the configuration covariantly, we define a spacelike vector as the basic defining direction of the coordinate system, and use the theory of induced representations to obtain irreducible representations of the Lorentz group with functions with support lying in this subspace.

The procedure for calculating cross sections for the scattering sector was worked out Cross sections, in general, reflect the effective area of interaction orthogonal to the incoming beam; since this is determined by a spacelike vector, the cross section would be three dimensional in spacetime. The time dimension corresponds to the time interval of the interaction (the three dimensional interaction region could be, in principle, measureable). The time interval may be divided out, resulting in the usual area formula.

This construction has important consequences for the theory of many body systems. If the potential functions between all pairs of particles are restricted to lie in the RMS oriented in the same way, the resulting dynamics would be consistent with the RMS dynamics of all pairwise subsystems. We show that if the wave functions describing the state of the system restrict relative distances between all pairs of particles to be in an RMS relation, as in the two body problem, the potential function becomes, in any computation with these states, also restricted in this way. Such calculations would be feasible for few body or well ordered systems, such as the example of the Stueckelberg string that we give here. We develop in the second part of the book a relativistic statistical mechanics applicable for a large number of particles.

For the scattering theory (in the continuous spectrum of the reduced motion of the two body problem), the standard form of the partial wave expansion is recovered if the RMS orientation is along the direction of the incoming beam. The treatment of the resulting partial wave decomposition in terms of analytic continuation of the partial wave amplitudes in the orbital angular momentum (as done by Regge) then becomes accessible, but is not treated in this book. Incorporating the treatment of relativistic spin given here would generalize such results to resonances with both orbital and spin angular momentum in a covariant way.

In our general treatment of relativistic formal quantum scattering theory, we give proof of the Gell-Mann Low theorem. This theorem was developed to provide the basis for computation of the stationary states of a many body system, such as a crystal. The idea was to assume that the interactions are turned off in the infinite future and the infinite past with a cutoff going exponentially to zero on the many body potential, and then treating it as a scattering system for which interactions vanish asymptotically.

The theorem states that the energy shift when interactions are turned on is given by the ratio of two infinite functions in the limit where the cutoff is removed. In terms of quantum field theory, the numerator contains both disconnected and connected diagrams, and the denominator just disconnected diagrams; the disconnected contributions cancel to give a presciption for which, perturbatively, one calculates just with connected diagrams.

The theorem was originally proved in the framework of nonrelativistic quantum field theory. The proof we give here, following the method of Gell-Mann and Low, is carried out in the framework of the relativistic formal scattering theory. Using the interaction picture in the framework of the relativistic quantum field theory we develop here, one could show that the diagrammatic conclusions are true in this formalism as well, but we do not work out the details in this book. It should be mentioned in this context that the Haag theorem states that there is no unitary transformation connecting the free fields to the interacting fields in the standard theory so that, although formal perturbation calculations give good results, the interaction picture rigorously does not exist in the usual quantum field theory. *E* Seidowitz has shown, however, that the SHP quantum fields admit a rigorous interaction picture (with the same structure as originally postulated by Feynman).

Our discussion of relativistic statistical mechanics rests strongly on the formulation originally given by Horwitz, Schieve and Piron (HSP). The idea was to consider a set of events occupying some finite region of spacetime. These events move according to the equations of motion and trace out world lines which correspond to particles. It was argued that the statistical mechanics of the set of events is therefore equivalent to the statistical mechanics of particles.

The microcanonical ensemble is defined in the nonrelativistic theory in terms of the volume of phase space in a narrow energy shell, and the entropy $S(E)$ is then defined as Boltzmann's constant times the logarithm of this volume; the derivative with respect to E then defines a temperature, which turns out to be an equilibrium property of the system with its surroundings in the canonical ensemble. The analog in the SHP theory would be to compute the volume in the phase space at a given value of the dynamical generator K. However, this volume is infinite because a finite value of the generator admits an unbounded phase space along the hyperbolas associated with each value of this invariant. The density of the microcanonical ensemble was therefore restricted as well to an *energy shell*, providing a convergent but non-invariant volume. The non-invariance was understood as a reflection of the fact that a motion of the system relative to the thermometer of the laboratory would give a false reading of temperature due to the additional velocity of the particles, and one must therefore choose a frame (say, center of momentum frame).

Since the particle masses are dynamically variable occuring in the generator of motion K, there is a possibility that after a large number of interactions, the asymptotic particle mass can drift significantly. We construct a statistical mechanical model of a particle here which provides a mechanism for which a 'mass temperature' and a 'mass chemical potential' can stabilize the asymptotic mass of the 'particle' (and in the case of more than one maximum in the Gibbs integral, provide for several mass states as phases through the Maxwell construction).

We have also discussed the possibility of a high temperature phase transition, according to the method of Haber and Weldon, which would result in a definite mass of the system, determined by the chemical potential. The resulting mass value is determined in a statistical sense, and therefore, in the dispersion width, leaves sufficient freedom for the full off-shell SHP theory.

In order to deal with particles with spin, we applied the method of induced representations of Wigner for the description of a relativistic particle with spin adapted to the requirements imposed by the relativistic quantum theory. The method requires that the representation be induced on a *covariant arbitrary timelike vector*, which we take to be universal for a system of identical particles, instead of the four-momentum used by Wigner. Each point on the orbit of this timelike vector is associated with a spin 1/2 representation of the rotation group, its stability subgroup. Two particles at the same point on their respective orbits then transform, under rotations in the spacelike surface orthogonal to the timelike vector, with the same $SU(2)$, and therefore their spins can be added with the usual Clebsch-Gordan coefficients. This induced representation implies a foliation of the Hilbert space of states. The pure states are represented by wave functions which transform under Lorentz transformation, for systems with spin degrees of freedom, under the little group representing spin, by our construction. This is also true in their coordinate or momentum representations, for which the generators are the generators of the Lorentz group projected into this foliation. We discuss this construction in the appendix to chapter 6, where we constructed a Pauli-Lubanski vector providing an angular momentum Casimir operator on the orbit.

The existence of the relation between spin and statistics in nature implies that the fermionic antisymmetry between any pair of identical particles, associated with a π rotation of the two-body subsystem can be valid only for particles on the same points of their respective orbits. This result introduces a foliation of the whole Fock space constructed from the many-body tensor product, and therefore of the corresponding quantum field theory for both bosons and fermions; we discuss the correspondence of this foliation with the structure of quantum field theory defined on a sequence of spacelike surfaces (as, for example, done by Schwinger and Tomonaga).

One can, moreover, compute the total spin state of a relativistic many-body system, provided all particles are at the same point on their respective orbits labelled by the timelike inducing vector, as required for identical particle systems, e.g. nuclei with particle constituents, hadrons with quark constituents.

Furthermore, as in the proposed experiment of Palacios *et al*, the spin entanglements induced in this way would exist for particles (such as the electrons in the outer shell of helium) ejected at equal world time τ, but not restricted, as in standard nonrelativistic mechanics, to equal time t, and on the same point of their orbit. These correlations should be seen, according to this theory, for particles at non-equal times within the support of the Stueckelberg wave functions, of the order of femtoseconds, as concluded from both the analysis of the Lindner *et al* experiment and the proposed experiment of Palacios *et al*. Both of these groups assumed coherence in time in the nonrelativisic quantum theory and used time dependent solutions of the nonrelativistic Schröodinger equation. This treatment is not, however, consistent with the basic foundations of the quantum theory, but may be expected to provide, as in the Lindner *et al* experiment, a good approximation.

The correlations implied by the existence of Cooper pairs, forming the foundation of the theory of superconductivity, existing, according to the nonrelativistic quantum theory, only at equal times, are predicted by the SHP theory to be maintained at unequal times. The theory can therefore be generalized to be consistent with relativistic covariance. In a similar way, we predict that the interference phenomena associated with the Josephson effect would be maintained if the two gates are open at slightly different times, or with a single gate opened at two times, with a result similar to that of the Lindner *et al* experiment. Such a result would be a significant generalization of the Josephson effect.

We finally remark that we study here as well the Boltzmann counting leading to the relativistic Bose-Einstein and Fermi-Dirac distributions.

Our discussion of the SHP covariant relativistic theory, with emphasis on the treatment of many body systems, constitutes a basic introduction to a subject with wide potentialities for further development.

We believe that there are new theoretical developments that may flow from this theory, posing as well interesting crucial experiments to be carried out as the technology develops.

Acknowledgments

We wish to thank our many colleagues and students for discussions of the topics treated in this book. In particular, one of us (LH) is grateful to Constantin Piron for his deep and insightful collaboration in the construction of a covariant classical and quantum theory based on the original ideas of E C G Stueckelberg in Geneva in 1971 and subsequent years. He also wishes to thank W C Schieve for his stimulation and participation in constructing the relativistic statistical mechanics discussed and further developed here, and S L Adler for inviting him to the Institute for Advanced Study for several visits where they worked on many other topics as well. Only the patience, support and encouragement of his wife, Ruth, made it possible to write this book.

The second author (RA) is grateful for his doctoral years at Tel Aviv University where he worked, upon recommendation from Y Ne'eman, with L Horwitz in the wide-ranging development of the subject, achieving many of the results reported here. He is grateful to Y Aharonov and S Elitzur for several discussions, and to his sister Ilana for her support and encouragement over the years.

Author biographies

Lawrence Paul Horwitz

Lawrence Paul Horwitz was born in New York City on October 14, 1930. He lived in Westchester County until 1934, then went to London where his father founded and managed a chain of women's wear shops, called the Richard Shops, and then returned to the United States in 1936. After a few years in Brooklyn, NY, his family moved to Forest Hills in Queens, NY, where he learned tennis and attended Forest Hills High School, a school dedicated to teaching students how to think, where he came to love physics. He then went to the College of Engineering, New York University, where he studied Engineering Physics and graduated summa cum laude with a Tau Beta Pi key and the S.F.B. Morse medal for physics. He met a young lady, Ruth Abeles, who arrived from Germany in the US in 1939 and became his wife before moving on to Harvard University in 1952 with a National Science Foundation Fellowship. He received his doctorate at Harvard working under the supervision of Julian Schwinger in 1957.

He then worked at the IBM Watson Research Laboratory where he met Herman Goldstine, a former assistant to John von Neumann, and among other things, explored with him octononic and quaternionic Hilbert spaces from both physical and mathematical points of view. He then moved on to the University of Geneva in 1964, becoming involved in scattering theory as well as continuing his studies of hypercomplex systems with L C Biedenharn and becoming involved in particle physics with Yuval Ne'eman at CERN. He became full professor at the University of Denver in 1966–1972; he then accepted a full professorship at Tel Aviv University. After stopping for a year to work with C Piron at the University of Geneva on the way to Israel, he has been at Tel Aviv University since 1973, with visits at University of Texas at Austin, Ilya Prigogine Center for Statistical Mechanics and Complex Systems in Brussels, and at CERN, ETH (Honggerberg, Zurich), University of Connecticut (Storrs, CT), IHES (Bures-sur-Yvette, Paris) and Institute for Advanced Study (Princeton, NJ), where he was a Member in Natural Sciences, 1993, 1996, 1999,2003 with short visits in August 1990, January 1991, working primarily with S L Adler. He is now Professor Emeritus at Tel Aviv University, Bar Ilan University and Ariel University.

His major interests are in particle physics, statistical mechanics, mathematical physics, theory of unstable systems, classical and quantum chaos, relativistic quantum mechanics, relativistic many body theory, quantum field theory, general relativity, representations of quantum theory on hypercomplex Hilbert modules, group theory and functional analysis, theories of irreversible quantum evolution, geometrical approach to the study of the stability of classical Hamiltonian systems, and to the dark matter problem, and classical and quantum chaos.

He is a member of the American Physical Society (Particle Physics), Swiss Physical Society, European Physical Society, International Association for

Mathematical Physics, Israel Physical Society, Israel Mathematics Union, European Mathematical Society, International Quantum Structures Association, Association of Members of the Institute for Advanced Study, and the International Association for Relativistic Dynamics.

Rafael Arshansky

Rafael Arshansky was born in Leningrad (St. Petersburg), USSR, in 1949. He received his Master's Degree for his work on 'Coherent States in Relativistic Quantum Theory' from the Lomonosov Moscow State University. In 1972, he emigrated to Israel, and obtained his PhD at Tel Aviv University for his work, *Topics in Relativistic Quantum Theory: Two Body Bound States and Scattering Theory*, under the supervision of L P Horwitz. His present fields of interest are the relativistic dynamics of events with any number of degrees of freedom in classical and quantum mechanics, and general relativity. He now lives in Jerusalem and independently, and with collaborations, pursues research in his home.

Lawrence P Horwitz and Rafael I Arshansky

Chapter 1

Introduction

In 1941, E C G Stueckelberg wrote a paper [1] based on ideas of V Fock [2] that established the foundations of a theory that could covariantly describe the classical and quantum relativistic mechanics of a single particle. Horwitz and Piron [3] extended the applicability of this theory—later named the Stueckelberg, Horwitz and Piron (SHP) theory—to the many-body problem. It is the purpose of this book to explain this development and provide examples of its applications.

Stueckelberg's basic idea [1] (see [4] for an extended discussion) was that the familiar particle worldline, representing the position in space as a function of time, consists of a sequence of *events*, spacetime points along a curve generated by a single point whose motion is guided by the dynamics of the system (figure 1.1). Thus, a free particle undergoing no interactions would generate a straight worldline on a spacetime diagram, with slope equal to its velocity. Under the influence of a small force, the event generating the worldline would no longer follow a straight path, but move along a more general, slightly different curve. He then imagined that a stronger force could turn the curve generated in this way to a direction running backwards in time (see figure 1.1).

As observed in the laboratory (all points along the curve represent the outcome of measurements made in the observer's laboratory, x corresponding to measuring sticks in the laboratory, and t corresponding to the detection of the particle's presence as recorded on the laboratory clock at the moment of observation), one sees a sequence developing forward in time according to the advance of the laboratory clock, even though the underlying dynamics leads the generating event monotonically along the path. The concepts underlying this statement are indeed not at all simple to grasp, but form the basis of the SHP theory, allowing the construction of a consistent covariant classical and quantum mechanics [4].

The interpretation of the segment of the curve running backwards in time was taken to be that of an *antiparticle*, known at that time [5] to correspond to a particle under time-reversed motion. One sees this in a very simple way by taking the complex conjugate of the Schrödinger equation (for a real valued Hamiltonian); the

doi:10.1088/978-1-6817-4948-8ch1

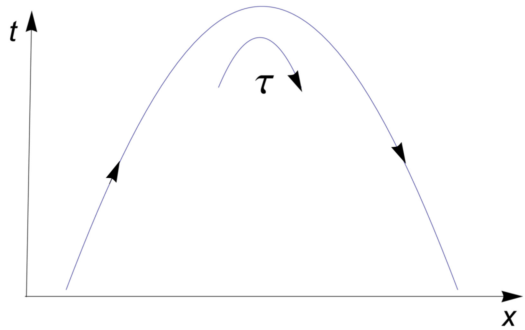

Figure 1.1. Stueckelberg's construction. Reprinted/adapted by permission from [4].

result is the identical Schrödinger equation for the complex conjugate wave function evaluated at $(-t)$. In the presence of an electromagnetic field, the gauge-invariant quantity $p - eA$, for p the particle momentum, e the electric charge and A the vector potential, is altered to the form $p + eA$ (which enters quadratically in the nonrelativistic theory) due to the fact that p is represented as $-i\frac{\partial}{\partial x}$. This is consistent with the idea that $-p$ is the time reversal of p.

However, the phenomenon that Stueckelberg described was completely *classical*; his construction, taking the interpretation from known results in quantum theory, therefore amounted to the description of *pair production in classical mechanics*.

He observed that due to the double valued property of this curve on the variable t, one must introduce a new invariant parameter along the curve to describe the motion in a single valued way. This new parameter, which he called τ, could then be used to construct a *covariant* dynamics by defining a Hamiltonian K with Hamilton equations ($\mu = 0, 1, 2, 3$; we take the signature to be $(-,+,+,+)$, in the diagonal metric $g_{\mu\nu}$, with, for some four-vector $a_\mu = g_{\mu\nu}a^\nu$)[1]

$$\begin{aligned} \dot{x}_\mu &= \frac{\partial K}{\partial p^\mu} \\ \dot{p}^\mu &= -\frac{\partial K}{\partial x_\mu} \end{aligned} \tag{1.1}$$

For example, for the free particle,

$$K = \frac{p^\mu p_\mu}{2M}, \tag{1.2}$$

where M is an intrinsic dimensional scale parameter of units of mass (not necessarily the measured mass of the particle), the Hamilton equations lead to

$$\dot{x}_\mu = \frac{p_\mu}{M}, \tag{1.3}$$

[1] Here $x^\mu = (ct, x^1, x^2, x^3)$ and $p^\mu = (E/c, p^1, p^2, p^3)$; we shall take the velocity of light $c = 1$, unless otherwise indicated.

with $\dot{p}^\mu = 0$. It follows from this result that

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{p^i}{Ec^2}, \tag{1.4}$$

the well-known result of Einstein's kinematics [6].

Stueckelberg was then in a position to postulate the quantum form of the theory by assuming commutation relations

$$[x^\mu, p^\nu] = \mathrm{i}\hbar g^{\mu,\nu} \tag{1.5}$$

and what has become known as the Stueckelberg–Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial\tau}\psi_\tau(x) = K\psi_\tau(x), \tag{1.6}$$

where $\psi_\tau(x)$ is a scalar wave function normalizable on integration over d^4x (an element of a Hilbert space over R^4), and here K is the operator form of the Hamiltonian.

As in Feynman's later discussion of covariant path integrals [7], one might imagine up to this point in our development that each particle may have its own τ. However, Horwitz and Piron [3] postulated that, in order to construct many-body theories, there is just one universal τ, in complete accordance with Newton's postulate [8] of a universal time. One can then proceed, as we shall do here, to formulate the many-body problem in this framework.

In the next chapter, we discuss the classical relativistic two-body problem and formulate the classical N-body problem. We also discuss the basis for electrodynamics in this framework. We discuss the quantum mechanical N-body problem in chapter 3, as well as further properties of the associated gauge fields and second quantization. We then discuss classical and quantum statistical mechanics, and some results and applications of the theory.

References

[1] Stueckelberg E C G 1941 *Helv. Phys. Acta* **14** 372 585
Stueckelberg E C G 1942 *Helv. Phys. Acta* **15** 23

[2] Fock F V 1937 *Phys. Zeit. Sowjetunion* **12** 404

[3] Horwitz L P and Piron C 1973 *Helv. Phys. Acta* **66** 316
Collins R E and Fanchi J R 1978 *Nuovo Cim.* **48 A** 314
Fanchi J R 1993 *Parametrized Relativistic Quantum Theory* (Dordrecht: Kluwer)

[4] Horwitz L P 2015 *Relativistic Quantum Mechanics (Fundamental Theories of Physics)* vol 180 (Dordrecht: Springer)

[5] Dirac P A M 1930 *Quantum Mechanics* 1st edn (Oxford: Oxford University Press)

[6] Einstein A 1922 *The Meaning of Relativity* (Princeton, NJ: Princeton University Press)

[7] Feynman R P 1949 *Phys. Rev.* **76** 769

[8] Newton I 1687 *Philosophia Naturalis Principia Mathematica* (London: Benjamin Motte)
Cohen I B and Whitman A 1999 *The Principia: Mathematical Principles of Natural Philosophy: A New Translation* (Berkeley, CA: University of California Press)

Full list of references

Chapter 1

[1] Stueckelberg E C G 1941 *Helv. Phys. Acta* **14** 372 585
Stueckelberg E C G 1942 *Helv. Phys. Acta* **15** 23
[2] Fock F V 1937 *Phys. Zeit. Sowjetunion* **12** 404
[3] Horwitz L P and Piron C 1973 *Helv. Phys. Acta* **66** 316
Collins R E and Fanchi J R 1978 *Nuovo Cim.* **48 A** 314
Fanchi J R 1993 *Parametrized Relativistic Quantum Theory* (Dordrecht: Kluwer)
[4] Horwitz L P 2015 *Relativistic Quantum Mechanics (Fundamental Theories of Physics)* vol 180 (Dordrecht: Springer)
[5] Dirac P A M 1930 *Quantum Mechanics* 1st edn (Oxford: Oxford University Press)
[6] Einstein A 1922 *The Meaning of Relativity* (Princeton, NJ: Princeton University Press)
[7] Feynman R P 1949 *Phys. Rev.* **76** 769
[8] Newton I 1687 *Philosophia Naturalis Principia Mathematica* (London: Benjamin Motte)
Cohen I B and Whitman A 1999 *The Principia: Mathematical Principles of Natural Philosophy: A New Translation* (Berkeley, CA: University of California Press)

Chapter 2

[1] Horwitz L P and Piron C 1973 *Helv. Phys. Acta* **46** 316
[2] Sommerfeld A 1921 *Atombau und Spektralinien* (Braunschweig: Vieweig and Sohn)
Ghoshal S K, Nandi K K and Ghoshal T K 1987 *Indian J. Pure Appl. Math.* **18** 194
Bergmann P D 1948 *Introduction to the Theory of Relativity* (New York: Prentice-Hall)
[3] Einstein A 1922 *The Meaning of Relativity* (Princeton, NJ: Princeton University Press)
[4] Yang C N 1974 *Phys. Rev. Lett.* **33** 445
Wu T T and Yang C N 1975 *Phys. Rev.* D **12** 3845
[5] Yang C N and Mills R 1954 *Phys. Rev.* **96** 191
[6] Saad D, Horwitz L P and Arshansky R I 1989 *Found. Phys.* **19** 125
[7] Horwitz L P 2015 *Relativisic Quantum Mechanics (Fundamental Theories of Physics)* vol 180 (Dordrecht: Springer)
[8] Jackson J D 1974 *Classical Electrodynamics* 2nd edn (New York: Wiley)
[9] Landau L and Lifshitz E 1951 *Classical Theory of Fields* (Boston, MA: Addison Wesley)
[10] Einstein A 1905 *Ann. Phys.* **17** 891
[11] Horwitz L P, Piron C and Reuse F 1975 *Helv. Phys. Acta* **48** 546
[12] Wigner E 1939 *Ann. Math.* **40** 149
Mackey G W 1968 *Induced Representations of Groups and Quantum Mechanics* (New York: Benjamin)
Bargmann V 1947 *Ann. Math.* **48** 568
[13] Horwitz L P and Lavie Y 1982 *Phys. Rev.* D **26** 819
[14] Land M 1998 *Found. Phys.* **28** 1479
[15] Lehmann H 1954 *Nuovo Cimento* **11** 342
Kallén G 1954 *Nuovo Cimento* **11** 342
[16] Schwinger J 1960 *Ann. Phys. NY* **9** 169
[17] Aharonovich I and Horwitz L P 2010 *J. Math. Phys.* **51** 052903
Aharonovich I and Horwitz L P 2011 *J. Math. Phys.* **52** 082901

Aharonovich I and Horwitz L P 2012 *J. Math. Phys.* **53** 032902
Aharonovich I and Horwitz L P 2006 *J. Math. Phys.* **47** 122902
[18] Newton T D and Wigner E 1949 *Rev. Mod. Phys.* **21** 400
[19] Hughston L P 1996 *Proc. R. Soc. Lon.* A **452** 953
Brody D C and Hughston L P 1998 *Proc. R. Soc. Lon.* A **454** 2445
Brody D C and Hughston L P 2001 *J. Geom. Phys.* **38** 19
[20] Strauss Y, Silman J, Machnes S and Horwitz L P 2011 *Int. J. Theor. Phys.* **50** 2179

Chapter 3

[1] Horwitz L P and Piron C 1973 *Helv. Phys. Acta* **66** 316
Collins R E and Fanchi J R 1978 *Nuovo Cimento* A **48** 314
Fanchi J R 1993 *Parametrized Relativistic Quantum Theory* (Dordrecht: Kluwer)
[2] Sommerfeld A 1921 *Atombau und Spektralinien* (Braunschweig: Vieweig and Sohn)
Ghoshal S K, Nandi K K and Ghoshal T K 1987 *Indian J. Pure Appl. Math.* **18** 194
Bergmann P D 1948 *Introduction to the Theory of Relativity* (New York: Prentice-Hall)
[3] Cook J L 1972 *Aust. Jour. Phys.* **25** 141
[4] Zmuidzinas J S 1966 *J. Math. Phys.* **7** 764
[5] Arshansky R I and Horwitz L P 1989 *J. Math. Phys.* **30** 60
[6] Horwitz L P 2015 *Relativisic Quantum Mechanics (Fundamental Theories of Physics)* vol 180 (Dordrecht: Springer)
[7] Naimark M A 1964 *Linear Representations of the Lorentz Group* (New York: Pergamon)
[8] Shapere A and Wilczek F 1980 *Geometric Phases in Physics* (Advanced Series in Mathematical Physics) vol 5 (Singapore: World Scientific)
[9] Gel'fand I M, Minlos R A and Shapiro Z Y 1963 *Representations of the Rotation and Lorentz Groups and Their Applications* (New York: Pergamon)
[10] Merzbacher E 1970 *Quantum Mechanics* 2nd edn (New York: Wiley)
[11] Gradshtyn I S and Rhyzik I M 2007 *Table of Integrals, Series and Products* 7th edn ed D Alan Jeffrey (London: Academic)
[12] Wigner E P 1931 *Gruppentheorie und ihre Anwendungen auf die Quantunmechanik der Atomspektren* (Braunschweig: Viewag Verlag)
Wigner E P 1959 *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (New York: Academic) (translation)
[13] Gottfried K 1966 *Quantum Mechanics* (New York: Benjamin)
[14] Chew G 1966 *The Analytic S-Matrix* (New York: Benjamin)
[15] Landau L D and Lifshitz E 1965 *Quantum Mechanics* (Oxford: Pergamon)
[16] Hofstadter R, Bumiller R and Yearian M R 1958 *Rev. Mod. Phys.* **30** 482
[17] Feynman R P, Kislinger M and Ravndal F 1971 *Phys. Rev.* D **3** 2706
[18] Kim Y S and Noz M E 1977 *Prog. Theor. Phys.* **57** 1373
[19] Leutwyler H and Stern J 1977 *Phys. Lett.* B **69** 207
[20] Itzykson C and Zuber J-B 1980 *Quantum Field Theory* (New York: McGraw-Hill)
[21] Dothan Y, Gell-Mann M and Ne'eman Y 1965 *Phys. Lett.* **17** 148
Ne'eman Y 1967 *Algebraic Theory of Particle Physics* (New York: Benjamin)
[22] Land M 2011 *J. Phys. Conf. Ser.* **330** 012014
[23] Arshansky R and Horwitz L P 1984 *Phys. Rev.* D **29** 2860
[24] Mahan G D 1990 *Many Particle Physics* (New York: Plenum)
[25] Suleymanov M, Horwitz L and Yahalom A 2017 *Front. Phys.* **12** 121103

[26] Huang K 1967 *Statistical Mechanics* (New York: Wiley)
[27] Nambu Y 1950 *Prog. Theor. Phys.* **5** 82
[28] Polchinsku J 1998 *String Theory I,II* (Cambridge: Cambridge University Press)
[29] Green M B, Schwartz J H and Witten E 1986 *Superstring Theory* (Cambridge: Cambridge University Press)
[30] Di Francesco P, Mathieu P and Sénéchal D 1997 *Conformal Field Theory* (Berlin: Springer)
[31] Wigner E 1939 *Ann. Math.* **40** 149
Mackey G W 1968 *Induced Representations of Groups and Quantum Mechanics* (New York: Benjamin)
Bargmann V 1947 *Ann. Math.* **48** 568
[32] Feynman R P 1950 *Phys. Rev.* **80** 440
[33] Arshansky R I and Horwitz L P 1989 *J. Math. Phys.* **30** 213
[34] Levinson N 1949 *Kgl. Danske Videnskab Mat.-fys. Medd.* **25** 9
[35] Horwitz L P and Lavie Y 1982 *Phys. Rev.* D **26** 819

Chapter 4

[1] Lehmann H, Symanzik K and Zimmermann W 1955 *Nuovo Cimento* **1** 1425
[2] Schweber S S 1964 *An Introduction to Relativistic Quantum Field Theory* (New York: Harper and Row)
[3] Jauch J and Rohrlich F 1955 *The Theory of Photons and Electrons* (Cambridge, MA: Addison-Wesley)
[4] Tomonaga S 1948 *Phys. Rev.* **74** 224
Schwinger J 1948 *Phys. Rev.* **73** 416
Schwinger J 1948 *Phys. Rev.* **74** 1439
[5] Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw-Hill)
[6] Feynman R P 1949 *Phys. Rev.* **76** 769
[7] Nambu Y 1950 *Prog. Theor. Phys.* **5** 82
[8] Feynman R P 1950 *Phys. Rev.* **80** 440
[9] Stueckelberg E C G 1941 *Helv. Phys. Acta* **14** 372, 585
Stueckelberg E C G 1942 *Helv. Phys. Acta* **15** 23
Horwitz L P and Piron C 1973 *Helv. Phys. Acta* **66** 316
Collins R E and Fanchi J R 1978 *Nuovo Cimento* A **48** 314
Fanchi J R 1993 *Parametrized Relativistic Quantum Theory* (Dordrecht: Kluwer)
[10] Schwinger J 1951 *Phys. Rev.* **82** 664
[11] Reed M and Simon B 1979 *Methods of Modern Mathematical Physics: III. Scattering Theory* (New York: Academic)
[12] Amrein W O, Jauch J M and Sinha K B 1977 *Scattering Theory in Quantum Mechanics* (New York: Benjamin)
[13] Taylor J R 1972 *Scattering Theory* (New York: Wiley)
[14] Newton R G 1967 *Scattering Theory of Waves and Particles* (New York: McGraw Hill)
[15] Sigal I M and Soffer A 1987 *Ann. Math.* **126** 35
[16] Horwitz L P and Soffer A 1980 *Helv. Phys. Acta* **53** 112
Soffer A 1984 *Lett. Math. Phys.* **8** 517
[17] Gell-Mann M and Low F 1951 *Phys. Rev.* **84** 350
[18] Fetter A L and Walecka J D 1971 *Quantum Theory of Many Particle Systems* (New York: McGraw Hill)

[19] Schwinger J 1961 *J. Math. Phys.* **2** 407
[20] Keldysh L 1965 *Sov. Phys. JETP* **20** 1018
[21] Horwitz L P and Engelberg E Z 2009 *Phys. Lett.* A **374** 40
[22] Borzdov G N 2017 *Phys. Rev.* A **96** 042117
[23] Lippmann B A and Schwinger J 1950 *Phys. Rev. Lett.* **79** 469
[24] Adler S L 1965 *Phys. Rev.* B **140** 376
Weissberger W I 1965 *Phys. Rev.* B **143** 1302

Chapter 5

[1] Horwitz L P 2015 *Relativistic Quantum Mechanics (Fundamental Theories in Physics)* vol 180 (Dordrecht: Springer)
[2] Horwitz L P 2017 *J. Phys. Conf. Ser.* **845** 012026
[3] Horwitz L P, Schieve W C and Piron C 1981 *Ann. Phys.* **137** 306
[4] Schweber S S 1964 *An Introduction to Relativistic Quantum Field Theory* (New York: Harper and Row)
[5] Arshansky R I and Horwitz L P 1989 *J. Math. Phys.* **30** 213 380
[6] Horwitz L P and Piron C 1973 *Helv. Phys. Acta* **46** 316
[7] Bjorken J D 1966 *Phys. Rev. Lett.* **16** 408
Bjorken J D 1969 *Phys. Rev.* **179** 1547
Feynman R P 1969 *Phys. Rev. Lett.* **23** 1415
[8] Horwitz L P, Shashoua S and Schieve W C 1989 *Physica* A **161** 300
[9] Huang K 1967 *Statistical Mechanics* (New York: Wiley)
[10] Land M C 2017 *Entropy* **19** 234

Chapter 6

[1] Horwitz L P, Schieve W C and Piron C 1981 *Ann. Phys.* **137** 306–40
[2] Baym G 1969 *Lectures on Quantum Mechanics* (New York: Benjamin)
[3] Wigner E 1939 *Ann. Math.* **40** 149
[4] Horwitz L P 2015 *Relativistic Quantum Mechanics (Fundamental Theories of Physics)* vol 180 (Dordrecht: Springer)
[5] Horwitz L P, Piron C and Reuse F 1975 *Helv. Phys. Acta* **48** 546
Arshansky R and Horwitz L P 1982 *J. Phys. A: Math. Gen.* **15** L659–62
[6] Horwitz L P and Zeilig-Hess M 2015 *J. Math. Phys.* **56** 092301
Lubanski J K 1942 *Physica* **9** 310 325 1942 see also [29]
[7] Huang K 1967 *Statistical Mechanics* (New York: Wiley)
[8] Haber H E and Weldon H A 1981 *Phys. Rev. Lett.* **46** 1497
[9] Burokovsky L, Horwitz L P and Schieve W C 1996 *Phys. Rev.* D **54** 4029
[10] Kirsten T 1994 *Proc. Int. Conf. on Non-Accelerator Particle Physics* ed R Cowsik (Singapore: World Scientific)
[11] Planck M 1914 *The Theory of Heat Radiation* (Philadelphia, PA: Blakiston)
[12] Horwitz L P and Shnerb N 1993 *Symmetries in Science VI: From the Rotation Group to Quantum Algebras* (Berlin: Springer), p 335 see also [4]
[13] Horwitz L P and Shnerb N 1998 *Found. Phys.* **28** 1509
[14] Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton, NJ: Princeton University Press)

[15] Tomczak S P and Haller K 1972 *Nuovo Cimento* B **8** 1
Haller K and Sohn R B 1979 *Phys. Rev.* A **20** 1541
Haller K 1975 *Acta Phys. Aust.* **42** 163
Haller K 1987 *Phys. Rev.* D **36** 1830
[16] Shnerb N and Horwitz L P 1993 *Phys. Rev.* A **48** 4068
[17] Fadeev L D and Popov V N 1967 *Phys. Lett.* B **25** 29
[18] Jauch J and Rohrlich F 1955 *The Theory of Photons and Electrons* (Cambridge, MA: Addison-Wesley)
[19] Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw Hill)
[20] Schwinger J S 1951 *Phys. Rev.* **82** 664
[21] Horwitz L P, Shashoua S and Schieve W C 1989 *Physica* A **161** 300
[22] Balescu R 1975 *Equilibrium and Non-Equilibrium Statistical Mechanics* (New York: Wiley)
[23] Horwitz L P ans Lavie Y 1982 *Phys. Rev.* D **26** 819
[24] Rarita W and Schwinger J 1941 *Phys. Rev.* **60** 61
[25] Biedenharn L C and Louck J D 1981 *The Racah–Wigner Algebra in Quantum Theory* (Reading, MA: Addison-Wesley)
[26] Einstein A, Podolsky B and Rosen N 1978 *Phys. Rev.* **41** 1881
[27] Mackey G W 1968 *Induced Representations of Groups and Quantum Mechanics* (New York: Benjamin)
[28] Stueckelberg E C G 1941 *Helv. Phys. Acta* **14** 322–3 588–94
Stueckelberg E C G 1942 *Helv. Phys. Acta* **15** 23–7
Horwitz L P and Piron C 1973 *Helv. Phys. Acta* **46** 316–26 see also [4]
[29] Weinberg S 1995 *The Quantum Theory of Fields* vol 1 (Cambridge: Cambridge University Press)
[30] Horwitz L P, Piron C and Reuse F 1975 *Helv. Phys. Acta* **48** 546
Arshansky R and Horwitz L P 1982 *J. Phys. A: Math. Gen.* **15** L659
[31] Newton T D and Wigner E 1949 *Rev. Mod. Phys.* **21** 400
[32] Palacios A, Rescigno T N and McCurdy C W 2009 *Phys. Rev. Lett.* **103** 253001
[33] Horwitz L P and Arshansky R 2017 arXiv: 1707.03294
[34] Land M C and Horwitz L P 1995 *J. Phys. A: Math. Gen.* **28** 3289
[35] Dirac P A M 1930 *Quantum Mechanics* 1st edn (Oxford: Oxford University Press)
[36] Saad D, Horwitz L P and Arshansky R I 1989 *Found. Phys.* **19** 1125–49
Aharonovich I and Horwitz L P 2006 *J. Math. Phys.* **47** 122902
[37] Bethe H 1931 *Z. Phys.* **71** 205
[38] Fetter A L and Walecka J D 1971 *Quantum Theory of Many Particle Systems* (New York: McGraw Hill)
[39] Cooper L N 1956 *Phys. Rev.* **104** 1189
[40] Josephson B D 1974 *Phys. Lett.* **1** 251
[41] Lindner F, Schätzel M G, Walther H, Baltuska A, Goulielmakis E, Krausz F, Miloševi D B, Bauer D, Becker W and Paulus G G 2005 *Phys. Rev. Lett.* **95** 040401
Horwitz L P 2006 *Phys. Lett.* A **355** 1
[42] Schwinger J 1951 *Proc. Natl Acad. Sci.* **37** 452
[43] Tomonaga S 1946 *Prog. Theor. Phys.* **1** 27
[44] Jauch J M and Rohrlich F 1976 *The Theory of Photons and Electrons* 2nd edn (New York: Springer)
[45] Horwitz L P and Zeilig-Hess M 2015 *J. Math. Phys.* **56** 092301

Chapter 7

[1] Stueckelberg E C G 1941 *Helv. Phys. Acta* **14** 322–3 588–94
Stueckelberg E C G 1942 *Helv. Phys. Acta* **15** 23–7
Horwitz L P and Piron C 1973 *Helv. Phys. Acta* **46** 316–26; see also [11]
[2] Arshansky R I and Horwitz L P 1989 *J. Math. Phys.* **30** 380
Arshansky R I and Horwitz L P 1989 *J. Math. Phys.* **30** 213
[3] Regge T 1959 *Nuovo Cimento* **24** 951
[4] Eden R J 1967 *High Energy Collisions of Elementary Particles* (Cambridge: Cambridge University Press)
Eden R J, Landshoff P V, Olive D I and Polkinghorne J C 1966 *The Analytic S-Matrix* (Cambridge: Cambridge University Press)
[5] Gell-Mann M and Low F 1951 *Phys. Rev.* **84** 350
Fetter A L and Walecka J D 1971 *Quantum Theory of Many Particle Systems* (New York: McGraw Hill)
[6] Haag R 1955 *Dan. Mat. Fys. Medd.* **29** 1 (1995)
Seidowitz E 2017 unpublished
[7] Horwitz L P, Schieve W C and Piron C 1981 *Ann. Phys.* **137** 306–40
[8] Synge J L 1957 *The Relativistic Gas* (Amsterdam: North Holland)
[9] Burakovsky L, Horwitz L P and Schieve W C 1996 *Phys. Rev.* D **54** 4029
Haber H E and Weldon H A 1981 *Phys. Rev. Lett.* **46** 1497
Haber H E and Weldon H A 1982 *Phys. Rev.* D **25** 502
[10] Wigner E 1939 *Ann. Math.* **40** 149
[11] Horwitz L P 2015 *Relativistic Quantum Mechanics (Fundamental Theories of Physics)* vol 180 (Dordrecht: Springer)
[12] Schwinger J 1948 *Phys. Rev.* **74** 1439
[13] Tomanaga S 1946 *Prog. Theor. Phys.* **1** 27
[14] Jauch J M and Rohrlich F M 1976 *The Theory of Photons and Electrons* 2nd edn (New York: Springer)
[15] Palacios A, Rescigno T N and McCurdy C W 2009 *Phys. Rev. Lett.* **103** 253001
[16] Horwitz L P 2017 arXiv:1707.03294
[17] Lindner F, Schätzel M G, Walther H, Baltuska A, Goulielmakis E, Krausz F, Milošević D B, Bauer D, Becker W and Paulus G G 2005 *Phys. Rev. Lett.* **95** 040401
[18] Horwitz L P 2006 *Phys. Lett.* A **355** 1
[19] Horwitz L P, Shashoua S and Schieve W C 1989 *Physica* A **161** 300
[20] Horwtiz L P and Rabin Y 1976 *Lett. Nuovo Cimento* **17** 501
[21] Davidson M 2015 *J. Phys. Conf. Ser.* **615** 012016
[22] Horwitz L P and Engelberg E Z 2009 *Phys. Lett.* A **374** 40 (2009)
Borzdov G N 2017 *Phys. Rev.* A **96** 042117
[23] Reed D 2018 Unified field mechanics II: formulations and empirical tests *Proc. 10th Int. Symp. Honoring Jean-Pierre Vigier (Porto Novo, Italy, 25–8 July 2016)* ed R L Amoroso, L H Kauffman, P Rowlands and G Albertini (Singapore: World Scientific)
[24] Ludwin D and Horwitz L P 2011 *J. Math. Phys.* **52** 012303