A Brief Introduction to Topology and Differential Geometry in Condensed Matter Physics

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To Rosangela, Henrique and Guilherme

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Preface

In recent years there have been great advances in the applications of topology and differential geometry to problems in condensed matter physics. Concepts drawn from topology and geometry have become essential to the understanding of several phenomena in the area. Physicists have been creative in producing models for actual physical phenomena which realize mathematically exotic concepts, and new phases have been discovered in condensed matter in which topology plays a leading role. An important classification paradigm is the concept of topological order, where the state characterizing a system does not break any symmetry, but it defines a topological phase in the sense that certain fundamental properties change only when the system passes through a quantum phase transition.

The main purpose of this book is to provide a brief, self-contained introduction to some mathematical ideas and methods from differential geometry and topology, and to show a few applications in condensed matter. It conveys to physicists the bases for many mathematical concepts, avoiding the detailed formality of most textbooks. The reader can supplement the description given here by consulting standard mathematical references such as those listed in the references.

There are many good books written about the subject, but they present a lot of material and demand time to gain a full understanding of the text. Here, I present a summary of the main topics, which will provide readers with an introduction to the subject and will allow them to read the specialized literature.

Very little in this text is my original contribution since the goal of the book is pedagogy rather than originality. It was mainly collected from the literature. Some time ago, I used to teach differential geometry in a graduate course about classical mechanics and wrote a book (in Portuguese) on the topic. Now, I have adapted that material and included ideas that appeared in the last years, to write the present book.

Chapter 1 is an introduction to path integrals and it can be skipped if the reader is familiar with the subject. Chapters 2–4 are the core of the book, where the main ideas of topology and differential geometry are presented. In chapter 5, I discuss the Dirac equation and gauge theory, mainly applied to electrodynamics. In chapters 6–8, I show how the topics presented earlier can be applied to the quantum Hall effect and topological insulators. I will be mainly interested in the technical details because there are already excellent books and review articles dealing with the physical aspects. In chapter 9, I treat the application of topology to one- and two-dimensional antiferromagnets and the XY model. The framework presented here can also be used to study other systems, such as topological superconductors and quasi-metals. The appendices, although important for the application of differential geometry to some problems in condensed matter, are more specific.

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Antonio Sergio Teixeira Pires (born 18 November 1948) is a Professor of Physics in the Physics Department at the Universidade Federal de Minas Gerais, Belo Horizonte, Brazil. He received his PhD in Physics from University of California in Santa Barbara in 1976. He works in quantum field theory applied to condensed matter. He is a member of the Brazilian Academy of Science, was an Editor of the *Brazilian Journal of Physics* and a member of the Advisory Board of the *Journal of Physics*: *Condensed Matter*.

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Chapter 1

Path integral approach

1.1 Path integral

A convenient tool to treat topological quantum effects in quantum field theory is the path integral technique, and in this chapter, I am going to present the basic ideas (following mainly Ashok 1993). For more details I refer the reader to the references (Altland and Simons 2010, Fradkin 2013, Kogut 1979, Schwartz 2014, Tsvelik 1996, Wen 2004). Readers familiar with the subject can skip this chapter. I will start by establishing the path integral approach for the single particle in quantum mechanics in one dimension. The formalism can then be easily generalized to arbitrary spatial dimensions.

In path integral formalism the aim is to calculate the probability amplitude that a particle that starts at the position x_i at a time t_i ends up at a position x_f at a time t_f, with $t_f > t_i$. From quantum mechanics we know that this is given by the time-evolution operator $U(t_f, x_f; t_i, x_i)$ which in the Heisenberg picture is written as

$$U(t_f, x_f; t_i, x_i) = \left\langle x_f, t_f \,\middle|\, x_i, t_i \right\rangle, \tag{1.1}$$

where $|x, t\rangle$ is a coordinate basis for every time t. We divide the time interval between the initial and final time into N infinitesimals steps of length

$$\Delta t = \frac{t_f - t_i}{N}. \tag{1.2}$$

Any intermediate time can be written as $t_n = t_i + n\Delta t$, with $n = 1, 2, \ldots, (N - 1)$. Considering time ordering from left to right, we can write equation (1.1) as (see figure 1.1)

$$\begin{aligned} U(t_f, x_f; t_i, x_i) = \lim_{\Delta t \to 0, N \to \infty} \int dx_1 \ldots dx_{N-1} \langle x_f, t_f \mid x_{N-1}, t_{N-1} \rangle \\ \langle x_{N-1}, t_{N-1} \mid x_{N-2}, t_{N-2} \rangle \ldots \langle x_1, t_1 \mid x_i, t_i \rangle. \end{aligned} \tag{1.3}$$

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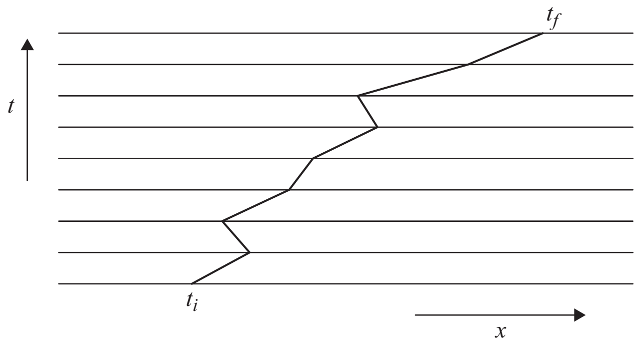

Figure 1.1. A discrete time axis and a path in quantum mechanics.

We know that

$$| x, t\rangle = e^{iHt} | x\rangle, \tag{1.4}$$

where I have set $\hbar = 1$, and we should remember to put it back if we are going to perform calculations. Therefore, we can write

$$\begin{aligned} \langle x_n, t_n | x_{n-1}, t_{n-1}\rangle &= \langle x_n | e^{-it_nH} e^{it_{n-1}H} | x_{n-1}\rangle = \langle x_n | e^{-i(t_n - t_{n-1})H} | x_{n-1}\rangle \\ &= \langle x_n | e^{-i\Delta t H} | x_{n-1}\rangle. \end{aligned} \tag{1.5}$$

Using the result

$$\langle x_2 | H | x_1\rangle = \int \frac{dp}{2\pi} e^{-ip(x_1 - x_2)} H(x, p), \tag{1.6}$$

we find

$$\langle x_n, t_n | x_{n-1}, t_{n-1}\rangle = \int \frac{dp_n}{2\pi} e^{ip_n(x_n - x_{n-1}) - i\Delta t H\left(\frac{x_n + x_{n-1}}{2}, p_n\right)}, \tag{1.7}$$

where to get a Weyl ordered Hamiltonian I wrote H using the mid-point prescription. Taking equation (1.7) into (1.3), and identifying $x_0 = x_i$, $x_n = x_f$ we can write

$$\begin{aligned} U(t_f, x_f; t_i, x_i) &= \lim_{\Delta t \to 0, N \to \infty} \int dx_1 \ldots dx_{N-1} \frac{dp_1}{2\pi} \ldots \frac{dp_N}{2\pi} \\ &\exp\left\{ i \sum_{n=1}^{N} \left[p_n(x_n - x_{n-1}) - \Delta t H\left(\frac{x_n + x_{n-1}}{2}, p_n\right)\right]\right\}. \end{aligned} \tag{1.8}$$

Let us now consider a Hamiltonian of the type

$$H(x, p) = \frac{p^2}{2m} + V(x). \tag{1.9}$$

This Hamiltonian covers a wide class of problems; however, some important applications, as will be shown in the next section, do not fit into this framework. Using equation (1.9) in (1.8) leads to

$$\begin{aligned} U(t_f, x_f; t_i, x_i) &= \lim_{\Delta t \to 0, N \to \infty} \int dx_1 \dots dx_{N-1} \frac{dp_1}{2\pi} \dots \frac{dp_N}{2\pi} \\ &\exp\left\{ i\Delta t \sum_{n=1}^{N} \left[p_n \left(\frac{x_n - x_{n-1}}{\Delta t} \right) - \frac{p_n^2}{2m} - V\left(\frac{x_n + x_{n-1}}{2} \right) \right] \right\}. \end{aligned} \tag{1.10}$$

Performing the momentum integrals using the result for Gaussian integration

$$\int_{-\infty}^{\infty} dp e^{-\frac{ap^2}{2} + bp} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}, \tag{1.11}$$

we obtain

$$\begin{aligned} U(t_f, x_f; t_i, x_i) &= \lim_{\Delta t \to 0, N \to \infty} \left(\frac{m}{2\pi i \Delta t} \right)^{N/2} \int dx_1 \dots dx_{N-1} \\ &\exp\left\{ i\Delta t \sum_{n=1}^{N} \left[\frac{m}{2} \left(\frac{x_n - x_{n-1}}{\Delta t} \right)^2 - V\left(\frac{x_n + x_{n-1}}{2} \right) \right] \right\}. \end{aligned} \tag{1.12}$$

Taking $N \to \infty$, while keeping $(t_f - t_i) = N\Delta t$ fixed, we can substitute the sum by an integral

$$\Delta t \sum_{n=1}^{N} \to \int_{t_i}^{t_f} dt, \tag{1.13}$$

and write equation (1.12) as

$$U(t_f, x_f; t_i, x_i) = \int Dx \exp\left\{ i \int_{t_i}^{t_f} dt \left[\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - V(x) \right] \right\} = \int Dx e^{iS[x]}, \tag{1.14}$$

where

$$S[x] = \int_{t_i}^{t_f} dt L\left(x, \frac{dx}{dt} \right), \tag{1.15}$$

L is the classical Lagrangian, $S[x]$ is the action, and we have introduced the integration measure

$$\int D[x(t)] = \lim_{N \to \infty} \prod_{n=1}^{N-1} \int \frac{dx_n}{\xi}, \tag{1.16}$$

with $\xi = (i2\pi\Delta t/m)^{1/2}$. In some cases, more care must be applied in taking the continuum limit, but here I am considering only the essential details. Equation (1.14) is the path integral for the probability amplitude of a particle in quantum mechanics. Feynman's idea of introducing the technique was that a particle going from A to B takes every possible trajectory, with each trajectory contributing with a complex factor e^{iS}.

Each path is weighted by its classical action, there are no quantum mechanical operators in the path integral. The quantum effects are present by the fact that the integration extends over all paths and is not just the subset of solutions of the classical equations of motion.

Following the same procedure, we can show that in quantum field theory with a Lagrangian density $L\ (\phi,\ \partial_\mu\phi)$ (where $\mu = t,\ x,\ y,\ z$) the amplitude transition from the state $\phi_i(r)$ to $\phi_f(r)$ is given by

$$\int D\phi(r,\ t)e^{iS[\phi(t,r)]}, \tag{1.17}$$

where the action is now given by

$$S[\phi] = \int d^4x\ L\ (\phi,\ \partial_\mu\phi) \tag{1.18}$$

In the path integral expression, the integration is performed over all possible paths in which ϕ, which at an initial time took the configuration $\phi_i(r)$, evolves at the final time t_f into the configuration $\phi_f(r)$. The field ϕ in condensed matter is in general an order parameter for a system, such as a superconductor or a ferromagnet.

1.2 Spin

One important application of the path integral approach in condensed matter is in magnetic systems. However, in the integrand of the path integral formalism one has an exponential of the classical action. But the spin is a fundamentally quantum object and the mechanics of a classical spin cannot be expressed within the standard formulation of Hamiltonian mechanics. We must resort to the coherent state formalism. I will illustrate this for the spin 1/2 case. For a spin 1/2 particle, we have only two states $|s_z\rangle$, $s_z = \pm 1$, with zero energy, and $s_z(t)$ is not a continuous function. To use the path integral approach, we use the coherent states $|\vec{n}\rangle$ where $\vec{n}$ is a unit vector and $|\vec{n}\rangle$ describes different states. $|\vec{n}\rangle$ is an eigenstate of the spin operator in the $\vec{n}$ direction: $\vec{n}.\ \vec{S}|\vec{n}\rangle = S|\vec{n}\rangle$.

We write

$$|\ \vec{n}\rangle = |\ z\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{1.19}$$

with $|z_1|^2 + |z_2|^2 = 1$. The total phase of z is not determined, so that we can write

$$z = \begin{pmatrix} e^{-i\phi}\cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}, \tag{1.20}$$

where (θ, ϕ) are the polar coordinates of $\vec{n}$. The coherent states $|\vec{n}\rangle$ are complete, so that we can write

$$\int \frac{d^2\vec{n}}{2\pi} \mid \vec{n}\rangle\langle\vec{n} \mid = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.21}$$

Now we can calculate the amplitude $\langle\vec{n}_2|U(t, 0)|\vec{n}_1\rangle$ that a state $|\vec{n}_1\rangle$ at a time $t = 0$ evolves to the state $|\vec{n}_2\rangle$ at time t. Since $H = 0$, we have $U(t, 0) = 1$. Inserting

$$\int \frac{d^2\vec{n}}{2\pi} \mid \vec{n}\rangle\langle\vec{n} \mid, \tag{1.22}$$

into $\langle\vec{n}_2|\vec{n}_1\rangle$ we obtain the path integral

$$\langle\vec{n}_2|\vec{n}_1\rangle = \lim_{N\to\infty} \int \prod_{i=1}^{N} \frac{d^2\vec{n}(t_i)}{2\pi} \langle\vec{n}(t)|\vec{n}(t_N)\rangle \ldots \langle\vec{n}(t_2)|\vec{n}(t_1)\rangle\langle\vec{n}(t_1)|\vec{n}(0)\rangle. \tag{1.23}$$

Now

$$\langle\vec{n}(\delta t)|\vec{n}(0)\rangle = z^+(\delta t)z(0), \tag{1.24}$$

but, $z^+(\delta t)z(\delta t) = 1$, so we can write

$$\begin{aligned} \langle\vec{n}(\delta t)|\vec{n}(0)\rangle &= 1 - z^+(\delta t)[z(\delta t) - z(0)] = 1 - z^+(\delta t)\left[\frac{z(\delta t) - z(0)}{\delta t}\right]\delta t \\ &= 1 - z^+(\delta t)\frac{\partial z(\delta t)}{\partial t}\delta t \approx \exp\left(-z^+\frac{\partial z}{\partial t}\delta t\right), \end{aligned} \tag{1.25}$$

which leads to

$$\langle\vec{n}_2(t)|\vec{n}_1(t)\rangle = \int D^2\left(\frac{\vec{n}(t)}{2\pi}\right)e^{iS[\vec{n}(t)]}, \tag{1.26}$$

(where D is the measure) with the action

$$S[\vec{n}(t)] = i\int_0^t dt z^+\frac{\partial z}{\partial t}. \tag{1.27}$$

This is an interesting result, despite $H = 0$, we have obtained a non-zero action. The term e^{iS} is here purely a quantum effect and is called the Berry phase. Berry phases will be treated in more detail in chapter 6. We can also write equation (1.27) as

$$S(\theta, \phi) = \frac{1}{2}\int dt(1 - \cos\theta)\frac{\partial\phi}{\partial t}. \tag{1.28}$$

If we have a spin $\vec{S}$ in a constant magnetic field $\vec{B} = -B\vec{n}$, and the ground state energy is denoted by E_0, the action in a time interval T is given by $-E_0T$. Let us consider what happens when the orientation of $\vec{B}$ changes slowly in time, writing

$\vec{B} = -B\vec{n}(t)$. The ground state now evolves as $|\vec{n}(t)\rangle$, and the amplitude probability is given by

$$\langle \vec{n} \mid \exp\left[-i\int_0^T dt\vec{B}(t).\ \vec{S}\right] \mid \vec{n}\rangle = e^{iS}. \tag{1.29}$$

Inserting many equation (1.22) terms into the time interval $[0,\ T]$ we find

$$\langle \vec{n} \mid \exp\left[-i\int_0^T dt\vec{B}(t).\ \vec{S}\right] |\vec{n}\rangle = e^{-iE_0T} \exp\left[i\int_0^T dti\langle\vec{n}(t)| \frac{d}{dt} |\vec{n}(t)\rangle\right], \tag{1.30}$$

and the action can be written as

$$S = -E_0T + i\int_0^T dtz^+\frac{dz}{dt}. \tag{1.31}$$

We can see there is an extra term given by the Berry phase. As we will see later, this is a topological term, and I will denote it by S_{top} to distinguish it from the spin S.

For general spin S, equation (1.28) can be written as

$$S_{\text{top}}[\theta,\ \phi] = iS\int d\text{t}(1 - \cos\theta)\frac{\partial\phi}{\partial\text{t}}. \tag{1.32}$$

If the motion of $\vec{n}(t)$ is such that its orientation coincides at the beginning and the end of the time interval, and considering that in the spherical coordinate system $(\hat{e}_r,\ \hat{e}_\theta,\ \hat{e}_\phi)$ we have

$$\frac{d\vec{n}}{dt} = \frac{d\theta}{dt}\hat{e}_\theta + \frac{d\phi}{dt}\sin\theta\hat{e}_\phi, \tag{1.33}$$

we can write equation (1.32) as

$$S_{\text{top}}[\theta,\ \phi] = iS\oint_\gamma d\text{t}\frac{d\vec{n}}{d\text{t}}.\ \vec{A} = iS\oint_\gamma d\vec{n}.\ \vec{A}, \tag{1.34}$$

where we have defined

$$\vec{A} = \frac{1 - \cos\theta}{\sin\theta}\hat{e}_\phi. \tag{1.35}$$

Using Stokes's theorem, we have

$$S_{\text{top}}[\vec{n}] = iS\oint_\gamma d\vec{n}.\ \vec{A} = iS\oint_{A_\gamma} d\vec{\sigma}.\ (\vec{\nabla}\times\vec{A}), \tag{1.36}$$

but $\vec{\nabla}\times\vec{A} = \hat{e}_r$, which leads to

$$S_{\text{top}}[\vec{n}] = iS\oint_{A_\gamma} d\vec{\sigma}.\ \vec{e}_r = iSA_\gamma, \tag{1.37}$$

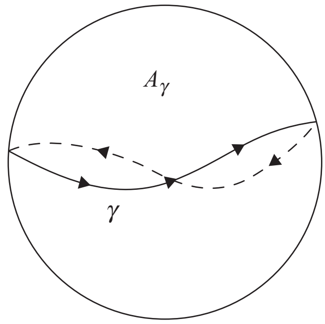

Figure 1.2. Region of integration in equation (1.37).

where A_γ is the region in the sphere S^2 which has the curve γ as its boundary and contains the north pole (see figure 1.2). The action S_{top} is thus a measure of the area bounded by the curve γ: $t\vec{\ }\vec{n}(t)$.

Using $\vec{B} \equiv \vec{\nabla} \times \vec{A}$, equation (1.37) can be interpreted as the action for a particle moving in a radial magnetic field of a magnetic monopole of strength 4π located at the origin of the sphere.

If we had taken $\vec{A} = -\frac{1-\cos\theta}{\sin\theta}\hat{e}_\phi$, the newly defined vector potential would be non-singular in the southern hemisphere, and we would have got

$$S_{\text{top}}[\vec{n}] = -iSA'_\gamma \tag{1.38}$$

where A'_γ is the area of a surface bounded by γ but covering the south pole of the sphere. The minus sign is due to the outward orientation of the surface A'_γ. We can see that the difference between the northern and the southern parts is given by $4\pi iS$, having in mind that the intersection between the two surfaces is the sphere. We will come back to this subject in chapter 9, when we will discuss magnetic models.

1.3 Path integral and statistical mechanics

In statistical mechanics, the equilibrium properties of a system can be obtained from the partition function $Z = \text{tr}\exp(-\beta H)$, where 'tr' denotes a summation over all possible configurations of the system. For a single particle we have

$$Z = \text{tr}[e^{-\beta H}] = \int dx \langle x \mid e^{-\beta H} \mid x\rangle. \tag{1.39}$$

The partition function can be interpreted as a trace over the transition amplitude $\langle x|e^{-iHt}|x\rangle$ evaluated at an imaginary time $t = -i\beta$. The transformation $t = -i\tau$ is called a Wick rotation. Although mathematically this can be a highly nontrivial procedure, the formal prescription is simple. First, we make the substitution $t = -i\tau$, and then we define the imaginary time action S_E using the real time action S_M through the correspondence

$$e^{iS_M}\big|_{t=-i\tau} \equiv e^{-S_E}, \tag{1.40}$$

where the subscripts E and M stand for Euclidean and Minkowskian space–time. For a field $\phi(t, r)$ in quantum field theory we have

$$Z = \int_{\phi_i=\phi_f} D\phi(\tau, r)e^{-S[\phi(\tau,r)]}. \tag{1.41}$$

Here we are summing over a path in which the field $\phi(\tau, r)$ obeys periodic boundary conditions in the imaginary-time direction. In equation (1.41) we integrate over all trajectories with the sole requirement $\phi_i = \phi_f$, with no constraint on what the starting point is. All we must impose is that the field comes back to where it started after Euclidean time τ. We can think of τ as parameterizing a circle.

While all bosonic fields are periodic in the time direction, fermionic fields should be made anti-periodic: they pick up a minus sign as we go around the circle.

Following Tanaka and Takayoshi (2015) we define a topological term S_{top} as the portion of the action which arises in addition to the kinetic action coming directly from the Hamiltonian H. When using the imaginary time, the term S_{top} is purely imaginary and hence contributes a phase factor to the Boltzmann weight e^{-S} (this leads to nontrivial quantum interference effects). The total action is generally of the form: $S = S_{\text{kin}} + S_{\text{top}}$.

Another way to introduce topological terms is the following. The symmetric stress–energy tensor $T_{\mu\nu}$ can be defined as a variation of the action with respect to the metric tensor $g^{\mu\nu}$. More precisely, an infinitesimal variation of the action can be written as

$$\delta S = \int dx\sqrt{g}\, T_{\mu\nu}\delta g^{\mu\nu}, \tag{1.42}$$

where $\sqrt{g}\,dx$ is an invariant volume of space (see chapter 4). We define topological terms as the metric-independent terms in the action. It follows that topological terms do not contribute to the stress–energy tensor. We will study topological terms in more detail later in the text.

1.4 Fermion path integral

A path integral over fermions is basically the same as for bosons, but we must consider that fermions anti-commute. However, we cannot directly write a Lagrangian for fermions, since they have no classical analogue. To implement the path integral, we need the notion of anti-commuting classical variables that are called *Grassmann* variables (Ashok 1993, Altland and Simons 2010).

A Grassmann algebra is a set of objects θ_i with the following properties:

(a) They anti-commute $\theta_i\theta_j + \theta_j\theta_i = 0$. This implies $\theta_i^2 = 0$ for any i.
(b) $\theta_i + \theta_j = \theta_j + \theta_i$.
(c) They can be multiplied by complex numbers $a \in C$.
(d) There is an element 0 such that $\theta_i + 0 = \theta_i$.

For any θ, the most general element of the algebra is

$$g = a + b\theta, \quad \text{with } a, b \in c. \tag{1.43}$$

For two θ the most general element is

$$g = a + b\theta_1 + c\theta_2 + d\theta_1\theta_2, \tag{1.44}$$

and so on. In defining a derivative, the direction in which the derivative operates must be specified. For a right derivative we have

$$\frac{\partial}{\partial\theta_i}(\theta_j\theta_k) = \theta_j\left(\frac{\partial\theta_k}{\partial\theta_i}\right) - \left(\frac{\partial\theta_j}{\partial\theta_i}\right)\theta_k = \delta_{ik}\theta_j - \delta_{ij}\theta_k. \tag{1.45}$$

For a left derivative the result is

$$\frac{\partial}{\partial\theta_i}(\theta_j\theta_k) = \left(\frac{\partial\theta_j}{\partial\theta_i}\right)\theta_k - \theta_j\left(\frac{\partial\theta_k}{\partial\theta_i}\right) = \delta_{ij}\theta_k - \delta_{ik}\theta_j. \tag{1.46}$$

Here I will use left derivatives. Note that we have

$$\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} + \frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i} = 0. \tag{1.47}$$

For a fixed i we have

$$\left(\frac{\partial}{\partial\theta_i}\right)^2 = 0. \tag{1.48}$$

If D represents the operation of differentiation with respect to one Grassmann variable and I represents the operation of integration, we must have

$$ID = DI = 0. \tag{1.49}$$

So, using equation (1.48) we see that the integration can be identified with differentiation: $I = D$.

For a function we have

$$\int d\theta f(\theta) = \frac{\partial f(\theta)}{\partial\theta}, \tag{1.50}$$

which gives

$$\int d\theta = \theta, \quad \int \theta d\theta = 1. \tag{1.51}$$

If we write $\theta' = a\theta$ with $a \neq 0$, we find

$$\int d\theta f(\theta) = \frac{\partial f(\theta)}{\partial\theta} = a\frac{\partial f(\theta'/a)}{\partial\theta'} = a\int d\theta' f(\theta'/a). \tag{1.52}$$

For many Grassmann variables, if $\theta'_i = a_{ij}\theta_j$ (where we sum over repeated indices) with $\det a_{ij} \neq 0$, we get

$$\int \prod_{i=1}^{n} d\theta_i f(\theta_i) = (\det a_{ij}) \int \prod_{i=1}^{n} d\theta'_i f(a_{ij}^{-1}\theta'_j). \tag{1.53}$$

We define a delta function as

$$\delta(\theta) = \theta. \tag{1.54}$$

We can verify that it satisfies

$$\int d\theta \delta(\theta) = \int d\theta \theta = 1. \tag{1.55}$$

For a function $f(\theta) = a + b\theta$, we have

$$\int d\theta \delta(\theta) f(\theta) = \int d\theta \theta f(\theta) = \int d\theta \theta (a + b\theta) = \int d\theta \theta a = \frac{\partial(\theta a)}{\partial \theta} = a = f(0). \tag{1.56}$$

For path integral calculations, we need Gaussian integrals. For two θ_i we have

$$\int d\theta_1 d\theta_2 e^{-\theta_1 A_{12} \theta_2} = \int d\theta_1 d\theta_2 (1 - A_{12}\theta_1\theta_2) = A_{12}, \tag{1.57}$$

where we have expanded the exponential in a Taylor series. The variable θ does not need to be small; rather the exponential is defined by its Taylor expansion.

Let us now consider two sets of independent Grassmann variables $(\theta_1, \ldots, \theta_n)$ and $(\bar{\theta}_1, \ldots, \bar{\theta}_n)$. We want to calculate the integral

$$I = \int \prod_{i,j} d\bar{\theta}_i d\theta_j e^{-\bar{\theta}_i A_{ij} \theta_j}. \tag{1.58}$$

We have

$$I = \int \prod_{i,j} d\bar{\theta}_i d\theta_j \left[1 - \bar{\theta}_i A_{ij} \theta_j + \frac{1}{2} (\bar{\theta}_i A_{ij} \theta_j)(\bar{\theta}_k A_{kl} \theta_l) + \ldots \right]. \tag{1.59}$$

The only non-zero term in this expansion is the one with all $n\theta_i$ and all $n\bar{\theta}_i$. This will give

$$I = \frac{1}{n!} \sum_{\text{permutations}\{i_n\}} \pm A_{i_1 i_2} \ldots A_{i_{n-1} i_n}. \tag{1.60}$$

If A_{ij} is a matrix, equation (1.59) is a sum over all elements $\{i, j\}$ where we choose each row and column once, with the sign from the ordering. But this is just the determinant. So the result is:

$$I = \det(A) \tag{1.61}$$

It is easy now to show that

$$\int \prod_{i,j} d\bar{\theta}_i d\theta_j e^{-\bar{\theta}_i A_{ij}\theta_j + c_i^* \theta_i + \bar{\theta}_i c_i} = \det A \exp(c_i^* A_{ij}^{-1} c_j). \tag{1.62}$$

That is all we need for the fermion path integral.

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