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Chapter 11

Modern physics

11.1 Relativity

On a conceptual basis, understanding relativity, especially at the intro level, is like trying to understand Dr Seuss[1]. It's just different and weird enough from the world you are accustomed to, so it's not going to be intuitive as classical physics can be intuitive. The only time students seriously struggle with relativity is when they fight it and decide they cannot understand it because it's too different from the classical world. Relax, accept the basic principles and you'll figure it out. You might even enjoy your new perspective on the world.

The two core principles of relativity are:

1. The speed of light is the same to all observers regardless of motion.
2. The laws of physics still hold in an inertial reference frame.

The first one gives that if you are looking for how fast light moves relative to an observer, the answer is always c. In addition, if you ever get an answer greater than c for any problem, you are doing it wrong. The second one is just saying that none of our usual laws change in a moving reference frame as long as that reference frame is not accelerating. This is why the so-called 'Twin Paradox' where one twin travels at a very high velocity and thus ends up being younger is not a paradox. Though they both see the other's clock ticking more slowly, the 'correct' one is the one who stayed in an inertial reference frame the entire time, namely the one who stayed on Earth. The other one had to accelerate.

[1] You don't need to know what a gardinka is to know why The Grinch was annoyed by them. It's a noisy toy of some sort and the details are unimportant.

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We can add three more to this that aren't formal principles but do apply:

1. The system should reduce down to what you expect from classical physics if the velocity is much lower than c.
2. You never see yourself changing.
3. We can disagree on when and where something happens, but we cannot disagree that it happens

(4) is important for keeping the symbols straight and figuring out how to set up the equations. If you are carrying a ruler and wearing a watch, you will never see that ruler change length nor will you see the watch tick slower or faster. In order to see relativistic effects, something must be moving **relative** to you. What follows next is a quick discussion on time dilation, length contraction, and relative velocities (Lorentz transformations). (5) helps you to keep yourself grounded in reality. When something happens or where it happens is relative to an observer, but it **always** happens. For an extreme example, you cannot dodge a bullet in one reference frame and not in another due to relativity so that you are alive according to one observer and dead according to another.

11.1.1 Time dilation

As the name implies, time dilation means time expands, or moves more slowly. You should be at least vaguely familiar with the concept of space-time now and you can think of this as the faster an object moves through space, the slower it moves through time. The equation that governs this is

$$\Delta t = \frac{\Delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{11.1}$$

where $\Delta\tau$ is called the proper time interval and Δt is the observer time interval. Both proper time and proper length are often given ridiculous definitions that are confusing ways of saying 'this is how an object views itself'. That means that if you are carrying a ruler and wearing a watch, the time the watch keeps and the length of the ruler are your proper time and proper length.

With that in mind, let's go through the classic muon decay problem **both** ways, using time dilation and length contraction, and show we get the same result from different perspectives.

Example: Muons have a half-life of 1.56 microseconds (1.56×10^{-6} s) and are created in the upper atmosphere traveling at about $0.99c$. Classically, virtually no muons should reach the surface of the Earth, a distance of about 10 km, because they will have decayed before they reach it. Due to relativistic effects, we do see quite a few muons. If 1000 muons are created in the upper atmosphere, determine how many will reach the surface of the Earth on average from the perspective of an observer on the surface of the Earth.

Solution:
First off we need to have a brief aside about half-life. The half-life of a substance is how long it takes for half of it to decay. So after one half-life there is half of it left, after two there is 1/4, after 3 there is 1/8, and so on. After n half-lives then, there are $\frac{1}{2^n} \times$ (original number of particles) remaining. If they are traveling at $0.99c$, then the time it takes will still just be given by

$$\begin{aligned} v &= \frac{d}{t}, \\ t &= \frac{d}{v} = \frac{10\ \text{km}}{0.99c} = \frac{10 \times 10^3\ \text{m}}{0.99 \times 3 \times 10^8\ \text{m s}^{-1}} = 3.367 \times 10^{-5}\ \text{s}, \end{aligned} \tag{11.2}$$

or better written as 33.67 microseconds. Divide this by microseconds per half-life (1.56) and we get number of half-lives, $\frac{33.67}{1.56} = 21.583$ half-lives. So this many half-lives pass between when the particles are created and when they hit the ground. The amount left after n half-lives is $\frac{1}{2^n}$(original number), with $n = 21.583$ and original number being 1000, is

$$\frac{1}{2^{21.583}}(1000) = 3.183\,2 \times 10^{-4}\ \text{particles}, \tag{11.3}$$

or less than one particle, which means none since you can't have a fraction of a particle remaining.

With time dilation though, the particles live longer *in our reference frame.* Remember we are talking about how we're seeing things so they still take the same time to hit the ground, it's just time passes more slowly for them (according to us). The proper time interval is then going to be the half-life and we can figure out how long the half-life of the muons appears to us by the following:

$$\begin{aligned} \Delta t &= \frac{\Delta\tau}{\sqrt{1 - \dfrac{v^2}{c^2}}} \\ \Delta t &= \frac{1.56 \times 10^{-6}\ \text{s}}{\sqrt{1 - \dfrac{(.99c)^2}{c^2}}} = 1.105\,9 \times 10^{-5}\ \text{s}. \end{aligned} \tag{11.4}$$

They still take the same amount of time to hit the ground, so do the same calculation we just did but with a new half-life

$$\begin{gathered} \frac{3.367 \times 10^{-5}\ \text{s}}{1.105\,9 \times 10^{-5} \dfrac{\text{s}}{\text{half-life}}} = 3.04\ \text{half-lives} \\ \frac{1}{2^{3.04}}(1000) = 121.6 \end{gathered} \tag{11.5}$$

So about 121 particles actually hit the ground. This is a drastically different result.

11.1.2 Length contraction

Length contraction means that if object A is moving relative to object B, object A will appear smaller. The equation governing this is

$$L = \sqrt{1 - \frac{v^2}{c^2}}\, l. \tag{11.6}$$

As before, proper length (l in this equation) is the length of the object as it appears to itself; it will never see itself changing. L is the contracted length measured by an observer who is stationary relative to it. Now to do the muon problem the other way

Example: Muons have a half-life of 1.56 microseconds (1.56×10^{-6} s) and are created in the upper atmosphere traveling at about $0.99c$. Classically, virtually no muons should reach the surface of the Earth, a distance of about 10 km, because they will have decayed before they reach it. Due to relativistic effects, we do see quite a few muons. If 1000 muons are created in the upper atmosphere, determine how many will reach the surface of the Earth on average from the perspective of the **muons**.

Solution:

The speed will remain the same in both cases. However, now we're talking about the perspective of the muons, which means we can't see time dilation effects for their half-life. They will not see a change in how fast they decay. What they will see is the distance to the surface of the Earth being contracted. In other words, we see the muons reaching the Earth because they live longer, they see themselves reaching the Earth because the Earth is actually closer to them. Finding the contracted length, we have

$$\begin{aligned} L &= \sqrt{1 - \frac{v^2}{c^2}}\, l, \\ L &= \sqrt{1 - \frac{(.99c)^2}{c^2}}\, 10 \times 10^3 \text{ m}, \\ &= 1410.7 \text{ m}. \end{aligned} \tag{11.7}$$

Velocity will remain the same and thus we can find the time it takes to hit the ground

$$t = \frac{d}{v} = \frac{1410.7 \text{ m}}{0.99 \times 3 \times 10^8 \text{ m s}^{-1}} = 4.75 \times 10^{-6} \text{ s}. \tag{11.8}$$

And just as before, determine how many half lives this is and use that to determine how many particles reach the ground

$$\begin{aligned} &\frac{4.75 \times 10^{-6} \text{ s}}{1.56 \times 10^{-6} \text{ s} \dfrac{s}{\text{half-life}}} = 3.04 \text{ half-lives}, \\ &\frac{1}{2^{3.04}}(1000) = 121.6. \end{aligned} \tag{11.9}$$

Which is the same number we got before. We can disagree on when something happens, where it happens or even why it happens, but we cannot disagree that it does happen.

11.1.3 Lorentz velocity transformations

The way most students get lost in the relative velocity problems is with keeping track of all the primed and unprimed variables, a convention that is well-intentioned but can be confusing. The best place to begin then is to carefully define everything. We begin with figure 11.1 for a visual description, and then give prose descriptions.

$$\begin{aligned} S &: \text{reference frame that you define to be stationary.} \\ S' &: \text{ reference frame that you define to be moving.} \\ v &: \text{relative velocity between frame } S \text{ and frame } S'. \\ u &: \text{velocity of an object as measured by someone in frame } S. \\ u' &: \text{velocity of an object as measured by someone in frame } S'. \end{aligned}$$

Now the two equations should make a little more sense. They are

$$\begin{aligned} u' &= \frac{u - v}{1 - \dfrac{uv}{c^2}}, \\ u &= \frac{u' + v}{1 + \dfrac{u'v}{c^2}}. \end{aligned} \tag{11.10}$$

Notice two important things about these equations. First, v doesn't have a prime on it anywhere, so this is the easy way to make sure you don't get lost in all the symbols. v is just the velocity between the two frames and they have to agree about that. What they disagree on is how fast objects within those frames are moving. Second, they'll

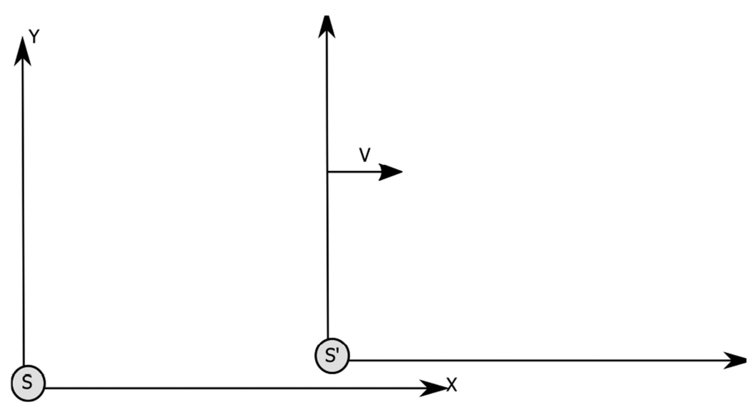

Figure 11.1. Visualization of the different reference frames and the velocity between them.

reduce down to exactly what you'd expect to see if the velocity is much lower than the speed of light, in which case they become

$$\begin{aligned} u' &= u - v, \\ u &= u' + v. \end{aligned} \tag{11.11}$$

In other words, if someone on a train moving at velocity v throws a ball with speed u', it will be moving at $u = u' + v$ according to someone on the ground. This is how you can check signs to make sure you've got the right equation and you've set it up right. Reduce it to a nonrelativistic case ($v \ll c$) and it should be what you expect to see.

Example: You are standing on the ground and see a train go by to your right at $0.9c$. Someone in the train throws a baseball to the left at $0.7c$ relative to him. How fast is the ball moving according to you?
Solution:
We'll define the ground to be the stationary frame and so we need to use

$$u = \frac{u' + v}{1 + \dfrac{u'v}{c^2}}. \tag{11.12}$$

v : relative velocity between train and ground.
u : velocity of an object as measured by someone on the ground.
u' : velocity of an object as measured by someone in the train.

As you may guess, just plugging in the numbers straight won't give us the correct answer because we need to be careful about signs. Define right as positive and left as negative and we get

$$u = \frac{-u' + v}{1 + \dfrac{-u'v}{c^2}}. \tag{11.13}$$

The way we can be sure we have got it right is to then reduce this down to a classical case. If someone is moving and throws a ball in the opposite direction to their own motion, you would expect an observer on the ground to see it move more slowly. Reducing this to that case we see that $u = -u' + v$ gives the correct answer since that will give a velocity of $0.2c$. Do it the other way and you get a velocity of $1.6c$, which means it would be going faster and that isn't what we would see even in the classical world. Now we can do the actual calculation

$$u = \frac{-.7c + .9c}{1 + \dfrac{(-.7c)(.9c)}{c^2}} = 0.540\,54c \tag{11.14}$$

11.2 Quantum mechanics

Despite what pop culture tells you, quantum is not physics-speak for cool, it merely means countable. The quantum world is simply then a world where we are dealing with individual particles, which have discrete descriptive qualities, such as energy. Everything you have learned still applies just fine as long as you *quantize* it, meaning that it must take on an integer multiple of some number.

The way this is linked into the math is with *boundary conditions*. There is always a physical reality that must be obeyed and that is where many of the equations come from. For instance, the solutions to the particle in a box problem come from noting that if it's stuck in the box, the wave amplitude must be zero at the edges of the box. This forces the wavelength to take on certain quantized values that will obey this boundary condition. You probably will not do too much with the actual math in an intro course but it will help your conceptual thinking considerably to use these boundary conditions as part of the link between the classical and quantum world.

11.2.1 The photoelectric effect

Your friend is sitting on the couch. You pull on him but not very hard and he doesn't move. You pull a little harder and you make him stand up. He sits back down, you pull **really** hard and he stands up and runs forward. This is exactly how you can think of the photoelectric effect. The actual equation describing it is

$$E = hf - \phi, \tag{11.15}$$

where the variables describe the following:

$$\begin{aligned} E &: \text{kinetic energy of the } \textit{ejected electron}, \\ hf &: \text{energy of the } \textit{incoming photon}, \\ \phi &: \text{work function of the metal.} \end{aligned}$$

So then what this says physically is that a photon with energy hf (which is always the energy of a photon) strikes an electron in a metal. If the energy of the photon is at least the work function of the metal, which is how much energy is required to break the electron out of the metal, then the electron will move. If there is more energy than is needed to break it out, it just means the electron moves faster (because it has extra kinetic energy). If the E in this equation is zero, that means there's just barely enough energy to break the electron free and no more.

Example: A photon with a wavelength of 310 nm strikes a metal and the electron released crosses a 3 V potential difference. What is the work function of the metal?
Solution:
Energy is energy, if the electron had enough energy to cross a 3 V potential difference, then that's how much was left over *after* it broke free from the metal. Since the electron-volt is defined as the amount of energy required to move 1 electron across a 1 V potential difference, it had 3 eV of 'leftover' kinetic energy, or

$$\begin{aligned} E &= hf - \phi, \\ 3\ \text{eV} &= hf - \phi. \end{aligned} \tag{11.16}$$

A very nice shortcut though is that since $c = \lambda f$ (this is light) you can write

$$\begin{aligned} hf &= \frac{hc}{\lambda}, \\ hc &= \text{constant} = 1240 \text{ eV nm}, \end{aligned} \tag{11.17}$$

so you do not have to convert everything to SI units. This gives

$$\begin{aligned} 3 \text{ eV} &= \frac{hc}{\lambda} - \phi, \\ 3 \text{ eV} &= \frac{1240 \text{ eV nm}}{310 \text{ nm}} - \phi. \end{aligned} \tag{11.18}$$

Notice how the units of nanometers cancel out and we are left with just energy in eV. Solving

$$\begin{aligned} 3 \text{ eV} &= 4 \text{ eV} - \phi, \\ \phi &= 1 \text{ eV}. \end{aligned}$$

Look back over it and make sure it seems right now. The incoming photon had an energy of 4 eV. The ejected electron had an energy of 3 eV. The difference has to go somewhere and so it must have gone into breaking the electron free (getting it off the couch).

11.2.2 Electron transitions

In an atom, electrons can occupy different quantized energy levels. This means they all must be some integer multiple of the same number. A subtle point that is often missed though is that atoms stay in their ground state unless something makes them get out of it. This is why the absorption spectrum is a subset of the emission spectrum. Not all wavelengths that are emitted can be absorbed because absorption requires the atom to be in the ground state. An example will be helpful here.

Example: An atom has a ground state of 1 eV and excited states of 2 eV, 3 eV, and 4 eV. How many lines are in the absorption and emission spectrums?
Solution:
Absorption requires that an electron goes from the ground state to a higher state, so the initial state must be 1 eV. It only has three places to go (2 eV, 3 eV, and 4 eV), so there are only three lines in the absorption spectrum.

The emission spectrum occurs when an electron drops down a level and it can drop down into any other lower energy level. So the possibilities are

$$\begin{aligned} &4 \longrightarrow 3,\ 4 \longrightarrow 2,\ 4 \longrightarrow 1, \\ &3 \longrightarrow 2,\ 3 \longrightarrow 1, \\ &2 \longrightarrow 1, \end{aligned}$$

which gives six lines in the emission spectrum.

11.2.3 Wavefunctions and probability

The usual definitions of a wave function (Ψ) are highly mathematical in nature. Though it is not a complete description, the best way to think of a wave function for now is that it is the particle itself. The wave function contains all the information about the particle, you just have to take a measurement on it to get that information. The problem of course is that it's not easy to get the wave function since it's buried in a second-order differential equation, but that is not a topic for this book.

The most common mistake made when dealing with wave functions is getting them confused with probability density functions, so here we will focus on the difference.

Wave functions (Ψ)**:**
This is a wave just like wave on a string and will probably look the same, being a sine or cosine function (or an exponential which can be written as sines or cosines). It has an amplitude just like any other wave and it must follow boundary conditions as discussed above. By itself it doesn't really tell you much and in fact it cannot correspond directly to a physical quantity because it can be imaginary. What the wavefunction does do is allow you to perform operations on it that will give you physical quantities, such as momentum, position, or probability.

Probability density ($|\Psi|^2$):
This is the probability per unit length, area, or volume, depending on how many dimensions you're using (most likely just length in an intro course) of finding the particle. Formally speaking it's finding any property of the particle if the wave function is a function of something other than position, but for our purposes here you will not be using anything but position. That's all this tells you.

The most common use of this will be normalization. Normalization is another boundary condition, what it says is that the particle has to be somewhere. An integral is just a sum and so if you add up all the probabilities of the particle existing at various places over all the possible places, it has to be 1, since the particle must exist somewhere. Mathematically this is written as

$$\int_{-\infty}^{\infty} |\Psi^2| dx = 1. \tag{11.19}$$

Normalization example: Normalize the wavefunction

$$\begin{aligned} \Psi &= Ae^{-ax}, \ \text{for } x \geqslant 0 \\ \Psi &= 0, \ \text{for } x < 0. \end{aligned} \tag{11.20}$$

Solution:
Normalizing a wavefunction means finding this constant A. Remember that an integral is just a sum, so even though it goes from negative infinity to positive infinity, all the negatives don't matter since the wavefunction is zero there.

So technically we're still adding them in but we're just adding zero. The integral then becomes

$$\begin{aligned} \int_0^\infty |\Psi^2| dx &= 1, \\ \int_0^\infty A^2 e^{-2ax} dx &= 1. \end{aligned} \tag{11.21}$$

This is an integral that is doable analytically, the result is

$$\begin{aligned} \int_0^\infty A^2 e^{-2ax} dx &= -\frac{A^2}{2a} e^{-2ax}|_0^\infty, \\ &= 0 + \frac{A^2}{2a}. \end{aligned} \tag{11.22}$$

This is equal to 1, so we can now find the normalization constant A

$$\begin{aligned} \frac{A^2}{2a} &= 1, \\ A &= \sqrt{2a}, \\ \Psi &= \sqrt{2a}\, e^{-ax}. \end{aligned} \tag{11.23}$$

Don't make normalization harder than it needs to be, it should never be difficult unless you have an integral that requires some calculus tricks to do.

11.3 Brief aside on energy equations

Modern physics usually comes fairly hard and fast at the end of the year when you're already tired and ready to be done with physics, then gives you several different equations that seem to all be the same thing. Here is a summary of most of them, giving what they mean and how they are different.

$$\begin{aligned} E &= hf \\ E &= hf - E_0 \\ E &= \frac{n^2 h^2}{8mL^2} \\ E &= -\frac{13.6 \text{ eV}}{n^2} \\ E &= mc^2 \\ E &= \gamma mc^2 \\ E &= \frac{p^2}{2\text{ m}} \end{aligned} \tag{11.24}$$

$E = hf$: energy of a photon, Planck's constant times frequency
$E = hf - E_0$: Photoelectric effect. Photon coming in with energy hf uses E_0
of it to break an electron free and whatever is left over goes to kinetic energy
of the electron(E in this case).
$E = \frac{n^2h^2}{8mL^2}$: Energy of a particle in a box of length L. The particle has mass m
and is in state n.
$E = -\frac{13.6 \text{ eV}}{n^2}$: Energy of an electron in the hydrogen atom in state$n$. (11.25)

$E = mc^2$: Rest energy of something with mass m, meaning the energy it
has simply by virtue of existing
$E = \gamma mc^2$: Total energy of something with mass m, kinetic energy plus rest energy.
$E = \frac{p^2}{2 \text{ m}}$: Standard kinetic energy of $\frac{1}{2}mv^2$ written in a more
convenient form for quantum calculations (11.26)

11.4 Summary and important notes

1. Both quantum and relativity only change a few things. Quantum physics requires things to be quantized and fit specific boundary conditions while relativity causes the numbers to change when you start getting very fast. Everything else you've learned still applies.
2. Nothing should ever exceed c and light itself always travels at c (in a vacuum) regardless of the motion of anything. If you get an answer that contradicts either of these, there's something wrong.
3. To keep proper time and proper length straight, remember that you never see yourself changing[2]. Proper time and proper length are just how the observer measures their own time and their own length.
4. There are quite a few equations for energy, be sure you know what each one means.

[2] Or my favorite analogy: relativity is like lifting; you never see yourself getting bigger, you just see other guys getting smaller.