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A Handbook of Mathematical Methods and Problem-Solving Tools for Introductory Physics

Joshua F Whitney
Heather M Whitney

SECOND
EDITION



A Handbook of Mathematical
Methods and Problem-Solving
Tools for Introductory Physics
(Second Edition)

A Handbook of Mathematical Methods and Problem-Solving Tools for Introductory Physics (Second Edition)

Joshua F Whitney and Heather M Whitney
Wheaton College, Wheaton, Illinois, USA

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For our students: past, present, and future.

Contents

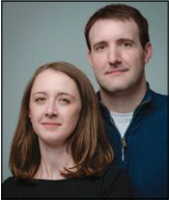
About the authors	x
1 Introduction	1-1
2 Kinematics	2-1
2.1 Equations needed with symbols defined	2-1
2.2 Understanding the equations	2-2
2.3 Keeping the signs straight	2-3
2.4 Motion in two dimensions: relating x and y	2-4
2.5 Summary and important notes	2-7
3 Newton's laws	3-1
3.1 Applying the first law	3-2
3.2 Applying the second law	3-2
3.3 Applying the third law	3-2
3.4 Examples	3-3
3.5 Summary and important notes	3-7
4 Energy and momentum	4-1
4.1 Energy	4-1
4.2 Example	4-2
4.3 Momentum	4-3
4.4 Types of collisions	4-4
4.5 Summary and important notes	4-4
5 Circular and rotational motion	5-1
5.1 Centripetal force	5-1
5.2 Torque	5-4
5.2.1 Moment of inertia	5-6
5.3 Waves	5-7
5.4 Oscillations	5-7
5.4.1 Wavelength and period (λ , T , or τ)	5-8
5.4.2 Frequency (f or ν)	5-8
5.4.3 Angular frequency (ω)	5-8
5.4.4 Wave number (k)	5-9

5.4.5	Phase angle (ϕ)	5-9
5.4.6	Quick reference guide for descriptions of oscillations	5-9
5.5	Traveling waves	5-10
5.6	Standing waves and boundary conditions	5-10
6	Basic optics	6-1
6.1	Diffraction and interference	6-1
6.2	Thin lenses	6-3
6.3	Summary and important notes	6-5
7	The right-hand rule	7-1
7.1	How to use it properly	7-2
7.2	Special case of magnetic fields from currents	7-3
7.3	Summary and important notes	7-3
8	Electric fields and electric potential	8-1
8.1	Finding fields and forces from point charges	8-1
8.1.1	Electric field integrals	8-4
8.2	Electric potential from point charges	8-6
8.3	Summary and important notes	8-7
9	Magnetic fields	9-1
9.1	Biot–Savart law	9-1
9.1.1	Right-hand rule for magnetic fields	9-2
9.1.2	Calculations with the Biot–Savart law	9-3
9.2	Magnetic forces	9-4
9.3	Trigonometry for electric and magnetic field integrals	9-6
9.4	Induced currents: Faraday’s and Lenz’s law	9-8
10	Circuits	10-1
10.1	Ohm’s law	10-1
10.2	Kirchhoff’s loop and junction laws	10-2
10.2.1	Loop law: the sum of the potential differences around a closed circuit is zero	10-2
10.2.2	Junction law: the sum of all currents into a junction must be equal to the sum of the currents out of the junction	10-4
10.3	Summary and important notes	10-6

11	Modern physics	11-1
11.1	Relativity	11-1
11.1.1	Time dilation	11-2
11.1.2	Length contraction	11-4
11.1.3	Lorentz velocity transformations	11-5
11.2	Quantum mechanics	11-6
11.2.1	The photoelectric effect	11-7
11.2.2	Electron transitions	11-8
11.2.3	Wavefunctions and probability	11-9
11.3	Brief aside on energy equations	11-10
11.4	Summary and important notes	11-11
12	General problem-solving tips	12-1
12.1	Break the problem down into as small pieces as possible	12-1
12.2	Write down everything you have and what you are trying to find	12-1
12.3	Check units	12-2
12.4	Pull all the 10^n factors out of your calculation before putting it in your calculator	12-2
12.5	Practice algebra	12-3
12.6	Keep everything symbolic as long as possible	12-3
12.7	Electron volts	12-4

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Chapter 1

Introduction

The purpose of this book is to ‘fill in the holes’, so to speak, that may be present after a student experiences lecture and lab on a given introductory physics topic and then tries to tackle problem solving on their own. In our experience, this independent effort of putting the pieces together is when the learning really happens, but students sometimes benefit from checking in with someone during that time. And sometimes that time comes when office hours are not available. So we wrote this little handbook with that situation in mind. It is comprised of a wide variety of insights, some big-picture, thinking-like-a-physicist kind of ideas, and some detail-oriented, such as the best way to put in positive or negative signs. Likewise, this handbook is written with both algebra- and calculus-based introductory physics students in mind. It has coverage of ideas that will apply to all students, regardless of which type of introductory physics class you are taking, and some calculus-specific ideas.

Thus, the handbook is best suited as a companion text when you begin to work through the actual homework problems after lectures. Each section will focus on a deeper understanding of the concept and equations you will be using along with problem-solving skills. Toward that end, most sections will begin with a very careful description of each concept and equation, what every symbol means and what the equation is saying in as simple terms as possible. Remember that equations themselves are merely tools for working with concepts and you will be in much better shape if you know what tool you are looking for before you start looking, thereby preventing you from wasting time, energy, and frustration trying to solve a problem with a hammer when you really need a wrench.

This book is intended to be as straightforward as possible so that it can be used as a study guide. Everything is explained thoroughly but we have tried to avoid being too funny or asking rhetorical questions or all the things that can contribute to a physics textbook being 1000+ pages long. Along these lines, we have also erred on the side of fewer figures than you might expect, because we anticipate that you will

want to refer to your course textbook as much as possible to reconcile what you see here with its presentation by your instructor and textbook. Every section will contain small summaries and important points that, in all honesty, you could probably just memorize and do reasonably well on your physics tests. But you might not really understand the physics with this approach. A much better course of action is to study these, apply them, and when you do not feel like you **need** to memorize them anymore, you are in good shape. That is to say, the summaries can be used as a test themselves in that when they feel intuitive, you have a fairly solid grasp of the material.

It is important to note this book is not exhaustive for an introductory physics course, nor is it intended to be. It is focused on the things that students usually struggle with due to, in our opinion, the disconnect that can sometimes happen between lecture and problem solving. You may not find the exact examples or explanations you are looking for, but the problem-solving techniques and conceptual explanations can be applied to almost anything in your introductory physics course. You can think of this handbook as a style manual of sorts for physics, but in a problem-solving sense. Just as with a writing handbook you would look at specific examples and then apply them more broadly as similar situations come up, you can take the ideas of this book and apply them to other problems in physics as well.

Our most significant piece of advice to students of introductory physics is that you should make time for trying problems that are new to you, without having a textbook open, and under some kind of time limit. Doing so gives you the opportunity to put all the pieces together that you are learning and make them your own. Any student we have ever had who takes this advice learns physics well. This book can support this process as a stand-alone resource for quick checks on a physics principle and how it is solved, which can then be used to inspire work on the problem at hand.

Above all, we wrote this handbook with our students, past, present, and future, in mind. We tried to consider the questions they have frequently had and what types of advice and suggestions from us seemed to be effective when implemented by them. A wonderful experience of teaching is that we teach but also learn from students. We have learned much from ours and dedicate this book to them.

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Chapter 2

Kinematics

2.1 Equations needed with symbols defined

The equations that get you through just about any kinematics problem are as follows:

$$\begin{aligned}x_f &= x_i + v_{i,x}\Delta t + \frac{1}{2}a_x(\Delta t)^2 \\v_{f,x}^2 &= v_{i,x}^2 + 2a_x\Delta x \\v_{f,x} &= v_{i,x} + a_x\Delta t.\end{aligned}\tag{2.1}$$

Now to completely define the symbols, we have

x_f : final position

x_i : initial position

$v_{f,x}$: final velocity

$v_{i,x}$: initial velocity

a_x : acceleration

Δt : change in time

Δx : change in position ($x_f - x_i$).

Here we have given the equations in terms of one direction of motion, the x -direction. We use the subscripts i and f to indicate initial and final condition. (Note that your textbook may use 0 for initial and no subscript at all for final conditions.) When you intend to apply the equations to other directions (y - and z -directions, e.g.), you can simply change out the subscripts.

2.2 Understanding the equations

Each component of these equations says something very specific about the motion of the object. So let's look at them all individually for the first equation

$$x_f = x_i + v_{i,x}\Delta t + \frac{1}{2}a_x(\Delta t)^2.$$

Reading left to right we define the components one by one. First we have

x_f : final position.

This is the final position of the object, after it has completed its motion during time Δt . Next we have

x_i : initial position.

This is simply where the object started, it can very often be set to zero for simplicity.

Next, how far the object has moved *because it started with some initial velocity* is given by

$$v_{i,x}\Delta t.$$

If it starts from rest, this term is zero. This can be positive or negative depending on which direction it is initially moving.

How far the object has moved *because it is accelerating* is given by

$$\frac{1}{2}a_x(\Delta t)^2.$$

If it is not accelerating, the entire term is zero. Like velocity, acceleration can be positive or negative. If it has the opposite sign from the initial velocity term, then the object is slowing down. If it is the same sign it is speeding up.

For the second equation the concepts are similar:

$$v_{f,x}^2 = v_{i,x}^2 + 2a_x\Delta x. \quad (2.2)$$

Reading left to right again, we first have

$v_{f,x}^2$: final velocity.

This is how fast you are moving at the end, squared. Don't let the fact that it is squared worry you: that simply comes from the derivation. You could, of course, write the expression just in terms of final velocity (and you will probably need to take the square root of the expression for some actual calculations of velocity).

Next, we have an expression for how fast the object was moving at the beginning of the situation, again squared:

$v_{i,x}^2$: initial velocity.

And finally,

$2a_x\Delta x$: Additional velocity gained or lost due to acceleration.

Just as you can figure out how much velocity changes because you are accelerating over some *time*, you can also figure out how much velocity changes because you are

accelerating over some *distance*, which is what this portion describes. This should not be surprising as distance and time are always very closely related.

Put it all together and this equation says ‘your final velocity depends on your initial velocity plus the product of acceleration and over which you were accelerating, with some math factors here and there’. So to recap how to use this, you have two equations for calculating final velocity, one that uses time and one that uses distance:

$$\begin{aligned}v_{f,x} &= v_{i,x} + a_x t: \text{ uses time} \\v_{f,x}^2 &= v_{i,x}^2 + 2a_x \Delta x: \text{ uses distance.}\end{aligned}\tag{2.3}$$

2.3 Keeping the signs straight

There are many sign rules suggested by various books and instructors, especially for the acceleration due to gravity. The only rules you really need are:

1. Set up a coordinate system.
2. Stick to it.

What this means is just that you decide what is positive and what is negative at the very beginning of each problem, then just put in the appropriate sign, keeping track of what each piece of each equation means.

Example: Three objects are dropped from a very high location. One is thrown straight up, one is thrown straight down and one is simply dropped. Set up equations that show where each one is after some time Δt .

Solution:

We are going to do this purely symbolically, but we could do it with numbers. Start the problem by declaring up is positive and down is negative. Then, use the subscripts 1, 2, and 3 for the objects thrown up, down, and just dropped, respectively. So let’s set up all the equations and explain them. Since we are dealing with a distance that depends on velocity, time, and acceleration, the expression that relates them is

$$x_f = x_i + v_{i,x} \Delta t + \frac{1}{2} a_x (\Delta t)^2.\tag{2.4}$$

So let’s set this up for the first object. Here, the full expression is given in its entirety; then we will customize it for each object.

$$x_{f,1} = x_{i,1} + v_{i,1} \Delta t + \frac{1}{2} a_{x,1} (\Delta t)^2.\tag{2.5}$$

All we are interested in is how far they went, so we can set $x_i = 0$ for all three. For the first object, this results in

$$x_{f,1} = v_{i,1} \Delta t + \frac{1}{2} a_{x,1} (\Delta t)^2.\tag{2.6}$$

The v_i in this case is the initial velocity with which it was thrown UP, so we'll change that to just some v_1 . This one is thrown up, so that will be positive, such that

$$x_{f,1} = v_{i,1}\Delta t + \frac{1}{2}a_{x,1}(\Delta t)^2. \quad (2.7)$$

Once the object is thrown it is in free fall, so the acceleration due to gravity will have the numerical value of g . Since we have defined up to be positive, that means $a = -g$, resulting in

$$x_{f,1} = v_{i,1}\Delta t - \frac{1}{2}g(\Delta t)^2: \text{ the trajectory of the first object.} \quad (2.8)$$

Now the second object. They are all in free fall so the last term remains the same. However, this one is thrown DOWN, so we have a minus sign on the velocity instead, giving

$$x_{f,2} = -v_{i,2}\Delta t - \frac{1}{2}g(\Delta t)^2: \text{ the trajectory of the second object.} \quad (2.9)$$

The last one is simply dropped, so there is no initial velocity, giving its distance traveled as

$$x_{f,3} = -\frac{1}{2}g(\Delta t)^2: \text{ the trajectory of the third object.} \quad (2.10)$$

Let us now look at them all together and make sure they make physical sense:

$$\begin{aligned} x_{f,1} &= v_{i,1}\Delta t - \frac{1}{2}g(\Delta t)^2 \\ x_{f,2} &= -v_{i,2}\Delta t - \frac{1}{2}g(\Delta t)^2 \\ x_{f,3} &= -\frac{1}{2}g(\Delta t)^2. \end{aligned} \quad (2.11)$$

Remember that up is defined as positive, so for a very small time the first one will have a positive displacement. This makes sense since it was thrown upwards and will need to stop and come back down. For the other two there is no way for them to ever have a positive displacement (meaning they went up), which again makes sense. One was thrown down and the other was dropped, so there is no physical way for them to go up.

2.4 Motion in two dimensions: relating x and y

The key to most 2D problems sounds simple: time progresses the same in both coordinates and binds them together. This is the **only** way that the x and y coordinates are related. In some ways, while this sounds simple, truly wrapping your mind around it and working from this premise is one of the first major

challenges of learning to think like a physicist that introductory students encounter. One way to keep everything straight is to carefully check conditions along the way. The best way to see this is with a conceptual example.

Example: You and a friend are playing catch on opposite sides of a house, both standing some distance d away from the house of height h and width w . What is the minimum velocity and angle needed to get over the house? Figure 2.1 below can help you to better visualize what is going on in the problem.

Solution:

To solve this problem, the two components of velocity will be required. Remember that everything is related through time, so your first goal needs to be identifying which coordinate (x or y) will give you something useful for time. Since there is no acceleration in the x direction and you do not know the initial velocity, this means the physical parameters from the y -direction will offer more information for problem solving. If it is just barely going to clear the house, then the minimum condition for that is for $v_y = 0$ at a height of h . Now we have something we can work with since we also know the acceleration (which is due to gravity). Set the direction up as positive for y and use the equation that relates velocity, distance and acceleration:

$$v_{f,y}^2 = v_{i,y}^2 + 2a_y\Delta y. \quad (2.12)$$

Final velocity is zero and the acceleration is negative g , so this develops with a little algebra as

$$\begin{aligned} 0 &= v_{i,y}^2 - 2g\Delta y \\ v_{i,y}^2 &= 2g\Delta y \\ v_{i,y} &= \sqrt{2g\Delta y}. \end{aligned} \quad (2.13)$$

The height of the house is Δy , so the final equation is

$$v_{i,y} = \sqrt{2gh}. \quad (2.14)$$

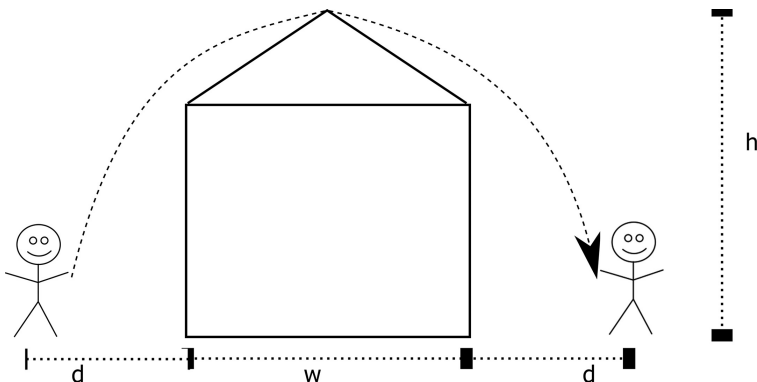


Figure 2.1. The scenario for trying to throw the ball over the house.

So now since we have the velocity in the y direction, we can find both the angle and the total velocity if we can find the velocity in the x direction. This is where relating these two through time becomes important. What we have to do now is find out how long it takes to get over to the other side. It will take the same amount of time to come down as it does to go up (which is always true for anything in free fall), so we need to make use of an expression that now relates initial velocity, acceleration, time, and distance. This is

$$x_f = x_i + v_{i,x}\Delta t + \frac{1}{2}a_x(\Delta t)^2. \quad (2.15)$$

Change symbols to y and set $a = -g$:

$$y_f = y_i + v_{i,y}\Delta t - \frac{1}{2}g(\Delta t)^2. \quad (2.16)$$

Conceptually speaking, what we are doing here is identical to just throwing the ball straight up and finding the time it takes to go up and then back down, so $y = y_i$. It starts and ends at the same vertical (but not horizontal) location. Now we can solve for the change in time, after a little algebra:

$$\begin{aligned} 0 &= v_{i,y}\Delta t - \frac{1}{2}g(\Delta t)^2 \\ v_{i,y}\Delta t &= \frac{1}{2}g(\Delta t)^2 \\ v_{i,y} &= \frac{1}{2}g\Delta t \\ 2\frac{v_{i,y}}{g} &= \Delta t. \end{aligned} \quad (2.17)$$

But we already know what $v_{i,y}$ is ($\sqrt{2gh}$), so the subsequent steps are

$$\begin{aligned} 2\frac{v_{i,y}}{g} &= \Delta t \\ 2\frac{\sqrt{2gh}}{g} &= \Delta t \\ 2\sqrt{2}\sqrt{\frac{h}{g}} &= \Delta t. \end{aligned} \quad (2.18)$$

Now that we have the travel time, all we have to do is figure out the x -direction velocity required to cover the distance in that time. There is no x -direction acceleration (as is typical for projectile motion) and the total distance is $2d + w$, since both people are standing d from the house and the house itself is a width w . So, get an equation out of the toolbox that relates time, velocity and distance:

$$x_f = x_i + v_{i,x}\Delta t + \frac{1}{2}a_x(\Delta t)^2. \quad (2.19)$$

Acceleration is zero since this is the x -direction and we'll set $x_i = 0$ since all we are interested in is how far it went. Making use of this customized equation, we have

$$\begin{aligned}
 x_f &= v_{i,x} \Delta t \\
 2d + w &= v_{i,x} \Delta t \\
 \frac{2d + w}{\Delta t} &= v_{i,x} \\
 \frac{2d + w}{2\sqrt{2} \sqrt{\frac{h}{g}}} &= v_{i,x}.
 \end{aligned} \tag{2.20}$$

The total velocity can be found using $v^2 = v_y^2 + v_x^2$ (which is the Pythagorean theorem applied to the vector components), with the angle given by trigonometry,

$\tan^{-1}\left(\frac{v_y}{v_x}\right)$. This would be altogether $\tan^{-1}\left(\frac{\sqrt{2gh}}{2d+w} \cdot \frac{2\sqrt{2}\sqrt{\frac{h}{g}}}{\sqrt{2gh}}\right)$. Note that this equation may look

unpleasant (or more hopefully, impressive) but that is only because we did not use numbers the entire time. In a problem with actual numbers it may be much less intimidating. But, by following this line of thinking, you have developed an expression that shows you the many factors upon which the necessary initial velocity depends, which is the real utility of physics.

2.5 Summary and important notes

1. Velocity changes displacement; acceleration changes velocity. This is a subtle thing that can trip you up easily. For instance, just because something is accelerating toward a point does not always mean that it is moving toward that point at that instant, just that the acceleration vector is pointed in that direction.
2. You must always pay attention to signs with velocity, acceleration, and displacement. A negative acceleration does not mean the object is slowing down nor does a positive acceleration mean it is speeding up. It is whether or not the acceleration and velocity vectors have the same sign that determines if it is slowing down or not.
3. Set up your positive and negative at the beginning of the problem *and stick to it*. Every time you write down a symbol, ask yourself if the sign matches the rest of the signs you have. Even if you are very good at this stuff, you will still make mistakes from time to time just writing down the wrong sign.
4. Pay attention to what each component of each equation means physically. Do not simply memorize them. If you do this it will become much easier to see which equation to use where. Equations are tools. Do not reach for the toolbox until you know which tool you need. If you try to accomplish a task that needs blunt force, such as one that would be helped by use of a hammer, and reach initially for sandpaper, you'll get nowhere.

5. The same amount of time passes for all coordinates in any problem. Use this to relate two or more coordinates to each other, but otherwise treat the coordinates completely separately as if you were doing two different problems.

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Chapter 3

Newton's laws

Let us look at their formal definitions and distill those down to something easier to digest.

1. An object at rest will remain at rest and a moving object will remain moving unless a force acts upon it.

This law, Newton's first law, is also known as the law of inertia. The simple takeaway is that if there is a net acceleration in some direction, there must also be a net force in that direction. Note carefully the word *net*. There can be all kinds of forces acting on something but it is still not accelerating if they are canceling each other out. This applies to any acceleration, regardless of source. So if an object is experiencing the same acceleration from an electric force as it subsequently experiences from a contact force (push or pull force), the object is experiencing the same quantity of force, regardless of its origin.

2.

$$\vec{F} = m\vec{a}. \quad (3.1)$$

This equation really should be written as

$$\vec{a} = \frac{\vec{F}}{m}, \quad (3.2)$$

which might seem like a trivial point, but it is extremely important: an acceleration is caused by a force and is directly proportional to the amount of force and indirectly proportional to the amount of inertia. $\vec{F} = m\vec{a}$ is easier to speak aloud, but in terms of cause and effect suggests that a force is present because of an acceleration, which is not physically true.

A prose description of Newton's second law can be given as 'a force on an object causes an acceleration that is inversely proportional to the mass of the object'. The

most important thing here is that this all applies to the same object, so it can also be read as ‘the force on an object is equal to the product of *its* mass and acceleration that *it* is experiencing due to the force’. Remembering this is extremely important when solving any problem with connected objects (like blocks and pulleys).

3. With every actionary *force* there exists an equal and opposite reactionary *force*.

The emphasis is there on Newton’s third law because the word ‘force’ is often left out, especially colloquially. This is the law that tells you why it hurts to punch a brick wall. Whenever two objects interact, no matter their size or shape, the force exerted by object A on object B is the same as the force by B on A, though in the opposite direction. So when a bug hits your windshield, the bug exerts the same force on the windshield as the windshield does on the bug. The bug goes splat and the window does not simply because it takes a lot more force to break a windshield than to splat a bug. Something very important about this law, a nuance that can be missed at first, is that it describes an *interaction* between two objects. In the case of gravity, it is an interaction due to the mass of the two objects. This law remains important for any force, including electrostatic forces, which exist due to charges. Do not wall off Newton’s third law to be applicable only to the gravitational interaction.

3.1 Applying the first law

The main use of this for problem-solving is that if there is an acceleration, there must be a *net* force, and vice versa, and in the same direction. So if an object is accelerating in the x direction but not the y , then you know that it only has a **net** force on it in the x direction. It may have forces on it in the y direction but they cancel out. This is used most commonly in static equilibrium problems, where the first thing you do is add up all the forces and set them equal to zero, as no acceleration means no **net** force.

3.2 Applying the second law

The simple thing to remember here is that it all applies to **one** object. If you have an object’s mass and its acceleration, you can find the **net** force on it, always. This is most commonly useful for block and rope and pulley problems. Remember, tension is a force itself so you can use the **net** tension on an object the same way you can use the **net** force.

3.3 Applying the third law

The problem-solving use of the third law is to be aware that the forces between two objects will always be equal and opposite. The key here, however, is to be aware of what is exerting the force on the object. It is a common mistake not to do so. If, for example, two objects are connected by a rope, then object A exerts a force *on the rope*, which then exerts a force on object B. This is a subtle difference that you can often get away with ignoring, but not always.

3.4 Examples

Applying the first and second laws

An object with a mass of 5 kg has an acceleration of 10 m s^{-2} in the $+x$ direction and is acted upon by three forces. Force 1 is 10 N and 30° above the positive x -axis, force 2 is 20 N and 10° below the positive x -axis and force 3 is unknown. Find the magnitude and direction of force 3. Figure 3.1 gives an idea of how to visualize the scenario.

Solution:

The sum of the forces acting on an object in a direction are equal to its mass times its acceleration *in that direction*. We have that the object only accelerates in the x -direction, so we can set up equations for the sum of the forces in each direction with the y acceleration being zero

$$\begin{aligned} F_{1x} + F_{2x} + F_{3x} &= ma_x, \\ F_{1y} + F_{2y} + F_{3y} &= 0. \end{aligned} \quad (3.3)$$

Going back to the method used earlier for directions, we'll just set everything positive for now—or rather, not assign signs yet—and worry about the signs later. This also allows us to not worry about the relative angles of above and below the x -axis and just find numbers for now. The adjacent side of both will be the x dimension, so

$$\begin{aligned} F_{1x} &= 10 \cos(30) \text{ N} = 8.66 \text{ N}, \\ F_{1y} &= 10 \sin(30) \text{ N} = 5 \text{ N}, \\ F_{2x} &= 20 \cos(10) \text{ N} = 19.7 \text{ N}, \\ F_{2y} &= 20 \sin(10) \text{ N} = 3.47 \text{ N}. \\ ma_x &= 5 \text{ kg} \times 10 \text{ m s}^{-2} = 50 \text{ N}. \end{aligned} \quad (3.4)$$

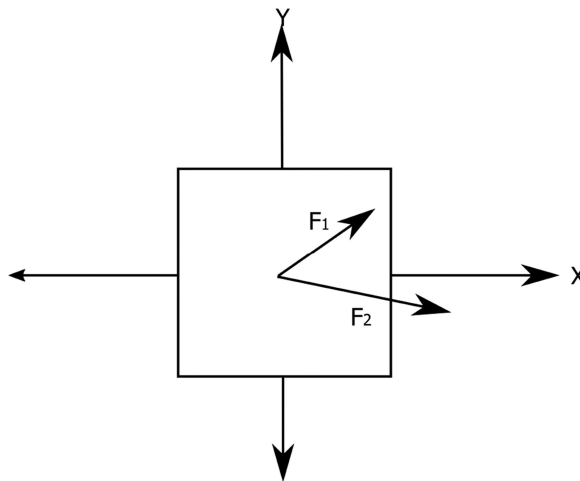


Figure 3.1. An object (usually a box in these kinds of problems!) being pulled by three forces. Two are given by the problem and drawn here.

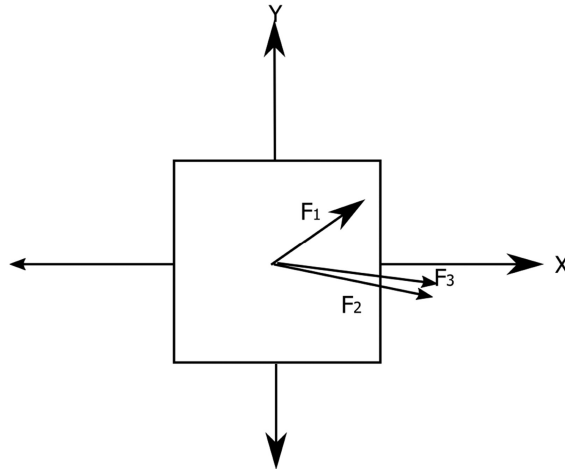


Figure 3.2. The full solution to the problem.

Now put them all together but be mindful of the signs. They both have the same horizontal direction but opposite vertical directions and we'll use standard convention of up being positive, which leads to

$$\begin{aligned}
 8.66 \text{ N} + 19.7 \text{ N} + F_{3x} &= 50 \text{ N}, \\
 F_{3x} &= 21.64 \text{ N}, \\
 5 \text{ N} - 3.47 \text{ N} + F_{3y} &= 0, \\
 F_{3y} &= -1.53 \text{ N}.
 \end{aligned}
 \tag{3.5}$$

Notice how the signs took care of themselves when interpreted from the scenario, after using the equations generically, and we got an expected answer for the vertical force. Figure 3.2 below shows the determined third force.

The forces should sum to zero in the y -direction and so we needed something going a little bit downward to get to zero, which we got. All that is left now is to find the magnitude and direction, which can be done by

$$\begin{aligned}
 |F_3| &= \sqrt{F_{3x}^2 + F_{3y}^2} = \sqrt{(21.64 \text{ N})^2 + (-1.53 \text{ N})^2} = 21.7 \text{ N}, \\
 \text{angle} &= \tan^{-1}\left(\frac{1.53}{21.64}\right) = 2.62^\circ.
 \end{aligned}
 \tag{3.6}$$

This will be below the positive x -axis because the y force is negative.

Applying the third law

Five boxes, all of mass 1 kg, are connected by ropes in a row and are being pulled to the right by a 10 N force. Find the tension in each rope. Figure 3.3 gives an example drawing of the scenario.



Figure 3.3. Five boxes being pulled by connecting ropes.

Solution:

The first thing we have to do is find the acceleration of the system. The 10 N force is acting on all five blocks, so the acceleration of the system is

$$\begin{aligned} F &= ma, \\ 10 \text{ N} &= (5 \text{ kg})a, \\ a &= 2 \text{ m s}^{-2}. \end{aligned} \quad (3.7)$$

Now to find the tension in each rope. Remember that the *net* force on any object is equal to *its* mass times *its* acceleration. Since we have the mass of each block and the acceleration of the system (which is the acceleration of each block as well), we know the net force is 2 N for all of them. Now we can go about finding the tension in each rope. Start from the leftmost block and work forward. The force on this one is 2 N and the only thing that can be exerting that force is the rope attached to it, so the tension in that rope must also be 2 N. Keep working forward and we see we have two tensions acting on the next block, the 2 N we just found and the one to the right of it. The sum of these must be 2 N because it is the net force on it and they are in opposite directions, or

$$\begin{aligned} T_2 - 2 \text{ N} &= 2 \text{ N}, \\ T_2 &= 4 \text{ N}. \end{aligned} \quad (3.8)$$

Do the same thing for the next block, the tension acting to the left is now 4 N from the previous block and the net force on it is 2 N, so the tension on the right side of that one is 6 N. Again for the last two blocks and we get 8 N and 10 N, respectively. 10 N is what we already knew the force had to be, so this gives confidence in the result.

Applying all three laws together

Let's look at Atwood's machine, a classic physics textbook problem which investigates the physics of a pulley with a block on either side. Problems involving Atwood machines may not seem exciting to students, but how you work on them is a good indicator of your understanding of the first month or so of an introductory physics course. Let's consider Atwood machines with a very general goal in mind: how do you find the acceleration of the system? figure 3.4 gives an example drawing of this machine, which is ubiquitous in introductory physics texts.

Solution:

Working an Atwood's machine pulley means taking on this line of thought: consider a pulley with a rope extended over it, with a mass on each end of the rope. We'll just

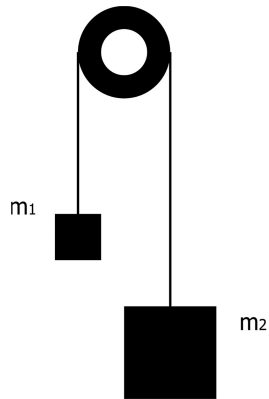


Figure 3.4. Atwood's machine.

call the left mass m_1 and the right mass m_2 , with the tension in the rope being T . We do not know the acceleration or even the direction of the acceleration, but we can just pick a direction to be positive. And it cannot be emphasized enough: each object has the same quantitative value for acceleration. We are going to say that the left block going up is a positive acceleration, which means the right block going down is also a positive acceleration. (One way to help this make a bit more sense is to consider how the system would move if you removed the pulley and laid the blocks and string out in a line on a table. When you pull on one block, the other moves in the same direction. In this case, you would call those both x or both y . When you curve the rope around the pulley, now a motion in the positive y direction for block 1 is tied, literally, to a motion in the negative y direction for block 2.) The tension in the rope will be the same for both blocks as will the acceleration since they are connected. So now we can write out equations for the sum of the forces on each block. Both blocks have *tension* and gravity acting on them, *not the weight of the other block*. As usual, write out the equations first with all positive signs and then put in the signs later:

$$\begin{aligned} T + m_1g &= m_1a, \\ T + m_2g &= m_2a. \end{aligned} \quad (3.9)$$

For both blocks, the tension is pulling up and gravity is pulling down, so these will need to have opposite signs. The question then is what is the sign of acceleration? Earlier we said we defined the left block (m_1) going up to be positive, which means the right block (m_2) going down is also positive. The tension is pointing up in both cases and the acceleration vector for gravity is pointing down, which gives

$$\begin{aligned} T - m_1g &= m_1a, \\ -T + m_2g &= m_2a. \end{aligned} \quad (3.10)$$

Now the problem needs some algebra, to solve for the acceleration. The easiest way to solve this is to add the two equations, which will cancel out T because it is

assumed to be the same on each side (so, note that you never have to identify the tension explicitly):

$$\begin{aligned} T - m_1g - T + m_2g &= m_1a + m_2a, \\ (m_2 - m_1)g &= (m_1 + m_2)a, \\ \frac{(m_2 - m_1)}{(m_1 + m_2)}g &= a. \end{aligned} \tag{3.11}$$

Let us examine this conceptually to see if it makes sense. If m_2 is bigger than m_1 , then the right block would be going down while the left is going up. This gives a positive acceleration, which is how we defined a positive acceleration in the first place. If the opposite is true, it would give a negative acceleration which also works since going the opposite way (right block up, left block down) would be the negative of how we originally defined it.

3.5 Summary and important notes

1. $F = ma$ at face value is not a functional expression. The physics of the relationship is that an acceleration is directly proportional to a force and inversely proportional to the inertia, and is caused by the force. Thus, it is better written $\vec{a} = \frac{\vec{F}}{m}$ when first considering its application to a problem.
2. Anytime you write $F = ma$, you must make sure that F , m , and a all refer to the same object.
3. Additionally with the previous point, everything must refer to the same object *and the same direction*. Be sure you have split up the force into x , y , and z components before you start writing down equations. This will not always be necessary if the object is only moving in one dimension but you should always be mindful of it.
4. Be sure you always write out force equations with the object that is actually exerting the force directly. Even though we almost always use the massless rope approximation, it is still the rope that is exerting the force between two object connected by said rope. Thus, if you have two blocks connected by a rope, the force on block 1 is the tension in the rope and the force on block 2 is also the tension in the rope. As with many things, this often simplifies down to the two blocks acting as a Newton's third law pair, but this is not always the case.

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Chapter 4

Energy and momentum

4.1 Energy

There are three primary equations that you use for describing systems in terms of energy and energy conservation:

$$\text{Kinetic energy : } \frac{1}{2}mv^2$$

$$\text{Gravitational potential energy : } mgh$$

$$\text{Spring potential energy : } \frac{1}{2}kx^2.$$

These equations all tell you something about *how the object possesses the energy*. It is common to think of only potential energy as being stored but it works the same way for kinetic energy. In simple terms, kinetic energy is the energy an object has because it is moving. Gravitational potential is the energy it has because it is in a gravitational field. Spring potential energy is the energy a spring (or spring-like object) has because it is compressed or stretched. In all these cases work was done in some way to get it to that state, requiring a transfer of energy from one source to another.

Now that this is established, we need to fully define conservation of energy. It is simply this: the amount of energy present in a system in an initial condition is equal to the amount of energy present in a system in its final condition. The key is in making sure you clearly define the initial and final conditions, what the system is, and what is relevant within them. **In any problem, it is prudent to identify and consider initial and final energies of all players.** There are shortcuts that you can get away with from time to time that may seem to work, but if you try (intentionally or unintentionally) to use them for others, they will not always work. What will always

work is considering initial and final states and properly considering the system. The best way to see this is with an example.

4.2 Example

An object of mass m is falling with some initial velocity v_1 and is at a height of h . If it falls onto an uncompressed spring of spring constant k ; how far does the spring compress?

Solution:

A common misstep to solving this problem is to try to identify velocities at different points of the motion and try to add them up. This would be wrong. Here's how such an approach might proceed: It is tempting to set this up as simply converting the potential energy into kinetic energy, getting velocity from that and then adding it to the initial velocity. This is wrong because velocity is squared in the energy equation and energy is what is conserved, not velocity. So if we do it the wrong way we come up with a final velocity (when it hits the spring) of

$$\begin{aligned}\frac{1}{2}mv_2^2 &= mgh \\ v_2 &= \sqrt{2gh} \\ v = \text{final velocity} &= v_1 + v_2 = v_1 + \sqrt{2gh} \quad (\text{this is wrong!}).\end{aligned}\tag{4.1}$$

Now, the right way to think about velocity. What we really need to do though is look at energies. So instead, consider initial and final energies at each point. Set initial as the top and final as the bottom (when it hits the spring again)

$$\begin{aligned}\frac{1}{2}mv_1^2 + mgh &= \frac{1}{2}mv^2 + 0 \\ v = \text{final velocity} &= \sqrt{v_1^2 + 2gh} \quad (\text{this is correct!}).\end{aligned}\tag{4.2}$$

This is a pretty significant difference. Note though that if the potential is much larger than the kinetic (or vice versa), you would get basically the same result. This is why it is so easy to do it wrong and think you are doing it right. Also, if you do not start with any initial velocity and thus no initial kinetic energy, you get the same result without being careful about using the initial and final states. Since it is very common for a problem to have either an object starting or ending at rest, you often do not have to worry about it, so be sure to set it up every time with initial and final energies so that you don't get into bad habits.

Now for the entire solution. All we need to worry about is initial and final energies here, so we can skip the middle step of converting potential to kinetic energy. The energy equation is then

$$\begin{aligned}
 K_i + U_i &= K_f + U_f \\
 \frac{1}{2}mv_1^2 + mgh &= \frac{1}{2}kx^2 \\
 x &= \sqrt{\frac{mv_1^2 + 2mgh}{k}}.
 \end{aligned}
 \tag{4.3}$$

There is no final kinetic energy because the velocity is zero when it reaches maximum compression. You might think at first we have to worry about compression of the spring for the gravitational potential energy and you'd be correct, but we can set our zero point of gravitational potential energy at the most compressed point of the spring and be done with it.

4.3 Momentum

The concept behind doing momentum problems correctly is very similar to energy (hence why they are often covered in close succession in a physics course): make sure you identify and use the initial and final states of everything. Momentum problems, like energy problems, often let you skip this if enough things are zero but that's not a good habit to have. More importantly, since they are vectors (unlike energy), you have to be careful to check the signs, which gives double motivation to give full consideration of initial and final conditions. Let's look at a classic example (and popular test question) that shows how this works:

Example: You are standing still on frictionless ice and your friend throws you a ball. To minimize your speed, should you catch the ball or deflect it back at him?

Solution:

Even though this is mostly conceptual, we can still figure it out with math. To do this, start with the equation you should *always* write first:

$$p_{1i} + p_{2i} = p_{1f} + p_{2f}. \tag{4.4}$$

Note that if you have more than two objects, you would have more than two momenta; this equation can extend out to infinity if needed, depending upon how you define the system. We'll call you p_1 and the ball p_2 . We know that you are standing still, so $p_{1i} = 0$. What we want to do is make p_{1f} as small as possible (because momentum is the product of mass and velocity, so minimizing momentum is minimizing velocity), so let's solve for it:

$$\begin{aligned}
 p_{2i} &= p_{1f} + p_{2f}, \\
 p_{2i} - p_{2f} &= p_{1f}.
 \end{aligned}
 \tag{4.5}$$

At first glance this may not seem to give anything. However, consider the two options: one is to catch it and one is to deflect it. Deflecting it would mean that p_{2i} and p_{2f} have opposite signs, so since they're subtracted from each other it would give a bigger number for p_{1f} . Catching it would mean they have the same sign and thus a smaller number. So the somewhat counterintuitive answer is that if you want to

minimize your speed, you would catch the ball, not bounce it back toward your friend.

4.4 Types of collisions

There are a few keywords you must be aware of in order to properly do any conservation problem. Formally speaking, energy is always conserved but it may be converted into forms you are not accounting for in the problem, such as heat. The exact form of the keywords may change based on your course; however the meaning remains the same. Important note: **momentum will always be conserved**. Here's what the keywords mean:

1. **Perfectly elastic: mechanical energy (kinetic and potential) is conserved.** What this means is you can assume there's no loss of energy to heat or anything else and just set the sum of initial kinetic and potential energy equal to the sum of the final kinetic and potential energy. Physically, this usually results in the objects bouncing apart.
2. **Elastic: mechanical energy is not conserved, and the objects bounce apart.** In this scenario, the objects don't stick together when they collide, but some energy may be transferred to another form, such as friction, when the collision occurs. You cannot assume mechanical energy is conserved.
3. **Inelastic: mechanical energy is not conserved, and the objects stick together.** This is an important one because this means that the velocities must be the same after the collision and the final mass is the sum of the original mass (they stuck together). Some textbooks or instructors call this perfectly inelastic.

You should check with your instructor as to what means what because they are not always the same depending on the textbook. This is the most common way to define them but it is not universal.

4.5 Summary and important notes

1. Until you develop a really solid intuition for conservation problems (which probably won't happen until after intro physics), always start by formally writing out initial energy = final energy or initial momentum = final momentum. Start with the concept and then fill in the players as appropriate.
 2. Remember that momentum is conserved for the system, not individual objects. An individual object can lose all of its momentum, energy, or both just as long as it goes somewhere else.
 3. Momentum is conserved as a vector, while energy is not a vector at all and it is simply the scalar value that is conserved. This means that while you can always just set initial energy equal to final energy; you have to make sure you set initial x momentum equal to final x momentum (and similarly for y and z).
- 2). Momentum must be conserved in each direction.

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Chapter 5

Circular and rotational motion

There are three main types of circular motion problems you will encounter: torque, centripetal force, and kinematic problems except going in a circle instead of straight. The best place to begin is with a solid definition of the equations as they relate to physical reality.

5.1 Centripetal force

If you take nothing else from this chapter, remember this important thing: centripetal force is the force **required** to make an object move in a circle. The common mistake is to think about it in terms of a counterbalancing force since we often set so many things equal to it to solve. This is not true. Centripetal force is created by something else and makes an object go in a circle. More formally, with the equation

$$F_c = \frac{mv^2}{r},$$

where F_c is the centripetal force, m is the mass, v is the linear velocity, and r is the radius of the circle, then this is better described as **centripetal force is the force required to make an object of mass m move with velocity v around a circle of radius r** . If whatever is creating the force is less than this number, then it will go in a straight line, not a circle. Let's do a few examples to see what I mean here

Example: A 5 kg ball is attached to a massless rope 1 m long and swung in a horizontal circle above your head with a velocity of 2 m s^{-1} . (a) What is the tension in the rope? (b) If you keep the tension the same and shorten the rope to 0.5 m, what is the velocity?

Solution:

- (a) The rope here is providing the centripetal force, as the tension in the rope is the only force acting on the ball. This problem then becomes very straightforward (we might even say, plug and chug) but take careful note of why: the tension is equal to the centripetal force not because it is counterbalancing it but because *it is* the centripetal force. So then we have

$$T = \frac{mv^2}{r} = 20 \text{ N.} \quad (5.1)$$

- (b) If we keep the tension the same, we're still providing the same centripetal force. This will in turn change either the velocity, the radius, or both. Since we can make the radius remain at a set point, it will only change the velocity and again it becomes a simple problem:

$$T = \frac{mv^2}{r}, \quad (5.2)$$

$$v = \sqrt{\frac{T}{m}r} = 1.41 \text{ m s}^{-1}.$$

At this point, you should take a step back and ask why this is a smaller number. The answer goes back to Newton's laws: it takes a force to make an object change direction and so since we're taking a sharper turn here (smaller radius of the circle), it takes more force to do that. Thus, the velocity will have to be lower if we cannot change the tension.

For a physical manifestation of this you are familiar with, think about driving your car in a circle. If the circle is very large, you can go around it fairly fast. If it is a tight curve, you have to go much more slowly. This is because there is a maximum amount of centripetal force that the friction between your tires and the road can provide. Go above that and you slide, which is not a good thing for a rather large hunk of metal. So when you decrease the radius of your circle, you must also decrease the velocity if the force provided is the same.

Conceptual example: If you fill a bucket with water and swing it in a vertical circle at a high enough speed, the water will not fall out. Why is this?

Solution:

This is where knowing the proper definition of centripetal force is key. Your intuition may tell you that centripetal force is holding the water in against the force of gravity, which is not entirely false but not really true either. What's going on again comes back to inertia: objects continue moving unless a force acts upon them. Gravity doesn't change, it is always acting straight down on the water. What you're seeing though is that as you are swinging the bucket and begin to swing it upward, you are now putting an upward force on it. When you reach the top of the circle there is no longer an upward force on it but it's still trying to move in that direction because it was already moving in that direction. The bottom of the bucket prevents

it from moving any farther up and the sides of the bucket exert a force to keep it moving in that circle.

Therefore, what you are actually seeing is not centripetal force working against gravity, but rather gravity **and** the force exerted by the bucket providing a centripetal force that keeps it going in a circle. The reason it doesn't fall out of the bucket when the bucket is upside down is just because the water was already moving with an upward velocity.

One more example to solidify these concepts, since they are very important.

Example: A rollercoaster of mass m is going around a circle of radius r and velocity v . What is the force on it at the top, bottom, and sides of the circle?

Solution:

As always, remember that centripetal force must be created by something. In this case, it is created by both the normal force and gravitational force. Whether or not these are in the same direction depends on where it is on the circle. At the top they are in the same direction and at the bottom they are in opposite directions. This is because the normal force always points toward the center and gravity always points down. The force we are actually looking for in this case is the normal force, since that will be the force that the rollercoaster 'feels'. In equation form, this looks like

$$\frac{mv^2}{r} = F_n + mg. \quad (5.3)$$

As usual, write it out with all positive signs and then determine if some of those need to be negative. In this case, at the top they are both pointed in the same direction (down) and at the bottom the normal force is pointed up while gravity is down. Since centripetal force always points toward the center of the circle, it should have the same sign as normal force in both cases,

$$\begin{aligned} \text{top: } \frac{mv^2}{r} &= F_n + mg, \\ \text{bottom: } \frac{mv^2}{r} &= F_n - mg. \end{aligned} \quad (5.4)$$

So now to find the force on the rollercoaster, which is normal force. Remember that what you feel as weight is actually normal force, hence why we're looking for it here. Solving

$$\begin{aligned} \text{top: } \frac{mv^2}{r} - mg &= F_n, \\ \text{bottom: } \frac{mv^2}{r} + mg &= F_n. \end{aligned} \quad (5.5)$$

As usual, you should check these against intuition. When you're at the top of the ride, you feel like you weigh less and at the bottom you feel like you weigh more. This follows directly from the equations.

A special case of this is the question of ‘what is the minimum velocity needed to make it around the loop?’ What this means in terms of equations is that the normal force is exactly zero because it’s just barely maintaining contact with the rails. From the equation for the top above then, we get

$$\begin{aligned}\frac{mv^2}{r} - mg &= 0, \\ \frac{mv^2}{r} &= mg.\end{aligned}\tag{5.6}$$

Here it is tempting to say again that centripetal force is counterbalancing gravity, which is not correct. Gravity is still providing the centripetal force but if it’s more than $\frac{mv^2}{r}$, then it will be providing *more* than what is needed. The smaller the circle, the greater the force required and so the reason it falls off if the velocity is lower than this is because gravity is providing enough force for a smaller circle. If you want to see this for yourself, just take something on a rope and swing it above your head. Get it going to a constant speed where it stays in a circle, then pull harder on the rope. You will then make it go in a smaller circle because you exerted more force on it.

5.2 Torque

The equation for torque is usually given as

$$\boldsymbol{\tau} = I\boldsymbol{\alpha} = \mathbf{r} \times \mathbf{F},\tag{5.7}$$

where τ is a force (described more below), I is the moment of inertia, α is the angular acceleration (the change in angular velocity with respect to time), r is the moment arm (the distance between the pivot point and applied force F).

To make sense of torque it’s best to think of it as angular force—a force that causes a rotation. Where a force causes an object to move (acceleration), a torque causes it to rotate (angular acceleration). With that in mind, we can rewrite Newton’s three laws in terms of rotational motion instead:

1. An object spinning will continue spinning unless a torque acts upon it.
2. $\tau = I\alpha$, or better written as $\alpha = \frac{\tau}{I}$. A torque causes an angular acceleration that is inversely proportional to the moment of inertia.
3. For every action torque, there is an equal and opposite reaction torque (which also gives conservation of angular momentum).

Therefore, **every problem-solving technique you learned from forces applies to torques as well**. The vector sum of the torques (taking into account both magnitude and direction) will always be $I\alpha$. If it has no angular acceleration, it has no **net** torque.

Dealing with torques by themselves usually isn’t too hard, especially if you have learned the principles of linear motion very well and look for parallels between how we describe linear and rotational motion and forces. What gets tough is combining forces and multiple torques, especially in an equilibrium problem. The key, as is almost always true with vectors, is to take it one step at a time identifying torques,

forces, and directions and writing them down in a set of equations. Again, as before, most textbooks will have a set sign convention that you can memorize for positive and negative, but all that really matters is that you keep your signs consistent. So if you define a positive torque to be one that causes a clockwise angular acceleration, then you must be sure that the force associated with that torque is also positive. We will now discuss two problems to show this, one involving a pulley and one static equilibrium problem.

Example: Reconsider Atwood's machine, just a pulley with a block on either side. Again call the left mass m_1 and the right mass m_2 , but now we'll also have the mass of the pulley which is M and its radius of R . Find the acceleration of the system.

Solution:

As before, set a direction to be positive and stick to it. We're going to choose the right block going down is a positive acceleration, which means the left block going up will also be positive *and* the pulley turning clockwise will be a positive angular acceleration. The equations end up being a little different this time though because the pulley is no longer frictionless, so the tensions may not be the same on both sides. However, Newton's second law still applies and the sum of the forces is still ma , such that

$$\begin{aligned} T_1 + m_1g &= m_1a, \\ T_2 + m_2g &= m_2a. \end{aligned} \quad (5.8)$$

For both blocks, the tension is pulling up and force due to gravity is pulling down, so these will need to have opposite signs. The question then is what is the sign of acceleration? Earlier we just said we defined the left block (m_1) going up to be positive, which means the right block (m_2) going down is also positive. The tension is pointing up in both cases and gravity is pointing down, which gives

$$\begin{aligned} T_1 - m_1g &= m_1a, \\ -T_2 + m_2g &= m_2a. \end{aligned} \quad (5.9)$$

However, we have to consider a third equation now that identifies the role of angular acceleration. Just as the sum of the forces is equal to ma , the sum of the torques is equal to $I\alpha$. There are two torques created by the tensions in the ropes (remember to always determine exactly what is creating the force) and they are at right angles, so the cross product ($\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$) reduces down to $\boldsymbol{\tau} = R\mathbf{F}$. The pivot point is right in the middle of the pulley and the force is exerted right on the edge, so the vector \mathbf{R} goes straight out from the middle to the edge and the vector \mathbf{F} points straight up or down, which is why they are at right angles. Writing out the torque equation now gives

$$T_1R + T_2R = I\alpha \quad (5.10)$$

but you should recognize here that they need to have opposing signs, since one will make it turn clockwise and the other counterclockwise. We've defined clockwise as positive already, which means that T_1 will need a negative sign, so

$$-T_1R + T_2R = I\alpha. \quad (5.11)$$

Now apply the condition of rolling without slipping ($\frac{a}{r} = \alpha$) to relate linear acceleration to angular and all that's left is to solve it

$$\begin{aligned} T_1 - m_1g &= m_1a, \\ -T_2 + m_2g &= m_2a, \\ -T_1R + T_2R &= I\alpha = \frac{I}{R}a. \end{aligned} \tag{5.12}$$

These may look complicated but can be solved with a little algebra. Whenever you're faced with a set of equations, look for a way to cancel variables out as it's usually the easiest way to do it. In this case, if we divide the third equation by R we can add all three together and cancel out both tensions. The division in the third equation gives

$$-T_1 + T_2 = \frac{I}{R^2}a. \tag{5.13}$$

Add this and the other two and get

$$\begin{aligned} T_1 - m_1g - T_2 + m_2g - T_1 + T_2 &= m_1a + m_2a + \frac{I}{R^2}a, \\ m_2g - m_1g &= m_1a + m_2a + \frac{I}{R^2}a, \\ (m_2 - m_1)g &= \left(m_1 + m_2 + \frac{I}{R^2}\right)a, \\ \frac{m_2 - m_1}{m_1 + m_2 + \frac{I}{R^2}}g &= a. \end{aligned} \tag{5.14}$$

For completeness, substitute the moment of inertia of a solid disk of radius R ($I = \frac{1}{2}MR^2$) and this becomes

$$\frac{m_2 - m_1}{m_1 + m_2 + \frac{1}{2}M}g = a. \tag{5.15}$$

You can check this against our previous answer to be confident. If the mass of the pulley is zero, it reduces down to what we had before, which is expected. Also, having a pulley with mass does not change anything about which direction it rotates, just how fast it does. Notice that all that happens as M gets bigger is acceleration goes down. You wouldn't expect that increasing the pulley mass would make it change direction.

5.2.1 Moment of inertia

The concept of moment of inertia is often a confusing one, which is why it has its own small section here. The best way to think of it is simply as *angular mass*. That is

to say, if the mass of an object tells you how hard it is to get it to move, moment of inertia tells you how hard it is to get it to rotate. The basic equation for moment of inertia is always the same, which is

$$I = \sum_i m_i R_i^2. \quad (5.16)$$

The moment of inertia I is mathematically the sum of the product of each individual point mass (m) and the square of its distance from the rotation point (R). This never changes, the only thing that does change is how we figure out what it is for something that's not a point mass. In terms of what you will need to do for problems, this means your total moment of inertia is always just the sum of individual moments of inertia. For example, the moment of inertia of a disk of mass M , radius R with two little globs of clay stuck to it of mass m_1 and m_2 at radii of r_1 and r_2 , respectively, is

$$I = \frac{1}{2}MR^2 + m_1r_1^2 + m_2r_2^2, \quad (5.17)$$

and that is all. Moment of inertia is a place where many students expect it to be much harder than it actually is. For completeness it is useful to know that the integral is used to determine expressions for the moments of inertia of materials made of up continuous distributions of mass (such as a disk), but the expressions for these are usually available in your course from a table in your textbook. There is no need to try to derive them on your own, although it can be fun to do so and this may be given to you as an exercise in a calculus course.

5.3 Waves

Wave mechanics are critical to virtually all of physics, but when you run into them at the introductory level, it can be difficult for students to visualize the phenomena and to properly sort out when to use which variable and what these describe.

It is not uncommon for students to lose track of what is moving where (or is not) when thinking about waves. Be sure that you are aware of the different 'stages', so to speak, of the phenomena as a whole. Waves start with disturbances. The disturbance then propagates through the material (such as air molecules, for sound) or the electromagnetic field (as in light). Sometimes the wave reflects back and adds up on itself, in the case of standing waves. There will likely be in your view a host of new terms and symbols that are easily confused as their meanings are close.

Let's first start with defining things in a useful, simple way. Those that correspond to symbols in equations will have the symbol next to the name in parentheses. Note that while the meaning is always the same, the symbols can change depending on the source material. For example, we usually use f for frequency in an introductory course but it is very common to see the ν (Greek letter nu) used in other materials.

5.4 Oscillations

Oscillation is the repetition of some kind of physical behavior due to the presence of some kind of restoring force. Repetition occurs in terms of both space and time.

A useful basic example of oscillation is a single swing of something, like a pendulum, in periodic motion. The important and easily-missed note here is that it is a swing out *and back again*. Physically, this would mean something like pulling a playground swing back, letting it go, then catching it again when it returns: this would be one full oscillation. Normally you hear this as ‘crest to crest or trough to trough’ on a graph but it can be any point as long as it is the same point in the oscillation. We normally use crest to crest because it is easier to see it on a plotted curve. To work with them we first need to define terms.

5.4.1 Wavelength and period (λ , T , or τ)

Wavelength is literally the length of the wave in meters (or any other length unit). This is measured over one full oscillation, which again can be measured at any point as long as it is the same point in the oscillation. It is rare that you want to use anything but crest to crest or trough to trough, but you can and it may make your math easier. Sometimes it is easier to visualize the point at which the motion crosses the equilibrium, for example. When that is the case, be careful that you do not accidentally report a half-wavelength, which would be the subsequent point at which the object crosses the equilibrium. You must look for the point at which the object crosses the equilibrium from the same direction as it did at the initial crossing.

Period then is the time it takes to complete one full oscillation. The reason these are together in this definition section is because they are different ways of measuring essentially the same thing. Wavelength is how long a wave is *in space*, period is how long a wave is *in time*. This way of thinking about wavelength and period can be very helpful for interpreting graphs.

5.4.2 Frequency (f or ν)

The everyday use of this word does not change when we apply it to physics. It still means the rate at which something occurs. That ‘something’ is now one full oscillation and thus frequency refers to how many oscillations occur per second, which is the units of Hertz (Hz). This is why both your computer’s processor and a child’s swingset can be measured in Hertz as it gives number of things that are happening per second. For physics, Hz will almost always refer to oscillation or cycles per second.

Frequency is then the inverse of period. Period can be thought of as seconds per cycle while frequency is cycles per second.

5.4.3 Angular frequency (ω)

Angular frequency, as the name implies, measures the rate of rotation. Rotations are one of the simplest forms of oscillations; imagine a single point on a rotating wheel rather than the entire wheel itself to see this. Mathematically it is $2\pi f$, so since a full circle is 2π radians, its main function (for basic oscillations at least) is to convert cycles/second into radians/second. Notice that this is the same units for angular velocity and why it uses the same symbol. Much of the formalism remains the same.

The reason behind this conversion (and the wave number k below) is to give us a consistent way to describe any oscillation in terms of sines and/or cosines. So you can learn how a pendulum works and apply exactly the same techniques to a spring, or a wheel, or anything else. More importantly, writing everything in terms of angles (radians) allows us to use sines and cosines and how to describe what the system is doing without needing to directly measure anything about the system itself. For example, saying a pendulum is at its equilibrium position does not tell you if it is going forwards or backwards (forwards being defined as the way it goes right after it is let go), but saying that it is at $\frac{\pi}{2}$ tells you it is at the equilibrium position going forwards.

5.4.4 Wave number (k)

Wave number is to wavelength what angular frequency is to frequency. Frequency gives the temporal (time-based) length of an oscillation and wavelength gives the spatial (space-based) length of the wave. Wave number then converts this to angular form just as angular frequency did and gives radians per unit distance. Note that some textbooks, especially those intended for students not majoring in physics, stay completely away from ever naming the wave number.

5.4.5 Phase angle (ϕ)

In science, the terms ‘in phase’ and ‘out of phase’ are used to describe relative positions of objects in oscillation (whether that be their repetition in space or their repetition in time). Something is ‘phase-shifted’ when you have two identical waves but they are not lined up directly on top of one another. Imagine running up and down a set of stadium stairs with your friend, but you wait until your friend is at the top before you begin. You would then be phase-shifted by a factor of π , as when you are at the top, they are at the bottom and vice-versa. Further, notice that if you were to average your positions at any time, you would always come out right in the middle, usually defined as zero in a graph.

5.4.6 Quick reference guide for descriptions of oscillations

This is how they will appear mathematically. Note carefully that there can be more than one symbol though in an introductory course the first one is almost always used

Symbol	Name	Meaning
f or ν	frequency	cycles per second
T or τ	period	seconds per cycle, length of a wave <i>in time</i>
λ	wavelength	length of a wave <i>in space (meters)</i>
ω	angular frequency	radians per second
k	wave number	radians per wavelength
ϕ	phase angle	offset from some standard of measure

5.5 Traveling waves

A traveling wave is exactly what it sounds like: a wave that travels¹. At the same time, although it sounds like an easy thing to grasp, there is a very essential note: in a traveling wave, the material itself is not traveling. It is the *disturbance* that is traveling through the material. These concepts can be tricky to deal with due to them changing so much in both time and space, but there is a simple strategy that can make a big difference: **check your axes**. Most of the time you will be dealing with them at least partially with a graph, and learning to interpret those graphs is essential.

5.6 Standing waves and boundary conditions

By contrast, standing waves are composed of multiple traveling waves interfering with one another. The vast majority of these will be a traveling wave reflected off a boundary and thus interfering with itself. Standing waves are used to describe things from a violin string to the hydrogen atom and beyond. This is a very important take-home point: standing waves both occur in the everyday phenomena that we are more familiar with (such as what happens when you pluck a string on an instrument) and they are used as models to explain phenomena that are beyond our direct observation with our eyes (such as atomic physics). Doing a good job learning standing waves on strings will help you understand quantum physics. What causes a standing wave to occur is the principles of **boundary conditions**. You are probably already familiar with initial conditions such as in a calculus problem. These are a subset of boundary conditions and the way we like to talk about them is that they ‘keep us grounded in reality’. This means that no matter what funky thing you have done with the math, it has to properly match the boundary conditions.

The boundary condition that we are most interested in then is the one that creates standing waves, and that is the property that when a wave hits a wall or wall-like thing, its amplitude must be zero². A standing wave is a wave effectively interfering with itself because it bounces off a surface and so **when the wave would be zero when it hits something that would force it to be zero anyway, you get a standing wave**.

¹ This is a big part of what makes waves a challenging section, many things are exactly what they sound like they should be but a few have names that tell you nothing at all (looking at you, wave number!).

² This is one of my (J Whitney’s) favorite properties in all of physics. Nothing esoteric or grandiose, it just has to be zero because it is a wall.

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Chapter 6

Basic optics

6.1 Diffraction and interference

Diffraction is governed by two primary equations, which are

$$\begin{aligned}d \sin \theta &= m\lambda \\ a \sin \theta &= m\lambda\end{aligned}\tag{6.1}$$

and most problems can be solved if you just know what all these symbols mean. To do that, we need a geometric diagram though, as shown in figure 6.1.

A very important note about any of these diagrams is that the distance d is almost always greatly exaggerated. It will usually be a thousand to a million or more times smaller than L . Now let's very carefully define what these equations physically mean

d : slit spacing,

a : slit width,

L : distance from the *slits* to the screen,

θ : diffraction angle, $\arctan\left(\frac{y}{L}\right)$,

m : order, (counting variable),

λ : wavelength of the light.

This is just a summary, though for studying purposes, here's what they all mean:

- d:** Distance between the slits for a double- or multiple-slit diffraction grating. Does not apply to single-slit.
- a:** Width of the slit itself. This really only matters for single-slit diffraction. It does technically exist for double slits too but it doesn't matter.

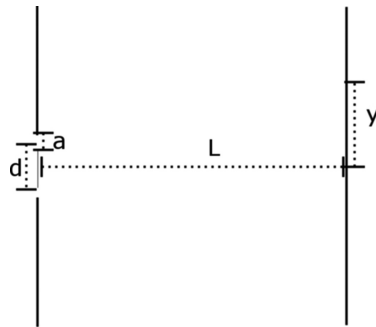


Figure 6.1. The important parameters in diffraction problems.

- L:** This is the distance from whatever is making the light diffract to where you see the interference pattern (usually a screen). Note carefully that it does not matter how far the light traveled before it hit the slits, only how far it traveled afterward.
- θ :** The angular distance between the central maximum and the point you're interested in. For example, θ for $m = 2$ for a double slit would give you the angular distance between the central maximum and the second bright fringe.
- m:** This is usually way more confusing than it needs to be, this is just a counting variable. We have more than one maximum and minimum and so we need a way to keep track of them and that's what m tells you. The one important note here is that it's on both sides, so you will have two $m = 1$ maxima, two $m = 2$, and so on, one on each side of the central maximum.
- λ :** Wavelength of the light. In order to see diffraction effects, the slit spacing or slit width needs to be in the same ballpark as the wavelength.

The equations themselves also need a little bit more explanation. In as few words as possible, here's the difference between the two:

$$d \sin \theta = m\lambda, \quad (6.2)$$

is for double-slit interference. m starts counting at zero and refers to *maxima* (*bright spots*).

$$a \sin \theta = m\lambda \quad (6.3)$$

is for single-slit interference. m starts counting at ONE and refers to *minima* (*dark spots*).

Both of these are the result of *path length difference*. The light striking the slits is just one beam and the grating causes it to split up and interfere with itself. The left-hand side of these two equations ($d \sin \theta$ or $a \sin \theta$) is that path length difference. The geometry of the single slits and double slits causes the equations to give maxima or minima but the principle is the same.

6.2 Thin lenses

Thin lenses follow one very simple equation, which is

$$\frac{1}{f} = \frac{1}{i} + \frac{1}{o} \quad (6.4)$$

where f is the focal length of the lens, i is the image distance of an image created by that lens of the object, which is at o . This may be written many different ways depending upon your textbook (e.g., you may see the letter p and q here) but this is generally the easiest to remember¹. Just be careful that you do not interpret the letter o to be zero. The hard part here is figuring out all of the positive and negative signs for the elements of the focal length and image and object positions. There are two rules that you can remember though to take care of all the signs.

1. The positions of all elements are measured relative to the lens in question.
2. Set up a rule for what is positive and negative and stick to it.

There are many conventions that you can memorize, but the easiest one is this: take as your default system the arrangement for a converging lens, where the object is on the left and the image is on the right. **When you work with any other arrangement, if the location of element changes, whatever changed gets a negative sign.** So let us look at some cases to be sure this is clear:

Default : object on left, image on right, converging lens : all values for the elements are positive.

Case 1 : object on left, image on left, converging lens: the position of the image is now negative.

Case 2 : object on right, image on right, converging lens: the position of the object is now negative.

Case 3 : object on left, image on right, diverging lens: the value for the focal length is now negative.

All of these also apply to multiple lens systems. Just as with vectors you have to pick a convention and stick to it, you have to do the same for lenses. So if you have defined an object to the left of a lens to be positive and you get a negative answer for an object distance, that means it's to the right instead.

One last thing needs to be defined to work with multiple lenses, and that is

In a multiple lens system, the image for the first becomes the object for the second.

The way to view this conceptually is how whenever you see a ray diagram, the rays converge and make a nice image. However, they don't stop there. As far as the second lens is concerned, there appears to be a light-emitting object at the point

¹This is also one of the few places we recommend putting in actual numbers as soon as possible, rather than solving symbolically. The algebra can get somewhat tedious when doing reciprocals.

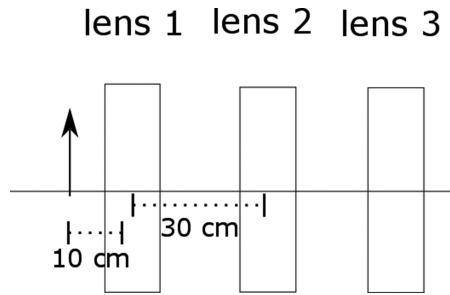


Figure 6.2. The setup for the three lenses. Because the problem does not specify what type of lens they are, a box is drawn to provide a general sense of location. The lenses are equally spaced.

where the image was formed from the first one, hence why the image for the first becomes the object for the second.

Example: A triple-lens system has three lenses all of which are 30 cm apart. (You might already be thinking, what type of lens is this? That will be determined by the end of the solution.) An object is placed 10 cm outside the focal length and to the left of the first one and parallel rays emerge to the right of the third. The second has a focal length of 40 cm and the third has a focal length of 50 cm. Determine the focal length of the first lens. Figure 6.2 gives you a basic idea of the setup.

Solution:

We have to work backwards here and recognize what things mean. Parallel rays emerging from the third mean that the image forms at infinity, so with the focal length of 50 cm, the thin lens equation reduces nicely as follows:

$$\begin{aligned}\frac{1}{f} &= \frac{1}{i} + \frac{1}{o}, \\ \frac{1}{50} &= \frac{1}{o_3}, \\ o_3 &= 50 \text{ cm.}\end{aligned}\tag{6.5}$$

Note that the term with the distance of the image from the lens has dropped out because 1 divided by infinity is zero. So now we know where the object is for the third lens. This is a positive number, so it's to the left of that lens. All of these are 30 cm apart, which means that this is 20 cm to the left of the second lens. Since the image for the second is the object for the third, $i_2 = -20$ cm, with the negative sign because the image is on the left of the second lens and we've already defined an image to the right to be positive. We have the focal length, so we can find the object distance for this one too. The algebra looks like the following:

$$\begin{aligned}\frac{1}{f_2} &= \frac{1}{i_2} + \frac{1}{o_2}, \\ \frac{1}{40} &= \frac{1}{-20} + \frac{1}{o_2}, \\ o_2 &= 13.33 \text{ cm.}\end{aligned}\tag{6.6}$$

This is the object for the second, which is the image for the first. This is a positive number so it was to the left of the second lens. The lenses are 30 cm apart, giving $i_1 = 16.7$ cm, a positive number because it's to the right of the first lens. Finally, though we're not told what the focal length or the object distance are, we are told that it's 10 cm outside the focal length, or $o_1 = 10 + f_1$. Going through the algebra now, we have

$$\begin{aligned}\frac{1}{f_1} &= \frac{1}{i_1} + \frac{1}{o_1}, \\ \frac{1}{f} &= \frac{1}{16.7} + \frac{1}{f + 10}, \\ 1 &= \frac{f}{16.7} + \frac{f}{f + 10}, \\ f + 10 &= \frac{f^2 + 10f}{16.7} + f, \\ 167 &= f^2 + 10f, \\ f &= 8.86, -18.9.\end{aligned}\tag{6.7}$$

We end up with a quadratic equation with two solutions, which is intentional for this example. How we know which one is the correct solution is that we already know that we placed the object to the left and it formed an image to the right. The lens that does this is a converging lens, so it has to be a positive focal length and 8.86 cm is the correct answer.

6.3 Summary and important notes

1. Interference and diffraction is based on the simple principle of path length difference, but most problem solving is merely keeping the equations straight. Carefully note what each symbol means in all of the equations and the differences between single- and double-slit diffraction.
2. Set your sign convention and stick to it. The one we have written here is probably the easiest, but you can have your own too. It must remain the same throughout the problem if you have more than one lens. If something is on the 'wrong' side, it gets a minus sign because that's how we say 'this is backwards' in math.
3. The image only forms at the focal length if the object is very far away (infinity). Otherwise, the focal length is just a property of the lens that gives information about how the light rays refract through the lens and does not give a pat answer for where the image forms.
4. In a multiple-lens system, the image for the first is the object for the second. But you still have to stick with your sign conventions, so if you define an object to the left as positive and you get the image of the first being to the right of the second, that means the object distance for the second must be negative.

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Chapter 7

The right-hand rule

A true understanding of this rule is often regarded as the most difficult thing in intro physics, but it does not have to be. There are a few simple keys that make it drastically easier to grasp¹.

1. **It is based on the cross product.**

Even if you haven't seen the cross product, all you really need to know for the right hand rule is that the cross product of two vectors is another vector *that is perpendicular to both of the original vectors*. So if you cross \mathbf{x} and \mathbf{y} , you get \mathbf{z} . This operation is designated by the symbol \times . In a table for cartesian coordinates, this looks like

$$\begin{aligned}\mathbf{x} \times \mathbf{y} &= \mathbf{z} \\ \mathbf{y} \times \mathbf{z} &= \mathbf{x} \\ \mathbf{z} \times \mathbf{x} &= \mathbf{y}\end{aligned}\tag{7.1}$$

$$\begin{aligned}\mathbf{y} \times \mathbf{x} &= -\mathbf{z} \\ \mathbf{z} \times \mathbf{y} &= -\mathbf{x} \\ \mathbf{x} \times \mathbf{z} &= -\mathbf{y}.\end{aligned}\tag{7.2}$$

Notice carefully the cyclical nature of the cross product and how if you reverse the order, you get a minus sign (e.g., $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$). Again, a minus sign is just how we say 'this is backwards' in math.

2. **It only gives you the sign (positive or negative).**

The most common mistake when trying to use the right-hand rule is waving your hand through the air randomly in an attempt to line up your fingers properly. Don't do this. When you use the right-hand rule, you should

¹We hope the humor here does not go unappreciated.

already know which direction the third vector is, you just don't know if it's positive or negative.

3. **It should be comfortable.**

If you find yourself doing yoga while trying to apply the right-hand rule, you are probably doing it wrong. Your hand should very easily and comfortably turn and point in the directions needed.

4. **Make sure your equations are written or memorized in the proper order.**

For instance, the equation for the force on a wire due to a magnetic field is

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B}, \quad (7.3)$$

where \mathbf{I} is the current, \mathbf{B} is the magnetic field and \mathbf{L} is the length of the wire. This means when you're doing the right-hand rule, you must be sure that the direction of the current is your first vector and the direction of the magnetic field is your second vector.

7.1 How to use it properly

Identify the first vector in your problem. If you are (e.g.) dealing with torque ($\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$), then this would be \mathbf{r} . Now identify the second vector, in this case \mathbf{F} . Before you do anything with your hand, figure out what vector is perpendicular to both of those. The simplest way to visualize this is to make an X with those two vectors and the third vector is one going straight through the center of the X, as shown in figure 7.1.

Your two authors have different ways of visualizing the right hand rule using your hand. Either works fine.

JW's view: Place your fingers straight out with your thumb up (like you are going to shake someone's hand) and point those fingers in the direction of \mathbf{r} . Now, turn your hand in whatever direction you need to so that your palm is facing the second vector (in the torque example this would be \mathbf{F}) and you are done, your thumb is now pointed in the direction of the resultant vector.

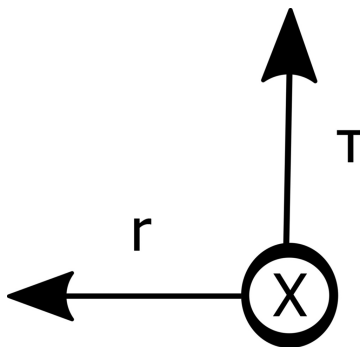


Figure 7.1. The basic scheme for the directions used by the right hand rule. The variables for its use with torque are given in this figure.

HW's view: Point your thumb in the direction of the first vector. Point your fingers in the direction of the second vector. Your palm points in the direction of the third. HW feels this is especially effective when using the right hand rule for expressions that determine force, because you can think of your palm applying force.

7.2 Special case of magnetic fields from currents

This is probably the worst victim of poor explanations when it comes to how to use the right hand rule. In order to find the magnetic field from a current, you point your thumb in the direction of the current and then wrap your fingers in the direction of the field. However, this goes both ways: if you have a straight field, it must come from a looping current, so your thumb now points in the direction of the field and your fingers wrap in the direction of the current.

The way to think of this is that, when dealing with magnetic fields and currents, **one will always be straight, the other circular**. So if you identify a straight current, you know it generates a circular magnetic field. Similarly, if you identify a straight magnetic field, you know it had to come from a circular current (current loop). This is why you can turn the page in your textbook feeling good about how to do this and then see it's totally backwards, because it is. Due to the way fields are generated by currents (described by the cross product), you can go either way depending on which is straight and which is circular.

7.3 Summary and important notes

1. You should already know the direction of the vector in question for the result of the cross product before you do anything with your hand; you just don't know if it's positive or negative and the right hand rule is used to determine that.
2. Make sure you do it in the order as written, $A \times B = -B \times A$, so if you do it backwards you'll get a minus sign.

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Chapter 8

Electric fields and electric potential

If you are going to struggle in an introductory physics course, this is probably where you will do it. Solving problems involving electric fields will often feel very circular in reasoning and it is easy to get lost. So let's focus on not getting lost. This chapter will cover finding electric fields and electric potentials from both point charges and charge distributions, the latter of which will require calculus.

8.1 Finding fields and forces from point charges

The two equations that govern fields and forces from any charge are

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \tag{8.1}$$
$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1q_2}{r^2} = q_1\mathbf{E}_2 = q_2\mathbf{E}_1.$$

We have written the second expression for force, its relation to the electric field, slightly differently from normal for a reason: it emphasizes that the force on q_1 is due to the electric field created by q_2 and vice versa (Newton's third law). This is because the first easy way to get lost here is to not be clear about what is *creating* the electric field and what is *affected by* the electric field. The field itself then is merely what force a charge would feel if it were there, hence the common use of the term 'test charge'.

With that out of the way, let's look at how to find fields from multiple charges. There is a five-step method that you can follow any time you need to find the resultant field of more than one charge:

1. Determine the magnitude of the electric field based on the *straight line distance* from the charge to the point where you are measuring the field.
2. Break this into its vector components, which will include some combination of E_x , E_y , and E_z .

3. Repeat step (2) for as many charges as you have.
4. Add all electric fields *as vectors*. This means adding all the x components, y components, and z components of the different fields from different charges.
5. Find the magnitude and direction of the field using addition in quadrature ($|E| = \sqrt{E_x^2 + E_y^2 + E_z^2}$) and use trigonometry to find the direction of the field.

Example: Four equal positive charges of charge q are at the corners of a square with sides 1 m long. Find the electric field at the center. Figure 8.1 shows the scenario.

Solution:

From the symmetry of the problem you may guess that the field is zero in the middle and you would be correct. But let's work it anyway. Go through the steps above:

1. Find the straight-line distance, which is the same for each charge. It's a 1 m square so the distance to the center from each is $r = \sqrt{0.5^2 + 0.5^2} = 0.707$ m. From this we find the magnitude, which is

$$|E| = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{q}{0.5^2} \quad (8.2)$$

2. Break it into components. Both sides of the triangle pointed at the center are the same length, so the angle is 45° , which conveniently means that sine and cosine are the same. So we can then write

$$\begin{aligned} E_x &= |E|\cos(45) = 0.707|E| \\ E_y &= |E|\sin(45) = 0.707|E| \end{aligned} \quad (8.3)$$

3. We have four charges all of equal magnitude, but the directions of the field from each will be different. Set right and up as our positive directions and

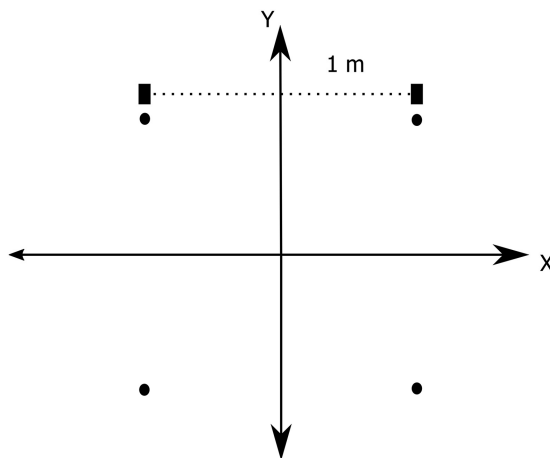


Figure 8.1. Four single charges for which we will find the electric field at the center of the configuration.

find the field directions too. Remember that field lines point outward from positive charges, so then we have

$$\begin{aligned}
 \text{Upper Left: } & E_x = 0.707|E|, & E_y = -0.707|E|, \\
 \text{Upper right: } & E_x = -0.707|E|, & E_y = -0.707|E|, \\
 \text{Lower left: } & E_x = 0.707|E|, & E_y = 0.707|E|, \\
 \text{Lower right: } & E_x = -0.707|E|, & E_y = 0.707|E|.
 \end{aligned}
 \tag{8.4}$$

4. Add them all up as vectors. As you can see now, adding them all will give zero, as expected. Which also means we have to change the problem a little to do step 5 (just because this is an example and it should be thorough).
5. For the purposes of the example, let's change the upper left to a negative charge. Since field lines point in to a negative charge, this means the signs are swapped for that charge, giving

$$\begin{aligned}
 \text{Upper Left: } & E_x = -0.707|E|, & E_y = 0.707|E|, \\
 \text{Upper right: } & E_x = -0.707|E|, & E_y = -0.707|E|, \\
 \text{Lower left: } & E_x = 0.707|E|, & E_y = 0.707|E|, \\
 \text{Lower right: } & E_x = -0.707|E|, & E_y = 0.707|E|,
 \end{aligned}
 \tag{8.5}$$

and everything no longer cancels out. Instead, we have $E_{x,\text{total}} = -1.414|E|$ and $E_{y,\text{total}} = 1.414|E|$. The magnitude of this field is then

$$|E_{\text{total}}| = \sqrt{E_{x,\text{total}}^2 + E_{y,\text{total}}^2} = \sqrt{2|E|^2 + 2|E|^2} = 2|E|,
 \tag{8.6}$$

and from trigonometry, we can find the direction by making the x and y components of the field two sides of a right triangle, or

$$\begin{aligned}
 \tan \theta &= \frac{\text{opposite}}{\text{adjacent}} = \frac{E_{y,\text{total}}}{E_{x,\text{total}}} = \frac{1.414|E|}{-1.414|E|} = -1, \\
 \theta &= 45^\circ.
 \end{aligned}
 \tag{8.7}$$

Note that I dropped the minus sign, this is intentional. You can keep all the signs all the way through the problem if you want to do so, but signs for vectors just indicate positive or negative. We already knew the x component was negative and the y component was positive, so we knew the basic direction without doing any trigonometry. If you had kept the sign and put it into your calculator as a negative 1, it would have given you negative 45° . This would have been fine but not necessarily intuitive as the convention is to set 0° as straight up, with positive being to the right and negative being to the left. So you would have gotten the same answer but it may be easier to get lost. You can do it either way but we strongly recommend doing it conceptually.

8.1.1 Electric field integrals

Making the jump to calculus problems is difficult for most students, usually because it's easy to get overwhelmed. The way to avoid this is to recognize that you are still doing exactly the same thing you were doing with point charges, just now when you add them together you have to integrate (an integral is just a sum). You will be dealing with charge distributions instead of point charges but you will still find the electric field at a distance r from a charge dq instead of q . The dq just represents an infinitely small point charge, rather than a point charge with a certain quantity q . With that in mind, we can write the infinitely small piece of electric field created by the infinitely small point charge as

$$dE = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2}. \quad (8.8)$$

Nothing has really changed; we're still dealing with an electric field generated by a charge, we've just written it as an infinitely small piece. If you want to find the total field then, you have to add all of these up as

$$dE_1 + dE_2 + dE_3 + \dots = \frac{1}{4\pi\epsilon_0} \frac{dq_1}{r_1^2} + \frac{1}{4\pi\epsilon_0} \frac{dq_2}{r_2^2} + \frac{1}{4\pi\epsilon_0} \frac{dq_3}{r_3^2} + \dots, \quad (8.9)$$

but your calculus classes have (hopefully) already taught you that there's a specific method to add up an infinitely large number of infinitely small things, and that is to integrate. So then we have

$$E = \int dE = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2}.$$

Now all that is left to do is to figure out how dq and r are related and you can do the integral.

Example: A uniformly charged rod with total charge Q and length L is on the y -axis with its center at the origin. Find the electric field at some distance x along the x -axis. A rough sketch of the scenario is shown in figure 8.2. Components of the resulting electric field are shown as possibilities, not necessarily the final result.

Solution:

First, write out the tiny piece of electric field from a tiny piece of charge:

$$dE = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2}. \quad (8.10)$$

Now we have to figure out what the distance r is from any tiny piece of charge. Since it's still the straight-line distance, we can make a triangle from any point and write it as $r = \sqrt{x^2 + y^2}$, where y is the distance from the origin to the charge and x is the distance to where we want to know the field. x remains a constant for all of these; only y changes. What we have to do with this is figure out how to write the qualities of a point charge with the information given. Since this rod has a one-dimensional

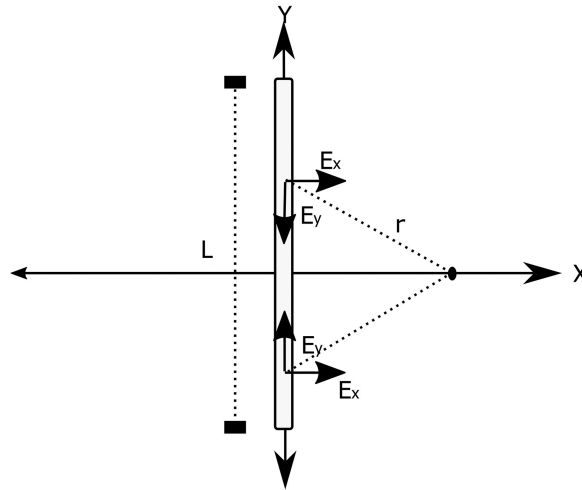


Figure 8.2. Rod of length L . The electric field components present at symmetric distances from the origin are shown, in order to emphasize what is similar and different between them.

distribution of charge, we will need the charge distribution over the length of the rod—a linear charge density. The entire rod has charge Q and length L , giving it a charge density $\lambda = \frac{Q}{L}$. Charge density is charge per unit length, so multiply the charge per unit length by a tiny piece of length and you get a tiny piece of charge. This looks like

$$dq = \lambda dy = \frac{Q}{L} dy. \quad (8.11)$$

(Note that problems involving two-dimensional distribution of charge, such as for circles, use a charge per unit area for charge density.) If you do not quite trust this you can quickly check it by integrating the charge density from 0 to L , which will give you total charge, as expected since you are just adding up all the little charges on the rod (which must be the total charge on the rod).

The tiny piece of electric field is then (plugging in the equations for r and dq)

$$\begin{aligned} dE &= \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2}, \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{L} dy \frac{1}{r^2}, \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{L} dy \frac{1}{x^2 + y^2}. \end{aligned} \quad (8.12)$$

We have intentionally left out one thing to emphasize its importance: the direction. Step (4) above was to add them up as vectors, which we still have to do. From the symmetry of the problem the y components all cancel out, but the x components add. The direction then depends on

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}} \quad (8.13)$$

multiply this by our above equation and we're ready to do the integral

$$\begin{aligned} dE &= \frac{1}{4\pi\epsilon_0} \frac{Q}{L} dy \frac{1}{x^2 + y^2} \frac{x}{\sqrt{x^2 + y^2}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \frac{x}{(x^2 + y^2)^{3/2}} dy. \end{aligned} \quad (8.14)$$

From here it is just a math problem, admittedly a cumbersome one. The only thing left to do physically is figure out the bounds of integration. The rod is centered on the origin, so it adds (integrates) from $-\frac{L}{2}$ to $\frac{L}{2}$

$$\begin{aligned} \int_0^E dE &= \int_{-L/2}^{L/2} \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \frac{x}{(x^2 + y^2)^{3/2}} dy, \\ E &= \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \frac{y}{x\sqrt{x^2 + y^2}} \Big|_{-L/2}^{L/2}, \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \frac{L/2}{x\sqrt{x^2 + \frac{L^2}{4}}} - \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \frac{-L/2}{x\sqrt{x^2 + \frac{L^2}{4}}}, \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{x\sqrt{x^2 + \frac{L^2}{4}}}. \end{aligned} \quad (8.15)$$

As a small check, see what happens if x becomes very large. You would expect this to look like a point charge, which it does since $\sqrt{x^2 + \frac{L^2}{4}}$ becomes just $\sqrt{x^2} = x$ and the entire thing reduces to $E = \frac{1}{4\pi\epsilon_0} \frac{Q}{x^2}$.

8.2 Electric potential from point charges

This works exactly the same way as before, only more easily since electric potential is **not a vector**. So the five-step method reduces down to a three-step method:

1. Determine the magnitude of the potential based on the *straight line distance* from the charge to the point where you are measuring it.
2. Repeat step (1) for as many charges as you have.
3. Add all potentials, being careful about signs. Direction does not matter but sign of the charge does.

Conceptually speaking, the hardest part of dealing with electric potentials is figuring out what the number means physically. It cannot be overemphasized that *the electric potential at a single point in space is usually relevant only within the*

context of the potential at another location. Because of this, measurements of potential, such as that of a battery, are usually of potential *difference*—the *change* in potential a charge would experience as it interacts with the chemicals in the battery. What will always matter is potential difference, so don't get stressed out if you get zero potential in a place where it doesn't feel like it should be zero. Sensemaking efforts should focus on what that zero means in the larger context of the problem.

Electric potential for a charged particle can be thought of the same way that height is for a particle with mass. The same ideas are present with mass as well, but we don't focus on them as explicitly in discussions about gravity. But forces do indeed have a quality of potential as well. Being at a height of zero doesn't tell you anything until you establish a coordinate system just as being at a potential of zero doesn't tell you anything. The only difference is that there can be negative charge while there cannot be negative mass.

Example: Four equal positive charges of charge q are at the corners of a square with sides 1 m long. Find the electric potential at the center.

Solution:

Follow the steps above the same way we did before.

1. Find the straight-line distance, which is the same for each charge. It is a 1 m square so the distance to the center from each is $r = \sqrt{0.5^2 + 0.5^2} = 0.707$ m. From this we find the magnitude, which is

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{r} = \frac{1}{4\pi\epsilon_0} \frac{q}{0.707 \text{ m}} \quad (8.16)$$

2. They all have the same distance and charge, so they're all this same V from above.
3. Adding them all up is just adding them all up. They're all the same, so it's just $4V$ or $\frac{1}{4\pi\epsilon_0} \frac{4q}{0.707} = \frac{1}{\pi\epsilon_0} \frac{4}{0.707}$ volts.

If we were to change one of the charges as before to a negative, all that would happen is this would become $2V$ instead of $4V$ because we would have to subtract one of them instead of add. If we changed two of them, the potential would be zero.

8.3 Summary and important notes

1. Finding electric fields from point charges will feel like going in circles because you must first find the magnitude of the field, break it into components, add those components, then find the magnitude and direction again. Be sure the r that you use in Coulomb's law is the straight-line distance.
2. Electric fields and potentials from charge distributions works exactly the same way except that you have to figure out what each 'point charge' dq is and add them up (integrate). Don't start a problem thinking 'oh no, calculus!'. Just think back to what you did before for point charges. If you get lost, especially with distances and directions, re-imagine the problem as

- a bunch of point charges that are very close together and then go back to the distribution
3. Electric potential at a point tells you nothing particularly useful on its own. You must have a potential difference to actually know what a charged particle will do.

A Handbook of Mathematical Methods and Problem-Solving Tools for Introductory Physics (Second Edition)

Joshua F Whitney and Heather M Whitney

Chapter 9

Magnetic fields

Magnetic fields are similar in many ways to electric fields. In relativity you even see that they can be the same thing depending on the observer. Nevertheless, the rules change a little bit for magnetic fields (just enough at first to feel confusing—or to at least require a little extra thinking). Whereas electric fields are generated by charges, magnetic fields are generated by *moving* charges (currents). This provides an important distinction as while an electric field was a vector, the charges generating it were not. However, the magnetic field and the thing that is making the magnetic field (current) are both vectors, which leads to some mathematical complications but also some very useful simplifications as well. Note carefully that the current itself is not usually written as a vector but rather the length of wire along which it travels. This makes no functional difference but can be confusing in notation.

9.1 Biot–Savart law

The most common way you will see this presented is

$$dB = \frac{\mu_0 Id\mathbf{l} \times \hat{r}}{4\pi r^2}, \quad (9.1)$$

which is easily in the contenders for most intimidating introductory physics equation. Let's break it down into pieces to physically explain what it is and how to use it. dB represents a tiny (infinitesimal) magnetic field. μ_0 and 4π are constants, with μ_0 representing the magnetic permeability of free space. I is the current flowing along some tiny line $d\mathbf{l}$. This is 'crossed with' (multiplied via the cross products as both are vectors) a unit vector \hat{r} , which is the *place at which the field is being measured*. We have emphasized this because, as the saying goes, there's the rub. This otherwise looks the same as Coulomb's Law if you were to remove this cross product and \hat{r} . The last piece then is r which is the scalar distance from the wire to the point

at which the field is being measured. Here is a quick reference table for all these variables:

dB : tiny piece of magnetic field

μ_0 : permeability constant

I : current that is generating the field

$d\mathbf{l}$: tiny piece of wire along which the current flows

\hat{r} : direction in which the field is being measured

r : distance from the wire to the point where the field is being measured

9.1.1 Right-hand rule for magnetic fields

The magnetic field is then **always** going to be perpendicular to the current generating it and this leads to a drastic simplification of the right-hand rule for fields. **A straight current generates a circular (looping) magnetic field. A circular (looping) current generates a straight magnetic field.** This is why it may seem you were told two conflicting things about using the right-hand rule for magnetic fields and it may have felt like the Emperor's New Clothes. On one page of a given textbook you would be finding the field by wrapping your fingers around the current and your thumb would point in the direction of the field and the next page you would find the field by pointing your thumb in the direction of the current and now your fingers wrap in the direction of the field. You can see how these come to be by looking at the Biot–Savart Law. Diagrams of how to do this in a 2D textbook are often not helpful, so we will describe what to do instead. Take your right hand flat in front of you with palm facing down and thumb pointed to the left. Now take a pencil (or any straight object) and place it lengthwise along your thumb. If a current were running from right to left in the pencil, then the magnetic field **at your fingers** would be in the direction of your palm (downward). Now notice what happens if you keep your thumb in the same place and rotate your hand. Your palm points in a different direction, giving you that circular field from a straight wire that you are used to seeing. For the other way around, draw a semicircle to represent current and point your thumb along it as before. As you follow it you should notice that your palm keeps the same direction, giving the straight field from a circular wire.

Let's now put it together with the Biot–Savart law. Your thumb was pointing along $I d\mathbf{l}$ and your fingers were pointing toward \hat{r} . Since this is a cross product the magnetic field generated must be perpendicular to **both** of these and the right-hand rule shows you which way it is. Since this perpendicular relationship always must exist and a wire is either straight or not straight, finding the direction of the field simplifies greatly to: **point your thumb in the direction of whichever one is straight (current or magnetic field) and your fingers will wrap in the direction of the other one.** So once you have done a few with the directions above, you can safely forget about them and just go to the simple (bolded) version. It was only intended to help you understand why the simple version works both ways.

9.1.2 Calculations with the Biot–Savart law

Problems involving this law are not very common in an introductory course and for good reason: they involve a cross product inside an integral, which may be beyond the scope of what your course entails in terms of mathematics. But it is still doable and good practice for dealing with pseudo-abstract cross products. If you are doing an algebra-based course then you can safely skip this section as it is beyond the scope of a course that does not use calculus (don't feel badly about it!)

Example 1

Show that what I just told you about the right-hand rule is correct and derive equations for a current loop.

Start with the law itself and then integrate both sides:

$$\begin{aligned} dB &= \frac{\mu_0}{4\pi} \frac{Id\mathbf{l} \times \hat{r}}{r^2}, \\ B &= \int \frac{\mu_0}{4\pi} \frac{Id\mathbf{l} \times \hat{r}}{r^2}. \end{aligned} \tag{9.2}$$

Right now this is general so we cannot actually do the integral, but we can figure something out from it. Do the loop of current first and set it on the page counterclockwise (just to make it more comfortable for your hand). An integral is just a fancy sum, so what we are doing is then adding up every little piece $Id\mathbf{l}$ ¹ crossed with the unit vector representing where we are measuring the field, which is at the middle. Since this is a symmetrical loop, all you need to do is figure it out for one spot and it will work for the rest. So look at a tiny piece at the top going left, which when crossed with a vector pointing toward the middle gives you one going either **into** or **out of** the page. Normally you would use the right-hand rule for this but we are trying to prove that, so instead use the cross product. With a standard coordinate system, going left at the top (current) is $-x$, going down is $-y$, so $-x \times -y = +z$. Again according to a standard coordinate system, this is coming out of the page, exactly what we'd get with the right-hand rule.

Now to actually do this we have to define the length of the circle, which is $2\pi R$, and set up the integral

$$\begin{aligned} B &= \int \frac{\mu_0}{4\pi} \frac{Id\mathbf{l} \times \hat{r}}{r^2}, \\ B &= \int \frac{\mu_0}{4\pi} \frac{Id\mathbf{l} \times \hat{r}}{r^2}. \end{aligned} \tag{9.3}$$

This little r in the denominator is a constant as the radius of the circle and everything is perpendicular, so the cross product goes away as well

¹Remember current is the same in all places in the wire.

$$\begin{aligned}
 B &= \int \frac{\mu_0}{4\pi} \frac{Id\mathbf{l}}{R^2}, \\
 &= \frac{\mu_0 I}{4\pi} \frac{1}{R^2} \int d\mathbf{l},
 \end{aligned}
 \tag{9.4}$$

but we are still in cylindrical coordinates so this integral is

$$\int d\mathbf{l} = \int_0^{2\pi} \int_0^R r dr d\theta = 2\pi R,
 \tag{9.5}$$

giving the familiar equation

$$B = \frac{\mu_0 I}{4\pi} \frac{1}{R^2} 2\pi R = \frac{\mu_0 I}{2R}.
 \tag{9.6}$$

Example 2

Show that what I just told you about the right-hand rule is correct and derive equations for a line of current.

This one is a bit simpler: choose the current wire coming out of the page and again, due to symmetry, we can choose any point and it works for all points. Choose then one directly to the left of the wire. The current is coming out (+z), the point is to the left (-x), so $z \times -x = -y$, which is down. Choose another directly below the wire for $z \times -y = x$, or right. One more to the right of the wire for $z \times x = y$, or up. Which you may notice by now is the same thing you get if you point your thumb in the direction of the current and wrap your fingers in the direction of the field.

The calculus of this example will be described in the last section of the fields called ‘dealing with the trig’. The last hurdle to overcome in actually doing these things is figuring out how that ‘extra’ piece comes in once you’ve got a decent grasp as if you look up many of these derivations, you’ll see a factor of $\frac{\text{something}}{\sqrt{R^2 + x^2}}$ pop out. The actual derivations are done in many textbooks and internet resources so they will not be rehashed here in their pure form, but rather it will be explained how to set them up.

9.2 Magnetic forces

Magnetic force requires two things: a magnetic field and a charge **moving in some way perpendicular to that field**. It does not have to be completely perpendicular but it cannot be 100% parallel. In other words, there needs to be at least some component of the charge’s velocity that is perpendicular to the field. This makes them both very simple and very complicated depending on the situation you happen to be dealing with. The force equation for a single charge moving in a magnetic field is

$$F = q\mathbf{v} \times \mathbf{B},
 \tag{9.7}$$

where q is the magnitude of the charge, v is the velocity of the charge relative to the field, and B is of course the magnitude of the field itself. The cross product here is

what makes this mathematically challenging since when you exert a force on something you change its velocity and thus trajectory, so you can be dealing with multiple angles at once. To keep it all straight, remember that you only care about the piece that is perpendicular to the magnetic field as that will be the only thing changing. If it is not moving perpendicular to the field then it is not experiencing a force in that direction and thus it is constant. As has been mentioned before in this text, a huge part of both understanding and working with physics is knowing what matters and what does not. The best way to see this is with a few examples.

Example 3

Spiraling electron. An electron with a velocity v at an angle of 30° above the xz plane enters a region of magnetic field \mathbf{B} in the y direction. Determine the radius of the resulting electron's spiral.

This problem² is tricky because the obvious thing to do would be to take the angle to be 30° and then you can reduce the cross product to $\sin(30^\circ)$. However, keep in mind this is the angle it makes with the xz plane, not with the magnetic field. I recall a student coming to me once having looked up an answer for a similar problem and he could not figure out why the answer used cosine instead of sine when it is obviously a cross product. The reason is that the angle it makes with the magnetic field is actually 60° and $\sin(60^\circ) = \cos(30^\circ)$, so using cosine of the angle given is mathematically the same as using sine of the actual angle between the velocity vector and the magnetic field. This is where the previous advice comes in, **check what you care about and what you don't**. The component parallel to the field does not really matter, so we are just interested in the perpendicular component. Thus, you can either let the math guide you and use $\sin(60^\circ)$ or you can find the component of the velocity vector that is already perpendicular by using $\cos(30^\circ)$ and then the cross product becomes 1. Formally then

$$F = qv \sin(60^\circ)B = qv \cos(30^\circ)B, \quad (9.8)$$

and it is making a circle, so this is providing the centripetal force required to make it not go straight

$$\frac{mv^2}{r} = qv \sin(60^\circ)B. \quad (9.9)$$

Ah, now we are hitting another snag, what is this v on the left side? Think carefully again about what matters and what's happening, which is the electron is spiraling upwards in the y direction. There are two components to the velocity vector (or at least it can be decomposed as such): one parallel to the field and one perpendicular. We are working with centripetal force and so we don't care about the parallel piece, which would be $v \sin(30^\circ)$. The piece that is always in the xz plane is the one we care about and that is given by $v \cos(30^\circ)$.

²I have intentionally left out a drawing as it will immediately lead you to the wrong conclusion, which we would then have to undo.

Now you can see why that student found an answer with only cosine, if we then substitute this piece for the centripetal force we get

$$\frac{m(v \cos(30^\circ))^2}{r} = qv \sin(60^\circ)B, \quad (9.10)$$

but $\cos(30^\circ) = \sin(60^\circ)$

$$\begin{aligned} \frac{m(v \cos(30^\circ))^2}{r} &= qv \cos(30^\circ)B, \\ \frac{mv \cos(30^\circ)}{r} &= qB, \\ r &= \frac{mv \cos(30^\circ)}{qB}. \end{aligned} \quad (9.11)$$

One last thing before we are done is to check this against what we know to be true. You've seen the equation for the radius of something in cyclotron motion, which is when it comes in purely perpendicular to the field (also known as the easy case)

$$r = \frac{mv}{qB},$$

so we want to check if this follows if the angle is zero and it does. Reduce the angle in the problem to zero so that it is fully perpendicular to the magnetic field and you come back to the equation you know is correct.

9.3 Trigonometry for electric and magnetic field integrals

This warrants its own section both because of its complexity and unexpected nature. The gist of it is that **when setting up any field integral, you have to remember they are still vectors**. This is why you will see factors like $\frac{1}{\sqrt{x^2 + R^2}}$ appear in examples in your main textbook when it seems the problem is almost solved. Let's do an example of this for an electric field (they make better examples since too much ends up zero with magnetic fields) and explain it carefully.

Example 4

A line of charge has total charge Q and length L and lies along the y -axis. Determine the electric field strength at a distance R from its midpoint.

First off, we can go back to our previous set of rules for point charges and see that we are not going to get anything in the y direction due to being at the midpoint (every charge above the midpoint is canceled by one below it). The ones in the x direction though will add, so just as before we'll add up all the electric fields created by all these charges, except now we have to add using an integral. The tiny piece of electric field from a tiny piece of charge is then

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \rightarrow dE = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2},$$

$$E_{\text{total}} = \sum \frac{1}{4\pi\epsilon_0} \frac{q_i}{r_i^2} \rightarrow E_{\text{total}} = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2}. \quad (9.12)$$

So right now we are at the point of finding the magnitude of the field from the straight line distance. The first catch though is we don't have dq , we only have Q and L , but we can get it through recognizing that $\frac{Q}{L}$ is charge per unit length, so if you multiply this by a tiny piece of length, you get a tiny piece of charge dq , or

$$\frac{Q}{L} dy = dq. \quad (9.13)$$

It is dy instead of dx because the charge is on the y axis. If it were on another axis it would be a different symbol. So now we are at

$$E_{\text{total}} = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \rightarrow E_{\text{total}} = \int \frac{1}{4\pi\epsilon_0} \frac{\frac{Q}{L} dy}{r^2},$$

but still missing two more pieces. The first is what to do with this r , which is still the straight line distance. It needs to be in terms of something we have, so make a triangle with the hypotenuse as r and one side being R (the point where we are trying to find the field) and the other side will have to be y , the location of charge dq . Then it is the usual method to get

$$r = \sqrt{y^2 + R^2},$$

$$E_{\text{total}} = \int \frac{1}{4\pi\epsilon_0} \frac{\frac{Q}{L} dy}{y^2 + R^2}. \quad (9.14)$$

This should all hopefully have been easy to follow to this point. Now for the part for which you have eagerly been waiting: the trigonometry. Recall the next step after finding the magnitudes of the fields is then adding them *as vectors*. How you do this with an integral is to include the angle. We already know the y components cancel out so it is just x now, which normally would be throw in a $\cos \theta$ and be done with it, but $\cos \theta$ itself is not simple. The side we are interested in is the x axis, which is going to be R and the adjacent side in the right triangle. Using whatever mnemonic you memorized ages ago, that gives

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{R}{r}, \quad (9.15)$$

but! we just figured out what this little r is:

$$\cos \theta = \frac{R}{\sqrt{y^2 + R^2}}, \quad (9.16)$$

which now we multiply by our electric field magnitude (the stuff currently inside the integral) to get...a lot of things

$$E_{\text{total}} = \int \frac{1}{4\pi\epsilon_0} \frac{\frac{Q}{L} dy}{y^2 + R^2} \frac{R}{\sqrt{y^2 + R^2}}, \quad (9.17)$$

and simplify to get

$$E_{\text{total}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{L} R \int \frac{dy}{(y^2 + R^2)^{2/2}} \frac{1}{(y^2 + R^2)^{1/2}}, \quad (9.18)$$

$$E_{\text{total}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{L} R \int \frac{dy}{(y^2 + R^2)^{3/2}}.$$

Here I wrote $(y^2 + R^2)^{2/2}$ to make it clear what we are doing when we are combining these and pulled out the R up top because it is in fact a constant. The one in the denominator is ‘protected’ so we cannot just pull it out but it is still a constant as well. But now this is an integral that is doable using standard integration techniques. We just need to add the bounds now, which since our reference point is at the middle (which we’ll set as 0), are $-\frac{L}{2}, \frac{L}{2}$

$$E_{\text{total}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{L} R \int_{-L/2}^{L/2} \frac{dy}{(y^2 + R^2)^{3/2}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \frac{1}{\sqrt{r^2 + \frac{L^2}{4}}}. \quad (9.19)$$

So to summarize, the entire purpose of this example is to emphasize how nothing has changed if you have to do an integration problem. A vector is still a vector which means you have to account for it. Note that you totally could do this off axis now for all possible directions but that is way harder than anything you would encounter in an introductory course. So if you want to impress your physics friends then go for it, but we are going to stop here.

9.4 Induced currents: Faraday’s and Lenz’s law

In physics, it is often helpful to break down a problem into as small components as possible. In the case of induced currents it is almost essential. There may be people who can glance at an induction problem and determine the magnetic field right away, but we are not of them. So instead we will give you a stepwise and nearly foolproof method for determining induced currents and fields.

First off, Faraday’s law itself is

$$\epsilon = - \frac{d\Phi}{dt} = - \frac{\Delta\Phi}{\Delta t}, \quad (9.20)$$

ϵ : induced emf

Φ : magnetic flux, or magnetic field through an area ($\Phi = BA$)

t : time over which the change occurs

where the minus sign out front is Lenz's Law, which gives you the direction. Formally speaking, Lenz's Law is that any change in magnetic *flux* will induce a current that then generates another magnetic *flux* to oppose that change. Notice carefully that it is magnetic flux that is changing, not just the field. This can lead to many tricky situations such as the flux getting smaller ($\Delta\Phi$), but time interval (Δt) being so small as to give a bigger emf than something with an increasing flux. In other words, **you have to be extra careful with problems involving this law to not make any assumptions about the answer until you've worked it all out.** This is (in our opinion) a huge part of what makes this difficult: in most other areas you want to guess a general idea of what the answer should look like, but with Lenz's Law you generally do not. At every step of the way you have to think of what's happening and what must follow and only then do you move on to the next step. We will give a list of how to do this and a few examples.

1. Determine the magnetic flux already present.
2. Determine how that flux is changing.
3. Determine what magnetic field is needed to exactly oppose that change.
4. Using the right-hand rule, determine the current that is generating that field.

Example 5

A magnetic field of magnitude B is going into a circular loop of wire placed flat on this page. This field is decreasing at a constant rate. In which direction is the induced current in the loop?

Let's use the list.

1. The flux already present is going into the page.
2. This flux is decreasing as the magnitude of the field is decreasing.
3. To oppose this change, we need a field going **into** the page that is **increasing**.
4. Point your thumb in the direction of the **induced** field (in) and your fingers wrap in the current, so clockwise.

Note here that we chose this example specifically because of a common pitfall: the induced field opposes the **change** in flux, not the flux itself. It will always be tempting to say that the induced field opposes the original field and you should be careful not to make this assumption, as mentioned above.

You may find yourself wondering how it is we can find these currents from fields since we have previously defined currents as creating fields, not the other way around. It can seem very circular reasoning that we are finding the induced fields and *then* the induced currents.

Where it eventually comes from is the force equation

$$F = qv \times \mathbf{B}, \quad (9.21)$$

where q is a charge, v is the velocity of that charge **relative to the magnetic field**, and B is that magnetic field. Think about this very piecewise now and imagine a circular current loop flat on the table in front of you. Now imagine there is a magnetic field in the upward direction in front of the loop, such that you can slide the loop forward

into the magnetic field, so there is now a charged particle ‘moving into’ a magnetic field (it is a conductor so there are many free charges). Using the right-hand rule now with v going forward and B going up, you’ll get a force to the right for a clockwise current. One more time with the right hand rule gives you a field from this current going downward, or opposite the direction of the initial field. Furthermore, notice that this field only exists as long as it is moving. Once there is no v , there is no F and thus no current.

On a bit more of a surface level, you can think of it as a consequence of a ‘double cross product’. You do one cross product to get the force (which will be the same direction as the current) and then another to get the field at some location (Biot–Savart law). So you have something like a velocity in $+x$ and a field in $+y$, giving a current in $+z$. If there’s a field in the y direction from a current in z , then it has to come from a cross product with x . If you are moving into the field (increasing B), then you have to measure backwards along x , giving $z \times -x = -y$.

Physically then, it is a consequence of how **a charge only feels a force from a field if it is moving perpendicular to that field** and **fields are perpendicular to the moving charges (currents) that create them**. So just as you can take a left turn by doing three right turns, if you keep making things perpendicular you are going to come back to the place you started.

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Chapter 10

Circuits

The actual equations that govern circuits are pretty simple, mostly just Ohm's law ($V = IR$). Just as with Newton's second law, the true physics of this expression is that $I = \frac{\Delta V}{R}$, because current—the movement of charge in a circuit—exists because a potential *difference* is present by which they can get energy. Problem-solving requires significant conceptual thinking and so this section will focus mostly on that. There is some math but it is trivial compared to electric and magnetic fields. Working with circuits then is mostly working with Ohm's law in increasingly complicated ways, so that's where we will start.

10.1 Ohm's law

Ohm's law is deceptively simple. In a very real sense, you can think of it as Newton's second law ($F = ma$) for circuits. The common way to write it is

$$V = IR$$

V : potential difference between two points in a circuit

I : current along those same two points

R : resistance along those same two points

but much like Newton's second law, it really should be written as

$$I = \frac{V}{R}$$

which physically is read as ‘**a potential difference will cause a current to flow that is inversely proportional to the resistance in the circuit**’, just as a force¹ causes an acceleration that is inversely proportional to the mass of the object accelerated. Ohm’s law is used in two main ways: finding V , I , or R when you have the other two for a complete circuit, OR finding what is called the ‘voltage drop’ across some sort of load (resistance). Anything that uses energy (at least for this course) can be modeled as a resistor, hence why we focus so much on resistors and more or less ignore things like light bulbs. For the purposes of our calculations, a light bulb is a resistor. Before we do this though we need a brief discussion of Kirchhoff’s laws.

10.2 Kirchhoff’s loop and junction laws

A thorough understanding of these is of the utmost importance to being able to work with anything involving circuits. They sound almost foolishly simple at first but applying them can be tricky when you get to anything reasonably complicated. So we’ll go through them and an example for each. The examples are intended to be difficult as to showcase that while these laws are incredibly simple to state, they can be difficult to use when the circuits get larger.

10.2.1 Loop law: the sum of the potential differences around a closed circuit is zero

The easiest way to understand this conceptually is with an analogy: potential difference to a charged particle is functionally the same thing as height to a massive² particle. So then, the loop law applied to your everyday surroundings would say that ‘if you start a ball at the top of a slide, let it roll to the bottom, then put it back at the top of the slide, the total change in height of the ball is zero’. Which seems strange since it is may be obvious, but it is identical to how you can think of potential differences. You start the ball at the top of the slide and it experiences a ‘height drop’ (analogous to a voltage drop), converting potential energy into something else (kinetic in this case). Then you pick it up and put it back at the top, which takes some energy from you (the battery) to reset it to its original height. Its total change in height is now zero. Now think of a simple circuit with just a battery and a resistor (or light bulb if you want). Starting at the positive terminal of the battery is starting at the top of the slide. The charged particle goes through the resistor and potential energy is converted into something else (heat in this case). The battery then does work on the particle to raise it back up to its original ‘height’ (voltage). The analogy becomes nearly perfect if instead of a standard playground slide you imagine a slide

¹ The term emf (\mathcal{E}), or electromotive force, comes from this concept. It is defined as a potential difference that gives rise to an electric current. You can have a potential difference without making a current but you cannot have a current without a potential difference. For the purposes of calculations, emf is the same thing as voltage and physically refers to the voltage provided by a battery or other power supply. The simplest way to think of emf is then ‘the voltage that makes the circuit go’.

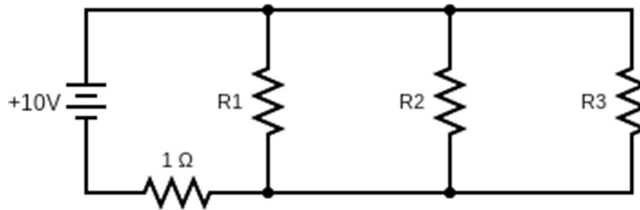
² In physics, the term ‘massive’ does not always mean huge. Sometimes it can mean a particle with mass. An electron has mass, a photon does not.

where, when the ball reaches the bottom, the slide becomes flat and wraps around to some sort of conveyor belt that brings it back to the top³. In terms of problem-solving, a critical note is that **if nothing happens between two points, the potential is the same**. Let's go through an example to...umm...exemplify this.

Example 6

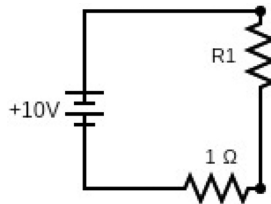
Any loop counts.

Determine the potential difference across each of the unknown resistors in the diagram



The current through resistor R1 is found to be 2.5 A and through the 1 Ω resistor is 5 A. R2 and R3 both have 1.25 A going through them. Determine the unknown resistors.

We'll do this piecewise in a way I recommend, which is to redraw it several times. In my case that's a lot of cut and paste, so let's take the first loop.

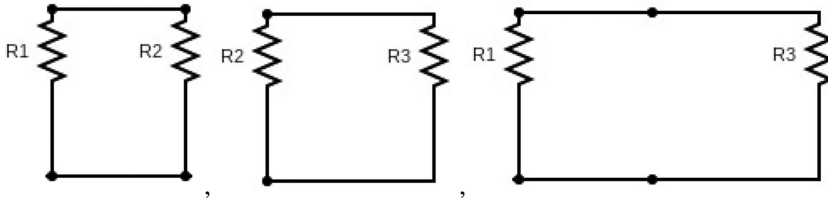


Note what was said earlier, any loop counts and every loop matters. If we can make a loop, we can use said loop. The sum of the voltages around this loop must be zero when it gets back to the battery and there is a 'voltage drop' of $V = IR$ every time it goes through a resistor, giving a drop of 5 V ($1 \Omega \times 5 \text{ A}$) and then we can solve for the other. It starts at 10 from the battery so

$$\begin{aligned}
 10 \text{ V} - 2.5(R1) - 5 \text{ V} &= 0 \\
 5 &= 2.5 R1 && (10.1) \\
 R1 &= 2 \Omega
 \end{aligned}$$

To find the others, just keep going and making loops, breaking it up into three parts

³ It is important to note that this one particle going around in circles is not what's actually happening since it is really an electron sea, but it is a very useful model. Remember that a model does not have to be completely correct; it just has to describe the phenomena correctly.



and you can see where I just erased the middle resistor (R_2) to emphasize the point that while it's there, if it's not part of the loop we are looking at, it doesn't matter (for now). Starting at the top left corner of each loop then and going clockwise

$$\begin{aligned}
 10 \text{ V} - 1.25 \text{ A} \times R_2 - 2.5 \text{ A} \times 2 \Omega &= 0 \\
 10 \text{ V} - 1.25 \text{ A} \times R_3 - 1.25 \text{ A} \times R_2 &= 0 \\
 10 \text{ V} - 1.25 \text{ A} \times R_2 - 2.5 \text{ A} \times 2 \Omega &= 0
 \end{aligned}
 \tag{10.2}$$

Now you just have to take a glance and figure out what to do with this. The first and third are the same equation, just for different resistors. This should not be surprising since the resistors have the same current through them and are both in a loop with R_1 . Solving for one gives both then

$$\begin{aligned}
 10 \text{ V} - 1.25 \text{ A} \times R_2 - 2.5 \text{ A} \times 2 \Omega &= 0 \\
 5 \text{ V} &= 1.25 \text{ A} \times R_2 \\
 R_2 = R_3 &= 4 \Omega
 \end{aligned}$$

which should not come as a surprise since they had the same current going through them and current splits proportionally to resistance.

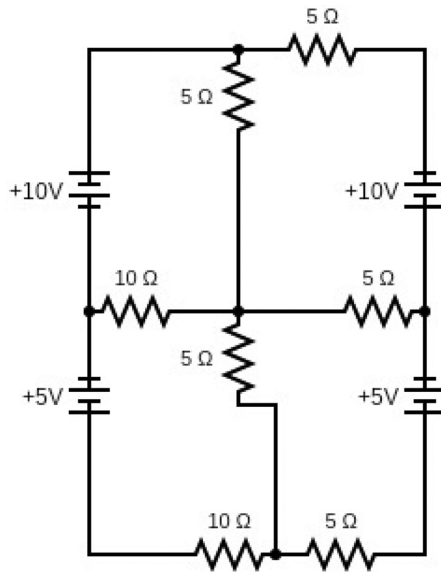
The point of this problem was to show that you can choose any loop and just keep making more loops as you need to figure out the problem. There were more we could have made but didn't and in fact we made more than we needed (we did not use the middle equation). As long as it comes back to where it started, it all sums to zero.

10.2.2 Junction law: the sum of all currents into a junction must be equal to the sum of the currents out of the junction

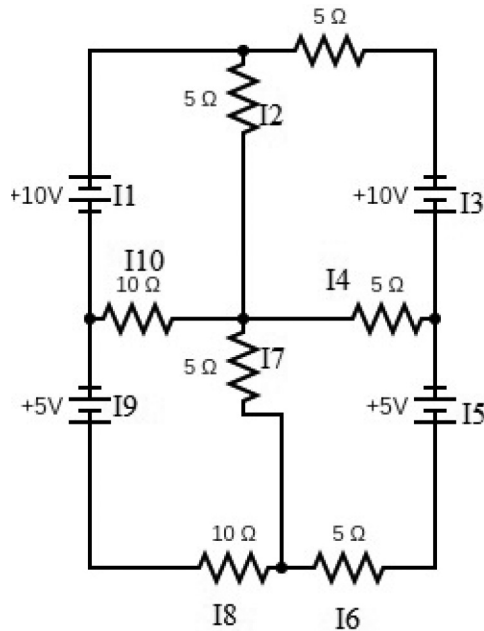
This is also sometimes called 'the law of conservation of current' which really is just an extension of the more fundamental law that charge is conserved. Current is charge per unit time, so the amount in must be equal to the amount out. In this sense, it is very helpful to think of electric current like water. Recall that water is an incompressible fluid so volume flow is conserved. The same thing applies to electric current. This is a simpler concept than the loop law, but it is still good to have a solid example.

Example 7

Many loops, handle it! Find the current through each of the batteries in the diagram



A neat ‘trick’ here is that you can choose whichever direction for the current you like and if you’re wrong, all that happens is your final answer will have a minus sign (indicating the current is going the other way). These problems look overwhelming at first but as long as you don’t make any assumptions, they work out. Remember here that **every time the current hits a junction, it becomes a different current**, and the sum of the new currents is the same as the old one. Let’s go through this image now and label them all



to emphasize a point, we're going to make everything go clockwise even if it's obvious it should not. This is still a lot to handle all together, so first we can write out equations for the currents. Going from I_1 and just following clockwise we get the first split into I_2 and I_3 , then I_3 splits into I_4 and I_5 , then I_5 splits again into I_7 and I_9 . Let's write those out

$$\begin{aligned} I_1 &= I_2 + I_3 \\ I_3 &= I_4 + I_5 \\ I_5 &= I_7 + I_9 \end{aligned} \tag{10.3}$$

But there's a problem: we said everything was going clockwise from the start. So that means I_4 isn't right and the middle equation should be

$$I_3 + I_4 = I_5. \tag{10.4}$$

Now to deal with the mess in the middle. Do it one at a time, starting with I_2

$$I_2 = I_4 + I_7 - I_{10} \tag{10.5}$$

this is because I_2 is coming down and everything is clockwise, so I_{10} is going to the right. Now that we have relationships among all the currents, we can write out equations for the voltages as well just as before. In this example it would end up with far more equations than you could reasonably work through (you would need matrix algebra), but it would be solvable. So that is where this example will end.

10.3 Summary and important notes

1. **Current and voltage are never 'used up' in a circuit. The only thing used up is energy.**

Current is just the flow of electrons, and this remains a constant. Voltage is like height so while there may be voltage drops, that's just like dropping from a high area to a low one. In a circuit, you'll eventually climb back up by means of something that will provide energy to the electrons. What does get used up is the ability of the electrons to do work. Now yes, if all the energy is gone (e.g., from a battery) then the current flow will stop, but that's because the battery can no longer create a potential difference. Current must remain the same on either side of any load (resistor, bulb, etc).

2. **Voltage is the same along things in parallel. Current is the same along things in series.**

Voltage is, again, like height. The voltage being the same along things in parallel is like seeing several waterfalls side by side. All the water falls from the same height. Current must be conserved and when two or more things are in series, any current going through one must go through the other(s) as well. So the current must be the same for any object in series.

3. **Similarly, the sum of the voltages in series must be the total voltage change. Sum of the currents in parallel must be the total current.**

If a current comes into a junction, you may not be able to tell how much will go down each side but you do know it must add together to sum to the current coming in. For voltage, if stuff in circuit elements is in series, then

crossing each one is like taking another step and when you add those steps up, they must equal the total ‘height’ of the ‘staircase’.

4. **To determine if something is in series or parallel, figure out what must happen.**

Follow the current in the normal direction (flow of positive charge) and determine if the current **must** flow through A in order to go through B. If it does, then A and B are in series. If it does not, then A and B are in parallel. There are no other options.

5. **Reduce and redraw.**

One of us (JW) is not a big fan of drawing. (HW is a big fan and usually starts problem solving by drawing.) But with circuits, always draw. Anytime you reduce down a circuit and are even remotely confused as to what to do next, redraw it with the equivalent capacitance or resistance.

6. **Don’t memorize too many equations for combining resistors and capacitors.**

The four equations are

$$\begin{aligned}
 R_{\text{series}} &= R_1 + R_2 + R_3 + \dots \\
 \frac{1}{R_{\text{parallel}}} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots \\
 \frac{1}{C_{\text{series}}} &= \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots \\
 C_{\text{parallel}} &= C_1 + C_2 + C_3 + \dots
 \end{aligned}
 \tag{10.6}$$

but you do not need to remember all of those. All you need to remember is the simplest one for resistors in series and that the others flip from there. You know it’s either series or parallel and either adding straight or adding reciprocals. So just remember the first one and build from there.

7. **Problem-solving with circuits has many ‘safeguards’, use them but do not rely on them.**

If there are two resistors in parallel and one has twice the resistance of the other, you know that it should also have half the current. If you set up your equations wrong when solving for resistances, you may get a negative resistance. Current is conserved so if it splits, it has to sum back to the original number. Checking off these boxes (so to speak) will not tell you if you’ve done it right, but it will often tell you if you’ve done it wrong, so pay attention to them.

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Chapter 11

Modern physics

11.1 Relativity

On a conceptual basis, understanding relativity, especially at the intro level, is like trying to understand Dr Seuss¹. It's just different and weird enough from the world you are accustomed to, so it's not going to be intuitive as classical physics can be intuitive. The only time students seriously struggle with relativity is when they fight it and decide they cannot understand it because it's too different from the classical world. Relax, accept the basic principles and you'll figure it out. You might even enjoy your new perspective on the world.

The two core principles of relativity are:

1. The speed of light is the same to all observers regardless of motion.
2. The laws of physics still hold in an inertial reference frame.

The first one gives that if you are looking for how fast light moves relative to an observer, the answer is always c . In addition, if you ever get an answer greater than c for any problem, you are doing it wrong. The second one is just saying that none of our usual laws change in a moving reference frame as long as that reference frame is not accelerating. This is why the so-called 'Twin Paradox' where one twin travels at a very high velocity and thus ends up being younger is not a paradox. Though they both see the other's clock ticking more slowly, the 'correct' one is the one who stayed in an inertial reference frame the entire time, namely the one who stayed on Earth. The other one had to accelerate.

¹ You don't need to know what a gardinka is to know why The Grinch was annoyed by them. It's a noisy toy of some sort and the details are unimportant.

We can add three more to this that aren't formal principles but do apply:

1. The system should reduce down to what you expect from classical physics if the velocity is much lower than c .
2. You never see yourself changing.
3. We can disagree on when and where something happens, but we cannot disagree that it happens

(4) is important for keeping the symbols straight and figuring out how to set up the equations. If you are carrying a ruler and wearing a watch, you will never see that ruler change length nor will you see the watch tick slower or faster. In order to see relativistic effects, something must be moving **relative** to you. What follows next is a quick discussion on time dilation, length contraction, and relative velocities (Lorentz transformations). (5) helps you to keep yourself grounded in reality. When something happens or where it happens is relative to an observer, but it **always** happens. For an extreme example, you cannot dodge a bullet in one reference frame and not in another due to relativity so that you are alive according to one observer and dead according to another.

11.1.1 Time dilation

As the name implies, time dilation means time expands, or moves more slowly. You should be at least vaguely familiar with the concept of space-time now and you can think of this as the faster an object moves through space, the slower it moves through time. The equation that governs this is

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (11.1)$$

where $\Delta\tau$ is called the proper time interval and Δt is the observer time interval. Both proper time and proper length are often given ridiculous definitions that are confusing ways of saying 'this is how an object views itself'. That means that if you are carrying a ruler and wearing a watch, the time the watch keeps and the length of the ruler are your proper time and proper length.

With that in mind, let's go through the classic muon decay problem **both** ways, using time dilation and length contraction, and show we get the same result from different perspectives.

Example: Muons have a half-life of 1.56 microseconds (1.56×10^{-6} s) and are created in the upper atmosphere traveling at about $0.99c$. Classically, virtually no muons should reach the surface of the Earth, a distance of about 10 km, because they will have decayed before they reach it. Due to relativistic effects, we do see quite a few muons. If 1000 muons are created in the upper atmosphere, determine how many will reach the surface of the Earth on average from the perspective of an observer on the surface of the Earth.

Solution:

First off we need to have a brief aside about half-life. The half-life of a substance is how long it takes for half of it to decay. So after one half-life there is half of it left, after two there is 1/4, after 3 there is 1/8, and so on. After n half-lives then, there are $\frac{1}{2^n} \times$ (original number of particles) remaining. If they are traveling at $0.99c$, then the time it takes will still just be given by

$$v = \frac{d}{t},$$

$$t = \frac{d}{v} = \frac{10 \text{ km}}{0.99c} = \frac{10 \times 10^3 \text{ m}}{0.99 \times 3 \times 10^8 \text{ m s}^{-1}} = 3.367 \times 10^{-5} \text{ s},$$
(11.2)

or better written as 33.67 microseconds. Divide this by microseconds per half-life (1.56) and we get number of half-lives, $\frac{33.67}{1.56} = 21.583$ half-lives. So this many half-lives pass between when the particles are created and when they hit the ground. The amount left after n half-lives is $\frac{1}{2^n}$ (original number), with $n = 21.583$ and original number being 1000, is

$$\frac{1}{2^{21.583}}(1000) = 3.183 \times 10^{-4} \text{ particles},$$
(11.3)

or less than one particle, which means none since you can't have a fraction of a particle remaining.

With time dilation though, the particles live longer *in our reference frame*. Remember we are talking about how we're seeing things so they still take the same time to hit the ground, it's just time passes more slowly for them (according to us). The proper time interval is then going to be the half-life and we can figure out how long the half-life of the muons appears to us by the following:

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Delta t = \frac{1.56 \times 10^{-6} \text{ s}}{\sqrt{1 - \frac{(0.99c)^2}{c^2}}} = 1.105 \times 10^{-5} \text{ s}.$$
(11.4)

They still take the same amount of time to hit the ground, so do the same calculation we just did but with a new half-life

$$\frac{3.367 \times 10^{-5} \text{ s}}{1.105 \times 10^{-5} \frac{\text{s}}{\text{half-life}}} = 3.04 \text{ half-lives}$$

$$\frac{1}{2^{3.04}}(1000) = 121.6$$
(11.5)

So about 121 particles actually hit the ground. This is a drastically different result.

11.1.2 Length contraction

Length contraction means that if object A is moving relative to object B, object A will appear smaller. The equation governing this is

$$L = \sqrt{1 - \frac{v^2}{c^2}} l. \quad (11.6)$$

As before, proper length (l in this equation) is the length of the object as it appears to itself; it will never see itself changing. L is the contracted length measured by an observer who is stationary relative to it. Now to do the muon problem the other way

Example: Muons have a half-life of 1.56 microseconds (1.56×10^{-6} s) and are created in the upper atmosphere traveling at about $0.99c$. Classically, virtually no muons should reach the surface of the Earth, a distance of about 10 km, because they will have decayed before they reach it. Due to relativistic effects, we do see quite a few muons. If 1000 muons are created in the upper atmosphere, determine how many will reach the surface of the Earth on average from the perspective of the **muons**.

Solution:

The speed will remain the same in both cases. However, now we're talking about the perspective of the muons, which means we can't see time dilation effects for their half-life. They will not see a change in how fast they decay. What they will see is the distance to the surface of the Earth being contracted. In other words, we see the muons reaching the Earth because they live longer, they see themselves reaching the Earth because the Earth is actually closer to them. Finding the contracted length, we have

$$\begin{aligned} L &= \sqrt{1 - \frac{v^2}{c^2}} l, \\ L &= \sqrt{1 - \frac{(0.99c)^2}{c^2}} 10 \times 10^3 \text{ m}, \\ &= 1410.7 \text{ m}. \end{aligned} \quad (11.7)$$

Velocity will remain the same and thus we can find the time it takes to hit the ground

$$t = \frac{d}{v} = \frac{1410.7 \text{ m}}{0.99 \times 3 \times 10^8 \text{ m s}^{-1}} = 4.75 \times 10^{-6} \text{ s}. \quad (11.8)$$

And just as before, determine how many half lives this is and use that to determine how many particles reach the ground

$$\begin{aligned} \frac{4.75 \times 10^{-6} \text{ s}}{1.56 \times 10^{-6} \text{ s} \frac{\text{s}}{\text{half-life}}} &= 3.04 \text{ half-lives}, \\ \frac{1}{2^{3.04}}(1000) &= 121.6. \end{aligned} \quad (11.9)$$

Which is the same number we got before. We can disagree on when something happens, where it happens or even why it happens, but we cannot disagree that it does happen.

11.1.3 Lorentz velocity transformations

The way most students get lost in the relative velocity problems is with keeping track of all the primed and unprimed variables, a convention that is well-intentioned but can be confusing. The best place to begin then is to carefully define everything. We begin with figure 11.1 for a visual description, and then give prose descriptions.

S : reference frame that you define to be stationary.

S' : reference frame that you define to be moving.

v : relative velocity between frame S and frame S' .

u : velocity of an object as measured by someone in frame S .

u' : velocity of an object as measured by someone in frame S' .

Now the two equations should make a little more sense. They are

$$\begin{aligned} u' &= \frac{u - v}{1 - \frac{uv}{c^2}}, \\ u &= \frac{u' + v}{1 + \frac{u'v}{c^2}}. \end{aligned} \tag{11.10}$$

Notice two important things about these equations. First, v doesn't have a prime on it anywhere, so this is the easy way to make sure you don't get lost in all the symbols. v is just the velocity between the two frames and they have to agree about that. What they disagree on is how fast objects within those frames are moving. Second, they'll

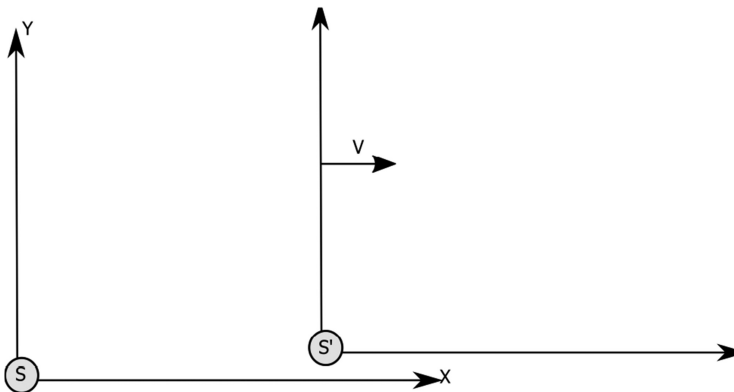


Figure 11.1. Visualization of the different reference frames and the velocity between them.

reduce down to exactly what you'd expect to see if the velocity is much lower than the speed of light, in which case they become

$$\begin{aligned} u' &= u - v, \\ u &= u' + v. \end{aligned} \tag{11.11}$$

In other words, if someone on a train moving at velocity v throws a ball with speed u' , it will be moving at $u = u' + v$ according to someone on the ground. This is how you can check signs to make sure you've got the right equation and you've set it up right. Reduce it to a nonrelativistic case ($v \ll c$) and it should be what you expect to see.

Example: You are standing on the ground and see a train go by to your right at $0.9c$. Someone in the train throws a baseball to the left at $0.7c$ relative to him. How fast is the ball moving according to you?

Solution:

We'll define the ground to be the stationary frame and so we need to use

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}. \tag{11.12}$$

v : relative velocity between train and ground.

u : velocity of an object as measured by someone on the ground.

u' : velocity of an object as measured by someone in the train.

As you may guess, just plugging in the numbers straight won't give us the correct answer because we need to be careful about signs. Define right as positive and left as negative and we get

$$u = \frac{-u' + v}{1 + \frac{-u'v}{c^2}}. \tag{11.13}$$

The way we can be sure we have got it right is to then reduce this down to a classical case. If someone is moving and throws a ball in the opposite direction to their own motion, you would expect an observer on the ground to see it move more slowly. Reducing this to that case we see that $u = -u' + v$ gives the correct answer since that will give a velocity of $0.2c$. Do it the other way and you get a velocity of $1.6c$, which means it would be going faster and that isn't what we would see even in the classical world. Now we can do the actual calculation

$$u = \frac{-.7c + .9c}{1 + \frac{(-.7c)(.9c)}{c^2}} = 0.54054c \tag{11.14}$$

11.2 Quantum mechanics

Despite what pop culture tells you, quantum is not physics-speak for cool, it merely means countable. The quantum world is simply then a world where we are dealing

with individual particles, which have discrete descriptive qualities, such as energy. Everything you have learned still applies just fine as long as you *quantize* it, meaning that it must take on an integer multiple of some number.

The way this is linked into the math is with *boundary conditions*. There is always a physical reality that must be obeyed and that is where many of the equations come from. For instance, the solutions to the particle in a box problem come from noting that if it's stuck in the box, the wave amplitude must be zero at the edges of the box. This forces the wavelength to take on certain quantized values that will obey this boundary condition. You probably will not do too much with the actual math in an intro course but it will help your conceptual thinking considerably to use these boundary conditions as part of the link between the classical and quantum world.

11.2.1 The photoelectric effect

Your friend is sitting on the couch. You pull on him but not very hard and he doesn't move. You pull a little harder and you make him stand up. He sits back down, you pull **really** hard and he stands up and runs forward. This is exactly how you can think of the photoelectric effect. The actual equation describing it is

$$E = hf - \phi, \quad (11.15)$$

where the variables describe the following:

E : kinetic energy of the *ejected electron*,
 hf : energy of the *incoming photon*,
 ϕ : work function of the metal.

So then what this says physically is that a photon with energy hf (which is always the energy of a photon) strikes an electron in a metal. If the energy of the photon is at least the work function of the metal, which is how much energy is required to break the electron out of the metal, then the electron will move. If there is more energy than is needed to break it out, it just means the electron moves faster (because it has extra kinetic energy). If the E in this equation is zero, that means there's just barely enough energy to break the electron free and no more.

Example: A photon with a wavelength of 310 nm strikes a metal and the electron released crosses a 3 V potential difference. What is the work function of the metal?

Solution:

Energy is energy, if the electron had enough energy to cross a 3 V potential difference, then that's how much was left over *after* it broke free from the metal. Since the electron-volt is defined as the amount of energy required to move 1 electron across a 1 V potential difference, it had 3 eV of 'leftover' kinetic energy, or

$$\begin{aligned} E &= hf - \phi, \\ 3 \text{ eV} &= hf - \phi. \end{aligned} \quad (11.16)$$

A very nice shortcut though is that since $c = \lambda f$ (this is light) you can write

$$hf = \frac{hc}{\lambda}, \quad (11.17)$$

$$hc = \text{constant} = 1240 \text{ eV nm},$$

so you do not have to convert everything to SI units. This gives

$$3 \text{ eV} = \frac{hc}{\lambda} - \phi, \quad (11.18)$$

$$3 \text{ eV} = \frac{1240 \text{ eV nm}}{310 \text{ nm}} - \phi.$$

Notice how the units of nanometers cancel out and we are left with just energy in eV. Solving

$$3 \text{ eV} = 4 \text{ eV} - \phi,$$

$$\phi = 1 \text{ eV}.$$

Look back over it and make sure it seems right now. The incoming photon had an energy of 4 eV. The ejected electron had an energy of 3 eV. The difference has to go somewhere and so it must have gone into breaking the electron free (getting it off the couch).

11.2.2 Electron transitions

In an atom, electrons can occupy different quantized energy levels. This means they all must be some integer multiple of the same number. A subtle point that is often missed though is that atoms stay in their ground state unless something makes them get out of it. This is why the absorption spectrum is a subset of the emission spectrum. Not all wavelengths that are emitted can be absorbed because absorption requires the atom to be in the ground state. An example will be helpful here.

Example: An atom has a ground state of 1 eV and excited states of 2 eV, 3 eV, and 4 eV. How many lines are in the absorption and emission spectrums?

Solution:

Absorption requires that an electron goes from the ground state to a higher state, so the initial state must be 1 eV. It only has three places to go (2 eV, 3 eV, and 4 eV), so there are only three lines in the absorption spectrum.

The emission spectrum occurs when an electron drops down a level and it can drop down into any other lower energy level. So the possibilities are

$$4 \longrightarrow 3, 4 \longrightarrow 2, 4 \longrightarrow 1,$$

$$3 \longrightarrow 2, 3 \longrightarrow 1,$$

$$2 \longrightarrow 1,$$

which gives six lines in the emission spectrum.

11.2.3 Wavefunctions and probability

The usual definitions of a wave function (Ψ) are highly mathematical in nature. Though it is not a complete description, the best way to think of a wave function for now is that it is the particle itself. The wave function contains all the information about the particle, you just have to take a measurement on it to get that information. The problem of course is that it's not easy to get the wave function since it's buried in a second-order differential equation, but that is not a topic for this book.

The most common mistake made when dealing with wave functions is getting them confused with probability density functions, so here we will focus on the difference.

Wave functions (Ψ):

This is a wave just like wave on a string and will probably look the same, being a sine or cosine function (or an exponential which can be written as sines or cosines). It has an amplitude just like any other wave and it must follow boundary conditions as discussed above. By itself it doesn't really tell you much and in fact it cannot correspond directly to a physical quantity because it can be imaginary. What the wavefunction does do is allow you to perform operations on it that will give you physical quantities, such as momentum, position, or probability.

Probability density ($|\Psi|^2$):

This is the probability per unit length, area, or volume, depending on how many dimensions you're using (most likely just length in an intro course) of finding the particle. Formally speaking it's finding any property of the particle if the wave function is a function of something other than position, but for our purposes here you will not be using anything but position. That's all this tells you.

The most common use of this will be normalization. Normalization is another boundary condition, what it says is that the particle has to be somewhere. An integral is just a sum and so if you add up all the probabilities of the particle existing at various places over all the possible places, it has to be 1, since the particle must exist somewhere. Mathematically this is written as

$$\int_{-\infty}^{\infty} |\Psi|^2 dx = 1. \quad (11.19)$$

Normalization example: Normalize the wavefunction

$$\begin{aligned} \Psi &= Ae^{-ax}, \text{ for } x \geq 0 \\ \Psi &= 0, \text{ for } x < 0. \end{aligned} \quad (11.20)$$

Solution:

Normalizing a wavefunction means finding this constant A . Remember that an integral is just a sum, so even though it goes from negative infinity to positive infinity, all the negatives don't matter since the wavefunction is zero there.

So technically we're still adding them in but we're just adding zero. The integral then becomes

$$\begin{aligned}\int_0^{\infty} |\Psi^2| dx &= 1, \\ \int_0^{\infty} A^2 e^{-2ax} dx &= 1.\end{aligned}\tag{11.21}$$

This is an integral that is doable analytically, the result is

$$\begin{aligned}\int_0^{\infty} A^2 e^{-2ax} dx &= -\frac{A^2}{2a} e^{-2ax} \Big|_0^{\infty}, \\ &= 0 + \frac{A^2}{2a}.\end{aligned}\tag{11.22}$$

This is equal to 1, so we can now find the normalization constant A

$$\begin{aligned}\frac{A^2}{2a} &= 1, \\ A &= \sqrt{2a}, \\ \Psi &= \sqrt{2a} e^{-ax}.\end{aligned}\tag{11.23}$$

Don't make normalization harder than it needs to be, it should never be difficult unless you have an integral that requires some calculus tricks to do.

11.3 Brief aside on energy equations

Modern physics usually comes fairly hard and fast at the end of the year when you're already tired and ready to be done with physics, then gives you several different equations that seem to all be the same thing. Here is a summary of most of them, giving what they mean and how they are different.

$$\begin{aligned}E &= hf \\ E &= hf - E_0 \\ E &= \frac{n^2 h^2}{8mL^2} \\ E &= -\frac{13.6 \text{ eV}}{n^2} \\ E &= mc^2 \\ E &= \gamma mc^2 \\ E &= \frac{p^2}{2m}\end{aligned}\tag{11.24}$$

$E = hf$: energy of a photon, Planck's constant times frequency

$E = hf - E_0$: Photoelectric effect. Photon coming in with energy hf uses E_0 of it to break an electron free and whatever is left over goes to kinetic energy of the electron (E in this case).

$E = \frac{n^2 h^2}{8mL^2}$: Energy of a particle in a box of length L . The particle has mass m and is in state n . (11.25)

$E = -\frac{13.6 \text{ eV}}{n^2}$: Energy of an electron in the hydrogen atom in state n .

$E = mc^2$: Rest energy of something with mass m , meaning the energy it has simply by virtue of existing

$E = \gamma mc^2$: Total energy of something with mass m , kinetic energy plus rest energy. (11.26)

$E = \frac{p^2}{2m}$: Standard kinetic energy of $\frac{1}{2}mv^2$ written in a more convenient form for quantum calculations

11.4 Summary and important notes

1. Both quantum and relativity only change a few things. Quantum physics requires things to be quantized and fit specific boundary conditions while relativity causes the numbers to change when you start getting very fast. Everything else you've learned still applies.
2. Nothing should ever exceed c and light itself always travels at c (in a vacuum) regardless of the motion of anything. If you get an answer that contradicts either of these, there's something wrong.
3. To keep proper time and proper length straight, remember that you never see yourself changing². Proper time and proper length are just how the observer measures their own time and their own length.
4. There are quite a few equations for energy, be sure you know what each one means.

²Or my favorite analogy: relativity is like lifting; you never see yourself getting bigger, you just see other guys getting smaller.

A Handbook of Mathematical Methods and Problem-Solving
Tools for Introductory Physics (Second Edition)

Joshua F Whitney and Heather M Whitney

Chapter 12

General problem-solving tips

This book attempts to cover the most common issues that students have with connecting in-class theory and discussion to actual problems that will show up on a test. Obviously it would be much too long if it was to cover every single type of problem you could encounter, but there are some general ideas to follow for virtually every problem. When you see someone ‘good at physics’ stop and think while working on a problem, it’s very likely they are doing one of the following things.

12.1 Break the problem down into as small pieces as possible

Physics, even at the highest and most fundamental levels in things like particle colliders, is all about taking complicated systems, breaking them into smaller and more understandable pieces, then putting those pieces back together in a useful way. A good example of this is how this book has emphasized that when dealing with vectors, it’s much easier to set up your equations and then worry about the signs and angles than to do it all at once. Faraday’s Law is another place where even those who have worked with it for a very long time (e.g., us) will get confused if they try to do it all at once. Working methodically pays off.

12.2 Write down everything you have and what you are trying to find

Do this even if a variable of interest is zero. In an introductory physics course, equations should be thought of as tools. It may be very obvious from the beginning that you need a screwdriver for a job, but do you need a Phillips or a flat head? Choosing the wrong tool will at best make your problem more complicated and often make it completely unworkable. Having all of the variables written down, even (and often especially) if they are zero, makes it much easier to figure out what’s

going on conceptually and then pick the right tool for the job, rather than plugging stuff into equations where it fits and hoping it works out.

12.3 Check units

You will hear this one over and over again from pretty much everybody trying to teach you physics. There is a reason for it. It's undoubtedly a nuisance but it's a very useful nuisance. Checking units won't assure you that you're right but it will confirm if you are wrong. More importantly, checking units will often tell you exactly *where* you went wrong. If you are looking for an answer in Newton's and you got one in $\text{kg s}^2 \text{m}^{-1}$, then you can identify that you divided something where you should have multiplied it or vice versa.

This also becomes a little bit more of a subtle check when you remember that the arguments of trig functions, exponentials, and logarithms must be dimensionless. Notice how you never see e^x in physics, it's always $e^{\frac{x}{\text{something}}}$. This something must have the same units as x so that they cancel out and it just becomes a number. Important note: radians and degrees are considered dimensionless, so they do not need to cancel.

12.4 Pull all the 10^n factors out of your calculation before putting it in your calculator

The king of careless mistakes (that cost a lot of points on tests) is calculator errors. When you're dealing with a whole lot of stuff in scientific notation it becomes highly cumbersome to put it all in the calculator and keep track of brackets. Instead, pull all those 10^n pieces out to the side and do the rest of the calculation. The easiest way to see this is with an example:

What is the electric force between two electrons at a distance of 20 nm? This is a very simple problem conceptually but you can mess it up with a lot of brackets in your calculator. The electric force is

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}, \quad (12.1)$$

and we have everything that goes into this equation

$$\begin{aligned} q_1 = q_2 &= 1.6 \times 10^{-19} \text{ C} \\ \epsilon_0 &= 8.85 \times 10^{-12} \text{ C}^2 \text{ Nm}^{-2} \\ r &= 20 \times 10^{-9} \text{ m}. \end{aligned} \quad (12.2)$$

Plugging all this in gives us a mess

$$F = \frac{(1.6 \times 10^{-19} \text{ C})(1.6 \times 10^{-19} \text{ C})}{4\pi(8.85 \times 10^{-12} \text{ C}^2 \text{ Nm}^{-2})(20 \times 10^{-9} \text{ m})^2}, \quad (12.3)$$

$$= \frac{(1.6 \times 10^{-19})(1.6 \times 10^{-19})}{4\pi(8.85 \times 10^{-12})(20 \times 10^{-9})^2} \text{ N}. \quad (12.4)$$

However, we can simplify it drastically by writing it as

$$F = \frac{(1.6)(1.6)}{4\pi(8.85)(20)^2} \times \frac{10^{-19} \times 10^{-19}}{10^{-12} \times (10^{-9})^2} \text{ N}, \quad (12.5)$$

and remembering your arithmetic that when you multiply stuff with exponents you add the exponents and when you divide, you subtract, this becomes

$$\begin{aligned} F &= \frac{(1.6)(1.6)}{4\pi(8.85)(20)^2} \times \frac{10^{-38}}{10^{-30}} \text{ N}, \\ &= \frac{(1.6)(1.6)}{4\pi(8.85)(20)^2} \times 10^{-8} \text{ N}. \end{aligned} \quad (12.6)$$

Now all this stuff to the left of the 10^{-8} is much easier to put in the calculator and the answer is

$$\begin{aligned} F &= 5.75 \times 10^{-5} \times 10^{-8} \text{ N} \\ &= 5.75 \times 10^{-13} \text{ N}. \end{aligned} \quad (12.7)$$

This seems like a lot of work but only because this is an example written out in great detail. It's much less work than typing all of those exponents into your calculator and watching brackets.

12.5 Practice algebra

Especially if you have trouble with time on tests, algebra practice is huge. Even if you're taking a calculus-based course, you'll spend most of your problem time doing algebra. One of the most prominent time (and error) saving methods is recognizing when you can add two equations together and cancel something out. For instance, the Atwood machine solution:

$$\begin{aligned} T - m_1g &= m_1a, \\ -T + m_2g &= m_2a. \end{aligned} \quad (12.8)$$

Add the left side of the first equation to the left side of the second and the right side of the first to the right side of the second to get tension to cancel

$$\begin{aligned} T - m_1g - T + m_2g &= m_1a + m_2a, \\ m_2g - m_1g &= m_1a + m_2a. \end{aligned} \quad (12.9)$$

12.6 Keep everything symbolic as long as possible

Introductory physics students hate doing this but it really, really helps. The exceptions to this are problems involving thin lenses, resistors and capacitors, since those involve many reciprocals that are annoying to deal with symbolically. But on the whole, if you plug in numbers too soon it becomes easy to get lost and if you're not getting the right answer, difficult to figure out where you went wrong. In addition, you will develop more physical intuition for what is actually going on in the problem instead of just coming up with a number. In the problem with the

Atwood machine with a massive pulley, for example, a number would have just been that, a number. The equation we derived though can be checked against what we know must be true¹.

This advice is probably the least-often taken of anything we ever tell students and yet we've never seen a student who does this well and also struggles with the course. There is the obvious question of correlation versus causation here but the link remains: those who are comfortable doing the math symbolically always do well.

12.7 Electron volts

In the photoelectric effect example we used electron volts, eV. This can make calculations in quantum and atomic physics much, much simpler. If you're using eV for energy instead of joules then you can also use $hc = 1240 \text{ eVnm}$ and $e = 1$ (the charge on an electron). If you really want to get fancy and save time, you can also use mass of the electron is $511 \times 10^3 \text{ eVc}^{-2}$ and mass of the proton is $938 \times 10^6 \text{ eVc}^2$. An example is helpful here.

Example: An electron is accelerated from rest through a 10 V potential difference. How fast is it moving?

Solution:

This is conservation of energy, so start as always

$$K_i + U_i = K_f + U_f. \quad (12.10)$$

$K_i = 0$ because it's at rest and so we can write this instead as

$$\Delta U = K_f. \quad (12.11)$$

ΔU is still $q\Delta V$, but now $q = 1$ in eV

$$\Delta U = (1 \text{ e})(10 \text{ V}) = 10 \text{ eV}, \quad (12.12)$$

which is equal to the final kinetic energy

$$10 \text{ eV} = K_f = \frac{1}{2}mv^2. \quad (12.13)$$

With this new system of units, $m = 511 \times 10^3 \text{ eV}/c^2$

$$10 \text{ eV} = \frac{1}{2}(511 \times 10^3 \text{ eV}/c^2)v^2. \quad (12.14)$$

Cancel out the units of eV and solve for v in terms of c

$$\frac{20 \text{ c}^2}{511 \times 10^3} = v^2, \quad (12.15)$$

$$v = \sqrt{\frac{20}{511 \times 10^3}} c = 6.256 \text{ 1} \times 10^{-3}c.$$

¹Your textbook authors shared a particularly comical moment in which they were discussing an Atwood's machine problem and convinced themselves that the acceleration would be infinite. That could not be true.

And now just multiply by the speed of light ($3 \times 10^8 \text{ m s}^{-1}$) to get an answer in SI units

$$\begin{aligned}v &= 6.256\ 1 \times 10^{-3}(3 \times 10^8 \text{ m s}^{-1}), \\ &= 1.876\ 8 \times 10^6 \text{ m s}^{-1}.\end{aligned}\tag{12.16}$$

If this is super confusing to you then you can skip it but it is recommended, especially if you struggle with time on tests.