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Modern Quantum Mechanics and Quantum Information

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Chapter 8

An alternative reality

8.1 Gazing in wonder

Up to this point, the discussion of the quantum theory in this book has followed the standard pedagogical practice of laying things out in a calm and methodical fashion, implying that the discoveries of Einstein, Bohr, Heisenberg, Schrödinger, etc are part of a predictable evolution. Even so, some readers may have found aspects of quantum theory to be strange and counterintuitive or, in a word, weird. The position taken in this chapter is that the reader who finds quantum phenomena to be weird is wrong in the sense that quantum phenomena are much weirder than anything he or she could imagine. Some well known experiments are reanalyzed, emphasizing the aspects that are most difficult to understand. More recent experiments that challenge the conventional understanding of quantum theory are also described

8.2 The Einstein–Podolsky–Rosen experiment

The Einstein–Podolsky–Rosen (EPR) experiment was described in chapter 1, but here it is analyzed from a different angle. Suppose two observers, Alice and Bob are a very long distance from each other. In principle, they could be light years apart. Suppose Alice has two spin 1/2 particles that are bound into a spin 0 pair

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle\,|-\rangle - |-\rangle\,|+\rangle). \tag{8.1}$$

They are split apart. Alice keeps one particle and sends the other to Bob.

Stern–Gerlach devices were discussed in chapter 1. Assume Alice and Bob have such devices and both are oriented in the z-direction, (SG_z). The experiment is done N times. The outcomes of their measurements are written as in table 8.1.

Experimentally it will be found that $N_1 = N_2$, so the probabilities of these events, calculated quantum mechanically, are

Table 8.1. The N_i in column 1 are the number of times that the measurements described in columns 2 and 3 occur.

Number of occurrences	Alice's measurement	Bob's measurement
N_1	$(\hat{\mathbf{z}}+)$	$(\hat{\mathbf{z}}-)$
N_2	$(\hat{\mathbf{z}}-)$	$(\hat{\mathbf{z}}+)$

$$P_1 = \frac{N_1}{N} = 0.5 \quad P_2 = \frac{N_2}{N} = 0.5. \tag{8.2}$$

This experiment appears to show that Alice is communicating the sign of her spin to Bob faster than the speed of light, because the instant she looks at her data and sees a $(\hat{\mathbf{z}}+)$, she knows Bob has a $(\hat{\mathbf{z}}-)$, even if Bob is millions of miles away. The wave function has collapsed from the one in equation (8.1) to

$$|\psi\rangle = |+\rangle\,|-\rangle. \tag{8.3}$$

The literal interpretation of the wave function, that the particles are actually wave like, leads to the conclusion that the instantaneous collapse of the wave function when one observer makes a measurement means 'stuff' is moving faster than the speed of light. The more sophisticated Born picture of the wave function leads to other difficulties. Einstein believed that it would be manifestly wrong for the EPR experiment to work because it would violate what he called the locality principle. The result of an experiment cannot be influenced by an action a long way off. He proposed the EPR as a gedanken experiment to prove that quantum mechanics is wrong.

Einstein's argument has been refuted. The direct proof is that the EPR experiment can actually be done today and, when it is, it agrees exactly with the orthodox quantum mechanical prediction. How, then, can Einstein's arguments that the laws of locality and relativity are violated be refuted? The answer is that the EPR cannot be used to send information in the usual sense. In the course of the experiment, Bob simply has an apparently random list of $|+\rangle$ and $|-\rangle$ results. It is only after he and Alice compare their results, with a signal that is subluminal, that they understand the correlations.

Although a message cannot be sent with the EPR method, it does have practical applications. As will be discussed later, it can be used for quantum key distribution.

8.3 Hidden variables

Einstein conceded that his claim that the EPR experiment violates the tenets of relativity does not hold up, but he still did not like the conclusions. He went back to an idea that he had proposed when the quantum theory first began to take shape, called the hidden variables theory. The proposal is that quantum theory is like the early theories of thermodynamics and fluid dynamics. The mathematics developed in those areas is very sophisticated and highly predictive of the phenomena that are observed. However, it is now known that there is an underlying reality and that the materials being treated as continua are actually made up of atoms and molecules. The more recent theory of statistical mechanics and molecular dynamics calculations obtain the same results on the basis of atomic theory. Einstein's argument is that Born's statistical interpretation of the wave function is just a statement of our ignorance of the underlying physics, and that, later, all quantum phenomena can be explained on the basis of variables that are presently hidden from us.

The simplest hidden variable explanation of the EPR experiment is that the spin-zero wave function in equation (8.2) is a fiction. For a reason that is not understood now, Alice's electrons must be created in pairs with one having spin up and the other having spin down. If Alice sends an up electron to Bob, then classical reasoning says the one she kept is down. Since Alice can't see which way the electron spin points before she measures it, statistically the probability she will send Bob an up or down electron is 0.5.

Although the hidden variable theory described above gives the correct answer for the EPR experiment, it requires a complete restructuring of classical theory in order to predict the hidden variable state for the electron pairs. There are still some physicists who find quantum theory so logically unsatisfying that they have attempted this. When put into practice, hidden variable theories become extremely convoluted and most physicists ignore them.

The state described in equation (8.1) is now called an entangled state. In chapter 10 it will be seen that entangled states are fundamental components in the devices used in the field called quantum information (QI). They are so important that experimentalists have developed numerous practical methods for making entangled states with photons, electrons, and other things. QI devices are already being used in cryptography and computing, and a growing number of high tech companies are selling them. It might be said that entangled states are becoming standard engineering practice.

It would appear that hidden variable theory is finished, but it is necessary to understand both sides of the argument because there are some who believe that they offer something to quantum theory. Papers are still being written that revive some of the arguments.

8.4 Bell's inequalities

The physicist John Stewart Bell was a great admirer of Einstein, and found Einstein's arguments for hidden variables compelling. He developed theories for more advanced forms of the EPR experiment with the hope of supporting hidden variables. His best known attempt is the following.

Suppose Alice and Bob have three Stern–Gerlach devices oriented in three directions specified by the unit vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$. Alice and Bob can measure the spin direction of a particle with any one of these devices. The notation $(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$ means that, if the observer uses the device with the orientation $\hat{\mathbf{a}}$ they will see the particle has spin up, if they use their second device that has orientation $\hat{\mathbf{b}}$ they will find it with spin down, or if they use their third device they

Table 8.2. The N_i in column 1 are the number of times that the measurements described in columns 2 and 3 occur.

Number of occurrences	Alice's measurement	Bob's measurement
N_1	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+)$	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-)$
N_2	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$
N_3	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$
N_4	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-)$	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+)$
N_5	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+)$	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-)$
N_6	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$
N_7	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$
N_8	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-)$	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+)$

will find that it has spin up. If the spins are entangled as in equation (8.1), if Alice has the possibilities $(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$ Bob must have the possibilities $(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$.

The possible observations that Bob and Alice can make and the number of such observations that will be made if the experiment is repeated many times are given in the Table 8.2. The number of occurrences for Alice having a given set of possibilities is hypothetical rather than experimental because it is not possible for her to know the results of all three experiments. It will be seen that the important properties of the N_i are only that they exist and are greater than zero. If hidden variables were assumed to exist, it would in principle be possible to calculate the N_i.

Suppose Alice chooses to use the device that measures in the $\hat{\mathbf{a}}$ direction and Bob chooses to use the device that measures in the $\hat{\mathbf{b}}$ direction. The probability that they will both see particles with spin up is $P(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+)$. By scanning through the table it is seen that Alice sees $\hat{\mathbf{a}}+$ at the same time Bob sees $\hat{\mathbf{b}}+$ only for the occurrences in the third and fourth row, so

$$P(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+) = \frac{N_3 + N_4}{N}, \tag{8.4}$$

where

$$N = N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8. \tag{8.5}$$

Simply from the fact that $N_i > 0$ it is possible to write

$$\frac{N_3 + N_4}{N} \leqslant \frac{N_3 + N_4 + N_2 + N_7}{N} = \frac{N_4 + N_2}{N} + \frac{N_3 + N_7}{N}, \tag{8.6}$$

and again from the table the measurements that are common for Alice and Bob in rows two and four are $\hat{\mathbf{a}}+$ and $\hat{\mathbf{c}}+$. The common measurements in rows three and seven are $\hat{\mathbf{c}}+$ and $\hat{\mathbf{b}}+$. It follows that the manipulation in equation (8.6) leads to

$$P(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+) \leqslant P(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+) + P(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+). \tag{8.7}$$

These probabilities can be calculated unambiguously by quantum mechanics because they are the absolute square of an inner product of two known states

$$P(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+) = |\langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}-\rangle|^2 \, P(\hat{\mathbf{a}}+;\hat{\mathbf{c}}+) = |\langle\hat{\mathbf{a}}+|\hat{\mathbf{c}}-\rangle|^2 \, P(\hat{\mathbf{c}}+;\hat{\mathbf{b}}+) = |\langle\hat{\mathbf{c}}+|\hat{\mathbf{b}}-\rangle|^2. \tag{8.8}$$

The reason Bob's plus spin states are replaced by minus states in the inner products is that those states belong to Alice. In order for Bob to see a plus state, Alice must see a minus. According to the rules of quantum mechanics, Alice can create all of the required states and take the inner products. She has Stern–Gerlach devices pointing in the $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ directions, so it is only a matter of sending enough electrons through the devices to find one with the spin in the proper direction.

All of the equations derived so far are true for any choice of $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, and the same result would be obtained. However, there is a choice that makes the calculations easier. The choice is to have the vectors in one plane, and to choose the axes in that plane as shown in the following drawing (figure 8.1).

As can be seen from this figure, $|\hat{\mathbf{a}}+\rangle = |+\rangle$ and $|\hat{\mathbf{a}}-\rangle = |-\rangle$. It was shown in chapter 1 that the spin eigenfunction for an arbitrary direction $\hat{\mathbf{n}}$ in the x-z plane is

$$|\,\hat{\mathbf{n}}+\rangle = \cos\frac{\theta}{2}\,|+\rangle + \sin\frac{\theta}{2}\,|-\rangle, \tag{8.9}$$

and

$$|\,\hat{\mathbf{n}}-\rangle = -\sin\frac{\theta}{2}\,|+\rangle + \cos\frac{\theta}{2}\,|-\rangle. \tag{8.10}$$

The angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is called θ_{ab} so

$$\langle\hat{\mathbf{a}}+|\hat{\mathbf{b}}-\rangle = -\sin\frac{\theta_{ab}}{2}. \tag{8.11}$$

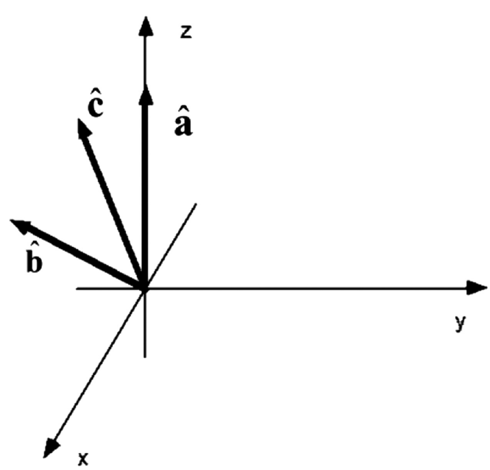

Figure 8.1. Bell's experiment.

It is also easy to see that

$$\langle \hat{\mathbf{a}}+|\hat{\mathbf{c}}-\rangle = -\sin\frac{\theta_{ac}}{2}. \tag{8.12}$$

Reorienting the z axis to be in the $\widehat{\mathbf{c}}$ direction gives

$$\langle \hat{\mathbf{c}}+|\hat{\mathbf{b}}-\rangle = -\sin\frac{\theta_{cb}}{2}. \tag{8.13}$$

Equation (8.7) then becomes

$$1/2(\sin\theta_{ab}/2)^2 \leqslant 1/2(\sin\theta_{ac}/2)^2 + 1/2(\sin\theta_{cb}/2)^2 \tag{8.14}$$

where the 1/2 comes from the probability that Alice will measure a spin up in the first place. Choosing the simplest case $\theta_{ac} = \theta_{cb} = 2\varphi$ leads to

$$1/2(\sin 2\phi)^2 = 2(\sin\phi\cos\phi)^2 \leqslant (\sin\phi)^2, \tag{8.15}$$

or

$$\cos^2\phi \leqslant \frac{1}{2}. \tag{8.16}$$

This is clearly not true for $0 < \phi < \frac{\pi}{4}$. Equation (8.15) is usually called Bell's inequality.

Bell's inequality is conceded to prove that the most common forms of the hidden variable picture are wrong. This caused him considerable discomfort, because it is opposite to what he was hoping for. He later wrote about the hidden variable explanation of a different experiment, the double slit interference experiment 'For me, it is so reasonable to assume that the photons in those experiments carry with them programs, which have been correlated in advance, telling them how to behave. This is so rational that I think that when Einstein saw that, and the others refused to see it, he was the rational man. The other people, although history has justified them, were burying their heads in the sand. I feel that Einstein's intellectual superiority over Bohr, in this instance, was enormous; a vast gulf between the man who saw clearly what was needed, and the obscurantist. So for me, it is a pity that Einstein's idea doesn't work. The reasonable thing just doesn't work'.

Although mortally wounded, this did not destroy the hidden variable program. There are those who simply added another twist to an already convoluted program in order to get around Bell's conclusions.

8.5 Double slit interference

Most physicists are familiar with Young's two slit interference experiment in optics. Feynman was fond of saying that all of quantum mechanics can be gleaned from carefully thinking through the implications of this experiment being applied to electrons. Feynman's statement becomes even more true when modern interference experiments are considered, as will be seen later.

Consider an incident plane wave of light generated, e.g. by a laser, passing through two slits. The ideal double slit interference pattern seen on the screen is $I_{ds} = C(1 + \cos(2\pi dy/\lambda L))$, where d is the distance between slits. The waves will interfere constructively at distances y from the center given by $y = n\frac{\lambda L}{d}$. An ideal interference pattern is shown in figure 8.2.

This is the aspect of the experiment that is of interest, but the experimental results do not look like the above because of the finite width of the slits. Light passing through a single slit produces a diffraction pattern on the screen that is described by

$$I_{ss} = I_0\left[\frac{\sin(2\pi ay/\lambda L)}{2\pi ay/\lambda L}\right]^2. \tag{8.17}$$

In this formula, a is width of slit, L is distance to screen, λ is the wave length, and y is the distance from the central line in the screen. The total interference and diffraction pattern is given by the product $I(y) = I_{ss}I_{ds}$ and shown in figure 8.3.

Young is famous for his experiment on light, and appeared to settle the debate as to whether light is a wave or a particle. It was later shown that Maxwell's equations can be manipulated to obtain a wave equation

$$\nabla^2 f(\mathbf{r}, t) = \frac{1}{c^2}\frac{\partial^2 f(\mathbf{r}, t)}{\partial t^2}, \tag{8.18}$$

that describes electromagnetic waves propagating with the speed of light c. This was interpreted to be the final proof of the wave nature of light, x-rays, radio waves, and all other electromagnet waves. Feynman's quantum electrodynamics (QED) reawakened this discussion, however, because the theory describes light as a collection of photons.

Schrödinger's equation predicts that electrons obey the wave equation

$$\nabla^2\psi(\mathbf{r}, t) = i\hbar\frac{\partial\psi(\mathbf{r}, t)}{\partial t}. \tag{8.19}$$

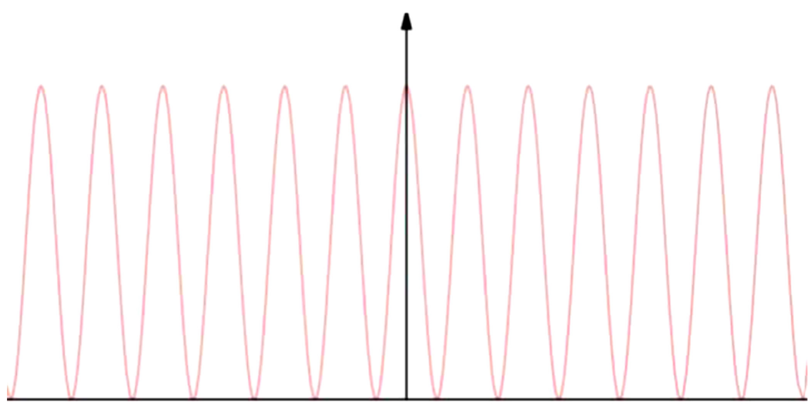

Figure 8.2. Ideal interference pattern.

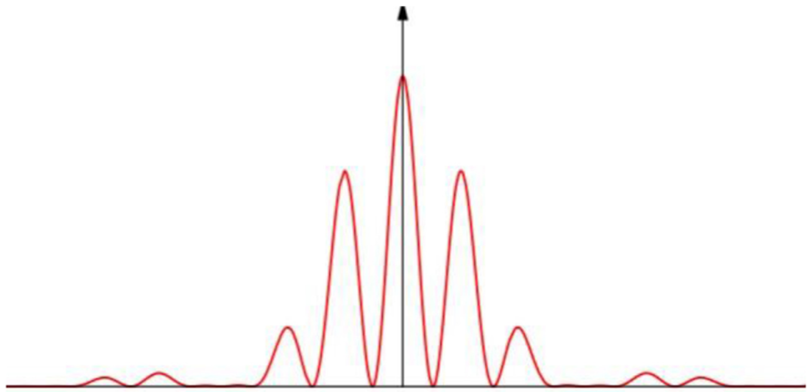

Figure 8.3. Experimental interference and diffraction pattern.

An obvious result of this is to try Young's experiment with electrons. The result is that they give the interference fringes expected for a wave with the de Broglie wave length $\lambda = h/p$. With modern equipment similar experiments can be done with atoms and even molecules.

These experiments seem to be definitive proof of the wave picture, but what happens if the intensity of the electron beam is made very low. A very sensitive experiment with one electron at a time was done at the Hitachi R & D Laboratory [1]. Using an advanced form of the kind of electron gun that was used in television sets at the time. A weak beam of electrons was allowed to pass through two slits. The sequence of spots that were found on the screen is shown in figure 8.4.

It is clear that electrons have a particle nature. All known detectors show the electron only when it interacts with matter, which is made of atoms. This means that they are localized, at least on the atomic scale. In photograph (a) the dots that indicate electrons appear to be random. They cannot be entirely random because, as more and more electrons hit the screen, the interference fringes develop.

There are areas of the screen where the probability for an electron spot to appear is very unlikely, and these regions don't change as the number of electrons increases or decreases. This is an illustration of Born's statistical interpretation of the wave function put forward in the early days of quantum mechanics. The wave function (or at least its absolute square) gives the probability for an event happening. This idea was carried further by J von Neumann who said the there is a state vector that evolves deterministically according to the Schrödinger equation, and that the process of measuring 'projects out' randomly one of the values that are allowed by the wave function. Each particle seems to strike at a random spot. It is only after a number of them have struck the screen that the diffraction pattern emerges.

Einstein noted in 1905 that the photoelectric effect, the ejection of electrons from a solid when light is shown on it, could only be explained by assuming that the light is made up of packets that have energy $h\nu$. These packets are the photons from QED. As with electrons, the photons are detected by their interaction with atoms in the screen, and are localized on the atomic scale. The results of double slit

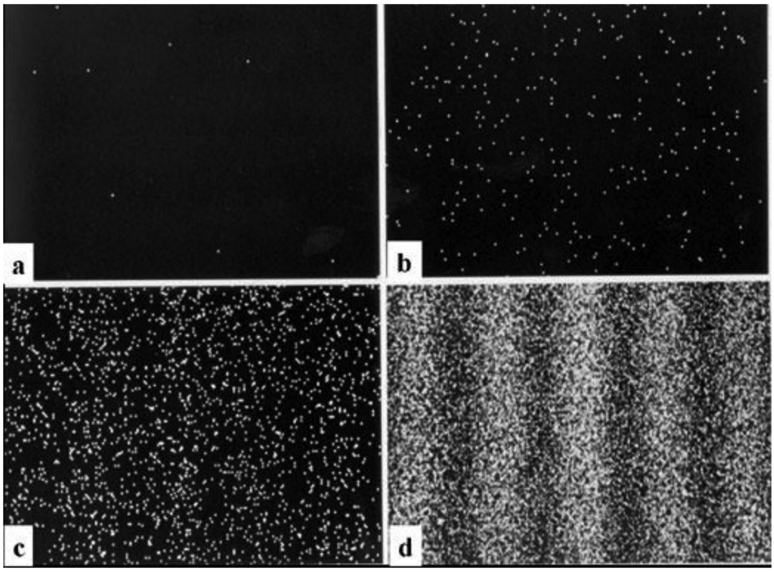

Figure 8.4. The number of electrons accumulated on the screen are; (a) 8 electrons; (b) 270 electrons; (c) 2000 electrons; (d) 160 000. The total exposure time from the beginning to the stage (d) is 20 min.

experiments with photons appear just like the electron results shown above if the beam intensity is low enough. The predictions of Huygens and Fresnel and the derivation from Maxwell's equations only hold in the limit of very many photons. This means that the questions arising in the interpretation of quantum mechanics are the same for photons and particles.

An aspect of quantum interference that can be seen from the low intensity measurements is that an electron or photon can only interfere with itself. If two electron guns are aimed at the slits, even though they fire the electrons simultaneously, there will be no interference. In addition, if an experiment is done to determine which slit the particle passes through, there is no interference. This will be made more clear in the quantum erasure experiments discussed later.

The concept of wave particle duality is introduced early in quantum mechanics courses. The idea is that sometimes an electron will act as a wave, and sometimes as a particle. The remarkable thing about the experiment illustrated in figure 8.4 is that it is showing wave and particle behavior simultaneously. As Bell pointed out, in order to explain this with hidden variables 'the photons in those experiments carry with them programs, which have been correlated in advance, telling them how to behave'. The electron that makes the first spot on the screen somehow has to know where all the other electrons are going to land. The quantum description is surprising, but the hidden variable description is unbelievable.

8.6 The adiabatic theorem

The next experiments to be discussed require a quantum theorem that was developed early. The standard reference for the adiabatic theorem is Volume II of

'*Quantum Mechanics*' by Albert Messiah [2]. He attributes a version derived for Heisenberg's matrix mechanics version of quantum theory to Ehrenfest. The theorem was first proved for Schrödinger's version of quantum mechanics by V Fock in 1928. T Kato and K O Friedrichs worked on a more mathematical form of the theorem, as have others. The derivation in Messiah is complicated and confusing because he talks about a Hamiltonian $H(t)$ that is time dependent without specifying how this time dependence comes about. As is well known, the Hamiltonian contains no explicit time dependence in the Schrödinger picture. It can only have a time dependence through the time dependence of parameters, such as the positions of the nuclei in a molecule in the Born–Oppenheimer approximation or external fields. The notation that makes this clear is $H(R(t))$.

In the Dirac notation, the time dependence of the state vector $|\psi(R(t),\, t)\rangle$ is given by

$$H(R(t))\,|\psi(R(t),\, t)\rangle = i\hbar\frac{d\,|\,\psi(R(t),\, t)\rangle}{dt}, \tag{8.20}$$

or

$$|\psi(R(t),\, t)\rangle = U(t,\, t_0)|\psi(R(t),\, t_0)\rangle. \tag{8.21}$$

The eigenvalues of $H(R(t))$ are time dependent through the time dependence of the parameters

$$H(R(t))|\psi(R(t),\, t)\rangle = \varepsilon_j(R(t))|\psi(R(t),\, t)\rangle. \tag{8.22}$$

The adiabatic theorem is that

$$U(t,\, t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t}\varepsilon_j(R(\tau))d\tau}. \tag{8.23}$$

The adiabatic theorem can be used to find the phase for an electron moving in a magnetic field. The Hamiltonian is

$$\frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{2m} + v(\vec{r}) = \frac{p^2}{2m} + v(\vec{r}) - \frac{e}{mc}\vec{A}\cdot\vec{p} + \frac{e^2}{2mc^2}A^2. \tag{8.24}$$

Ignoring the last term and using first order perturbation theory

$$\varepsilon = \varepsilon_0 - \frac{e}{mc}\vec{A}\cdot\langle\vec{p}\,\rangle \approx \varepsilon_0 - \frac{e}{c}\vec{A}\cdot\frac{d\langle\vec{r}\,\rangle}{dt}, \tag{8.25}$$

the phase in equation (8.23) is

$$-\frac{i}{\hbar}\int_{t_0}^{t}\varepsilon dt = -\frac{i}{\hbar}\varepsilon_0(t - t_0) + \frac{ie}{\hbar c}\int_{\vec{r}(t_0)}^{\vec{r}(t)}\vec{A}\cdot d\vec{r}. \tag{8.26}$$

Dirac [3] pointed out that if space is divided into cubes that are small on a macroscopic scale but large on the quantum scale, then $\vec{A}(\vec{r})$ is a constant in each cube so equation (8.25) holds. However, it changes from cube to cube so equation (8.26) leads to

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\varepsilon_0(t-t_0)+\frac{ie}{\hbar c}\int_{\vec{r}(t_0)}^{\vec{r}(t)}\vec{A}(\vec{r})\cdot d\vec{r}}\,|\psi(t_0)\rangle. \tag{8.27}$$

8.7 The Bohm–Aharanov phase

The Bohm–Aharanov phase appears when a charged particle moves around a localized magnetic flux. The effect was analyzed theoretically by D Bohm and his student Y Aharanov [4], and was put forward as a gedanken experiment to demonstrate that orthodox quantum mechanics is wrong. A more modern analysis of the phase is given below.

If one imagines that the electron is in a wave-packet state that is highly localized, it is possible to talk about the position of the electron. Bohm and Aharonov noted that, if the electron follows a trajectory C that encircles a solenoid containing a magnetic field **B**, the wave function at the screen will take on the net phase that is the difference between the one from the upper path and the lower path, as illustrated in figures 8.1 and 8.5

$$\varphi(\vec{r}_0, \vec{r}_s) = \frac{e}{\hbar c}\int_{\vec{r}_0}^{\vec{r}_s}\vec{A}(upper)\cdot d\vec{r} - \frac{e}{\hbar c}\int_{\vec{r}_0}^{\vec{r}_s}\vec{A}(lower)\cdot d\vec{r} = \frac{e}{\hbar c}\int_C \vec{A}\cdot d\vec{r}. \tag{8.28}$$

This phase has been calculated with the adiabatic theorem, equation (8.27).

The magnetic field is constrained to be within a cylinder that is so small that it fits behind the part of the middle screen that separates the two slits. From classical electromagnetism it is known that the magnitude of the vector potential falls off like $1/r$ outside of the cylinder.

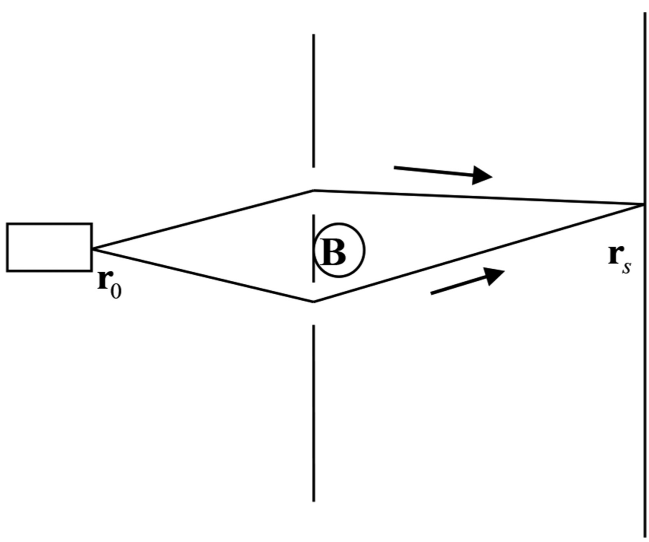

Figure 8.5. Trajectories of an electron in Bohm–Aharanov double slit experiment.

Using Stokes theorem and the fact that the magnetic field is the curl of $\vec{A}$, the phase can be manipulated

$$\varphi(\vec{r}_0, \vec{r}_s) = \frac{e}{\hbar c}\int_C \vec{A} \cdot d\vec{r} = \frac{e}{\hbar c}\int_S (\nabla \times \vec{A}) \cdot \hat{\mathbf{n}} dS = \frac{e}{\hbar c}\int_S \vec{B} \cdot \hat{\mathbf{n}} dS = \frac{e}{\hbar c}\Phi, \quad (8.29)$$

where Φ is the total magnetic flux in the cylinder. The electrons passing through the slits do not touch the cylinder, so every possible trajectory from $\vec{r}_0$ to $\vec{r}_s$ will have the same phase shift.

If there is no magnetic field, the wave function at the screen will manifest the ordinary double slit interference pattern as shown in figure 8.3. The addition of a magnetic field will cause a shift in the interference pattern as illustrated in figure 8.6.

Bohm and Aharanov's argument that this result cannot be correct is that the path of a particle cannot be modified by a potential alone. In classical electromagnetism, potentials are a mathematical convenience. Forces are caused by fields. For example, the force on an electron is

$$\vec{F} = e\vec{v} \times \vec{B}. \quad (8.30)$$

In this experiment, the electrons pass through a region in which $\vec{B} = 0$ but $\vec{A} \neq 0$. Quantum theory in both the Heisenberg and Schrödinger formulations makes use of Hamiltonians, and Hamiltonians necessarily contain potential functions.

Somewhat surprisingly this experiment can be carried out [5]. Bohm and Aharanov were visiting the University of Bristol when they proposed it. A professor at that university, R G Chambers, was aware of a newly discovered form of matter known as iron whiskers. These are single crystals that are long but only nanometers wide. They are even better at constraining a magnetic field than soft iron. They can be grown using a technique in which ferrous chloride is reduced in a hydrogen gas flow at high temperature. Their small diameter and high susceptibility make them the perfect material for constructing a Bohm–Aharanov device.

Chambers observed exactly the shift in interference fringes predicted by quantum theory. The argument that only fields can modify the behavior of particles is simply another illustration of the difference between the classical world and the quantum world.

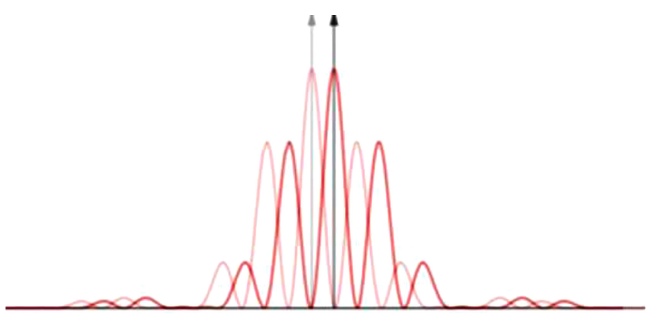

Figure 8.6. Shift in pattern caused by presence of magnetic flux.

8.8 The Berry phase

The Berry geometrical phase [6] appears when time-dependent parameters in a quantum Hamiltonian change around a closed path in parameter space. It is an addition to the standard dynamical phase $\varphi = \int_{t_0}^{t} \varepsilon(\tau) d\tau$ predicted by the adiabatic theorem described above. Traditionally, any observable in quantum mechanics is regarded as the eigenvalue of a Hermitean operator. Berry showed by construction that there are observables of a completely different nature. The Berry phase is a well defined gauge-invariant phase of the state vectors that can be measured experimentally, but it cannot be expressed as the eigenvalue of any operator because it depends on the particular path in parameter space that one chooses to traverse. This is the reason for the term 'geometrical phase'.

It will be seen that the Berry phase only exists when the Hamiltonian that is used to describe the system contains time-dependent parameters. Such a Hamiltonian does not describe an isolated system. The time-dependent parameters describe the rest of the Universe that is not included in the Hilbert space that is being considered, e.g., external fields that are not being treated quantum mechanically. In a truly isolated system, there will be no time-dependent parameters and hence no Berry phase. In this sense, the Berry phase is an unnecessary semiclassical concept. Its value stems from the fact that on many occasions a parameterized Hamiltonian is the most convenient way to treat a physical problem. On such occasions, a Berry phase will also be useful and unavoidable.

The Berry phase is based on the concepts of holonomy and anholonomy [7] in the abstract space of the parameters in the quantum Hamiltonian. Anholonomy is the failure of certain variables to return to their original values in a system that appears to be periodic. Anholonomy can be illustrated in a non quantum mechanical context by considering the parallel transport of a vector around a closed path in curved space. A sphere can be considered to be a curved two-dimensional space, for example, a globe that shows the map of the Earth. Put a pencil on the north pole of such a globe pointing along any longitude. Move the pencil in the direction of its point along the longitude until it reached the equator. Now move the pencil along the equator with the eraser end always pointing toward the north pole. When it reaches some other longitude, move it in the direction of the eraser back to the north pole. Although the pencil is kept parallel throughout this circuit in the sense that the eraser always points in the same direction, the curvature of the space leads to the result that the final position of the pencil makes an angle ϕ with the original position. This experiment is shown in figure 8.7.

In order to use this concept in a quantum mechanical context, consider a Hamiltonian that depends on the momentum operators for N particles $\vec{p}_i$, the position operators $\vec{r}_i$, the spins of the particles s_i, and some number d of time-dependent parameters $R_\alpha(t)$,

$$H\big(\vec{p}_1, \vec{p}_2, \dots, \vec{p}_N, \vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, s_1, s_2, \dots, s_N, R_1(t), R_2(t), \dots, R_d(t)\big). \quad (8.31)$$

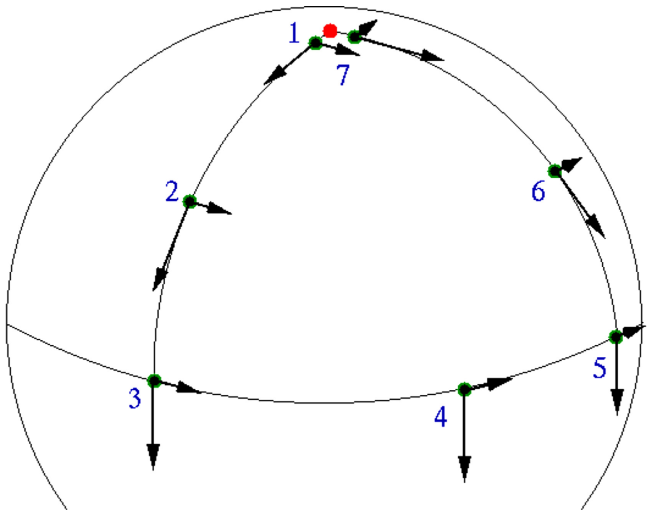

Figure 8.7. Sphere to illustrate anholonomy.

In the following equations, the parameters that are not of immediate interest will be suppressed, and the ones that appear are abbreviated. The wave function

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots, \mathbf{r}_N, s_1, s_2, s_3, \dots, s_N, R_1(t), R_2(t), R_3(t), \dots, R_d(t)), \tag{8.32}$$

is a solution of

$$H\Psi(R(t), t) = i\hbar\frac{\partial\Psi(R(t), t)}{\partial t}, \tag{8.33}$$

and hence

$$\Psi(R(t), t) = U(t, t_0)\Psi(R(t_0), t_0). \tag{8.34}$$

The Hamiltonian has eigenvalues and eigenvectors that depend on time through the variation of the $R_\alpha(t)$

$$H\psi_j(R(t), t) = \varepsilon_j(R(t))\psi_j(R(t), t). \tag{8.35}$$

According to the adiabatic approximation

$$\psi_j(R(t), t) = U(t, t_0)\psi_j(R(t_0), t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t}\varepsilon_j(R(t'))dt'}\psi_j(R(t), t_0). \tag{8.36}$$

Suppose the system is cyclic in the sense that the time-dependent parameters all return to their original values $R_\alpha(t_1) = R_\alpha(t_0)$ at time t_1. Then the preceding formula becomes

$$\psi_j(R(t_0), t_1) = U(t_1, t_0)\psi_j(R(t_0), t_0) = e^{i\varphi}\psi_j(R(t_0), t_0), \tag{8.37}$$

with

$$\varphi = -\frac{1}{\hbar}\int_{t_0}^{t_1} \varepsilon(R(t'))dt'. \tag{8.38}$$

The dynamical phase factor φ can be measured experimentally by interference if the cycled system is recombined with another that was separated from it at an earlier time and for which $R_\alpha(t) = R_\alpha(t_0)$ for all t. The dynamical phase factor for the system for which the $R_\alpha(t)$ are constant is obviously

$$\varphi_0 = -\frac{1}{\hbar}\varepsilon(R(t_0))(t_1 - t_0) \tag{8.39}$$

and the experiment will measure $\varphi - \varphi_0$.

Berry showed that, for certain physical systems, there will be a geometrical phase factor $\gamma(C)$ that depends on the circuit C that the $R_k(t)$ trace out in parameter space, which is in addition to the dynamical phase factor φ. It follows that the interference experiment will measure $\gamma(C) + \varphi - \varphi_0$. He presented a derivation of $\gamma(C)$, and studied the conditions that are necessary for it to be non-zero.

If Berry's postulate is correct, the standard adiabatic formula must be modified to read

$$\psi_j(R(t), t) = U(t, t_0)\psi_j(R(t_0), t_0) = e^{i\gamma_j - \frac{i}{\hbar}\int_{t_0}^{t} \varepsilon_j(R(t'))dt'}\psi_j(R(t), t_0). \tag{8.40}$$

Insert this wave function into the time-dependent Schrödinger equation

$$H\psi_j(R(t), t) = \varepsilon_j(R(t))\psi_j(R(t), t) = i\hbar\frac{\partial\psi_j(R(t), t)}{\partial t}, \tag{8.41}$$

leads to

$$\begin{aligned} i\hbar\frac{\partial\psi_j(R(t), t)}{\partial t} &= \varepsilon_j e^{i\gamma_j - \frac{i}{\hbar}\int_{t_0}^{t} \varepsilon_j(R(t'))dt'}\psi_j(R(t), t_0) \\ &+ i\hbar e^{i\gamma_j - \frac{i}{\hbar}\int_{t_0}^{t} \varepsilon_j(R(t'))dt'}\left\{ i\frac{d\gamma_j}{dt}\psi_j(R(t), t_0) + \frac{\partial\psi_j(R(t), t_0)}{\partial t}\right\}. \end{aligned} \tag{8.42}$$

The Berry form for the wave function can only be consistent with the Schrödinger equation if

$$\frac{d\gamma_j}{dt}\psi_j(R(t), t_0) = i\frac{\partial\psi_j(R(t), t_0)}{\partial t}. \tag{8.43}$$

The normalization condition $\int\psi_j{}^*\psi_j dv = 1$ leads to

$$\frac{d\gamma_j}{dt} = i\int\psi_j{}^*(R(t), t_0)\frac{\partial\psi_j(R(t), t_0)}{\partial t}dv, \tag{8.44}$$

or, because the time dependence comes from the time-dependent parameters

$$\frac{d\gamma_j}{dt} = i\int \psi_j{}^*(R(t),\, t_0)\sum \frac{\partial \psi_j(R(t),\, t_0)}{\partial R_\alpha}\frac{dR_\alpha}{dt}dv. \tag{8.45}$$

Integrating the left side of this equation from t_0 to t_1 obviously gives

$$\int_{t_0}^{t_1} \frac{d\gamma_j}{dt}dt = \gamma_j(t_1) - \gamma_j(t_0). \tag{8.46}$$

The integral on the right is the same as a line integral in d-dimensional space. The line follows a path defined by the values the $R_\alpha(t)$ take as t increases from t_0 to t_1. The case for which all of the $R_\alpha(t_1)$ equal $R_\alpha(t_0)$ is focused on. That is, a closed contour C in R_α space. Thus,

$$\gamma_j(C) = i\oint_C \int \psi_j{}^*(R(t),\, t_0)\sum \frac{\partial \psi_j(R(t),\, t_0)}{\partial R_\alpha}dvdR_\alpha \tag{8.47}$$

where $\gamma_j(C) = \gamma_j(t_1) - \gamma_j(t_0)$.

At this point, the notation will be switched to a version that is similar to the vector notation used in the conventional vector analysis that appears in elementary physics texts. In the field of differential geometry it is shown that operations like gradients, curls, etc can be extended to multidimensional space. With this notation,

$$\gamma_j(C) = i\oint_C \int \psi_j{}^*(R(t),\, t_0)\nabla_\alpha \psi_j(R(t),\, t_0)dv \cdot d\vec{R} \tag{8.48}$$

where a generalization of the gradient concept to the d-dimensional space of the parameters R_α has been used. The subscript on the gradient operator is to remind us that the derivatives are with respect to the R_α. The gradient of the normalization condition can be written

$$\nabla_\alpha \int \psi_j{}^*\psi_j dv = 0 = \int \nabla_\alpha \psi_j{}^*\psi_j dv + \int \psi_j{}^*\nabla_\alpha \psi_j dv, \tag{8.49}$$

so it follows that $\int \psi_j{}^*\nabla_\alpha \psi_j dv$ is pure imaginary.

The equation for $\gamma_j(C)$ may be rewritten

$$\gamma_j(C) = \oint_C \vec{A}_j \cdot d\vec{R}, \tag{8.50}$$

where

$$\vec{A}_j = -\mathrm{Im}\int \psi_j{}^*(R(t),\, t_0)\nabla_\alpha \psi_j(R(t),\, t_0)dv. \tag{8.51}$$

Stokes theorem in three dimensions is

$$\int_S (\nabla_\alpha \times \vec{A}) \cdot \hat{\mathbf{n}}ds = \oint \vec{A} \cdot d\vec{R} = \oint \sum_\alpha A_\alpha dR_\alpha, \tag{8.52}$$

and this is another operation that generalizes to multiple dimensions. A vector quantity $\vec{B}_j$ can be defined

$$\vec{B}_j = \nabla_\alpha \times \vec{A}_j = -\mathrm{Im}\int (\nabla_\alpha \psi_j^* \times \nabla_\alpha \psi_j) dv. \tag{8.53}$$

Using the preceding equations

$$\gamma_j(C) = \int_S \vec{B}_j \cdot \hat{\mathbf{n}} ds. \tag{8.54}$$

The normalization integral can be written

$$\int \psi_j^* \psi_j dv = \langle j | j \rangle, \tag{8.55}$$

in the Dirac notation where $|j, R(t)\rangle$ is an abstract vector that depends on the R_α. With this notation and using the fact that the eigenvectors are a complete set

$$\sum |k\rangle\langle k| = I, \tag{8.56}$$

leads to

$$\vec{B}_j = -\mathrm{Im}\sum_k \nabla_\alpha \langle j|k\rangle \times \langle k|\ \nabla_\alpha\ |j\rangle. \tag{8.57}$$

Taking the derivative of the eigenvalue equation leads to

$$\vec{B}_j = -\mathrm{Im}\sum_k \nabla_\alpha \langle j|k\rangle \times \langle k|\ \nabla_\alpha\ |j\rangle. \tag{8.58}$$

Premultiplying by $\langle k|$ leads to

$$\langle k|\ \nabla_\alpha\ |j\rangle = -\frac{\langle k|\ \nabla_\alpha H\ |j\rangle}{\varepsilon_k - \varepsilon_j}, \tag{8.59}$$

if $k \neq j$, so

$$\vec{B}_j = -\mathrm{Im}\sum_{k \neq j} \frac{\langle j|\ \nabla_\alpha H |k\rangle \times \langle k|\ \nabla_\alpha H\ |j\rangle}{(\varepsilon_k - \varepsilon_j)^2}. \tag{8.60}$$

The advantage to this equation is that it eliminates the need for differentiating all of the eigenvectors and replaces it with finding the derivatives of $H(R(t))$ with respect to the R_α. It can be shown that $\nabla \cdot \vec{B}_j = 0$. From this it follows that integrating over any surface S that has the trajectory C as an edge will lead to the same Berry phase.

If there are no singularities, the Berry phase will be zero. The typical way that the Berry phase will be non-zero is that a surface S with a boundary C passes through a point for which $\varepsilon_j = \varepsilon_k$ for some value of k. Of course, this cannot be the case for a point on the trajectory C because then the adiabatic theorem would not hold.

This sounds like Cauchy's theorem in complex variable theory that leads to the result that the value of a contour integral is given by the poles of the singularities included within the contour. This is an analogy, but Berry and others have pointed out that it should not be pushed too far. The underlying mathematics is not the same.

As an example, consider the case of a spin 1/2 particle in a time-dependent magnetic field. Calling the magnetic field $\vec{R}$, the Hamiltonian is

$$H = -\frac{\mu}{2}\vec{\sigma} \cdot \vec{R}(t), \tag{8.61}$$

where $\mu = \frac{e\hbar}{mc}$. If it is assumed that $\vec{R}$ is in the $\hat{\mathbf{z}}$ direction, then

$$H = -\frac{\mu}{2}\sigma_z R_z = \begin{pmatrix} \varepsilon_- & 0 \\ 0 & \varepsilon_+ \end{pmatrix} \tag{8.62}$$

and the two eigenvalues of this system are

$$\varepsilon_\pm = \pm\frac{\mu}{2}R_z. \tag{8.63}$$

They are degenerate when $R_z = 0$. Obviously, the gradient of the Hamiltonian in equation (8.61) is

$$\nabla_\alpha H = \frac{\mu}{2}\vec{\sigma} \tag{8.64}$$

where

$$\vec{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\mathbf{x}} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\mathbf{y}} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\mathbf{z}}. \tag{8.65}$$

The eigenvectors for the eigenvalues $\varepsilon_\pm$ are $|+\rangle$ and $|-\rangle$. The matrix elements for equation (8.60) are

$$\begin{aligned} \langle + | \vec{\sigma} | - \rangle &= \hat{\mathbf{x}} - i\hat{\mathbf{y}} \\ \langle - | \vec{\sigma} | + \rangle &= \hat{\mathbf{x}} + i\hat{\mathbf{y}}. \end{aligned} \tag{8.66}$$

If the state $|j\rangle$ is chosen to be $|+\rangle$, then the vector $\vec{B}_+$ is

$$\vec{B}_+ = -\mathrm{Im}\frac{\frac{\mu^2}{4}\langle + | \vec{\sigma} | - \rangle \times \langle - | \vec{\sigma} | + \rangle}{(\mu R_z)^2} \tag{8.67}$$

or

$$\vec{B}_+ = -\frac{\hat{\mathbf{z}}}{2R_z^2} = -\frac{\hat{\mathbf{R}}}{2\,|\mathbf{R}|^2} \tag{8.68}$$

where, more generally, the magnetic field is chosen to point in the direction $\hat{\mathbf{R}}$. Assume that the trajectory C lies on the surface of a sphere of radius R that is centered at the origin of parameter space. The easiest surface S to use in the integral

$$\gamma_+ = \int_S \vec{B}_+ \cdot \hat{\mathbf{n}}ds, \tag{8.69}$$

is the portion of the surface of that sphere inside of the trajectory. The area of that surface is $R^2\Delta\Omega$, where $\Delta\Omega$ is the solid angle subtended by the surface. Then

$$\gamma_+ = \int_S \vec{B}_+ \cdot \hat{\mathbf{R}} R^2 d\Omega = -\frac{1}{2}\Delta\Omega. \tag{8.70}$$

Several conclusions can be drawn from this derivation. It provides a concrete example of a model for which the Berry phase is non-zero. It demonstrates that the singularity at $R = 0$ in parameter space where $\varepsilon_+ = \varepsilon_-$ is crucial to the existence of $\gamma_+(C)$. Finally, the physical example of the rotation of spins in a magnetic field can be realized in the laboratory.

There are two experiments in which the spins of neutrons under the influence of a rotating magnetic field are studied [8]. Neutrons are the particle of choice for this experiment because they have no charge but they have the same spin as an electron. Neutron diffraction devices are a ready source of a monoenergetic stream of neutrons that can be sent through a field. As a neutron with fixed velocity in the z-direction travels through a cylinder wrapped in a helical pattern with superconducting wires carrying very large currents it sees a time-dependent magnetic field perpendicular to the z-direction $B_\perp(t)$.

There is also a fixed field in the z-direction B_z although the solenoid that creates that is not shown in figure 8.8. The easiest way to calculate the Berry phase for this experiment is to use equation (8.69), which leads to

$$\gamma_+ = \pi B_z B_\perp^2. \tag{8.71}$$

When the neutrons were sent through the device illustrated in figure 8.8 they measured the phase shift predicted by Berry's theory. This differs from ordinary precession discussed previously which takes place in a magnetic field that is fixed in direction.

There have been hundreds of papers on the Berry phase, and many other experiments have been done that demonstrate its existence. The Bohm–Aharanov phase described above can be looked upon as a special case of the Berry phase. There is an effect known as the molecular Aharonov–Bohm effect that can best be treated as a Berry phase. In this effect, the Jahn–Teller distortion in a molecule

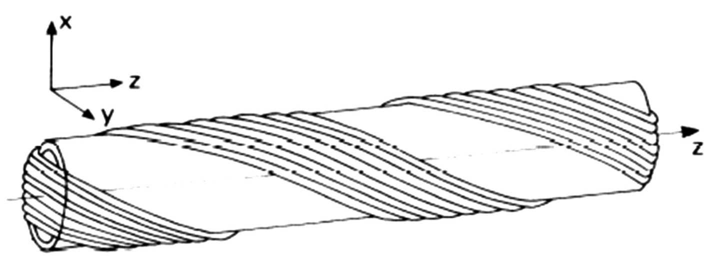

Figure 8.8. Sketch of a cylinder wrapped with wires in a helical pattern.

treated in the Born–Oppenheimer approximation causes a linkage between the electronic states and the vibrational states. The effect is observed in metallic trimmers such as Na_3 and Li_3. Many phenomena that were known before Berry's work have been reanalyzed using his theory. The modern theory of polarization in solids and ferroelectricity is based on the Berry phase. The effects of anholonomy in the scattering of polarized optical waves in a crystal was recognized by S Pancharatnam before Berry did his work.

John Hannay set out to answer a question posed by Berry, namely, what is the classical limit $\hbar \to 0$ of the Berry phase? Instead of accomplishing that, Hannay derived a different anholonomy effect that arises when the parameters in a classical Hamiltonian trace out a closed trajectory in parameter space adiabatically. The Hannay phase that appears in the angle variable in an action-angle analysis of an integrable classical system. There are limits on the systems for which it will occur that do not exist in the quantum case. Berry showed the semiclassical connection between the Hannay phase and the Berry phase. The precession of a Foucault pendulum is an example of the Hannay phase.

8.9 Quantum erasure

8.9.1 First experiment

This is the standard Young's double slit experiment. Light leaves the source one photon at a time (figure 8.9). The photons that pass through the upper slit are said to be on path one, and the ones that pass through the lower slit are on path two. The wave function at the screen for a photon seen by a counter at the position shown may be written as a superposition

$$|\psi(\mathbf{r}\text{onscreen})\rangle = Ae^{i\delta_1}|\alpha_1\rangle + Ae^{i\delta_2}|\alpha_2\rangle, \tag{8.72}$$

where $\delta_1 = \frac{2\pi(l_1 - l_0)}{\lambda}$ and $\delta_2 = \frac{2\pi(l_2 - l_0)}{\lambda}$ and the kets are vectors that describe the polarization of the wave. The intensity at the screen is the absolute value of this wave function

$$I = \langle\psi|\psi\rangle = |A|^2(1 + 1 + \langle\alpha_1|\alpha_2\rangle e^{i\delta} + \langle\alpha_2|\alpha_1\rangle e^{-i\delta}), \tag{8.73}$$

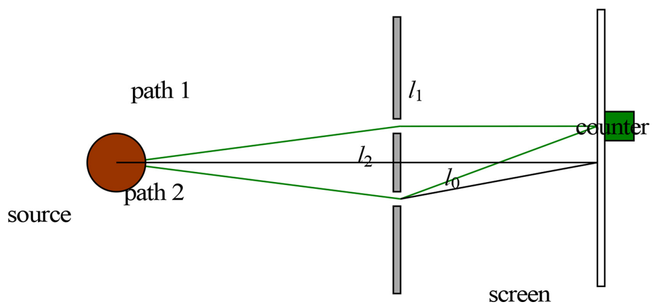

Figure 8.9. First experiment

where $\delta = \frac{2\pi(l_2 - l_1)}{\lambda}$. Normally, the polarization vector that applies to both paths is the same, the one that the photon had when it left the source. For this case, the intensity simplifies to

$$I = \langle\psi|\psi\rangle = 2\,|A|^2(1 + \cos\delta). \tag{8.74}$$

This is the standard equation for two slit interference, ignoring the diffraction due to the finite widths of the slits.

8.9.2 Quarter-wave plate

In order to understand the next experiments the reader must be familiar with an experimental tool called a quarter-wave plate. This device consists of a carefully adjusted thickness of a birefringent material such that the light associated with the larger index of refraction is retarded by 90° in phase (a quarter wave length) with respect to that associated with the smaller index. The material is cut so that the optic axis is parallel to the front and back sides of the plate. Any linearly polarized light which strikes the plate will be divided into two components with different indices of refraction. One of the useful applications of this device is to convert linearly polarized light to circularly polarized light and vice versa by adjusting the plane of the incident light so that it makes 45° angle with the optic axis. This gives equal amplitude o- and e-waves. When the o-wave is slower, as in calcite, the o-wave will fall behind by 90° in phase, producing circularly polarized light.

$$\begin{aligned} |R\rangle &= e^{i\delta}\,|x\rangle + e^{i(\delta-\pi/2)}\,|y\rangle = e^{i\delta}(|x\rangle - i\,|y\rangle) \\ |L\rangle &= e^{i\delta}\,|x\rangle + e^{i(\delta+\pi/2)}\,|y\rangle = e^{i\delta}(|x\rangle + i\,|y\rangle) \end{aligned}. \tag{8.75}$$

8.9.3 Second experiment

This starts out as a standard interference experiment, but quarter-wave plates are placed behind the two slits (figure 8.10). They are oriented so that a photon with polarization in the x direction is converted to one with right circular polarization by the red plate and left polarized by the blue. A y-polarized photon is transformed in the opposite way, as illustrated in the figure above

$$\begin{aligned} &\text{Path1:}\quad |x\rangle \to |L\rangle \qquad |y\rangle \to i\,|R\rangle \\ &\text{Path2:}\quad |x\rangle \to |R\rangle \qquad |y\rangle \to -i\,|L\rangle \end{aligned}. \tag{8.76}$$

These may be looked upon as experimental results.

If a linear polarizer is put behind the source so that only x-polarized photons pass through

$$\psi_x = Ae^{i\delta_1}\,|x\rangle + Ae^{i\delta_2}\,|x\rangle \to Ae^{i\delta_1}\,|L\rangle + Ae^{i\delta_2}\,|R\rangle, \tag{8.77}$$

and the intensity is

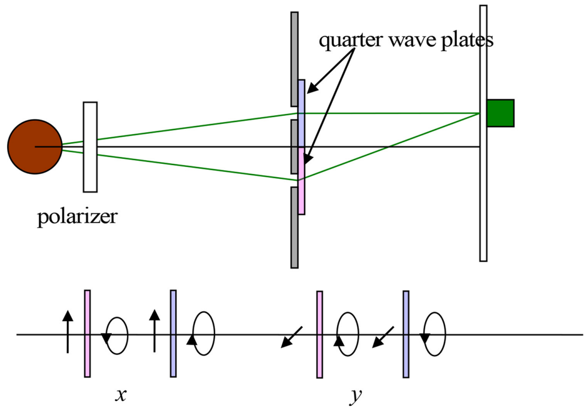

Figure 8.10. Second experiment.

$$I_x = \langle\psi_x|\psi_x\rangle = |A|^2(1 + 1 + \langle L|R\rangle e^{i\delta} + \langle R|L\rangle e^{-i\delta}) = 2\,|A|^2. \tag{8.78}$$

If a y-polarizer is put behind the source,

$$\psi_y = Ae^{i\delta_1}\,|y\rangle + Ae^{i\delta_2}\,|y\rangle \rightarrow iAe^{i\delta_1}\,|R\rangle - iAe^{i\delta_2}\,|L\rangle \tag{8.79}$$

and the intensity is

$$I_y = \left\langle \psi_y \middle| \psi_y \right\rangle = |A|^2(1 + 1 + \langle R|L\rangle e^{i(\delta-\pi)} + \langle L|R\rangle e^{-i(\delta-\pi)}) = 2\,|A|^2. \tag{8.80}$$

Obviously, there is no interference pattern for these cases.

Suppose now the polarizer is oriented so that the polarization of the photon is at an angle α away from the x direction, which is the state

$$|\alpha+\rangle = \cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle. \tag{8.81}$$

Then

$$\psi_{\alpha+} = Ae^{i\delta_1}(\cos\alpha\,|L\rangle + i\sin\alpha\,|R\rangle) + Ae^{i\delta_2}(\cos\alpha\,|R\rangle - i\sin\alpha\,|L\rangle), \tag{8.82}$$

and the intensity of this state is

$$I_{\alpha+} = 2\,|A|^2(1 + \sin 2\alpha\,\sin\delta). \tag{8.83}$$

The polarization state vector orthogonal to $|\alpha+\rangle$ is

$$|\alpha-\rangle = -\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle, \tag{8.84}$$

and the intensity from this state is

$$I_{\alpha-} = 2\,|A|^2(1 - \sin 2\alpha\,\sin\delta). \tag{8.85}$$

These intensities are consistent with the I_x and I_y derived before because $\sin 2\alpha$ is zero for $\alpha = 0$ or $\pi/2$.

The quantum mechanical explanation for these experiments is that in an interference experiment a photon can only interfere with itself. The quarter-wave plates make it possible to distinguish the path that the photon takes and this information destroys the interference.

8.9.4 Third experiment

In this experiment, the information given by using quarter-wave plates in the previous experiment is now erased by inserting another polarizer after the two slits as shown in figure 8.11.

Suppose the first polarizer is oriented in the x direction. This light is passed through the two slits where it is converted to circularly polarized light of two different kinds. As shown above, light in this condition will show no interference fringes. Passing the light through a linear polarizer after the two slits filters out the linearly polarized light. Experiments show that this light shows interference fringes when it reaches the screen

$$\begin{aligned}\psi_x = Ae^{i\delta_1}|x\rangle + Ae^{i\delta_2}|x\rangle \rightarrow Ae^{i\delta_1}|L\rangle + Ae^{i\delta_2}|R\rangle \\ \rightarrow Ae^{i\delta_1}|x\rangle + Ae^{i\delta_2}|x\rangle.\end{aligned} \tag{8.86}$$

The effect of the quarter-wave plates has thus been erased.

Whoa! We claimed above that the conversion to circularly polarized light introduced information about the slit that the photon went through, and this information makes it impossible for the photon to interfere with itself. It is intuitively obvious that when interference is destroyed, it cannot be brought back. The quantum mechanical answer to this conundrum is that light, which was described above with simple equations, is made up of a huge number of photons. The quarter-wave plates put the photons into a statistical state, but, as seen in equation (8.75), circularly polarized light can be looked on as a superposition of light linearly polarized in two directions. Some photons were always in a state in which they could interfere with themselves. The last linear polarizer filters out these photons so that the interference pattern can be seen.

The relation of photons to light is analogous to the relation of molecules to liquid. If there is a high enough density of molecules, they behave according to the fluid

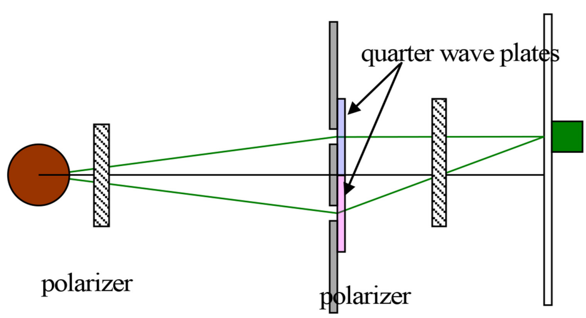

Figure 8.11. Third experiment.

dynamics laws like the Euler equations. A large enough number of photons will behave according the classical laws of optics.

8.9.5 Fourth experiment

A more interesting quantum erasure experiment using photons and a Young's double slit apparatus was done by S P Walborn *et al* from the Universidade Federal de Minas Gerais in Brazil [9].

In order to understand this experiment, the reader must familiarize themselves with two devices that have not been described above. Entangled particles are used in the EPR experiment, but this experiment requires entangled photons. Present research on QI has led to devices to produce them. Walborn *et al* used the following. The entangled photons are produced by a process called spontaneous parametric down conversion. This takes place in a special nonlinear crystal called beta-barium borate (BBO). A photon from an argon ion pump laser (351.1 nm) is converted to two longer wave length (702.2 nm) photons. The two photons go off in two different directions.

The other device is a coincidence counter. Coincidence counting has been used for years in experimental particle physics. The determination that two events occur at the same time is made electronically with a coincidence system. This unit operates on standardized pulses and determines whether events occur within a certain time interval, called the resolving time. The standard pulses from any single channel analyzer are used as input, with one input from each detector.

The ordinary laser is replaced by a BBO crystal which will emit entangled photons with wave length 2λ when it is excited by a photon with wave length λ. The photons are sent on two different paths. Path s leads from the source to Alice's friend Bob. The signal that Bob has counted a photon after processing it is sent to the coincidence counter. Photons on path p reach the coincidence counter at a point on the screen by passing through double slits with half-wave plates as in the second experiment above. This set of operations is shown in figure 8.12.

8.9.5.1 Double slit experiment

There is no polarizer in the path from Alice to Bob and there are no quarter-wave plates next to the slits. The entangled s and p photons are in the state

$$|\psi\rangle = 1/\sqrt{2}\left[\psi_s\,|x\rangle\psi_p\,|y\rangle + \psi_s\,|y\rangle\psi_p\,|x\rangle\right]. \tag{8.87}$$

The counting at the screen is done as follows. The counter is placed at the top of the screen and counts the number of pulses that it senses for x seconds. The number of counts is put in a bin. It is then moved down by an increment, and counts the pulses for another x seconds. Those counts are put in the next bin. This process is repeated until it reaches the bottom of the screen. It is assumed that the spatial increments are small so that plotting the number of counts in the bins gives an approximately continuous curve.

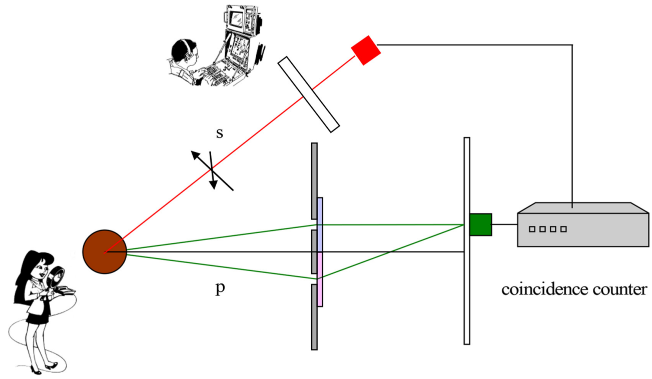

Figure 8.12. Fourth experiment. The red box is Bob's counter. The green box measures photons that go through path p and reach the screen.

If the coincidence counter is ignored, the counting described above gives a curve that shows no interference.

Leaving everything else the same, the counting is done differently. Any pulse that is not in coincidence with a pulse that reaches Bob's counter is ignored. This does not determine the polarization of the photon that left the source, but it has the effect of determining it. The resulting curve shows interference as in the first experiment.

8.9.5.2 Which-way experiment

There is still no polarizer in the path from Alice to Bob and there are quarter-wave plates next to the slits. There is no interference pattern. The only pulses counted are the ones that are coincident with the ones seen by Bob. This caused an interference pattern in the preceding case, but the presence of the quarter-wave plates destroys it. The reason for this was discussed in detail in connection with the second experiment described above.

8.9.5.3 Quantum erasure

There is a linear polarizer in the path from Alice to Bob and there are quarter-wave plates next to the slits. The direction of polarization makes no difference. There is no interference pattern in the counts at the screen but counting only the pulses that are coincident with the ones counted by Bob brings back the interference pattern. The reason for this is that the photons being counted at the screen are in the same orientation state as they were when they left the source. Using the coincidence counter has the same effect as the polarizer after the slits in the third experiment.

The explanation for this effect is the same as for the third experiment, but it is made more interesting because it has the extra wrinkle of entanglement. In the first experiment, all of the photons are allowed to interfere with themselves. Even with the quarter-wave plates, some will. The coincidences with photons that all have the same polarization filters out the ones that will interfere.

8.9.5.4 Delayed erasure

To this point the implication has been that Bob measures the pulses that hit his counter at exactly the same time as they hit the counter at the screen. This does not have to be the case. The time stamps on all of the electrons in a bin can be retained. The path from the source to Bob's counter can be lengthened so that his copy of the entangled photons is measured later than the copy that hit the screen. The time increment for the delay Δt is the same for all the photons, and it can be calculated from the experimental parameters. The photons from a given entangled pair can thus be found by combining Bob's counts with the ones on the screen even when they are not counted at the same time.

It is thus possible to reproduce the preceding experiments with delayed photons. Somehow the fact that interference is identified with information that did not exist when the photons struck the screen makes the erasure experiment seem even more mysterious.

8.10 Resume

It could be argued that the EPR effect and the Bohm–Aharanov effect are discoveries of interesting quantum phenomena that were found for the wrong reason. The predictions of quantum theory seemed so outlandish that very good scientists proposed experiments that they were convinced would not work. The experiments suggested by Berry and Walborn also seem to contradict intuition, but they expected quantum theory to be validated.

Problems

P8.1 Is the idea of a collapsing wave function a general feature of the measurement process?

P8.2 Draw a cartoon showing what Alice putting electrons in her box must look like according to hidden variable theory.

P8.3 Work out the Bell's inequalities for $P(\hat{\mathbf{b}}+;\hat{\mathbf{c}}+)$.

P8.4 Suppose there were four men and four women in a quantum mechanics class. They are standing in the hall outside the classroom. There is a row of eight chairs in the classroom. One at a time, they enter the classroom, pick a chair, write their name on a piece of tape, and place it on the bottom of the seat. When the process is finished, they go into the classroom and sit in the chair they picked. What is the chance that no two students picked the same chair and that the arrangement of the students would be alternating man, woman, man, woman, …?

P8.5 Why is the Hamiltonian in quantum mechanics normally time independent?

P8.6 Is the Bohm–Aharanov effect gauge invariant?

P8.7 Hold a cat with his feet pointing toward the ceiling. Drop the cat. Hopefully the cat will land with his feet on the floor. Is angular momentum conserved? Is this an example of anholonomy?

P8.8 When entangled electrons are used in a two slit experiment, is the rule that an electron can only interfere with itself broken?

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