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Quantum Mechanics
Lecture notes
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Appendix A

Selected mathematical formulas

This appendix lists selected mathematical formulas that are used in this lecture course series, but not always remembered by students (and some instructors :-).

A.1 Constants

- Euclidean circle's *length-to-diameter ratio*:

$$\pi = 3.141\,592\,653\ldots; \qquad \pi^{1/2} \approx 1.77. \tag{A.1}$$

- *Natural logarithm base*:

$$e \equiv \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718\,281\,828\ldots; \tag{A.2a}$$

from that value, the logarithm base conversion factors are as follows ($\xi > 0$):

$$\frac{\ln \xi}{\log_{10}\xi} = \ln 10 \approx 2.303, \qquad \frac{\log_{10}\xi}{\ln \xi} = \frac{1}{\ln 10} \approx 0.434. \tag{A.2b}$$

- The *Euler* (or 'Euler–Mascheroni') *constant*:

$$\gamma \equiv \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n}-\ln n\right) = 0.577\,156\,649\,0\ldots; \\ e^{\gamma} \approx 1.781. \tag{A.3}$$

A.2 Combinatorics, sums, and series

(i) *Combinatorics*

- The number of different *permutations*, i.e. *ordered* sequences of k elements selected from a set of n distinct elements ($n \geqslant k$), is

$$^{n}P_k \equiv n\cdot(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}; \tag{A.4a}$$

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in particular, the number of different permutations of *all* elements of the set ($n = k$) is

$$^{k}P_{k} = k \cdot (k-1) \cdots 2 \cdot 1 = k!. \tag{A.4b}$$

- The number of different *combinations*, i.e. *unordered* sequences of k elements from a set of $n \geqslant k$ distinct elements, is equal to the binomial coefficient

$$^{n}C_{k} \equiv \binom{n}{k} \equiv \frac{^{n}P_{k}}{^{k}P_{k}} = \frac{n!}{k!(n-k)!}. \tag{A.5}$$

In an alternative, very popular 'ball/box language', $^{n}C_{k}$ is the number of different ways to put in a box, in an arbitrary order, k balls selected from n distinct balls.

- A generalization of the binomial coefficient notion is the multinomial coefficient,

$$^{n}C_{k_1,k_2,\ldots k_l} \equiv \frac{n!}{k_1!k_2!\ldots k_l!}, \qquad \text{with } n = \sum_{j=1}^{l} k_j, \tag{A.6}$$

which, in the standard mathematical language, is a number of different permutations in a multiset of l distinct element types from an n-element set which contains k_j ($j = 1, 2,\ldots l$) elements of each type. In the 'ball/box language', the coefficient (A.6) is the number of different ways to distribute n distinct balls between l distinct boxes, each time keeping the number (k_j) of balls in the jth box fixed, but ignoring their order inside the box. The binomial coefficient $^{n}C_{k}$ (A.5) is a particular case of the multinomial coefficient (A.6) for $l = 2$ - counting the explicit box for the first one, and the remaining space for the second box, so that if $k_1 \equiv k$, then $k_2 = n - k$.

- One more important combinatorial quantity is the number $M_n^{(k)}$ of ways to place n *indistinguishable* balls into k distinct boxes. It may be readily calculated from Eq. (A.5) as the number of different ways to select $(k-1)$ partitions between the boxes in an imagined linear row of $(k - 1 + n)$ 'objects' (balls in the boxes *and* partitions between them):

$$M_n^{(k)} = {}^{n-1+k}C_{k-1} \equiv \frac{(k-1+n)!}{(k-1)!n!}. \tag{A.7}$$

(ii) *Sums and series*

- *Arithmetic progression*:

$$r + 2r + \cdots + nr \equiv \sum_{k=1}^{n} kr = \frac{n(r+nr)}{2}; \tag{A.8a}$$

in particular, at $r = 1$ it is reduced to the sum of n first natural numbers:

$$1 + 2 + \cdots + n \equiv \sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \qquad \text{(A.8}b\text{)}$$

- Sums of squares and cubes of n first natural numbers:

$$1^2 + 2^2 + \cdots + n^2 \equiv \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \qquad \text{(A.9}a\text{)}$$

$$1^3 + 2^3 + \cdots + n^3 \equiv \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}. \qquad \text{(A.9}b\text{)}$$

- The *Riemann zeta function*:

$$\zeta(s) \equiv 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \equiv \sum_{k=1}^{\infty} \frac{1}{k^s}; \qquad \text{(A.10}a\text{)}$$

the particular values frequently met in applications are

$$\begin{aligned} &\zeta\left(\frac{3}{2}\right) \approx 2.612, \qquad \zeta(2) = \frac{\pi^2}{6}, \qquad \zeta\left(\frac{5}{2}\right) \approx 1.341, \\ &\zeta(3) \approx 1.202, \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(5) \approx 1.037. \end{aligned} \qquad \text{(A.10}b\text{)}$$

- Finite geometric progression (for real $\lambda \neq 1$):

$$1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1} \equiv \sum_{k=0}^{n-1} \lambda^k = \frac{1-\lambda^n}{1-\lambda}; \qquad \text{(A.11}a\text{)}$$

in particular, if $\lambda^2 < 1$, the progression has a finite limit at $n \to \infty$ (called the *geometric series*):

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \lambda^k = \sum_{k=0}^{\infty} \lambda^k = \frac{1}{1-\lambda}. \qquad \text{(A.11}b\text{)}$$

- *Binomial sum* (or the 'binomial theorem'):

$$(1+a)^n = \sum_{k=0}^{n} {}^nC_k a^k, \qquad \text{(A.12)}$$

where ${}^nC_k$ are the binomial coefficients defined by Eq. (A.5).

- The *Stirling formula*:

$$\lim_{n\to\infty} \ln(n!) = n(\ln n - 1) + \frac{1}{2}\ln(2\pi n) + \frac{1}{12n} - \frac{1}{360n^3} + \ldots; \qquad \text{(A.13)}$$

for most applications in physics, the first term[1] is sufficient.

- The *Taylor* (or 'Taylor–Maclaurin') *series*: for any infinitely differentiable function $f(\xi)$:

$$\begin{aligned}\lim_{\tilde{\xi}\to 0} f(\xi+\tilde{\xi}) &= f(\xi) + \frac{df}{d\xi}(\xi)\ \tilde{\xi} + \frac{1}{2!}\frac{d^2 f}{d\xi^2}(\xi)\ \tilde{\xi}^2 + \cdots \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\frac{d^k f}{d\xi^k}(\xi)\ \tilde{\xi}^k;\end{aligned} \tag{A.14a}$$

note that for many functions this series converges only within a limited, sometimes small range of deviations $\tilde{\xi}$. For a function of several arguments, $f(\xi_1,\xi_2,\ldots,\xi_N)$, the first terms of the Taylor series are

$$\begin{aligned}\lim_{\tilde{\xi}_k\to 0} f(\xi_1+\tilde{\xi}_1,\ \xi_2+\tilde{\xi}_2,\ \cdots) = f(\xi_1,\ \xi_2,\ \cdots) \\ +\sum_{k=1}^{N}\frac{\partial f}{\partial \xi_k}(\xi_1,\ \xi_2,\ \cdots)\ \tilde{\xi}_k \\ +\frac{1}{2!}\sum_{k,k'=1}^{N}\frac{\partial^2 f}{\partial_k\xi\ \partial\xi_{k'}}\tilde{\xi}_k\tilde{\xi}_{k'} + \cdots\end{aligned} \tag{A.14b}$$

- The *Euler–Maclaurin formula*, valid for any infinitely differentiable function $f(\xi)$:

$$\begin{aligned}\sum_{k=1}^{n} f(k) = \int_0^n f(\xi)d\xi + \frac{1}{2}[f(n)-f(0)] + \frac{1}{6}\cdot\frac{1}{2!}\left[\frac{df}{d\xi}(n) - \frac{df}{d\xi}(0)\right] \\ -\frac{1}{30}\cdot\frac{1}{4!}\left[\frac{d^3 f}{d\xi^3}(n) - \frac{d^3 f}{d\xi^3}(0)\right] \\ +\frac{1}{42}\cdot\frac{1}{6!}\left[\frac{d^5 f}{d\xi^5}(n) - \frac{d^5 f}{d\xi^5}(0)\right] + \cdots;\end{aligned} \tag{A.15a}$$

the coefficients participating in this formula are the so-called *Bernoulli numbers*[2]:

$$\begin{aligned}&B_1 = \frac{1}{2},\quad B_2 = \frac{1}{6},\quad B_3 = 0,\quad B_4 = \frac{1}{30},\quad B_5 = 0, \\ &B_6 = \frac{1}{42},\quad B_7 = 0,\quad B_8 = \frac{1}{30},\quad \cdots\end{aligned} \tag{A.15b}$$

[1] Actually, this leading term was derived by A de Moivre in 1733, before J Stirling's work.

[2] Note that definitions of B_k (or rather their signs and indices) vary even among the most popular handbooks.

A.3 Basic trigonometric functions

- Trigonometric functions of the sum and the difference of two arguments[3]:

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b, \tag{A.16a}$$

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b. \tag{A.16b}$$

- Sums of two functions of arbitrary arguments:

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{b-a}{2}, \tag{A.17a}$$

$$\cos a - \cos b = 2\sin\frac{a+b}{2}\sin\frac{b-a}{2}, \tag{A.17b}$$

$$\sin a \pm \sin b = 2\sin\frac{a \pm b}{2}\cos\frac{\pm b - a}{2}. \tag{A.17c}$$

- Trigonometric function products:

$$2\cos a\ \cos b = \cos(a+b) + \cos(a-b), \tag{A.18a}$$

$$2\sin a\ \cos b = \sin(a+b) + \sin(a-b), \tag{A.18b}$$

$$2\sin a\ \sin b = \cos(a-b) - \cos(a+b); \tag{A.18c}$$

For the particular case of equal arguments, $b = a$, these three formulas yield the following expressions for the squares of trigonometric functions, and their product:

$$\begin{aligned}\cos^2 a &= \frac{1}{2}(1 + \cos 2a), \qquad \sin a \cos a = \frac{1}{2}\sin 2a,\\ \sin^2 a &= \frac{1}{2}(1 - \cos 2a).\end{aligned} \tag{A.18d}$$

- Cubes of trigonometric functions:

$$\cos^3 a = \frac{3}{4}\cos a + \frac{1}{4}\cos 3a, \qquad \sin^3 a = \frac{3}{4}\sin a - \frac{1}{4}\sin 3a. \tag{A.19}$$

- Trigonometric functions of a complex argument:

$$\begin{aligned}\sin(a + ib) &= \sin a \cosh b + i\cos a \sinh b,\\ \cos(a + ib) &= \cos a \cosh b - i\sin a \sinh b.\end{aligned} \tag{A.20}$$

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[3] I am confident that the reader is quite capable of deriving the relations (A.16) by representing the exponent in the elementary relation $e^{i(a \pm b)} = e^{ia}e^{\pm ib}$ as a sum of its real and imaginary parts, Eqs. (A.18) directly from Eqs. (A.16), and Eqs. (A.17) from Eqs. (A.18) by variable replacement; however, I am still providing these formulas to save his or her time. (Quite a few formulas below are included because of the same reason.)

- Sums of trigonometric functions of n equidistant arguments:

$$\sum_{k=1}^{n} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} k\xi = \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \left(\frac{n+1}{2} \xi \right) \sin \left(\frac{n}{2} \xi \right) \Big/ \sin \left(\frac{\xi}{2} \right). \tag{A.21}$$

A.4 General differentiation

- Full differential of a product of two functions:

$$d(fg) = (df)g + f(dg). \tag{A.22}$$

- Full differential of a function of several independent arguments, $f(\xi_1, \xi_2, \ldots, \xi_n)$:

$$df = \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_k} d\xi_k. \tag{A.23}$$

- Curvature of the Cartesian plot of a 1D function $f(\xi)$:

$$\kappa \equiv \frac{1}{R} = \frac{| d^2 f / d\xi^2 |}{[1 + (df/d\xi)^2]^{3/2}}. \tag{A.24}$$

A.5 General integration

- Integration *by parts* - immediately follows from Eq. (A.22):

$$\int_{g(A)}^{g(B)} f \ dg = fg \Big|_A^B - \int_{f(A)}^{f(B)} g \ df. \tag{A.25}$$

- Numerical (approximate) integration of 1D functions: the simplest *trapezoidal rule*,

$$\begin{aligned} \int_a^b f(\xi) d\xi &\approx h \left[f\left(a + \frac{h}{2}\right) + f\left(a + \frac{3h}{2}\right) + \cdots + f\left(b - \frac{h}{2}\right) \right] \\ &= h \sum_{n=1}^{N} f\left(a - \frac{h}{2} + nh\right), \quad h \equiv \frac{b-a}{N}. \end{aligned} \tag{A.26}$$

has relatively low accuracy (error of the order of $(h^3/12)d^2f/d\xi^2$ per step), so that the following *Simpson formula*,

$$\begin{aligned} \int_a^b f(\xi) d\xi \approx \frac{h}{3} [f(a) + 4f(a+h) + 2f(a+2h) + \cdots + 4f(b-h) + f(b)], \\ h \equiv \frac{b-a}{2N}, \end{aligned} \tag{A.27}$$

whose error per step scales as $(h^5/180)d^4f/d\xi^4$, is used much more frequently[4].

A.6 A few 1D integrals[5]

(i) *Indefinite integrals*:

- Integrals with $(1 + \xi^2)^{1/2}$:

$$\int (1 + \xi^2)^{1/2} d\xi = \frac{\xi}{2}(1 + \xi^2)^{1/2} + \frac{1}{2}\ln|\xi + (1 + \xi^2)^{1/2}|, \tag{A.28}$$

$$\int \frac{d\xi}{(1 + \xi^2)^{1/2}} = \ln|\xi + (1 + \xi^2)^{1/2}|, \tag{A.29a}$$

$$\int \frac{d\xi}{(1 + \xi^2)^{3/2}} = \frac{\xi}{(1 + \xi^2)^{1/2}}. \tag{A.29b}$$

- Miscellaneous indefinite integrals:

$$\int \frac{d\xi}{\xi(\xi^2 + 2a\xi - 1)^{1/2}} = \arccos\frac{a\xi - 1}{|\xi|(a^2 + 1)^{1/2}}, \tag{A.30a}$$

$$\int \frac{(\sin\xi - \xi\cos\xi)^2}{\xi^5} d\xi = \frac{2\xi\sin 2\xi + \cos 2\xi - 2\xi^2 - 1}{8\xi^4}, \tag{A.30b}$$

$$\int \frac{d\xi}{a + b\cos\xi} = \frac{2}{(a^2 - b^2)^{1/2}}\tan^{-1}\left[\frac{(a - b)}{(a^2 - b^2)^{1/2}}\tan\frac{\xi}{2}\right], \quad \text{for } a^2 > b^2. \tag{A.30c}$$

$$\int \frac{d\xi}{1 + \xi^2} = \tan^{-1}\xi. \tag{A.30d}$$

(ii) *Semi-definite integrals*:

- Integrals with $1/(e^\xi \pm 1)$:

$$\int_a^\infty \frac{d\xi}{e^\xi + 1} = \ln(1 + e^{-a}), \tag{A.31a}$$

[4] Higher-order formulas (e.g. the *Bode rule*), and other guidance including ready-for-use codes for computer calculations may be found, for example, in the popular reference texts by W H Press *et al* [1]. In addition, some advanced codes are used as subroutines in the software packages listed in the same section. In some cases, the Euler–Maclaurin formula (A.15) may also be useful for numerical integration.

[5] A powerful (and free) interactive online tool for working out indefinite 1D integrals is available at http://integrals.wolfram.com/index.jsp.

$$\int_{a>0}^{\infty} \frac{d\xi}{e^{\xi} - 1} = \ln \frac{1}{1 - e^{-a}}. \tag{A.31b}$$

(iii) *Definite integrals*:

- Integrals with $1/(1 + \xi^2)$:[6]

$$\int_0^{\infty} \frac{d\xi}{1 + \xi^2} = \frac{\pi}{2}, \tag{A.32a}$$

$$\int_0^{\infty} \frac{d\xi}{(1 + \xi^2)^{3/2}} = 1; \tag{A.32b}$$

more generally,

$$\int_0^{\infty} \frac{d\xi}{(1 + \xi^2)^n} = \frac{\pi}{2}\frac{(2n - 3)!!}{(2n - 2)!!} \equiv \frac{\pi}{2}\frac{1 \cdot 3 \cdot 5 \ldots (2n - 3)}{2 \cdot 4 \cdot 6 \ldots (2n - 2)}, \quad \text{for } n = 2,\ 3,\ \ldots \tag{A.32c}$$

- Integrals with $(1 - \xi^{2n})^{1/2}$:

$$\int_0^{1} \frac{d\xi}{(1 - \xi^{2n})^{1/2}} = \frac{\pi^{1/2}}{2n}\Gamma\left(\frac{1}{2n}\right) \Big/ \Gamma\left(\frac{n + 1}{2n}\right), \tag{A.33a}$$

$$\int_0^{1} (1 - \xi^{2n})^{1/2} d\xi = \frac{\pi^{1/2}}{4n}\Gamma\left(\frac{1}{2n}\right) \Big/ \Gamma\left(\frac{3n + 1}{2n}\right), \tag{A.33b}$$

where $\Gamma(s)$ is the *gamma-function,* which is most often defined (for $\text{Re}\, s > 0$) by the following integral:

$$\int_0^{\infty} \xi^{s-1} e^{-\xi}\, d\xi = \Gamma(s). \tag{A.34a}$$

The key property of this function is the recurrence relation, valid for any $s \neq 0, -1, -2, \ldots$:

$$\Gamma(s + 1) = s\Gamma(s). \tag{A.34b}$$

Since, according to Eq. (A.34*a*), $\Gamma(1) = 1$, Eq. (A.34*b*) for non-negative integers takes the form

$$\Gamma(n + 1) = n!, \quad \text{for } n = 0,\ 1,\ 2,\ \cdots \tag{A.34c}$$

[6] Eq. (A.32*a*) follows immediately from Eq. (A.30*d*), and Eq. (A.32*b*) from Eq. (A.29*b*)—a couple more examples of the (intentional) redundancy in this list.

(where $0! \equiv 1$). Because of this, for integer $s = n + 1 \geqslant 1$, Eq. (A.34*a*) is reduced to

$$\int_0^\infty \xi^n e^{-\xi} d\xi = n!. \tag{A.34d}$$

Other frequently met values of the gamma-function are those for positive semi-integer arguments:

$$\begin{aligned}&\Gamma\left(\frac{1}{2}\right) = \pi^{1/2}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\pi^{1/2}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\pi^{1/2},\\ &\Gamma\left(\frac{7}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\pi^{1/2}, \quad \ldots.\end{aligned} \tag{A.34e}$$

- Integrals with $1/(e^\xi \pm 1)$:

$$\int_0^\infty \frac{\xi^{s-1}d\xi}{e^\xi + 1} = (1 - 2^{1-s})\,\Gamma(s)\zeta(s), \quad \text{for } s > 0, \tag{A.35a}$$

$$\int_0^\infty \frac{\xi^{s-1}d\xi}{e^\xi - 1} = \Gamma(s)\zeta(s), \quad \text{for } s > 1, \tag{A.35b}$$

where $\zeta(s)$ is the Riemann zeta-function—see Eq. (A.10). Particular cases: for $s = 2n$,

$$\int_0^\infty \frac{\xi^{2n-1}d\xi}{e^\xi + 1} = \frac{2^{2n-1} - 1}{2n}\pi^{2n}B_{2n}, \tag{A.35c}$$

$$\int_0^\infty \frac{\xi^{2n-1}d\xi}{e^\xi - 1} = \frac{(2\pi)^{2n}}{4n}B_{2n}. \tag{A.35d}$$

where B_n are the Bernoulli numbers—see Eq. (A.15). For the particular case $s = 1$ (when Eq. (A.35*a*) yields uncertainty),

$$\int_0^\infty \frac{d\xi}{e^\xi + 1} = \ln 2. \tag{A.35e}$$

- Integrals with $\exp\{-\xi^2\}$:

$$\int_0^\infty \xi^s e^{-\xi^2} d\xi = \frac{1}{2}\Gamma\left(\frac{s+1}{2}\right), \quad \text{for } s > -1; \tag{A.36a}$$

for applications the most important particular values of s are 0 and 2:

$$\int_0^\infty e^{-\xi^2} d\xi = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\pi^{1/2}}{2}, \tag{A.36b}$$

$$\int_0^\infty \xi^2 e^{-\xi^2} d\xi = \frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{\pi^{1/2}}{4}, \tag{A.36c}$$

although we will also run into the cases $s = 4$ and $s = 6$:

$$\int_0^\infty \xi^4 e^{-\xi^2} d\xi = \frac{1}{2}\Gamma\left(\frac{5}{2}\right) = \frac{3\pi^{1/2}}{8},$$
$$\int_0^\infty \xi^6 e^{-\xi^2} d\xi = \frac{1}{2}\Gamma\left(\frac{7}{2}\right) = \frac{15\pi^{1/2}}{16}; \tag{A.36d}$$

for odd integer values $s = 2n + 1$ (with $n = 0, 1, 2, \ldots$), Eq. (A.36*a*) takes a simpler form:

$$\int_0^\infty \xi^{2n+1} e^{-\xi^2} d\xi = \frac{1}{2}\Gamma(n+1) = \frac{n!}{2}. \tag{A.36e}$$

- Integrals with cosine and sine functions:

$$\int_0^\infty \cos(\xi^2)\, d\xi = \int_0^\infty \sin(\xi^2)\, d\xi = \left(\frac{\pi}{8}\right)^{1/2}. \tag{A.37}$$

$$\int_0^\infty \frac{\cos \xi}{a^2 + \xi^2} d\xi = \frac{\pi}{2a} e^{-a}. \tag{A.38}$$

$$\int_0^\infty \left(\frac{\sin \xi}{\xi}\right)^2 d\xi = \frac{\pi}{2}. \tag{A.39}$$

- Integrals with logarithms:

$$\int_0^1 \ln \frac{a + (1-\xi^2)^{1/2}}{a - (1-\xi^2)^{1/2}} d\xi = \pi[a - (a^2 - 1)^{1/2}], \quad \text{for } a \geqslant 1. \tag{A.40}$$

$$\int_0^1 \ln \frac{1 + (1-\xi)^{1/2}}{\xi^{1/2}} d\xi = 1. \tag{A.41}$$

- Integral representations of the Bessel functions of integer order:

$$J_n(\alpha) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{i(\alpha \sin\xi - n\xi)} d\xi,$$
$$\text{so that } e^{i\alpha \sin\xi} = \sum_{k=-\infty}^{\infty} J_k(\alpha) e^{ik\xi}; \tag{A.42a}$$

$$I_n(\alpha) = \frac{1}{\pi}\int_0^\pi e^{\alpha \cos\xi} \cos n\xi \;\; d\xi. \tag{A.42b}$$

A.7 3D vector products

(i) *Definitions*:

- *Scalar* ('dot-') *product*:

$$\mathbf{a}\cdot\mathbf{b} = \sum_{j=1}^{3} a_j b_j, \tag{A.43}$$

where a_j and b_j are vector components in any orthogonal coordinate system. In particular, the vector squared (the same as the norm squared):

$$a^2 \equiv \mathbf{a}\cdot\mathbf{a} = \sum_{j=1}^{3} a_j^2 \equiv \|\mathbf{a}\|^2. \tag{A.44}$$

- *Vector* ('cross-') *product*:

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &\equiv \mathbf{n}_1(a_2b_3 - a_3b_2) + \mathbf{n}_2(a_3b_1 - a_1b_3) + \mathbf{n}_3(a_1b_2 - a_2b_1)\\ &= \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},\end{aligned} \tag{A.45}$$

where $\{\mathbf{n}_j\}$ is the set of mutually perpendicular unit vectors[7] along the corresponding coordinate system axes[8]. In particular, Eq. (A.45) yields

$$\mathbf{a}\times\mathbf{a} = 0. \tag{A.46}$$

(ii) *Corollaries* (readily verified by Cartesian components):

- Double vector product (the so-called *bac minus cab* rule):

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}). \tag{A.47}$$

- Mixed scalar–vector product (the *operand rotation rule*):

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}). \tag{A.48}$$

- Scalar product of vector products:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}); \tag{A.49a}$$

[7] Other popular notations for this vector set are $\{\mathbf{e}_j\}$ and $\{\hat{\mathbf{r}}_j\}$.

[8] It is easy to use Eq. (A.45) to check that the direction of the product vector corresponds to the well-known 'right-hand rule' and to the even more convenient *corkscrew rule*: if we rotate a corkscrew's handle from the first operand toward the second one, its axis moves in the direction of the product.

in the particular case of two similar operands (say, $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$), the last formula is reduced to

$$(\mathbf{a} \times \mathbf{b})^2 = (ab)^2 - (\mathbf{a} \cdot \mathbf{b})^2. \tag{A.49b}$$

A.8 Differentiation in 3D Cartesian coordinates

- Definition of the *del* (or 'nabla') vector-operator ∇:[9]

$$\nabla \equiv \sum_{j=1}^{3} \mathbf{n}_j \frac{\partial}{\partial r_j}, \tag{A.50}$$

where r_j is a set of linear and orthogonal (*Cartesian*) coordinates along directions $\mathbf{n}_j$. In accordance with this definition, the operator ∇ acting on a *scalar* function of coordinates, $f(\mathbf{r})$,[10] gives its gradient, i.e. a new *vector*:

$$\nabla f \equiv \sum_{j=1}^{3} \mathbf{n}_j \frac{\partial f}{\partial r_j} \equiv \mathbf{grad}\, f. \tag{A.51}$$

- The *scalar product* of del by a *vector* function of coordinates (a *vector field*),

$$\mathbf{f}(\mathbf{r}) \equiv \sum_{j=1}^{3} \mathbf{n}_j f_j(\mathbf{r}), \tag{A.52}$$

compiled formally following Eq. (A.43), is a *scalar* function—the *divergence* of the initial function:

$$\nabla \cdot \mathbf{f} \equiv \sum_{j=1}^{3} \frac{\partial f_j}{\partial r_j} \equiv \mathrm{div}\, \mathbf{f}, \tag{A.53}$$

while the *vector product* of ∇ and $\mathbf{f}$, formed in a formal accordance with Eq. (A.45), is a new vector - the *curl* (in European tradition, called rotor and denoted **rot**) of $\mathbf{f}$:

$$\begin{aligned} \nabla \times \mathbf{f} \equiv \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ \frac{\partial}{\partial r_1} & \frac{\partial}{\partial r_2} & \frac{\partial}{\partial r_3} \\ f_1 & f_2 & f_3 \end{vmatrix} &= \mathbf{n}_1\left(\frac{\partial f_3}{\partial r_2} - \frac{\partial f_2}{\partial r_3}\right) + \mathbf{n}_2\left(\frac{\partial f_1}{\partial r_3} - \frac{\partial f_3}{\partial r_1}\right) \\ &+ \mathbf{n}_3\left(\frac{\partial f_2}{\partial r_1} - \frac{\partial f_1}{\partial r_2}\right) \equiv \mathbf{curl}\, \mathbf{f}. \end{aligned} \tag{A.54}$$

[9] One can run into the following notation: $\nabla \equiv \partial/\partial \mathbf{r}$, which is convenient is some cases, but may be misleading in quite a few others, so it will be not used in these notes.

[10] In this, and four next sections, all scalar and vector functions are assumed to be differentiable.

- One more frequently met 'product' is $(\mathbf{f}\cdot\nabla)\mathbf{g}$, where $\mathbf{f}$ and $\mathbf{g}$ are two arbitrary vector functions of $\mathbf{r}$. This product should be also understood in the sense implied by Eq. (A.43), i.e. as a vector whose jth Cartesian component is

$$[(\mathbf{f}\cdot\nabla)\,\mathbf{g}]_j = \sum_{j'=1}^{3} f_{j'}\frac{\partial g_j}{\partial r_{j'}}. \tag{A.55}$$

A.9 The Laplace operator $\nabla^2 \equiv \nabla\cdot\nabla$

- Expression in Cartesian coordinates—in the formal accordance with Eq. (A.44):

$$\nabla^2 = \sum_{j=1}^{3}\frac{\partial^2}{\partial r_j^2}. \tag{A.56}$$

- According to its definition, the Laplace operator acting on a *scalar* function of coordinates gives a new scalar function:

$$\nabla^2 f \equiv \nabla\cdot(\nabla f) = \mathrm{div}(\mathbf{grad}\, f) = \sum_{j=1}^{3}\frac{\partial^2 f}{\partial r_j^2}. \tag{A.57}$$

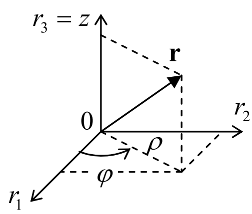
- On the other hand, acting on a *vector* function (A.52), the operator ∇^2 returns another *vector*:

$$\nabla^2\mathbf{f} = \sum_{j=1}^{3}\mathbf{n}_j\nabla^2 f_j. \tag{A.58}$$

Note that Eqs. (A.56)–(A.58) are only valid in Cartesian (i.e. orthogonal and linear) coordinates, but generally not in other (even orthogonal) coordinates—see, e.g. Eqs. (A.61), (A.64), (A.67) and (A.70) below.

A.10 Operators ∇ and ∇^2 in the most important systems of orthogonal coordinates[11]

(i) *Cylindrical*[12] *coordinates* $\{\rho, \varphi, z\}$ (see figure below) may be defined by their relations with the Cartesian coordinates:



$$r_1 = \rho\cos\varphi,\quad r_2 = \rho\sin\varphi,\quad r_3 = z. \tag{A.59}$$

[11] Some other orthogonal curvilinear coordinate systems are discussed in *Part EM*, section 2.3.
[12] In the 2D geometry with fixed coordinate z, these coordinates are called *polar*.

- Gradient of a scalar function:

$$\nabla f = \mathbf{n}_\rho \frac{\partial f}{\partial \rho} + \mathbf{n}_\varphi \frac{1}{\rho}\frac{\partial f}{\partial \varphi} + \mathbf{n}_z \frac{\partial f}{\partial z}. \tag{A.60}$$

- The Laplace operator of a scalar function:

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}, \tag{A.61}$$

- Divergence of a vector function of coordinates ($\mathbf{f} = \mathbf{n}_\rho f_\rho + \mathbf{n}_\varphi f_\varphi + \mathbf{n}_z f_z$):

$$\nabla \cdot \mathbf{f} = \frac{1}{\rho}\frac{\partial(\rho f_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial f_\varphi}{\partial \varphi} + \frac{\partial f_z}{\partial z}. \tag{A.62}$$

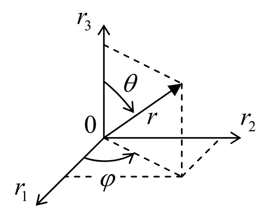
- Curl of a vector function:

$$\nabla \times \mathbf{f} = \mathbf{n}_\rho\left(\frac{1}{\rho}\frac{\partial f_z}{\partial \varphi} - \frac{\partial f_\varphi}{\partial z}\right) + \mathbf{n}_\varphi\left(\frac{\partial f_\rho}{\partial z} - \frac{\partial f_z}{\partial \rho}\right) + \mathbf{n}_z \frac{1}{\rho}\left(\frac{\partial(\rho f_\varphi)}{\partial \rho} - \frac{\partial f_\rho}{\partial \varphi}\right). \tag{A.63}$$

- The Laplace operator of a vector function:

$$\nabla^2 \mathbf{f} = \mathbf{n}_\rho\left(\nabla^2 f_\rho - \frac{1}{\rho^2} f_\rho - \frac{2}{\rho^2}\frac{\partial f_\varphi}{\partial \varphi}\right) + \mathbf{n}_\varphi\left(\nabla^2 f_\varphi - \frac{1}{\rho^2} f_\varphi + \frac{2}{\rho^2}\frac{\partial f_\rho}{\partial \varphi}\right) + \mathbf{n}_z \nabla^2 f_z. \tag{A.64}$$

(ii) *Spherical coordinates* $\{r, \theta, \varphi\}$ (see figure below) may be defined as:



$$\begin{aligned} r_1 &= r\sin\theta\cos\varphi, \\ r_2 &= r\sin\theta\sin\varphi, \\ r_3 &= r\cos\theta. \end{aligned} \tag{A.65}$$

- Gradient of a scalar function:

$$\nabla f = \mathbf{n}_r \frac{\partial f}{\partial r} + \mathbf{n}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{n}_\varphi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}. \tag{A.66}$$

- The Laplace operator of a scalar function:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{(r\sin\theta)^2}\frac{\partial^2 f}{\partial \varphi^2}. \tag{A.67}$$

- Divergence of a vector function $\mathbf{f} = \mathbf{n}_r f_r + \mathbf{n}_\theta f_\theta + \mathbf{n}_\varphi f_\varphi$:

$$\nabla \cdot \mathbf{f} = \frac{1}{r^2}\frac{\partial\left(r^2 f_r\right)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial\left(f_\theta \sin\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial f_\varphi}{\partial\varphi}. \tag{A.68}$$

- Curl of a similar vector function:

$$\begin{aligned}\nabla \times \mathbf{f} = {} & \mathbf{n}_r \frac{1}{r\sin\theta}\left(\frac{\partial\left(f_\varphi \sin\theta\right)}{\partial\theta} - \frac{\partial f_\theta}{\partial\varphi}\right) + \mathbf{n}_\theta \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial f_r}{\partial\varphi} - \frac{\partial\left(r f_\varphi\right)}{\partial r}\right) \\ & + \mathbf{n}_\varphi \frac{1}{r}\left(\frac{\partial\left(r f_\theta\right)}{\partial r} - \frac{\partial f_r}{\partial\theta}\right).\end{aligned} \tag{A.69}$$

- The Laplace operator of a vector function:

$$\begin{aligned}\nabla^2 \mathbf{f} = {} & \mathbf{n}_r\left(\nabla^2 f_r - \frac{2}{r^2} f_r - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(f_\theta \sin\theta\right) - \frac{2}{r^2\sin\theta}\frac{\partial f_\varphi}{\partial\varphi}\right) \\ & + \mathbf{n}_\theta\left(\nabla^2 f_\theta - \frac{1}{r^2\sin^2\theta} f_\theta + \frac{2}{r^2}\frac{\partial f_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial f_\varphi}{\partial\varphi}\right) \\ & + \mathbf{n}_\varphi\left(\nabla^2 f_\varphi - \frac{1}{r^2\sin^2\theta} f_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial f_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial f_\theta}{\partial\varphi}\right).\end{aligned} \tag{A.70}$$

A.11 Products involving ∇

(i) *Useful zeros*:

- For any scalar function $f(\mathbf{r})$,

$$\nabla \times (\nabla f) \equiv \mathbf{curl}(\mathbf{grad} f) = 0. \tag{A.71}$$

- For any vector function $\mathbf{f}(\mathbf{r})$,

$$\nabla \cdot (\nabla \times \mathbf{f}) \equiv \mathrm{div}(\mathbf{curl} f) = 0. \tag{A.72}$$

(ii) The *Laplace operator* expressed via the curl of a curl:

$$\nabla^2 \mathbf{f} = \nabla(\nabla \cdot \mathbf{f}) - \nabla \times (\nabla \times \mathbf{f}). \tag{A.73}$$

(iii) Spatial differentiation of a product of a *scalar* function by a *vector* function:x

- The scalar 3D generalization of Eq. (A.22) is

$$\nabla \cdot (f\ \mathbf{g}) = (\nabla f) \cdot \mathbf{g} + f(\nabla \cdot \mathbf{g}). \tag{A.74a}$$

- Its vector generalization is similar:

$$\nabla \times (f\,\mathbf{g}) = (\nabla f) \times \mathbf{g} + f(\nabla \times \mathbf{g}). \tag{A.74b}$$

(iv) Spatial differentiation of products of *two vector* functions:

$$\nabla \times (\mathbf{f} \times \mathbf{g}) = \mathbf{f}(\nabla \cdot \mathbf{g}) - (\mathbf{f} \cdot \nabla)\mathbf{g} - (\nabla \cdot \mathbf{f})\mathbf{g} + (\mathbf{g} \cdot \nabla)\,\mathbf{f}, \tag{A.75}$$

$$\nabla(\mathbf{f} \cdot \mathbf{g}) = (\mathbf{f} \cdot \nabla)\mathbf{g} + (\mathbf{g} \cdot \nabla)\,\mathbf{f} + \mathbf{f} \times (\nabla \times \mathbf{g}) + \mathbf{g} \times (\nabla \times \mathbf{f}), \tag{A.76}$$

$$\nabla \cdot (\mathbf{f} \times \mathbf{g}) = \mathbf{g} \cdot (\nabla \times \mathbf{f}) - \mathbf{f} \cdot (\nabla \times \mathbf{g}). \tag{A.77}$$

A.12 Integro-differential relations

(i) For an *arbitrary surface* S limited by closed contour C:

- The *Stokes theorem*, valid for any differentiable vector field $\mathbf{f}(\mathbf{r})$:

$$\int_S (\nabla \times \mathbf{f}) \cdot d^2\mathbf{r} \equiv \int_S (\nabla \times \mathbf{f})_n d^2r = \oint_C \mathbf{f} \cdot d\mathbf{r} \equiv \oint_C f_\tau\, dr, \tag{A.78}$$

where $d^2\mathbf{r} \equiv \mathbf{n}d^2r$ is the elementary area vector (normal to the surface), and $d\mathbf{r}$ is the elementary contour length vector (tangential to the contour line).

(ii) For an *arbitrary volume* V limited by closed surface S:

- *Divergence* (or 'Gauss') *theorem*, valid for any differentiable vector field $\mathbf{f}(\mathbf{r})$:

$$\int_V (\nabla \cdot \mathbf{f})\, d^3r = \oint_S \mathbf{f} \cdot d^2\mathbf{r} \equiv \oint_S f_n\, d^2r. \tag{A.79}$$

- *Green's theorem*, valid for two differentiable scalar functions $f(\mathbf{r})$ and $g(\mathbf{r})$:

$$\int_V (f\,\nabla^2 g - g\nabla^2 f)\, d^3r = \oint_S (f\,\nabla g - g\nabla f)_n d^2r. \tag{A.80}$$

- An identity valid for any two scalar functions f and g, and a vector field $\mathbf{j}$ with $\nabla\cdot\mathbf{j} = 0$ (all differentiable):

$$\int_V [f(\mathbf{j} \cdot \nabla g) + g(\mathbf{j} \cdot \nabla f)]\; d^3r = \oint_S fgj_n\, d^2r. \tag{A.81}$$

A.13 The Kronecker delta and Levi-Civita permutation symbols

- The *Kronecker delta symbol* (defined for integer indices):

$$\delta_{jj'} \equiv \begin{cases} 1, & \text{if } j' = j, \\ 0, & \text{otherwise.} \end{cases} \tag{A.82}$$

- The *Levi-Civita permutation symbol* (most frequently used for 3 integer indices, each taking one of values 1, 2, or 3):

$$\varepsilon_{jj'j''} \equiv \begin{cases} +1, & \text{if the indices follow in the 'correct' ('even')} \\ & \text{order: } 1 \to 2 \to 3 \to 1 \to 2\ldots, \\ -1, & \text{if the indices follow in the 'incorrect' ('odd')} \\ & \text{order: } 1 \to 3 \to 2 \to 1 \to 3\ldots, \\ 0, & \text{if any two indices coincide.} \end{cases} \tag{A.83}$$

- Relation between the Levi-Civita and the Kronecker delta products:

$$\varepsilon_{jj'j''}\varepsilon_{kk'k''} = \sum_{l,l',l''=1}^{3} \begin{vmatrix} \delta_{jl} & \delta_{jl'} & \delta_{jl''} \\ \delta_{j'l} & \delta_{j'l'} & \delta_{j'l''} \\ \delta_{j''l} & \delta_{j''l'} & \delta_{j''l''} \end{vmatrix}; \tag{A.84a}$$

summation of this relation, written for 3 different values of $j = k$, over these values yields the so-called *contracted epsilon identity*:

$$\sum_{j=1}^{3} \varepsilon_{jj'j''}\varepsilon_{jk'k''} = \delta_{j'k'}\delta_{j''k''} - \delta_{j'k''}\delta_{j''k'}. \tag{A.84b}$$

A.14 Dirac's delta-function, sign function, and theta-function

- Definition of 1D *delta-function* (for real $a < b$):

$$\int_a^b f(\xi)\delta(\xi)d\xi = \begin{cases} f(0), & \text{if } a < 0 < b, \\ 0, & \text{otherwise,} \end{cases} \tag{A.85}$$

where $f(\xi)$ is any function continuous near $\xi = 0$. In particular (if $f(\xi) = 1$ near $\xi = 0$), the definition yields

$$\int_a^b \delta(\xi)d\xi = \begin{cases} 1, & \text{if } a < 0 < b, \\ 0, & \text{otherwise.} \end{cases} \tag{A.86}$$

- Relation to the *theta-function* $\theta(\xi)$ and *sign function* $\operatorname{sgn}(\xi)$

$$\delta(\xi) = \frac{d}{d\xi}\theta(\zeta) = \frac{1}{2}\frac{d}{d\xi}\operatorname{sgn}(\xi), \tag{A.87a}$$

where

$$\theta(\xi) \equiv \frac{\operatorname{sgn}(\xi) + 1}{2} = \begin{cases} 0, & \text{if } \xi < 0, \\ 1, & \text{if } \xi > 1, \end{cases}$$
$$\operatorname{sgn}(\xi) \equiv \frac{\xi}{|\xi|} = \begin{cases} -1, & \text{if } \xi < 0, \\ +1, & \text{if } \xi > 1. \end{cases} \tag{A.87b}$$

- An important integral[13]:

$$\int_{-\infty}^{+\infty} e^{is\xi} ds = 2\pi\delta(\xi). \tag{A.88}$$

- 3D generalization of the delta-function of the radius-vector (the 2D generalization is similar):

$$\int_V f(\mathbf{r})\delta(\mathbf{r})d^3r = \begin{cases} f(0), & \text{if } 0 \in V, \\ 0, & \text{otherwise;} \end{cases} \tag{A.89}$$

  it may be represented as a product of 1D delta-functions of Cartesian coordinates:

$$\delta(\mathbf{r}) = \delta(r_1)\delta(r_2)\delta(r_3). \tag{A.90}$$

A.15 The Cauchy theorem and integral

Let a complex function $f(z)$ be analytic within a part of the complex plane z, that is limited by a closed contour C and includes point z'. Then

$$\oint_C f(z)dz = 0, \tag{A.91}$$

$$\oint_C f(z)\frac{dz}{z - z'} = 2\pi i f(z') \tag{A.92}$$

The first of these relations is usually called the *Cauchy integral theorem* (or the 'Cauchy–Goursat theorem'), and the second one—the *Cauchy integral* (or the 'Cauchy integral formula').

A.16 Literature

(i) Properties of some *special functions* are briefly discussed at the relevant points of the lecture notes; in the alphabetical order:
- Airy functions: *Part QM* section 2.4;
- Bessel functions: *Part EM* section 2.7;
- Fresnel integrals: *Part EM* section 8.6;
- Hermite polynomials: *Part QM* section 2.9;
- Laguerre polynomials (both simple and associated): *Part QM* section 3.7;

[13] The coefficient in this relation may be readily recalled by considering its left-hand part as the Fourier-integral representation of function $f(s) \equiv 1$, and applying Eq. (A.85) to the reciprocal Fourier transform

$$f(s) \equiv 1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-is\xi}[2\pi\delta(\xi)]d\xi.$$

- Legendre polynomials, associated Legendre functions: *Part EM* section 2.8, and *Part QM* section 3.6;
- Spherical harmonics: *Part QM* section 3.6;
- Spherical Bessel functions: *Part QM* sections 3.6 and 3.8.

(ii) For *more formulas*, and their discussion, I can recommend the following handbooks[14]:

- *Handbook of Mathematical Formulas* [2];
- *Tables of Integrals, Series, and Products* [3];
- *Mathematical Handbook for Scientists and Engineers* [4];
- *Integrals and Series* volumes 1 and 2 [5];
- A popular textbook *Mathematical Methods for Physicists* [6] may be also used as a formula manual.

Many formulas are also available from the symbolic calculation modules of the commercially available software packages listed in section (iv) below.

(iii) Probably the most popular collection of *numerical calculation codes* are the twin manuals by W Press *et al* [1]:

- *Numerical Recipes in Fortran 77*;
- *Numerical Recipes* [in C++—KKL].

My lecture notes include very brief introductions to numerical methods of differential equation solution:

- ordinary differential equations: *Part CM*, section 5.7;
- partial differential equations: *Part CM* section 8.5 and *Part EM* section 2.11, which include references to literature for further reading.

(iv) The following are the most popular *software packages* for numerical and symbolic calculations, all with plotting capabilities (in the alphabetical order):

- Maple (www.maplesoft.com/products/maple/);
- MathCAD (www.ptc.com/engineering-math-software/mathcad/);
- Mathematica (www.wolfram.com/mathematica/);
- MATLAB (www.mathworks.com/products/matlab.html).

References

[1] Press W *et al* 1992 *Numerical Recipes in Fortran 77* 2nd edn (Cambridge: Cambridge University Press)
Press W *et al* 2007 *Numerical Recipes* 3rd edn (Cambridge: Cambridge University Press)

[2] Abramowitz M and Stegun I (eds) 1965 *Handbook of Mathematical Formulas* (New York: Dover), and numerous later printings. An updated version of this collection is now available online at http://dlmf.nist.gov/.

[14] On a personal note, perhaps 90% of all formula needs throughout my research career were satisfied by a tiny, wonderfully compiled old book [7], used copies of which, rather amazingly, are still available on the Web.

[3] Gradshteyn I and Ryzhik I 1980 *Tables of Integrals, Series, and Products* 5th edn (New York: Academic)

[4] Korn G and Korn T 2000 *Mathematical Handbook for Scientists and Engineers* 2nd edn (New York: Academic)

[5] Prudnikov A *et al* 1986 *Integrals and Series* vol 1 (Boca Raton, FL: CRC Press)
Prudnikov A *et al* 1986 *Integrals and Series* vol 2 (Boca Raton, FL: CRC Press)

[6] Arfken G *et al* 2012 *Mathematical Methods for Physicists* 7th edn (New York: Academic)

[7] Dwight H 1961 *Tables of Integrals and Other Mathematical Formulas* 4th edn (London: Macmillan)

Appendix B

Selected physical constants

The listed numerical values of the constants are from the most recent (2014) International CODATA recommendation (see, e.g. http://physics.nist.gov/cuu/Constants/index.html), besides a newer result for k_B*—see* [1]. *Please note the recently announced (but, by this volume's press time, not yet official) adjustment of the SI values - see, e.g. https://www.nist.gov/si-redefinition/meet-constants. In particular, the Planck constant will also get a definite value (within the interval specified in table B.1), enabling a new, fundamental standard of the kilogram.*

Table B.1.

Symbol	Quantity	SI value and unit	Gaussian value and unit	Relative rms uncertainty
c	speed of light in free space	$2.99\,792\,458 \times 10^{8}$ m s^{-1}	$2.99\,792\,458 \times 10^{10}$ cm s^{-1}	0 (defined value)
G	gravitation constant	6.6741×10^{-11} m^{3} kg^{-1} s^{-2}	6.6741×10^{-8} cm^{3} g^{-1} s^{-2}	$\sim 5 \times 10^{-5}$
$\hbar$	Planck constant	$1.05\,457\,180 \times 10^{-34}$ J s	$1.05\,457\,180 \times 10^{-27}$ erg s	$\sim 2 \times 10^{-8}$
e	elementary electric charge	$1.6\,021\,762 \times 10^{-19}$ C	$4.803\,203 \times 10^{-10}$ statcoulomb	$\sim 6 \times 10^{-9}$
m_e	electron's rest mass	$0.91\,093\,835 \times 10^{-30}$ kg	$0.91\,093\,835 \times 10^{-27}$ g	$\sim 1 \times 10^{-8}$
m_p	proton's rest mass	$1.67\,262\,190 \times 10^{-27}$ kg	$1.67\,262\,190 \times 10^{-24}$ g	$\sim 1 \times 10^{-8}$
μ_0	magnetic constant	$4\pi \times 10^{-7}$ N A^{-2}	–	0 (defined value)
ε_0	electric constant	$8.854\,187\,817 \times 10^{-12}$ F m^{-1}	–	0 (defined value)
k_B	Boltzmann constant	$1.380\,649 \times 10^{-23}$ J K^{-1}	$1.3\,806\,490 \times 10^{-16}$ erg K^{-1}	$\sim 2 \times 10^{-6}$

doi:10.1088/2053-2563/aaf3a3ch12

Comments:

1. The fixed value of c was defined by an international convention in 1983, in order to extend the official definition of the second (as 'the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom') to that of the meter. The values are back-compatible with the legacy definitions of the meter (initially, as 1/40 000 000th of the Earth's meridian length) and the second (for a long time, as $1/(24 \times 60 \times 60) = 1/86\,400$th of the Earth's rotation period), within the experimental errors of those measures.
2. ε_0 and μ_0 are not really the fundamental constants; in the SI system of units one of them (say, μ_0) is selected arbitrarily[1], while the other one is defined via the relation $\varepsilon_0\mu_0 = 1/c^2$.
3. The Boltzmann constant k_B is also not quite fundamental, because its only role is to comply with the independent definition of the kelvin (K), as the temperature unit in which the triple point of water is exactly 273.16 K. If temperature is expressed in energy units k_BT (as is done, for example, in *Part SM* of this series), this constant disappears altogether.
4. The dimensionless *fine structure* ('Sommerfeld's') *constant* α is numerically the same in any system of units:

$$\alpha \equiv \left\{ \begin{array}{cc} e^2/4\pi\varepsilon_0\hbar c & \text{in SI units} \\ e^2/\hbar c & \text{in Gaussian units} \end{array} \right\} \approx 7.297\,352\,566 \times 10^{-3}$$
$$\approx \frac{1}{137.035\,999\,14},$$

and is known with a much smaller relative rms uncertainty (currently, $\sim 3 \times 10^{-10}$) than those of the component constants.

References

[1] Gaiser C *et al* 2017 *Metrologia* **54** 280
[2] Newell D 2014 *Phys. Today* **67** 35–41

[1] Note that the selected value of μ_0 may be changed (a bit) in a few years—see, e.g., [2].

Bibliography

This section presents a partial list of textbooks and monographs used in the work on the EAP series[1,2].

Part CM: Classical Mechanics

Fetter A L and Walecka J D 2003 *Theoretical Mechanics of Particles and Continua* (New York: Dover)
Goldstein H, Poole C and Safko J 2002 *Classical Mechanics* 3rd edn (Reading, MA: Addison Wesley)
Granger R A 1995 *Fluid Mechanics* (New York: Dover)
José J V and Saletan E J 1998 *Classical Dynamics* (Cambridge: Cambridge University Press)
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Sommerfeld A 1964 *Mechanics of Deformable Bodies* (New York: Academic)

Part EM: Classical Electrodynamics

Batygin V V and Toptygin I N 1978 *Problems in Electrodynamics* 2nd edn (New York: Academic)
Griffiths D J 2007 *Introduction to Electrodynamics* 3rd edn (Englewood Cliffs, NJ: Prentice-Hall)
Jackson J D 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley)
Landau L D and Lifshitz E M 1984 *Electrodynamics of Continuous Media* 2nd edn (Auckland: Reed)
Landau L D and Lifshitz E M 1975 *The Classical Theory of Fields* 4th edn (Oxford: Pergamon)

[1] The list does not include the sources (mostly, recent original publications) cited in the lecture notes and problem solutions, and the mathematics textbooks and handbooks listed in section A.16.
[2] Recently several high-quality teaching materials on advanced physics became available online, including R. Fitzpatrick's text on *Classical Electromagnetism* (farside.ph.utexas.edu/teaching/jk1/Electromagnetism.pdf), B Simons' 'lecture shrunks' on *Advanced Quantum Mechanics* (www.tcm.phy.cam.ac.uk/~bds10/aqp.html), and D Tong's lecture notes on several advanced topics (www.damtp.cam.ac.uk/user/tong/teaching.html).

Panofsky W K H and Phillips M 1990 *Classical Electricity and Magnetism* 2nd edn (New York: Dover)
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