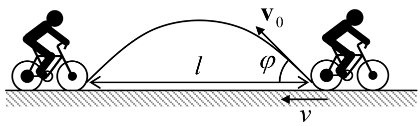
IOP Publishing

Classical Mechanics
Problems with solutions
Konstantin K Likharev

Chapter 1

Review of fundamentals

***Problem* 1.1.** A bicycle, ridden with velocity v on a wet pavement, has no mudguards on its wheels. How far behind should the following biker ride to avoid being splashed over? Neglect the effects of air resistance.



Solution: The easiest way to solve this problem is to use a reference frame moving with the cyclists. Assuming that their speed is constant, in this reference frame the bike frames are at rest, but the ground moves back with speed v (see the arrow in the figure above), and hence the rim of each wheel moves around its axis with that speed. Because of this, the speed of each water drop immediately after detachment from the tire is the same: $|\mathbf{v}_0| = v$. Since this moving reference frame is inertial, we may write Newton's laws in it and hence use all their corollaries. In particular, this means that after its detachment, each drop follows the well-known parabolic trajectory and before returning to the initial height travels the distance[1]

$$l = \frac{v^2}{g} \sin 2\varphi, \tag{*}$$

[1] I hope that the reader knows how to derive this formula, but just in case… Since the drop's acceleration during its flight equals g = const, and is directed downward, placing the reference frame origin at the point of the drop's detachment from the tire, we may spell out Eq. (1.18) of the lecture notes as follows:

$$x(t) = -v \cos \varphi \; t, \quad y(t) = -gt^2/2 + v \sin \varphi \; t.$$

Now, requiring the drop to return to the initial height, $y(t) = 0$, for the time of flight we obtain: $t = 2v \sin \varphi / g$. Plugging this expression into the above formula for $x(t)$, we obtain $x(t) = -l$, where l is given by Eq. (*).

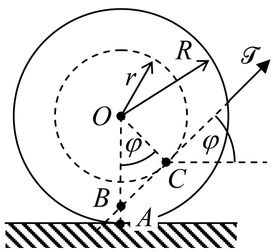
doi:10.1088/978-0-7503-1401-5ch1

where φ is the take-off angle—see the figure above. The distance is largest for drops with $\varphi = \pi/4$:[2]

$$l_{\max} = \frac{v^2}{g}.$$

As the figure above shows, this is the smallest distance to be absolutely safe from splashing, although this expression may be corrected for bike shape details (for example, for a different radius R of the wheel), and for what exactly is meant by the distance between the bikes. For realistic bike velocities, $v \gg (gR)^{1/2} \sim 2\ \mathrm{m\ s^{-1}} \sim 5$ mph, these corrections are minor, because $l_{\max} \gg R$.

***Problem* 1.2.** Two round disks of radius R are firmly connected with a coaxial cylinder of a smaller radius r, and a thread is wound on the resulting spool. The spool is placed on a horizontal surface, and thread's end is being pooled out at angle φ—see the figure below. Assuming that the spool does not slip on the surface, what direction would it roll?



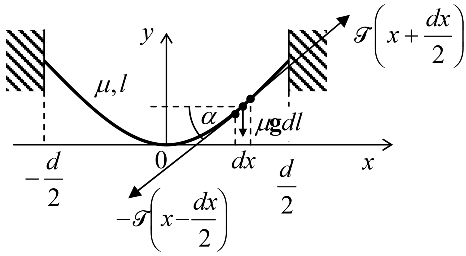
Solution: The no-slip roll of the spool may be considered as its rotation about the instantaneous axis which coincides with the spool–surface contact line. (In the figure above, it is perpendicular to the plane of drawing and passes through point A.) Thus the direction of rotation depends on whether the line of the applied force $\mathscr{T}$ passes above or below the axis, i.e. whether point B (where that line crosses the vertical line OA) is located above or below point A. From the right triangle OBC we readily obtain $OB = OC/\cos\varphi \equiv r/\cos\varphi$, while $OA \equiv R$. So, if

$$\frac{r}{\cos\varphi} < R, \qquad \text{i.e. if } \cos\varphi > \frac{r}{R},$$

the spool will roll in the direction of the applied force (in the figure above, to the right), but otherwise it will roll back. In particular, if the thread is being pulled horizontally ($\varphi = 0$, $\cos\varphi = 1$), the spool will roll to the right, while if it us pulled up ($\varphi = \pi/2$, $\cos\varphi = 0$) it will roll to the left, for any $r < R$.

[2] Note that, curiously enough, in the reference frame of the ground, these 'most splash-dangerous' drops have the horizontal velocity $v(1 - 1/\sqrt{2}) \approx +0.293v > 0$, i.e. move in the same direction as the bikes.

Problem* 1.3. Calculate the equilibrium shape of a flexible, heavy rope of length l, with a constant mass μ per unit length, if it is hung in a uniform gravity field between two points separated by a horizontal distance d—see the figure below.



Solution: Let us introduce the Cartesian coordinates as shown in the figure above, with the origin at the lowest point of the rope. In equilibrium, the vector sum of the forces acting on each small rope fragment, of length dl, should vanish, so that for the vector $\mathscr{T}$ of the rope tension force as a function of coordinate x we may write

$$\mathscr{T}\left(x + \frac{dx}{2}\right) - \mathscr{T}\left(x - \frac{dx}{2}\right) + \mu\mathbf{g}dl = 0. \qquad (*)$$

Here $dx = dl \cos\alpha$ (where α is the rope's slope at this particular point, see the figure above) is the horizontal axis fragment corresponding to dl, so that

$$dl = \frac{dx}{\cos\alpha} \equiv (1 + \tan^2\alpha)^{1/2}\, dx = (1 + y'^2)^{1/2} dx,$$

$$\text{where} \quad y' \equiv \frac{dy}{dx} = \tan\alpha.$$

Due to the smallness of dx, we may expand the function $\mathscr{T}(x)$ in the Taylor series in dx, and keep only the first (linear) term of the tension difference participating in Eq. (*):

$$\frac{d\mathscr{T}}{dx}dx + \mu\mathbf{g}(1 + y'^2)^{1/2} dx = 0.$$

After the cancellation of $dx \neq 0$, two Cartesian components of this vector equation yield two scalar equations for two unknown scalar functions: $y(x)$, describing the shape of the rope, and $\mathscr{T}(x)$, the magnitude of its tension:

$$\frac{d\mathscr{T}_x}{dx} \equiv \frac{d}{dx}(\mathscr{T}\cos\alpha) \equiv \frac{d}{dx}\left(\frac{\mathscr{T}}{(1 + y'^2)^{1/2}}\right) = 0,$$

$$\frac{d\mathscr{T}_y}{dx} \equiv \frac{d}{dx}(\mathscr{T}\sin\alpha) \equiv \frac{d}{dx}\left(\frac{\mathscr{T}\, y'}{(1 + y'^2)^{1/2}}\right) = \mu g(1 + y'^2)^{1/2}.$$

The first of these equations yields $\mathscr{T}/(1 + y^2)^{1/2} = \text{const} \equiv \mathscr{T}_0$, where $\mathscr{T}_0$ has the sense of the rope's tension at its lowest point (where $y' = 0$). Plugging this relation into the second equation, we obtain the following second-order differential equation for the function we are interested in, $y(x)$:

$$\mathscr{T}_0 y'' = \mu g(1 + y'^2)^{1/2}, \qquad \text{where} \quad y'' \equiv \frac{d^2y}{dx^2}.$$

It is straightforward to integrate this equation. First, we may represent the second derivative as[3]

$$y'' \equiv \frac{dy'}{dx} = \frac{dy'}{dy}\frac{dy}{dx} = y'\frac{dy'}{dy} = \frac{1}{2}\frac{d(y'^2)}{dy},$$

so that our equation becomes

$$\frac{\mathscr{T}_0}{2}\frac{d(y'^2)}{dy} = \mu g(1 + y'^2)^{1/2}, \quad \text{or equivalently:} \quad \frac{\mathscr{T}_0}{2}\frac{d(1 + y'^2)}{(1 + y'^2)^{1/2}} = \mu g dy.$$

Now we may integrate both parts, obtaining

$$\mathscr{T}_0(1 + y'^2)^{1/2} = \mu g y + \text{const}.$$

Since we have selected the origin of y at the lowest point of the rope, where $y' = 0$, this constant also equals $\mathscr{T}_0$, so that

$$\mathscr{T}_0(1 + y'^2)^{1/2} = \mu g y + \mathscr{T}_0.$$

Solving this equation for $y' \equiv dy/dx$, and then separating variables x and y, we get

$$y' = \pm[(1 + (\mu g/\mathscr{T}_0)y)^2 - 1]^{1/2}, \qquad \text{giving} \quad \frac{dy}{[(1 + (\mu g/\mathscr{T}_0)y)^2 - 1]^{1/2}} = \pm dx.$$

It is convenient to integrate both parts of this equation from the lowest point, where $x = 0$ and $y = 0$, to some point $x > 0$, because at this interval $dy/dx > 0$ (see the figure above), and me may select positive sign on the right-hand side of the equation. Introducing dimensionless variable $\xi \equiv 1 + (\mu g/\mathscr{T}_0)y$, so that $dy = (\mathscr{T}_0/\mu g)\, d\xi$, we may bring the integral of the left-hand side to a simpler form:

$$\int_{y=0}^{y} \frac{d\xi}{(\xi^2 - 1)^{1/2}} = \frac{\mu g}{\mathscr{T}_0}x.$$

This integral may be readily worked out using one more substitution: $\xi \equiv \cosh\beta$, so that the nominator, $d\xi = \sinh\beta\, d\beta$, and denominator, $(\xi^2-1)^{1/2} = (\cosh^2\beta-1)^{1/2} = \sinh\beta$, are proportional to the same function, $\sinh\beta$, which cancels. As a result, this integral is just $\int d\beta = \beta$, by the definition of β equal to $\cosh^{-1}\xi \equiv \cosh^{-1}[1 + (\mu g/\mathscr{T}_0)y]$, and we obtain

$$\cosh^{-1}\left(1 + \frac{\mu g y}{\mathscr{T}_0}\right) = \frac{\mu g x}{\mathscr{T}_0}, \quad \text{i.e.} \quad y = \frac{\mathscr{T}_0}{\mu g}\left(\cosh\frac{\mu g x}{\mathscr{T}_0} - 1\right). \qquad (**)$$

[3] This is a very popular transformation, which was already used (for other variables) for the derivation of Eq. (1.20) of the lecture notes, and will be repeatedly used later in the course.

So, the free-hanging, uniform ropes or chains have the form of the plot of the hyperbolic cosine function[4]. Due to this fact, this curve is sometimes called the *chainette*. (A more popular term for this curve is 'catenary', but the terms 'alysoid' and 'funicular' may be also encountered.) What remains now is to find the constant $\mathscr{T}_0$. This may be done by the requirement that the sum of all elementary lengths $dl = (1 + y'^2)^{1/2}dx$ equals its actual length l:

$$l \equiv \int_l dl = \int_{-d/2}^{+d/2} (1 + y'^2)^{1/2} dx = 2\int_0^{d/2} (1 + y'^2)^{1/2} dx. \qquad (***)$$

From Eq. (**), we obtain

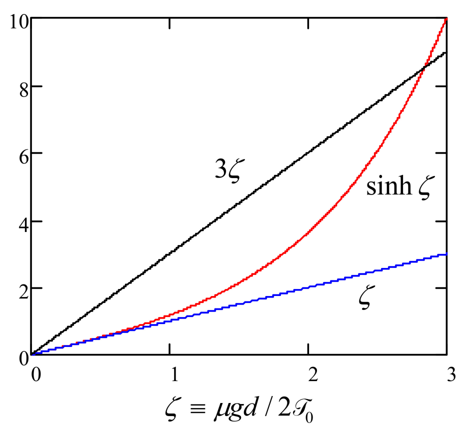
$$y' = \sinh \frac{\mu g x}{\mathscr{T}_0}, \qquad \text{so that} \quad (1 + y'^2)^{1/2} = \cosh \frac{\mu g x}{\mathscr{T}_0};$$

due to the last equality, the integration in Eq. (***) is elementary, giving

$$l = \frac{2\mathscr{T}_0}{\mu g} \sinh \frac{\mu g d}{2\mathscr{T}_0}, \qquad \text{i.e.} \ \frac{\mu g l}{2\mathscr{T}_0} = \sinh \frac{\mu g d}{2\mathscr{T}_0},$$

or in a convenient dimensionless form:

$$\frac{l}{d}\zeta = \sinh \zeta, \quad \text{where} \quad \zeta \equiv \frac{\mu g d}{2\mathscr{T}_0}.$$



This is a transcendental equation for ζ (and hence for $\mathscr{T}_0$); from the plot of its both sides as functions of this variable (see the figure above) it is evident that the equation has a single positive root for any $l/d > 1$. Using the well-known asymptotic

[4] Additional question: is this solution a good approximation for suspension bridge cables? If not, why?

behaviors of the sine hyperbolic for small and large values of its argument, it is straightforward to show that

$$\mathscr{T}_0 \to \mu g l \times \begin{cases} \dfrac{1}{2\sqrt{6}}\left(\dfrac{l}{l-d}\right)^{1/2} \to \infty, & \text{at } l/d \to 1, \\ \dfrac{d/l}{2\ln(2l/d)} \to 0, & \text{at } l/d \to \infty. \end{cases}$$

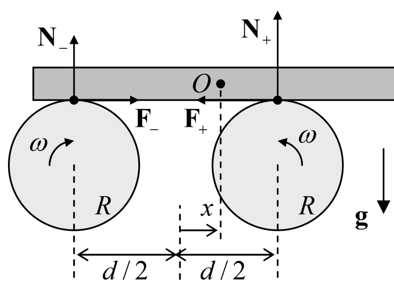
In the former limit, $\mathscr{T}_0$ is much larger than the weight $\mu g l$ of the whole rope, while in the latter limit, is much less than the weight.

In conclusion, let me note that this problem may be also solved (or rather the differential equation for the function $y(x)$ derived) by the calculus of variations, from the condition that the total potential energy of the rope,

$$U = \int_l \mu g y dl = \mu g \int_{-d/2}^{+d/2} y(1 + y'^2)^{1/2} dx,$$

has to be minimal at equilibrium, upon the condition of constancy of rope's length l, i.e. of the integral (***). Although such solution is lengthier, it is highly recommended to the reader, in particular because we would need the calculus of variations several times in this course, starting from the derivation of the Lagrange equations in the next chapter.

***Problem* 1.4.** A uniform, long, thin bar is placed horizontally on two similar round cylinders rotating toward each other with the same angular velocity ω and displaced by distance d—see the figure below. Calculate the laws of relatively slow horizontal motions of the bar within the plane of drawing for both possible directions of cylinder rotation, assuming that the friction force between the slipping surfaces of the bar and each cylinder obeys the simple *Coulomb approximation*[5] $|F| = \mu N$, where N is the normal pressure force between them, and μ is a constant (velocity-independent) coefficient. Formulate the condition of validity of your result.



[5] It was suggested in 1785 by the same C-A de Coulomb who discovered the famous *Coulomb law* of electrostatics, and hence pioneered the whole qualitative science of electricity.

Solution: Let the current horizontal displacement of the bar's center-of-mass (point O) from the symmetry plane of the system equal x—see the figure above. Then we may write the following two equations for the normal pressure forces $\mathbf{N}_\pm$,

$$N_- + N_+ = Mg,$$
$$N_-\left(\frac{d}{2} + x\right) - N_+\left(\frac{d}{2} - x\right) = 0,$$

where M is bar's mass. These equations express, correspondingly, the balances of vertical forces and their torques, necessary to avoid the vertical and angular accelerations of the bar. (Note that contributions of friction forces $\mathbf{F}_\pm$ into the torque balance may be ignored only because of small thickness of the bar.) Solving this simple system of two linear equations, we obtain

$$N_\pm = Mg\frac{d/2 \pm x}{d}.$$

If the bar motion is relatively slow, $|v| < \omega R$, its surface slips relatively to those of both cylinders, so it is legitimate to use the kinetic-friction approximation $|F_\pm| = \mu N_\pm$ for each of the friction forces, and for the total horizontal force we may write

$$|F| = |F_+ - F_-| = 2\mu Mg\frac{|x|}{d}.$$

What follows depends on the direction of the cylinders' rotation. If their top points, on which the bar rests, move toward each other (as shown in the figure above), then the force F_+ is always directed to the left, so that taking the shown direction of displacement x for the positive one, we may write $F_+ = -2\mu Mg(d/2 - x)/d < 0$, while the counterpart force is positive: $F_- = 2\mu Mg(d/2 + x)/d$. As a result,

$$F = F_+ - F_- = -2\mu Mg\frac{x}{d}.$$

In this case, the horizontal component of Newton's second law for the bar reads

$$M\ddot{x} = -2\mu Mg\frac{x}{d}. \qquad (*)$$

This is the well-known equation of 1D motion of a body on an elastic spring with spring constant $\kappa = 2\mu Mg/d$, and its solutions are sinusoidal oscillations of frequency

$$\omega_0 = \left(\frac{\kappa}{M}\right)^{1/2} = \left(\frac{2\mu g}{d}\right)^{1/2}.$$

Note that this sinusoidal solution is only valid if the displacement amplitude $A \equiv x_{max}$ is lower than $\omega R/\omega_0$, so that the velocity amplitude, $\omega_0 A$, is below the cylinder's top speed, ωR. What happens at larger amplitudes depends on the static friction coefficient μ_s or, more exactly, its relation with the kinetic friction coefficient μ. The reader is encouraged to carry out a semi-quantitative analysis of the various cases.

In the second case, when the cylinders rotate in the direction opposite to that shown in the figure above (with their top parts moving away from each other), both friction forces have opposite directions, and we need to change the sign in the expression for the total horizontal force F. This gives, instead of Eq. (*), the following equation:

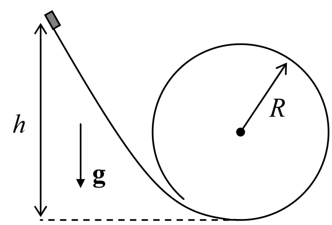
$$M\ddot{x} = 2\mu Mg\frac{x}{d}. \qquad (**)$$

Its general solution is a sum of either two exponents, or two hyperbolic functions of time[6]:

$$x(t) = C_+e^{\lambda t} + C_-e^{-\lambda t} \equiv C_c \cosh \lambda t + C_s \sinh \lambda t, \qquad \text{with} \quad \lambda = \left(\frac{2\mu g}{d}\right)^{1/2}, \quad (***)$$

where constants $C_\pm$ (or alternatively, $C_{c,s}$) are determined by the initial conditions—the initial position and velocity of the bar. Note that whatever the conditions are, according to Eq. (***), the displacement x and velocity $v = dx/dt$ of the bar will grow exponentially at $t \gg 1/\lambda$. So, at this direction of cylinder rotation, our solution (***) will eventually run out of its validity range $|v| < \omega R$.

***Problem* 1.5.** A small block slides, without friction, down a smooth slide that ends with a round loop of radius R—see the figure to the right. What smallest initial height h allows the block to make its way around the loop without dropping from the slide, if it is launched with negligible initial velocity?



Solution: The most critical point of the motion is evidently the highest point of the round loop, where the block's velocity v is smallest, and the block's weight force, $m\mathbf{g}$, is directed exactly along the possible direction of the detachment from the slide's surface. This velocity value may be readily calculated from the mechanical energy conservation law written for the initial and the critical points:

$$mgh = \frac{mv^2}{2} + 2mgR, \qquad \text{giving} \quad v^2 = 2g(h - 2R), \qquad (*)$$

[6] This fact may either be verified by its substitution to Eq. (**), or obtained in the regular fashion by looking for the solution in the form $C \exp\{\lambda t\}$, as is discussed in detail in the lecture notes, section 3.2.

where m is the mass of the block. In order to avoid the detachment from the slide, this velocity should be so high that the block weight mg could not, alone (without slide's reaction), provide the necessary centripetal acceleration $a = v^2/R$:

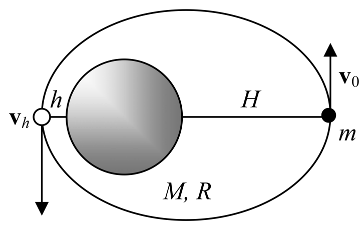
$$mg < m\frac{v^2}{R}.$$

Plugging the last form of Eq. (*) into this condition, we may reduce it to a very simple form:

$$h > h_{\min} = \frac{5}{2}R.$$

Note that the result is independent not only of the block's mass m (which is, due to the weak equivalence principle, common for all problems where the only substantial force is that of gravity), but also of the gravity acceleration g.

***Problem* 1.6.** A satellite of mass m is being launched from height H over the surface of a spherical planet with radius R and mass $M \gg m$—see the figure below. Find the range of initial velocities $\mathbf{v}_0$ (normal to the radius) providing closed orbits above the planet's surface.



Solution: The simplest way to solve this problem is to write the laws of conservation of the angular momentum and the energy, for two opposite points of the elliptical orbit (see the figure above):

$$mv_0(H + R) = mv_h(h + R), \qquad \frac{m}{2}v_0^2 - G\frac{mM}{H + R} = \frac{m}{2}v_h^2 - G\frac{mM}{h + R}.$$

Solving this system of equations for v_0 and v_h, we obtain, in particular:

$$v_0^2 = 2GM\frac{h + R}{(H + R)(h + H + 2R)}.$$

For the two boundaries of the velocity interval of our interest ($h = 0$ and $h \to \infty$), we obtain, respectively:

$$(v_0^2)_{\min} = 2GM\frac{R}{(H + R)(H + 2R)}, \quad (v_0^2)_{\max} = 2GM\frac{1}{H + R}.$$

For the particular case of satellite launch from planet's surface ($H = 0$), these formulas are reduced to the well-known expressions for the so-called *first and second space velocities*[7].

$$v_1 = \left(\frac{GM}{R}\right)^{1/2}, \quad v_2 = \left(\frac{2GM}{R}\right)^{1/2} = \sqrt{2}\ v_1 \approx 1.41\ v_1.$$

For our Earth ($M = M_E \approx 6.0 \times 10^{24}$ kg, $R = R_E \approx 6.4 \times 10^6$ m), these velocities are close, respectively, to 7.9 and 11.2 km s^{-1}.

***Problem* 1.7.** Prove that the thin-uniform-disk model of a galaxy describes small harmonic oscillations of stars inside it along the direction normal to the disk, and calculate the frequency of these oscillations in terms of the Newton's gravitational constant G and the average density ρ of the star/dust matter of the Galaxy.

Solution: Let us calculate the net gravitational force $\mathbf{F}$ exerted on the star, of mass m, by the whole galactic disk. This may be done by the direct summation of Newton's law of gravity (see, e.g. Eq. (1.15) of the lecture notes) for two point-masses m and m',

$$\mathbf{F}_{\text{point}} = -G\frac{mm'}{R^3}\mathbf{R}, \qquad \text{where } \mathbf{R} \equiv \mathbf{r} - \mathbf{r}', \tag{*}$$

over all elementary masses $dm' = \rho(\mathbf{r}')d^3r'$ of the disk:

$$\mathbf{F}(\mathbf{r}) = -Gm\int \frac{\rho(\mathbf{r}')}{|\,\mathbf{r} - \mathbf{r}'|^3}(\mathbf{r} - \mathbf{r}')\, d^3r'.$$

However, even in our simple case of constant density ρ, such integration is a bit cumbersome, because of the vector nature of the integral. It is helpful here (and in many other problems) to use the analogy of the Newton law (*) with the Coulomb law of the electrostatic interaction of two point charges q and q',[8]

$$\mathbf{F}_{\text{point}} = \frac{q}{4\pi\varepsilon_0}\frac{q'}{R^3}\mathbf{R}.$$

Now we may use the well-known Gauss law of electrostatics (which follows from the Coulomb law)[9],

$$\oint_S F_n d^2r = \frac{q}{\varepsilon_0}\int_V \rho(\mathbf{r}')d^3r',$$

to write its gravitational analog (with $q \leftrightarrow m$, and $1/4\pi\varepsilon_0 \leftrightarrow -G$, i.e. $1/\varepsilon_0 \leftrightarrow -4\pi G$):

$$\oint_S F_n d^2r = -4\pi Gm\int_V \rho(\mathbf{r}')d^3r'. \tag{**}$$

[7] Alternatively, v_2 is called the 'escape velocity'.
[8] See, e.g. *Part EM* Eq. (1.1).
[9] See, e.g. *Part EM* Eq. (1.16), with both sides multiplied by q, so that $\mathbf{E} \to q\mathbf{E} = \mathbf{F}$.

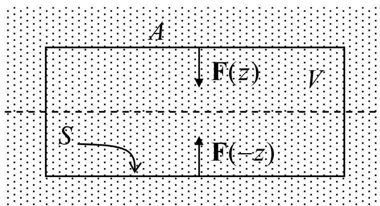
Here V is an arbitrary 'Gaussian' volume, S is the closed surface limiting the volume, and F_n is the component of force $\mathbf{F}$ along the outer normal $\mathbf{n}$ to the surface: $F_n = \mathbf{F}\cdot\mathbf{n}$.

For our current problem, it is beneficial to consider the Gaussian volume V in the form of a flat 'pillbox', with a thickness $2z$ smaller than that of the galactic disk, and planar 'lids' of area A parallel to the disk's plane—see the figure below, where the dashed line indicates the plane of disk's symmetry (from which the perpendicular coordinate z will be measured). Taking the pillbox lid area A to be much smaller that the galactic disk area, we may use problem's symmetry to argue that the force $\mathbf{F}$ should be:

(i) directed perpendicular to the galactic disk plane, and hence to the pillbox lids: $\mathbf{F} = F_z\mathbf{n}_z$;

(ii) independent of the 'horizontal' (in our figure) position: $F_z = F_z(z)$; and

(iii) symmetric relative to the symmetry plane: $F_z(-z) = -F_z(z)$.



With these assumptions, the gravity force flux through the lateral sides of the pillbox vanishes (because on these sides $\mathbf{F}\perp\mathbf{n}$, so that $\mathbf{F}\cdot\mathbf{n} = 0$), while the flux $\int F_n d^2r$ through each of the two lids is just $F_z(z)A$, so that Eq. (**) yields

$$2F(z)A = -4\pi Gm\rho(2zA),$$

giving, finally,

$$F(z) = -\kappa z, \qquad \text{with} \quad \kappa \equiv 4\pi Gm\rho.$$

Such an attractive force, trying to return the star to the disk's symmetry plane and proportional to its deviation from the plane, is similar to that provided by the usual elastic spring, and hence causes harmonic oscillations of the star about the symmetry plane, with frequency

$$\omega = \left(\frac{\kappa}{m}\right)^{1/2} = (4\pi G\rho)^{1/2},$$

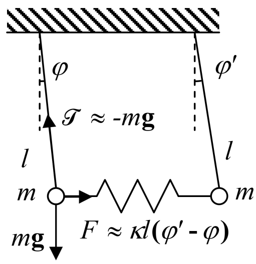
independent of the star's mass.

For our galaxy (the Milky Way) in the vicinity of our Sun, $\rho \approx 1.4 \times 10^{-20}$ kg m^{-3}, and the above formula yields $\omega \approx 3.3 \times 10^{-15}$ s^{-1}, corresponding to the oscillation period $\mathcal{T} = 2\pi/\omega \approx 60$ million years[10]. The amplitude of our Sun's oscillations (which cannot be calculated from the problem's data, but may be deduced from the

[10] Just for the reader's reference, this oscillation period is much shorter that the period, ~ 240 million years, of the Sun's rotation about the galactic center.

experimentally measured Sun's velocity relative to the neighboring stars) is about 2×10^{18} m, i.e. an order of magnitude smaller than the Milky Way disk's thickness ($\sim 2 \times 10^{19}$ m). On the other hand, the amplitude is much larger than the average distance between the stars in our vicinity, $\sim 10^{16}$ m. These two strong relations make this simple model valid for an approximate but very reasonable description of the Sun's motion.

***Problem* 1.8.** Derive the differential equations of motion for small oscillations of two similar pendula coupled with a spring (see the figure below), within the vertical plane. Assume that at the vertical position of both pendula, the spring is not stretched ($\Delta L = 0$).



Solution: If the deviations of the pendula from their vertical positions are small, $|\varphi|$, $|\varphi'| \ll 1$ (see the figure above), in the linear approximation in φ and φ' the magnitude of the supporting rod tension $\mathscr{T}$ equals mg, and its horizontal component equals $(-mg\varphi)$. In the same approximation, the linear displacements of the pendula from the equilibrium (vertical) positions are, respectively, $l\varphi$ and $l\varphi'$, and the spring extension ΔL is $l(\varphi'-\varphi)$, so that the force acting on each pendulum equals $\pm\kappa l(\varphi'-\varphi)$, where κ is the spring constant. As a result, in the linear approximation, the horizontal components of Newton's second law for the two pendula are:

$$m(l\ddot{\varphi}) = \kappa l(\varphi' - \varphi) - mg\varphi,$$

$$m(l\ddot{\varphi}') = -\kappa l(\varphi' - \varphi) - mg\varphi'.$$

The solution of this system of equations will be the subject of problem 6.1.

***Problem* 1.9.** One popular futuristic concept of travel is digging a straight railway tunnel through the Earth and letting a train go through it, without initial velocity—driven only by gravity. Calculate the train's travel time through such a tunnel, assuming that the Earth's density ρ is constant, and neglecting the effects of friction and planetary rotation.

Solution: Let us apply the gravitational analog of the Gauss law, given by Eq. (**) in the solution of problem 1.7,

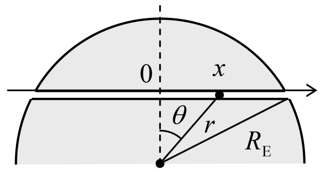
$$\oint_S F_n d^2r = -4\pi Gm \int_V \rho(\mathbf{r}')d^3r',$$

to a sphere of radius $r \leqslant R_E$, taking into account that due to the system's symmetry, $\mathbf{F} = \mathbf{n}_r F(r)$ and $F_n = F$. The result shows that the net gravity force felt by the train at distance r from the Earth's center is determined only by the planet's mass inside a sphere of this radius,

$$\mathbf{F} = -G\frac{M(r)m}{r^3}\mathbf{r}, \quad \text{with} \quad M(r) = \rho\frac{4\pi}{3}r^3,$$

where m is the train's mass. With the notation used in the figure below, the force's component directed along the tunnel is

$$F_x = -F\sin\theta = G\frac{M(r)m}{r^2}\sin\theta = -\frac{4\pi}{3}Gm\rho\, r\sin\theta.$$



But the product $r\sin\theta$ is nothing more than the linear displacement x of the train from the middle of the tunnel, so that F_x depends on x linearly, similarly to the force of the usual elastic spring with the equilibrium point at $x = 0$:

$$F_x = -\kappa x, \qquad \text{with} \quad \kappa = \frac{4\pi}{3}Gm\rho.$$

The spring constant κ looks simpler if expressed via the gravity acceleration g on the Earth's surface and its radius R_E. Indeed, by the definition of g,

$$g = G\frac{M(R_E)}{R_E^2} = G\frac{\rho}{R_E^2}\frac{4\pi}{3}R_E^2 = \frac{4\pi}{3}G\rho R_E, \qquad \text{so that} \quad \kappa = m\frac{g}{R_E}.$$

As a result of this analogy, the equation of train's motion along the tunnel, $m\ddot{x} = -\kappa x$, is similar to that of the mass on a spring; it describes periodic, sinusoidal oscillations of x in time, with period

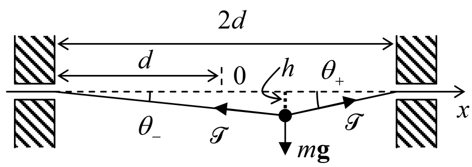
$$\mathcal{T} = \frac{2\pi}{\omega_0}, \qquad \text{where} \quad \omega_0 = \left(\frac{\kappa}{m}\right)^{1/2} = \left(\frac{g}{R_E}\right)^{1/2}.$$

Evidently the time Δt of a one-way journey of the train through the tunnel, with no initial velocity, is just a half of this period:

$$\Delta t = \frac{\mathcal{T}}{2} = \frac{\pi}{\omega_0} = \pi\left(\frac{R_E}{g}\right)^{1/2}.$$

Perhaps the most curious feature of this result is that it is independent of the tunnel's length. The reason is that, the longer the tunnel, the steeper is its average incline toward the Earth's center, and hence the larger is the train's acceleration. So, if our Earth were uniform, the travel time from any point of its surface to any other point would be the same (about 42 min and 13 s). In reality, ρ grows toward the Earth's center, so that the above result is accurate only for relatively short tunnels, with length $l \ll R$, while for longer tunnels the travel would be even faster.

***Problem* 1.10.** A small bead of mass m may slide, without friction, along a light string, stretched with a force $\mathscr{T} \gg mg$ between two points separated by a horizontal distance $2d$—see the figure below. Calculate the frequency of horizontal oscillations of the bead about its equilibrium position.



Solution: Due to the given condition $\mathscr{T} \gg mg$, the string remains nearly horizontal even under the weight of the bead, so that both angles $\theta_\pm$ (see the figure above) are small. As a result, the horizontal motion of the bead is much slower than its vertical oscillations, and the vertical displacement h may be calculated ignoring its dynamics. Then from the requirement that the sum of two vertical components, $\mathscr{T}$ sin $\theta_\pm \approx \mathscr{T}\theta_\pm$, of the string tension $\mathscr{T}$ counterbalances its weight $m\mathbf{g}$:

$$\mathscr{T}(\theta_- + \theta_+) = mg,$$

plus the geometric relations evident from the figure above:

$$\theta_- = \frac{h}{d+x}, \qquad \theta_+ = \frac{h}{d-x}, \tag{*}$$

where x is the horizontal displacement of the bead from its equilibrium position at the center of the string—see the figure above. Solving this simple system of three equations for h and $\theta_\pm$, we obtain, in particular,

$$h = \frac{mg}{2\mathscr{T}d}(d^2 - x^2),$$

so that Eqs. (*) become

$$\theta_- = \frac{mg}{2\mathscr{T}d}(d - x), \qquad \theta_+ = \frac{mg}{2\mathscr{T}d}(d + x).$$

Now we may use these results to calculate the net horizontal component of the tension forces exerted on the bead:

$$F_x = \mathscr{T}\cos\theta_+ - \mathscr{T}\cos\theta_- \approx \mathscr{T}\left(1 - \frac{\theta_+^2}{2}\right) - \mathscr{T}\left(1 - \frac{\theta_-^2}{2}\right)$$
$$= \frac{\mathscr{T}}{2}\left(\frac{mg}{2\mathscr{T}d}\right)^2[(d-x)^2 - (d+x)^2] = -\frac{m^2g^2}{2\mathscr{T}d}x.$$

This force may be represented as $F_x = -\kappa x$, with

$$\kappa = \frac{m^2g^2}{2\mathscr{T}d} > 0,$$

i.e. is always directed toward the equilibrium point $x = 0$, and is similar to the one provided by the usual elastic spring. Hence the frequency of the bead's oscillations may be found from the well-known formula for the frequency of a mass on a spring:

$$\omega = \left(\frac{\kappa}{m}\right)^{1/2} = g\left(\frac{m}{2\mathscr{T}d}\right)^{1/2}. \qquad (**)$$

This result shows, in particular, that $\omega \to 0$ at $\mathscr{T} \to \infty$. This is natural because in this limit the string becomes virtually horizontal, and the returning horizontal force, which results from the string's slopes, vanishes. Note also that:

- The calculated frequency (**) of the horizontal oscillations of the bead is much smaller than that, $\Omega \sim (2\mathscr{T}/md)^{1/2}$, of its vertical oscillations[11]. This relation confirms the validity of our approach.
- Our result, while being conditioned by the strong inequality $\mathscr{T} \gg mg$, is valid for an arbitrary oscillation amplitude $A \equiv x_{\max}$, while it is less than d.

***Problem* 1.11.** For a rocket accelerating due to a working jet motor (and hence spending its fuel), calculate the relation between its velocity and the remaining mass. *Hint*: For the sake of simplicity, consider 1D motion.

Solution: Let us write the law of conservation of the net momentum P of the rocket and a small portion dm of its exhaust gases, ejected with the relative velocity u during a small time interval dt, in the so-called *instantaneous rest frame*—an inertial reference frame moving, in the particular instant under consideration, with the same velocity v as the body under consideration—in our case, the accelerating rocket:

$$dP \equiv m\,dv + dm\ \ u = 0. \qquad (*)$$

[11] For small, purely vertical oscillations, the formula $\Omega = (2\mathscr{T}/md)^{1/2}$ is exact (prove this!). The coexistence of various oscillations in this system, at arbitrary ratio $\mathscr{T}/mg$, will be discussed in problem 3.1.

Dividing all terms of this equation by dt, and moving the term proportional to u into the right-hand side, we obtain the following equation:

$$m\frac{dv}{dt} = -u\frac{dm}{dt}.$$

The equation shows that the magnitude of the effective force (in engineering, called *thrust*) of the rocket engine is

$$F_{\text{ef}} = \mu u,$$

where $\mu \equiv (-dm/dt) > 0$ is the fuel mass burn rate. Assuming that the rate, as well as the exhaust velocity u are constant in time (meaning that $m(t) = m(0) - \mu t$), the resulting equation of motion,

$$[m(0) - \mu t]\frac{dv}{dt} = \mu u,$$

may be readily integrated to find the velocity and coordinate of the rocket as functions of time (a useful exercise, highly recommended to the reader).

However, since we are only interested in the relation between the remaining rocket mass and the achieved velocity, we may directly integrate Eq. (*),

$$\int \frac{dm}{m} = -\frac{1}{u}\int dv,$$

obtaining

$$\ln m = -\frac{v}{u} + \text{const}.$$

Now using the initial conditions to find the integration constant, we obtain the famous formula[12]

$$v(t) = v(0) + u \ln \frac{m(0)}{m(t)}.$$

It shows that, a bit counter-intuitively, a rocket may reach velocities much higher than the relative velocity u of the exhaust gases. However, for this the initial mass of the fuel, contributing to $m(0)$, has to be much larger than that of the ship itself, including the useful payload. This result is the basis for all rocket engineering, notably including multi-stage designs.

***Problem* 1.12.** Prove the following *virial theorem*[13]. For a set of N particles performing a periodic motion,

$$\bar{T} = -\frac{1}{2}\sum_{k=1}^{N}\overline{\mathbf{F}_k \cdot \mathbf{r}_k},$$

[12] It was derived, in an implicit form, by W Moore in 1813, and then re-discovered (and used to discuss the rocket motion and space travel) by K Tsiolkovsky in 1903.

[13] It was first stated by R Clausius in 1870. The term *virial* was derived by him from *vis*, the Latin for 'force'.

where the top bar means time averaging, in this case over the motion period. What does the virial theorem say about:
(i) the 1D motion of a particle in a confining potential $U(x) = ax^{2s}$, with $a > 0$ and $s > 0$, and
(ii) the orbital motion of a particle moving in a central potential $U(r) = -C/r$?

Hint: Explore the time derivative of the following scalar function of time: $G(t) \equiv \sum_{k=1}^{N} \mathbf{p}_k \cdot \mathbf{r}_k$.

Solution: Differentiating the function $G(t)$ by parts,

$$\frac{dG}{dt} \equiv \sum_{k=1}^{N} \dot{\mathbf{p}}_k \cdot \mathbf{r}_k + \sum_{k=1}^{N} \mathbf{p}_k \cdot \dot{\mathbf{r}}_k,$$

and using Eqs. (1.3), (1.9), and (1.13) of the lecture notes, we obtain

$$\frac{dG}{dt} = \sum_{k=1}^{N} \mathbf{F}_k \cdot \mathbf{r}_k + \sum_{k=1}^{N} m_k \dot{\mathbf{r}}_k \cdot \dot{\mathbf{r}}_k.$$

The term under the last sum is just twice the kinetic energy (1.19) of the kth particle, so that the sum of these terms is twice the total kinetic energy T of the system, and hence

$$\frac{dG}{dt} = \sum_{k=1}^{N} \mathbf{F}_k \cdot \mathbf{r}_k + 2T. \qquad (*)$$

If system's motion is periodic with some time period $\mathcal{T}$, so is the function G: $G(t + \mathcal{T}) = G(t)$, and the time average of its derivative over the period equals zero[14]:

$$\overline{\frac{dG}{dt}} \equiv \frac{1}{\mathcal{T}} \int_{t}^{t+\mathcal{T}} \frac{dG(t')}{dt'} dt' = \frac{1}{\mathcal{T}} \int_{t'=t}^{t'=t+\mathcal{T}} dG(t') = \frac{1}{\mathcal{T}} [G(t + \mathcal{T}) - G(t)] = 0,$$

so that the averaging of Eq. (*) yields

$$0 = \sum_{k=1}^{N} \overline{\mathbf{F}_k \cdot \mathbf{r}_k} + 2\bar{T},$$

thus proving the virial theorem.

(i) For the 1D motion of a particle in a time-independent potential $U(x)$, the radius-vector **r**, the velocity **v**, and the force **F** have single Cartesian components, with $F_x = -dU/dx$, so that the virial theorem is reduced to

[14] Actually, this statement (and hence the virial theorem) is asymptotically (i.e. in the limit $\mathcal{T} \to \infty$) valid even if the system is not periodic, but is *stably bound*, meaning that the particles stay together in a limited region of space, and their velocities remain finite.

$$\bar{T} = \frac{1}{2}\overline{x\frac{dU}{dx}}, \qquad \text{with} \quad T \equiv \frac{m}{2}v^2 \equiv \frac{m}{2}\dot{x}^2.$$

For the particular case $U(x) = ax^{2s}$,

$$x\frac{dU}{dx} = 2sax^{2s} \equiv 2sU,$$

so that the theorem yields

$$\bar{T} = s\bar{U},$$

for any a and s. (Conditions $a > 0$ and $s > 0$ are necessary to ensure that the particle's motion is periodic.)

Note that for the most important case of the quadratic confining potential ($s = 1$), this result is reduced to the equality of the average values of the kinetic and potential energies—a fact well-known from the analysis of the sinusoidal motion of such a *harmonic oscillator*.

(ii) For a particle moving in a central potential $U(r) = -C/r$, the force is directed toward the center:

$$\mathbf{F}(\mathbf{r}) = -\nabla U = -\frac{C}{r^3}\mathbf{r},$$

so that the (only) term, $\mathbf{F}\cdot\mathbf{r}$, on the right-hand side of the virial theorem may be expressed as

$$\mathbf{F} \cdot \mathbf{r} = -\frac{C}{r^3}\mathbf{r} \cdot \mathbf{r} = -\frac{C}{r} = U,$$

and the theorem is reduced to a very simple (and powerful) equality

$$\bar{T} = -\frac{1}{2}\bar{U}.$$

This equality is valid, in particular, for the elliptical orbits of the planetary motion, which will be discussed in chapter 3.