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EPRL/FK group field theory

J. Ben Geloun[1(a)], R. Gurau[1] and V. Rivasseau[2]

[1] *The Perimeter Institute for Theoretical Physics - Waterloo, ON, N2L 2Y5, Canada*
[2] *Laboratoire de Physique Théorique, CNRS UMR 8627, Université Paris XI - F-91405 Orsay Cedex, France, EU*

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Abstract – The purpose of this note is to clarify the group field theory vertex and propagators corresponding to the EPRL/FK spin foam models and to detail the subtraction of leading divergences of the model.

Introduction. – Group field theories (GFTs) (see [1–3]) are the higher-dimensional generalization of random matrix models. Like in matrix models, the Feynman graphs of group field theory are dual to triangulations (gluing of simplices). The combinatorics of a Feynman graph encodes the topology of the gluing while its amplitude encodes a sum over metrics compatible with a fixed gluing. The correlation functions of GFTs sum over both metrics and topologies giving rise to a new fundamental approach for quantum gravity.

In [4], a locality requirement[1] was proposed for GFTs, namely to restrict their vertices to simple products of δ-functions which identify group elements in strands crossing the vertex. In dimension D the simplest and most natural vertex with this locality property represents $D+1$ subsimplices of dimension $D-1$ bounding a D-dimensional simplex (hence connected through $D(D+1)/2$ such δ-functions). The fields are functions on $SO(D)^D$, and the D-stranded propagators represent the gluing of D-dimensional simplices along $(D-1)$-dimensional subsimplices. Using as propagator a diagonal $SO(D)$ gauge averaging projection T (ensuring flatness of the holonomies), the amplitude of a Feynman graph equals the partition function of a BF theory discretized on the dual gluing of simplices. Recently such models have received increased attention and various partial power counting results have been established, either for generic three-dimensional models [5,6] or for colored and linearized models [7–9].

Besides, gravity can be seen as a constrained version of BF theory. In line with this approach, new spin foam rules have been proposed to implement the so-called Plebanski simplicity constraints and reproduce the partition function of fully fledged 4D gravity [10–12]. These new models (referred to as EPRL/FK in this paper) mix the left and right part of $SO(4) \simeq SU(2) \times SU(2)$ in a novel way and give a central rôle to the Immirzi parameter. Amplitudes of particular spin foams in the EPRL/FK models, revealing improved UV behavior, have been derived in [13] and recovered in [4].

But, as spin foams are only Feynman graphs of the GFT, one still needs to identify an appropriate GFT propagator which generates the EPRL/FK spin foam amplitudes. A first GFT formulation of the EPRL/FK propagator was given in the coherent-state representation basis in [11]. A second step has been performed in [4], were the propagator (written still in terms of coherent states) was computed as a product of gauge (T) and simplicity (S) projection operators, $C = TST$ (see footnote [2]). Note that C has a non trivial spectrum, hence is suited for a Renormalization Group (RG) analysis. Here we propose another equivalent formulation, free of explicit sums over coherent states, and which might lead to a transparent saddle point analysis for estimating graph amplitudes.

In this paper we obtain the EPRL/FK propagator in group space and consequently more compact formulas for both the propagator and the Feynman amplitudes of the GFT underlying EPRL/FK spin foams, trading the coherent-states integrals for integrals over group

(a) International Chair in Mathematical Physics Applications, ICMPA-UNESCO Chair - 072BP50 Cotonou, Republic of Benin; E-mail: jbengeloun@perimeterinstitute.ca

[1] In any quantum field theory, a vertex can be dressed by an arbitrary fraction of the propagator without changing the bulk theory, provided we amputate each propagator by the square of that fraction. The locality principle on vertices fixes this ambiguity.

[2] Beware that different letters are used in [4].

elements (while the discrete sums over spin indices remain unchanged). Our formulas are well defined for irrational values of the Immirzi parameter and constitute a better starting point for slicing the propagator according to its spectrum (and subsequently a fully fledged RG analysis). Furthermore, using this direct space formulation, we show that the leading divergence of the mass kind can be extracted and reproduces previous results [4,13].

Note that with the definition of locality we use, the GFT vertex corresponds to an abstract simplex, and not to a geometric simplex. If one were to incorporate a gauge averaging operator in the vertex, one would enforce the flatness of the wedges (the portion of the dual faces inside each simplex) and thus of the geometric D simplices. Although we lose the interpretation of the vertex as a geometric simplex, the definition of locality we use has many advantages. First it is independent of the particular classical theory one models (BF theory, EPRL/FK models, etc.). Second, it separates the topological and metric information at the level of the GFT action: the vertex encodes only topological data, and all the geometrical data is encoded in the propagator kernel (*i.e.* in the gluing of simplices). Third, it allows arbitrary quantum fluctuations of the GFT field, even if they do not correspond to geometric simplices (in the BF and EPRL/FK models such fluctuations do not propagate). Fourth, as we will see in the sequel, this definition of locality leads to a well-defined and simple prescription to identify divergences.

This paper is organized as follows: the second section details the simplicity projector S in direct space in terms of characters and the third section presents the EPRL/FK propagator and computes the Feynman amplitudes of arbitrary graphs. Finally, the fourth section explains the subtraction of leading divergences.

The simplicity projector S. – In [11] (and subsequently in [4]), the kernel of the EPRL/FK simplicity projector S is taken to be

$$S=\sum_{j^+,j^-}\delta_j^\gamma d_{j^++j^-}\int \mathrm{d}n\,|j^+,n\rangle\otimes|j^-,n\rangle\langle j^+,n|\otimes\langle j^-,n|\,, \tag{1}$$

where $d_J:=2J+1$, $\delta_j^\gamma:=\delta_{|1-\gamma|j^+=(1+\gamma)j^-}$, γ is the Immirzi parameter and, given the spin j representation space of $SU(2)$, $H_j=\{|j,m\rangle, |m|\leqslant j\}$, $SU(2)$ coherent states [14] are indexed by a normal vector $\vec{n}$ of the sphere S^2, and write

$$|j,n\rangle\equiv\sum_p D^j_{pj}(\alpha,\beta,0)|j,p\rangle\,, \tag{2}$$

with the Wigner matrix element $D^j_{pq}(g)=\langle j,p|g^j|j,q\rangle=e^{-i\alpha p}d^j_{pq}(\beta)e^{-i\psi q}$ representing a $SU(2)$ group element g in terms of its Euler angles (α,β,ψ) in the $z-y-z$ order.

Although, as S is a projector, $S^2=S$, the square of eq. (1) is

$$S\,=\,S^2=\sum_{j^+,j^-}\delta_j^\gamma\,d^2_{j^++j^-}\int \mathrm{d}n\mathrm{d}n'|j^+,n\rangle\otimes|j^-,n\rangle \times\langle j^++j^-,n|j^++j^-,n'\rangle\langle j^+,n'|\otimes\langle j^-,n'|\,, \tag{3}$$

which looks quite different. This discrepancy is explained by the over completeness of the coherent-states basis[3]. It the sequel, we choose the representation provided in eq. (3) as it is better suited for explicit computations.

Remark that the δ_j^γ does not really make sense (*e.g.*, if γ is irrational) but should be understood in an asymptotic sense as $j_\pm\to\infty$. This will be detailed later, and the formulas we will derive for the amplitudes of the theory ultimately make sense for any γ.

It is important to realize that eq. (3) is in fact only a shortened (and somewhat confusing) notation. The operator S acts on functions defined on $SO(4)$ which decompose in Fourier modes as $f(g)=\sum d_j f^j_{pm}D^j_{pm}(g)$. Matrix elements of the operator S therefore join a $D^{j_1}_{p_1m_1}(g_1)$ to a $D^{j_2}_{p_2m_2}(g_2)$, hence have two groups of indices j_1, p_1, m_1 and j_2, p_2, m_2. To make matters worse, over $SO(4)\simeq SU(2)\times SU(2)$ each of the above six indices is in fact a double index, $\vec{j}_i=(j_i^+,j_i^-)$, corresponding, respectively, to each of the two copies of $SU(2)$. In full detail S writes

$$\begin{aligned}S^{\vec{j}_1,\vec{j}_2}_{(\vec{p}_1,\vec{m}_1);(\vec{p}_2,\vec{m}_2)}&=\delta_{\vec{j}_1,\vec{j}_2}\delta^\gamma_{j_1}\,d^2_{j_1^++j_1^-}\delta_{\vec{p}_1,\vec{p}_2}\\&\times\int \mathrm{d}n\mathrm{d}n'\,\langle\vec{j}_1,\vec{m}_1|(|j_1^+,n\rangle\otimes|j_1^-,n\rangle)\\&\times\langle j_1^++j_1^-,n|j_1^++j_1^-,n'\rangle\\&\times(\langle j_1^+,n'|\otimes\langle j_1^-,n'|)|\vec{j}_2,\vec{m}_2\rangle,\end{aligned} \tag{4}$$

where $|\vec{j},\vec{m}\rangle=|j^+,m^+\rangle\otimes|j^-,m^-\rangle$. Denoting the matrix elements of unitary representations of $SU(2)\times SU(2)$ as $D^{\vec{j}}_{\vec{p}\,\vec{m}}(g):=D^{j^+}_{p^+m^+}(g^+)D^{j^-}_{p^-m^-}(g^-)$, we find in the direct (group) space

$$\begin{aligned}S(g_1,g_2)\,=\,&\sum d_{j_1^+}d_{j_1^-}S^{\vec{j}_1\vec{j}_2}_{(\vec{p}_1\vec{m}_1);(\vec{p}_2\vec{m}_2)}\\&\times D^{\vec{j}_1}_{\vec{p}_1\vec{m}_1}(g_1)\overline{D^{\vec{j}_2}_{\vec{p}_2\vec{m}_2}(g_2)}\,.\end{aligned} \tag{5}$$

Substituting (4) in eq. (5), and summing over $\vec{p}_2$ and $\vec{j}_2$ (and renaming $\vec{j}_1=\vec{j}$), we get

$$\begin{aligned}S(g_1,g_2)\,=\,&\sum d_{j^+}d_{j^-}\,\delta_j^\gamma\,d^2_{j^++j^-}\\&\times D^{\vec{j}}_{\vec{m}_2\vec{m}_1}\big((g_2)^{-1}g_1\big)\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2)\,,\end{aligned} \tag{6}$$

[3]In order to conclude that S is a projector, in [4] one proves that $S^3=S^2$, rather than proving $S^2=S$.

with

$$\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2)=\int \mathrm{d}n\mathrm{d}k\,\langle\vec{j},\vec{m}_1|\Big(|j^+,n\rangle\otimes|j^-,n\rangle\Big)$$
$$\times\langle j^++j^-,n|j^++j^-,k\rangle\Big(\langle j^+,k|\otimes\langle j^-,k|\Big)|\vec{j},\vec{m}_2\rangle. \tag{7}$$

The integral $\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2)$ is evaluated as follows. Substituting the definition of coherent states eq. (2) in eq. (7), then inserting judiciously phases in the new fictitious variables such that $D^j_{pj}(\phi,\psi,0)e^{-i\chi j}=D^j_{pj}(\phi,\psi,\chi)$, one is able to translate integrals over the sphere $\int \mathrm{d}n$ to $SU(2)$ (Haar) group integrals $\int \mathrm{d}g$. Then $\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2)$ (7) can be rewritten as

$$\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2)=$$
$$\times\sum_r\int \mathrm{d}g\mathrm{d}g'\,D^{j^+}_{m_1^+j^+}(g)D^{j^-}_{m_1^-j^-}(g)\overline{D^{j^++j^-}_{r(j^++j^-)}(g)}$$
$$\times D^{j^++j^-}_{r(j^++j^-)}(g')\overline{D^{j^+}_{m_2^+j^+}(g')}\,\overline{D^{j^-}_{m_2^-j^-}(g')}\,. \tag{8}$$

Using the Hermitian conjugation of the Wigner matrices, $D^j_{mn}(g^{-1})=\overline{D^j_{nm}(g)}=(-1)^{n-m}D^j_{-n-m}(g)$, the following holds:

$$\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2)=\sum_r(-1)^{r-j^+-j^-+m_2^+-j^++m_2^--j^-}$$
$$\times\int \mathrm{d}g D^{j^+}_{m_1^+j^+}(g)D^{j^-}_{m_1^-j^-}(g)D^{j^++j^-}_{-r-(j^++j^-)}(g)$$
$$\times\int \mathrm{d}g' D^{j^++j^-}_{r(j^++j^-)}(g')D^{j^+}_{-m_2^+-j^+}(g')D^{j^-}_{-m_2^--j^-}(g')\,. \tag{9}$$

The group integrals of products of three Wigner matrices compute in terms of Wigner $3j$ symbols [15], thus (using symmetry properties of these symbols)

$$\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2)=\begin{pmatrix} j^+ & j^- & j^++j^-\\ j^+ & j^- & -(j^++j^-)\end{pmatrix}^2$$
$$\times\sum_r(-1)^{r+m_2^++m_2^-}\begin{pmatrix} j^+ & j^- & j^++j^-\\ m_1^+ & m_1^- & -r\end{pmatrix}$$
$$\times\begin{pmatrix} j^+ & j^- & j^++j^-\\ -m_2^+ & -m_2^- & r\end{pmatrix}. \tag{10}$$

Taking into account the evaluation of particular $3j$ symbols according to [15],

$$\begin{pmatrix} j^+ & j^- & j^++j^-\\ j^+ & j^- & -(j^++j^-)\end{pmatrix}=\frac{(-1)^{2j^+}}{\sqrt{2(j^++j^-)+1}}, \tag{11}$$

$$\begin{pmatrix} j^+ & j^- & j^++j^-\\ -m_2^+ & -m_2^- & r\end{pmatrix}=(-1)^{-2r}\begin{pmatrix} j^+ & j^- & j^++j^-\\ m_2^+ & m_2^- & -r\end{pmatrix}, \tag{12}$$

and by the selection rules, the $3j$ symbol is zero unless $m_2^++m_2^--r=0$, hence we get

$$\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2)=\frac{1}{d_{j^++j^-}}\sum_r\begin{pmatrix} j^+ & j^- & j^++j^-\\ m_1^+ & m_1^- & -r\end{pmatrix}$$
$$\times\begin{pmatrix} j^+ & j^- & j^++j^-\\ m_2^+ & m_2^- & -r\end{pmatrix}, \tag{13}$$

which can be finally rewritten as an integral over the group

$$\mathcal{I}(\vec{j},\vec{m}_1,\vec{m}_2)=\frac{1}{d_{j^++j^-}}\sum_r\int \mathrm{d}h\,D^{j^+}_{m_1^+,m_2^+}(h)$$
$$\times D^{j^-}_{m_1^-,m_2^-}(h)D^{j^++j^-}_{-r,-r}(h)\,. \tag{14}$$

Substituting (14) in eq. (6) yields

$$S(g_1,g_2)=\sum_{j^+,j^-}d_{j^+}d_{j^-}d_{j^++j^-}\,\delta^\gamma_j$$
$$\times\sum_{\vec{m}_1,\vec{m}_2}D^{j^+}_{m_2^+,m_1^+}\big((g_2^+)^{-1}g_1^+\big)\,D^{j^-}_{m_2^-,m_1^-}\big((g_2^-)^{-1}g_1^-\big)$$
$$\times\sum_r\int \mathrm{d}h\,D^{j^+}_{m_1^+m_2^+}(h)\,D^{j^-}_{m_1^-m_2^-}(h)\,D^{j^++j^-}_{-r-r}(h), \tag{15}$$

where the integral in the last line is performed over only one group element $h\in SU(2)$. Summing over $\vec{m}_1,\vec{m}_2$ and r in eq. (15), we infer the compact expression

$$S(g_1,g_2)=\sum d_{j^+}d_{j^-}\,d_J\,\delta^\gamma_j\delta_{J=j^++j^-}$$
$$\times\int \mathrm{d}h\,\chi^{j^+}\big(g_1^+h(g_2^+)^{-1}\big)\,\chi^{j^-}\big(g_1^-h(g_2^-)^{-1}\big)\,\chi^J(h), \tag{16}$$

with $\chi^j(g)=\mathrm{Tr}_j(g)=\sum_k D^j_{kk}(g)$ the character of g in the representation j. Using $\overline{\chi(h)}=\chi(h^\dagger)$ and the orthogonality of characters, one can check directly using eq. (16) that S is a projector. Note now that eq. (16) makes sense for any value of γ by virtue of the property $\chi^j(g)=\sin[(j+1/2)\theta]/[\sin(\theta/2)]$ is well defined for all values of j, half integer or not.

The simplicity projector S admits several limiting cases: 1) $\gamma=1$ sets $j^-=0$, leading to a BF theory for the $+$ copy of $SU(2)$. 2) Ignoring both $\delta^\gamma_j\,\delta_{J=j^++j^-}$ we recover the $SO(4)$ BF theory. 3) Finally, $\gamma\to 0$ leads to $j^+=j^-$, which is the EPR spin foam model [10].

The EPRL/FK propagator and Feynman amplitudes. – To build the EPRL/FK propagator one needs to compose four simplicity projectors, one for each strand of the 4D GFT line, with two gauge invariance projectors, common to all four strands. The ordinary $SO(4)$ gauge invariance propagator, T, corresponding to "diagonal right" invariant fields *i.e.* fields satisfying $\phi(g_1h,g_2h,g_3h,g_4h)=\phi(g_1,g_2,g_3,g_4)$, $\quad h\in SO(4)$, has

kernel

$$T(\{g_s\},\{g'_s\}) = \int \mathrm{d}h^+\mathrm{d}h^- \prod_{s=1}^{4} \delta\big(g_s^+ h^+ (g_s'^+)^{-1}\big)\delta\big(g_s^- h^- (g_s'^-)^{-1}\big), \tag{17}$$

where $\{g_s\}$ denotes a collection of four group elements associated to the strands. The pair of integration variables (h^+, h^-) is common to all four strands of a line. The propagator writes

$$C(\{g_s\};\{g'_s\}) = \int \prod_s \big(\mathrm{d}u_s \mathrm{d}v_s\big)\, T(\{g_s\},\{u_s\}) \times \Big[\prod_s S(u_s,v_s)\Big]\, T(\{v_s\},\{g'_s\}), \tag{18}$$

or in detail, denoting $\delta_J = \delta_{J=j^+ + j^-}$ and $h^\pm_{\rm in}, h^\pm_{\rm out}$ the dummy variables corresponding to the two T operators

$$\begin{aligned} C(\{g_s\};\{g'_s\}) = &\sum_{j_s^+, j_s^-, J_s} d_{j_s^+} d_{j_s^-} d_{J_s}\, \delta^\gamma_{j_s}\, \delta_{J_s} \\ &\times \int \mathrm{d}h^\pm_{\rm in}\mathrm{d}h^\pm_{\rm out} \int \prod_s \mathrm{d}h_s \\ &\times \int \prod_s \big(\mathrm{d}u_s^\pm \mathrm{d}v_s^\pm\big) \prod_s \delta\big(g_s^+\, h^+_{\rm in}\, (u_s^+)^{-1}\big)\delta\big(g_s^-\, h^-_{\rm in}\, (u_s^-)^{-1}\big) \\ &\times \prod_s \chi^{j_s^+}(u_s^+\, h_s\, (v_s^+)^{-1})\, \chi^{j_s^-}(u_s^-\, h_s\, (v_s^-)^{-1})\, \chi^{J_s}(h_s) \\ &\times \prod_s \delta\big(v_s^+\, h^+_{\rm out}\, (g_s'^+)^{-1}\big)\delta\big(v_s^-\, h^-_{\rm out}\, (g_s'^-)^{-1}\big). \end{aligned} \tag{19}$$

Integrating over $u_s^\pm, v_s^\pm$, we get

$$\begin{aligned} C(\{g_s\};\{g'_s\}) = &\sum_{j_s^+, j_s^-, J_s} d_{j_s^+} d_{j_s^-} d_{J_s}\, \delta^\gamma_{j_s}\, \delta_{J_s} \\ &\times \int \mathrm{d}h^\pm_{\rm in}\mathrm{d}h^\pm_{\rm out} \int \prod_s \mathrm{d}h_s \\ &\times \prod_s \chi^{j_s^+}(g_s^+\, h^+_{\rm in}\, h_s\, h^+_{\rm out}\, (g_s'^+)^{-1})\, \chi^{j_s^-} \\ &\times (g_s^-\, h^-_{\rm in}\, h_s\, h^-_{\rm out}\, (g_s'^-)^{-1})\, \chi^{J_s}(h_s)\,. \end{aligned} \tag{20}$$

A EPRL/FK GFT line is represented together with all its associated group elements in fig. 1.

A Feynman graph of the EPRL/FK group field theory is made of propagators (eq. (20)) and vertices made of trivial conservation δ-functions. The integrand is usually factored into contributions associated either to closed strands (called faces) or to open (external) strands.

To write the full amplitude of a graph $\mathcal{G}$ we introduce some notations. We denote the two couples of "in" and "out" variables of a line l by $h^\pm_{{\rm in};l}$ and $h^\pm_{{\rm out};l}$. We denote ∂f the set of lines of the boundary of the face f and $|\partial f|$

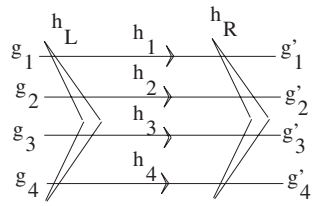

Fig. 1: A EPRL/FK line.

its cardinal. For each line $l \in \partial f$ we have a variable h_{lf} (corresponding to h_s in eq. (20)). Furthermore, we denote ϵ_{lf} the incidence matrix of lines within faces [4,8], which is 0 if $l/\in \partial f$ and 1 (or -1) if $l \in \partial f$ and the orientations of l and f coincide (or not). Finally, denoting $\mathcal{L}_\mathcal{G}$ the set of lines and $\mathcal{F}_\mathcal{G}$ the set of faces of $\mathcal{G}$, the amplitude writes

$$\begin{aligned} A_\mathcal{G}(\{g_s^+\},\{g_s^-\}) = &\sum_{j_f^+, j_f^-, J_{lf}} \left(\prod_{f\in\mathcal{F}_\mathcal{G}} d_{j_f^+} d_{j_f^-}\, \delta^\gamma_{j_f} \left(\prod_{l\in\partial f} d_{J_{lf}} \delta_{J_{lf}=j_f^+ + j_f^-}\right)\right) \\ &\times \int \Big[\prod_{l\in\mathcal{L}_\mathcal{G}} \mathrm{d}h^\pm_{{\rm in};\,l}\mathrm{d}h^\pm_{{\rm out};\,l}\Big] \\ &\times \int \Big[\prod_{\substack{l\in\mathcal{L}_\mathcal{G}, f\in\mathcal{F}_\mathcal{G}\\ l\in\partial f}} \mathrm{d}h_{lf}\Big]\Big[\prod_{\substack{l\in\mathcal{L}_\mathcal{G}, f\in\mathcal{F}_\mathcal{G}\\ l\in\partial f}} \chi^{J_{lf}}(h_{lf})\Big] \\ &\times \prod_{f\in\mathcal{F}_\mathcal{G}} \left[\chi^{j_f^+}\left(\prod_{l\in\partial f} (h^+_{{\rm in};\,l}\, h_{lf}\, h^+_{{\rm out};\,l})^{\epsilon_{lf}}\right)\right. \\ &\left.\times \chi^{j_f^-}\left(\prod_{l\in\partial f} (h^-_{{\rm in};\,l}\, h_{lf}\, h^-_{{\rm out};\,l})^{\epsilon_{lf}}\right)\right], \end{aligned} \tag{21}$$

where for external, open faces, with group elements at the endpoints $g_s^\pm$ and $g_s'^\pm$, the argument in the last line is replaced by

$$\begin{aligned} &\chi^{j_f^+}\left[(g_s^+)^{\epsilon_{ef}}\left(\prod_{l\in\partial f} (h^+_{{\rm in};\,l}\, h_{lf}\, h^+_{{\rm out};\,l})^{\epsilon_{lf}}\right)(g_s'^+)^{\epsilon_{ef}}\right] \\ &\times \chi^{j_f^-}\left[(g_s^-)^{\epsilon_{ef}}\left(\prod_{l\in\partial f} (h^-_{{\rm in};\,l}\, h_{lf}\, h^-_{{\rm out};\,l})^{\epsilon_{lf}}\right)(g_s'^-)^{\epsilon_{ef}}\right], \end{aligned} \tag{22}$$

with ϵ_{ef} the incidence matrix of external points with faces.

Subtraction, locality, and all that. – Starting from the Feynman amplitude of a graph (21), one can address the subtraction of divergences in this theory following the definition of locality proposed in [4]. Take the example of a two point function. The amplitude of a connected graph writes in terms of the amplitude of the amputated graph as

$$A_\mathcal{G}(\phi) = \int \mathrm{d}g_s \mathrm{d}g'_s\, \phi(\{g_s\})\, \phi(\{g'_s\})\, A_\mathcal{G}(\{g_s\},\{g'_s\})\,. \tag{23}$$

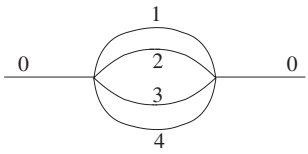

Fig. 2: A graph exhibiting a mass divergence.

The leading ("mass") divergence is immediately identified by Taylor developing "at zeroth order" the field $\phi(\{g'_s\})$[4] around $\{g'_s\}=\{g_s\}$

$$\mu_{\mathcal{G}} = \int \mathrm{d}g'_s\, A_{\mathcal{G}}(\{g_s\},\{g'_s\})\,. \tag{24}$$

Taking into account eq. (22), the integration over the external field $g'^{\pm}_s$ fixes all j^+_f, j^-_f and J_{lf} to 0 and the external strand contribution drops out of eq. (21). In general, the leading divergence of any graph $\mathcal{G}$ is therefore obtained by integrating eq. (21) ignoring the external strands.

Take the example of the graph $\mathcal{G}$ drawn schematically in fig. 2. All lines have parallel strands, and are oriented from left to right. We denote the lines 1 to 4 (which can be interpreted as colors in a colored model [16]), and the face by the couple of labels of the lines composing them. The set of internal faces of this graph is therefore $f=\{f_{12},f_{13},f_{14},f_{23},f_{24},f_{34}\}$.

The mass divergence of $\mathcal{G}$ writes

$$\begin{aligned}
&\delta\mu_{\mathcal{G}} = \\
&\sum_{j^+_{12},j^-_{12},J_{1,12},J_{2,12},\ldots} \left(d_{j^+_{12}} d_{j^-_{12}} \delta^{\gamma}_{j_{12}} \left(d_{J_{1,12}}\delta_{J_{1,12}} d_{J_{2,12}}\delta_{J_{2,12}}\right)\right)\ldots \\
&\times \int \mathrm{d}h^{\pm}_{\mathrm{in},1}\mathrm{d}h^{\pm}_{\mathrm{out},1}\mathrm{d}h^{\pm}_{\mathrm{in},2}\mathrm{d}h^{\pm}_{\mathrm{out},2}\ldots \\
&\times \int \mathrm{d}h_{1,12}\mathrm{d}h_{2,12}\ldots \chi^{J_{1,12}}(h_{1,12})\chi^{J_{2,12}}(h_{2,12}) \\
&\times \ldots \chi^{j^+_{12}}(h^+_{\mathrm{in},1}h_{1,12}h^+_{\mathrm{out},1}(h^+_{\mathrm{in},2}h_{2,12}h^+_{\mathrm{out},2})^{-1}) \\
&\times \chi^{j^-_{12}}(h^-_{\mathrm{in},1}h_{1,12}h^-_{\mathrm{out},1}(h^-_{\mathrm{in},2}h_{2,12}h^-_{\mathrm{out},2})^{-1})\ldots.
\end{aligned} \tag{25}$$

In eq. (25) we have 6 independent sums, 16 integrals over line $h^{\pm}_{\mathrm{in,out}}$ variables, 12 integrations over $h_{i,ij}$ strand variables of a product of 24 characters.

Divergences arise for large values of the spin labels $j^{\pm}, J$, thus we cutoff all the sums by some sharp cutoff Λ. Each $d^{\pm}_j, d_J$ factor will bring a factor Λ. Using the parametrization of a $SU(2)$ group element as $g=e^{i\frac{\theta}{2}\vec{k}\cdot\vec{\sigma}}$, where $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ are Pauli matrices, the $SU(2)$ Haar measure admits the representation $dg=(1/2\pi)\mathrm{d}\theta\sin^2(\theta/2)\mathrm{d}k$, with $\theta\in[0,4\pi]$ and $k\in S^2$. From this point, the integrals over the characters are of the form

$$\int\prod_{j=1}^{n}\mathrm{d}\theta_j\left(\sin\frac{\theta_j}{2}\right)^2\int_{S^2}\mathrm{d}q_j\, F(\Lambda,\theta_j,q_j)\,. \tag{26}$$

[4] As always, sub leading divergences are more difficult to extract (one needs to push further the Taylor development of the external fields), and is deferred for further work.

The integrals over the normals q_j are bounded by 1 and will be ignored. The integrals over θ_j will be evaluated by some saddle point approximation. The saddle point equations are $\theta_j=\theta^s_j$ with

$$\theta^s_p=0 \quad \forall\, p\leqslant k\,, \qquad \theta^s_p\neq 0 \quad \forall p>k\,. \tag{27}$$

The behavior of eq. (26) is strongly dependent of k. In fact, when translating at the saddle point $x_j=\theta_j-\theta^s_j$, and performing the rescaling $x_j=u_j/\sqrt{\Lambda}$ close to the saddle point, eq. (26) writes

$$\begin{aligned}
&\int\prod_{j=1}^{k}\left(\sin\frac{u_j}{2\sqrt{\Lambda}}\right)^2\frac{\mathrm{d}u_j}{\sqrt{\Lambda}}\prod_{j=k+1}^{n}\left(\sin\frac{\frac{u_j}{\sqrt{\Lambda}}+\theta^s_j}{2}\right)^2 \\
&\times\frac{\mathrm{d}u_j}{\sqrt{\Lambda}}F\left(\Lambda,\frac{u}{\sqrt{\Lambda}}+\theta^s\right)\approx\frac{1}{(\sqrt{\Lambda})^{3k}}\frac{1}{(\sqrt{\Lambda})^{n-k}}\int\prod_{j=1}^{k}\frac{u_j^2}{4}\mathrm{d}u_j \\
&\times\prod_{j=k+1}^{n}\left(\sin\frac{\theta^s_j}{2}\right)^2\mathrm{d}u_j\, F\left(\Lambda,\frac{u}{\sqrt{\Lambda}}+\theta^s\right),
\end{aligned} \tag{28}$$

and the remaining integral gives no extra scaling in Λ. Therefore the scaling of eq. (26) is fixed by n (the number of integration variables) and k (the number of directions with saddle point equation $\theta_j=0$).

For the graph of fig. 2, we change variables to

$$\begin{aligned}
(\tilde{h}^+_{\mathrm{in};2})^{-1} &= h^+_{\mathrm{in},1}h_{1,12}h^+_{\mathrm{out},1}(h^+_{\mathrm{out},2})^{-1}h^{-1}_{2,12}(h^+_{\mathrm{in},2})^{-1}, \\
(\tilde{h}^+_{\mathrm{in};3})^{-1} &= h^+_{\mathrm{in},1}h_{1,13}h^+_{\mathrm{out},1}(h^+_{\mathrm{out},3})^{-1}h^{-1}_{3,13}(h^+_{\mathrm{in},3})^{-1}, \\
(\tilde{h}^+_{\mathrm{in};4})^{-1} &= h^+_{\mathrm{in},1}h_{1,14}h^+_{\mathrm{out},1}(h^+_{\mathrm{out},4})^{-1}h^{-1}_{4,14}(h^+_{\mathrm{in},4})^{-1},
\end{aligned} \tag{29}$$

and similarly for the "minus" variables. This brings the contribution of the faces f_{12},f_{13},f_{14} into the form

$$\begin{aligned}
&\chi^{j^+_{12}}\left((\tilde{h}^+_{\mathrm{in};2})^{-1}\right)\chi^{j^+_{13}}\left((\tilde{h}^+_{\mathrm{in};3})^{-1}\right)\chi^{j^+_{14}}\left((\tilde{h}^+_{\mathrm{in};4})^{-1}\right) \\
&\times\chi^{j^-_{12}}\left((\tilde{h}^-_{\mathrm{in};2})^{-1}\right)\chi^{j^-_{13}}\left((\tilde{h}^-_{\mathrm{in};3})^{-1}\right)\chi^{j^-_{14}}\left((\tilde{h}^-_{\mathrm{in};4})^{-1}\right),
\end{aligned} \tag{30}$$

while the (+ part) contribution of the face f_{23} becomes

$$\begin{aligned}
&\tilde{h}^+_{\mathrm{in};2}\, h^+_{\mathrm{in},1}h_{1,12}h^+_{\mathrm{out},1}(h^+_{\mathrm{out},2})^{-1}h^{-1}_{2,12} \\
&\cdot h_{2,23}h^+_{\mathrm{out},2}(h^+_{\mathrm{out},3})^{-1}h^{-1}_{3,23} \\
&\cdot h_{3,13}h^+_{\mathrm{out},3}(h^+_{\mathrm{out},1})^{-1}h^{-1}_{1,13}(h^+_{\mathrm{in},1})^{-1}(\tilde{h}^+_{\mathrm{in};3})^{-1},
\end{aligned} \tag{31}$$

and similarly for the faces f_{24} and f_{34}. Note that all the remaining variables, ($h^+_{\mathrm{in};1}$ and $h^+_{\mathrm{out};1}$, $h^+_{\mathrm{out};2}$, $h^+_{\mathrm{out};3}$, $h^+_{\mathrm{out};4}$) appear always in pairs h, h^{-1}.

The integration variables h_{lf} and $\tilde{h}$ appear explicitly as arguments of some character $\chi^j(h)F(h,\ldots)$. For all these

variables, and the associated $\theta_h^s \neq 0$ as

$$\int \mathrm{d}h\, \chi^j(h) F(h) = \int \mathrm{d}\theta_h\, \sin\frac{\theta_h}{2}\, \sin\frac{(2j+1)\theta_h}{2}\, F(\theta_h, \ldots)\,, \qquad (32)$$

the integrand is exactly zero at $\theta_h = 0$. It is easy to check that the remaining group elements, as they appear only in pairs h, h^{-1} have $\theta_h = 0$ at the saddle. We therefore have $12 \times h_{lf} + 3 \times \tilde{h}^+ + 3 \times \tilde{h}^-$ variables with $\theta^s \neq 0$ and $5 \times h^+ + 5 \times h^-$ variables with $\theta^s = 0$. The scaling at the saddle point is, according to eq. (28), $1/[\sqrt{\Lambda}^{3\times 10}\, \sqrt{\Lambda}^{18}] = \Lambda^{-24}$. In eq. (25) we count 6 independent sums and 24 factor d_{j^+}, d_{j^-} and d_J, hence

$$\delta\mu_{\mathcal{G}} \approx \sum_{6\times} \Lambda^{24}\Lambda^{-24} \approx \Lambda^6, \qquad (33)$$

which coincides with the results of [4,13].

Using a similar power counting argument, for the same graph $\mathcal{G}$ for the BF model with $SU(2)$ group ($\gamma = 1$), taking into account that "minus" variables are absent and we have $n = 20$, $k = 5$ and we recover the well-known scaling $\sum_{6\times} \Lambda^{18}/[\sqrt{\Lambda}^{3\times 5}\, \sqrt{\Lambda}^{15}] = \Lambda^9$. For an arbitrary graph the saddle point analysis becomes more involved, and the scaling is influenced both by the position of the saddle point in the θ space and by the presence of degenerate directions. A precise analysis is in progress [17].

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REFERENCES

[1] Boulatov D. V., *Mod. Phys. Lett. A*, **7** (1992) 1629 (arXiv:hep-th/9202074).
[2] Freidel L., *Int. J. Theor. Phys.*, **44** (2005) 1769 (arXiv:hep-th/0505016).
[3] Oriti D., *The group field theory approach to quantum gravity,* arXiv:gr-qc/0607032.
[4] Krajewski T., Magnen J., Rivasseau V., Tanasa A. and Vitale P., *Quantum Corrections in the Group Field Theory Formulation of the EPRL/FK Models,* arXiv:1007.3150 [gr-qc].
[5] Freidel L., Gurau R. and Oriti D., *Phys. Rev. D*, **80** (2009) 044007 (arXiv:0905.3772 [hep-th]).
[6] Magnen J., Noui K., Rivasseau V. and Smerlak M., *Class. Quantum Grav.*, **26** (2009) 185012 (arXiv:0906.5477 [hep-th]).
[7] Ben Geloun J., Magnen J. and Rivasseau V., *Bosonic Colored Group Field Theory,* arXiv:0911.1719 [hep-th].
[8] Ben Geloun J., Krajewski T., Magnen J. and Rivasseau V., *Class. Quantum Grav.*, **27** (2010) 155012 (arXiv:1002.3592 [hep-th]).
[9] Bonzom V. and Smerlak M., *Bubble divergences from cellular homology,* arXiv:1004.5196 [gr-qc] .
[10] Engle J., Pereira R. and Rovelli C., *Phys. Rev. Lett.*, **99** (2007) 161301 (arXiv:0705.2388 [gr-qc]).
[11] Freidel L. and Krasnov K., *Class. Quantum Grav.*, **25** (2008) 125018 (arXiv:0708.1595 [gr-qc]).
[12] Engle J., Livine E., Pereira R. and Rovelli C., *Nucl. Phys. B*, **799** (2008) 136 (arXiv:0711.0146 [gr-qc]).
[13] Perini C., Rovelli C. and Speziale S., *Phys. Lett. B*, **682** (2009) 78 (arXiv:0810.1714 [gr-qc]).
[14] Perelomov A., *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin) 1986.
[15] Messiah A., *Quantum Mechanics*, Vol. **II** (North Holland, Amsterdam) 1962, appendix C, pp. 1053–1078.
[16] Gurau R., *Colored Group Field Theory,* arXiv:0907.2582 [hep-th].
[17] Ben Geloun J., Gurau R. and Rivasseau V., in preparation.