LETTER

Stochastic thermodynamics of resetting

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Stochastic thermodynamics of resetting

Jaco Fuchs(a), Sebastian Goldt(a) and Udo Seifert(b)

II. Institut für Theoretische Physik, Universität Stuttgart - 70550 Stuttgart, Germany

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Abstract – Stochastic dynamics with random resetting leads to a non-equilibrium steady state. Here, we consider the thermodynamics of resetting by deriving the first and second law for resetting processes far from equilibrium. We identify the contributions to the entropy production of the system which arise due to resetting and show that they correspond to the rate with which information is either erased or created. Using Landauer’s principle, we derive a bound on the amount of work that is required to maintain a resetting process. We discuss different regimes of resetting, including a Maxwell demon scenario where heat is extracted from a bath at constant temperature.

Introduction. – Stochastic processes with resetting, that is a sudden transition to a single preselected state or region in phase space, have attracted a lot of interest recently. First considered from a mathematical point of view [1], they arise in a range of problems from the optimisation of search strategies (where resetting events restart the search and can lead to shorter search times [2–7]) to kinetic proofreading [8–11] or even population dynamics [12,13] (where a reset corresponds to a catastrophic event), to name but a few.

From a thermodynamic perspective, resetting raises interesting issues for three reasons. First, resetting changes the information content of the system: once the system is reset, all knowledge about its previous state is lost. Information, however, is physical and deleting it comes at a thermodynamic cost, an idea that goes back to Landauer’s principle and Bennett’s work [14,15]. This connection between information and thermodynamics has attracted a lot of attention from stochastic thermodynamics over the last decade, both theoretically [16,17] and experimentally [18,19]. Here, the question is what thermodynamic cost is at least required to implement resetting in a steady state. Second, a particularly intriguing application of stochastic thermodynamics is analysing the efficiency of computation in biological systems [20–22]. The key idea here is that the informational efficiency of, say, a search strategy has to be weighted against the thermodynamic costs of implementing it. Given the apparent role of resetting processes in nature, an understanding of the dissipation involved is thus paramount. Finally, the competition between the stochastic exploration of states and resetting, which constrains the system to a particular region of its phase space, gives rise to a non-equilibrium steady state (NESS). Such a state is characterised by a stationary probability distribution and a non-vanishing probability current and is an intriguing object of study in its own right.

In this letter, we address these points by analysing both, the thermodynamics of resetting a single colloidal particle, arguably the paradigm for the field [2,23,24], and resetting a discrete system [25]. We will derive the first and second law of thermodynamics in both cases and identify contributions to the well-established total entropy production rate of stochastic thermodynamics [16] which arise due to resetting. We illustrate the physical interpretation of these rates and their properties using several examples, including an implementation of a Maxwell demon that extracts heat from a heat bath at constant temperature using resetting.

Thermodynamics of resetting. – We consider an overdamped colloidal particle along a spatial coordinate $x$ immersed in a heat bath at temperature $T$. The particle experiences a systematic force $F(x) = -\partial_x V(x)$ and is randomly reset to a fixed position $x_0$ with a space-dependent rate $r(x) \geq 0$, as shown schematically in fig. 1.
The dynamic of the particle is captured by an augmented Fokker-Planck equation [26],

\[ \partial_t p(x) = -\partial_x j(x) - r(x)p(x) + \delta(x - x_0) \int dx' r(x')p(x') = 0, \]  

where

\[ j(x) \equiv F(x)p(x) - \partial_x p(x) \]  

is the probability current. The second and third term on the right-hand side of eq. (1) add the probability flux out of each point \( x \) and into the reset position \( x_0 \) to the standard Fokker-Planck equation, ensuring that probability is conserved. This interplay of drift and diffusion on the one hand and resetting on the other leads to a non-equilibrium steady state, where the probability distribution \( p(x) \) of the particle’s position is stationary, but there still is a non-vanishing current \( j(x) \). Here and for the remainder of the letter, we set \( T = 1 \) and choose dimensionless units, without loss of generality. For general \( r(x) \), eq. (1) has analytical solutions only in a few cases.

To obtain the first law for the system, we multiply eq. (1) by \( V(x) \) and integrate by parts. We impose natural boundary conditions such that \( p(x) \) as well as \( j(x) \) vanish for \( x \to \pm \infty \). Using \( \partial_x V(x) = -F(x) \), we find that in the steady state

\[ \int dx \frac{j(x)^2}{p(x)} + \int dx r(x)p(x) [V(x) - V(x_0)] = 0. \]  

Following [16], we identify the first term in eq. (3) as the rate of heat dissipation in the medium and the second term as the work which is extracted from the system in the steady state,

\[ \dot{Q} \equiv \int dx j(x)F(x), \]  

\[ \dot{W}^{\text{out}} \equiv \int dx r(x)p(x) [V(x) - V(x_0)] \]  

such that eq. (3) can be written as

\[ \dot{Q} + \dot{W}^{\text{out}} = 0, \]  

which we interpret as the first law of thermodynamics for resetting in the continuous case.

The (Shannon) entropy of the system is defined as

\[ S^{\text{sys}} = -\int dx \ p(x) \ln p(x) \]  

and its derivative \( \dot{S}^{\text{sys}} = 0 \) in the steady state. Differentiating eq. (7) with respect to time and inserting eq. (1) yields

\[ \int dx \frac{j(x)^2}{p(x)} = \int dx j(x)F(x) - \int dx r(x)p(x) \ln p(x) \]  

\[ + \ln p(x_0) \int dx r(x)p(x) \geq 0. \]  

Following [16], we identify

\[ \dot{S}^{\text{abs}} \equiv \dot{Q} = \int dx j(x)F(x) \]  

as the entropy production in the surrounding medium because it equals the dissipated heat in eq. (4). The last two terms on the right-hand side of eq. (8) arise from the resetting terms in the Fokker-Planck equation (1), suggesting the definition

\[ \dot{S}^{\text{ins}} \equiv -\ln p(x_0) \int dx r(x)p(x) \]  

is the insertion entropy rate. It depends on the probability density at the position \( x_0 \) to which the particle is reset and the probability flux out of every other \( x \). We call their sum the resetting entropy production rate,

\[ \dot{S}^{\text{rst}} \equiv \dot{S}^{\text{abs}} + \dot{S}^{\text{ins}} = \int dx r(x)p(x) \ln \left( \frac{p(x)}{p(x_0)} \right) \]  

Equality in eq. (8) is reached for a vanishing current only. However, as soon as \( r(x) \) is non-zero somewhere, there will be resetting which directly leads to a non-zero current. Thus,

\[ \dot{S}^{\text{m}} - \dot{S}^{\text{abs}} - \dot{S}^{\text{ins}} = \dot{S}^{\text{m}} - \dot{S}^{\text{rst}} > 0 \]  

for a non-vanishing reset rate \( r(x) \), which we interpret as the second law of thermodynamics including resetting.

**Discrete dynamics.** Resetting can also be implemented in discrete systems. We distinguish two types of transition rates from state \( m \) to state \( n \). For any connected states \( m \) and \( n \), transitions occur at a rate \( w_{mn} \), which for thermodynamic consistency obey

\[ \ln \frac{w_{mn}}{w_{nm}} = E_m - E_n, \]
where $E_n$ is the energy of state $n$. Furthermore, resetting from any state $m$ to a fixed state $n_0$ is performed at a state-dependent reset rate $r_m$. The master equation is then given by

$$
\partial_t p_n = \sum_m \left[ p_m w_{mn} - p_n w_{nm} \right]
- p_n r_n + \delta_{n0} \sum_m p_m r_m = 0,
$$

where the last two terms on the right-hand side account for the resetting, similarly to the augmented Fokker-Planck equation (1). However, we can reduce eq. (15) to a standard master equation by introducing transition rates $w_{mn} \equiv w_{mn} + r_m \delta_{n0}$.

Multiplying eq. (15) by $E_n$ and summing over all states, we obtain

$$
\sum_m p_m w_{mn} \ln \frac{w_{mn}}{w_{nm}} + \sum_n r_n p_n (E_n - E_{n0}) = 0,
$$

where we have used eq. (14). We identify the first term with the rate of heat dissipation and, hence, the rate of entropy production in the medium as [16]

$$
\dot Q = \dot S^m = \sum_m p_m w_{mn} \ln \frac{w_{mn}}{w_{nm}}.
$$

We consequently define the work which is extracted from the system as

$$
\dot W^{\text{out}} = \sum_n r_n p_n (E_n - E_{n0})
$$

such that eq. (17) becomes the first law, eq. (6), for resetting processes in the discrete case.

Starting from the discrete Shannon entropy of the system, we have

$$
\dot S^\text{sys} = - \sum_n \dot p_n \ln p_n = 0.
$$

Using the master equation (15) and similar arguments as before, it follows that

$$
\sum_m p_m w_{mn} \ln \frac{p_m w_{mn}}{p_n w_{nm}} = \sum_m p_m w_{mn} \ln \frac{w_{mn}}{w_{nm}} - \sum_n r_n p_n \ln p_n + \sum_n r_n p_n \ln p_{n0} \geq 0,
$$

where we have applied the log sum inequality. We define the rate of entropy production due to absorption and insertion as

$$
\dot S^{\text{abs}} = \sum_n r_n p_n \ln p_n,
$$

$$
\dot S^{\text{ins}} = - \sum_n r_n p_n \ln p_{n0},
$$

which should be compared to (10) and (11). Introducing the resetting entropy production for discrete systems,

$$
\dot S^{\text{rst}} = \sum_n r_n p_n \ln \frac{p_n}{p_{n0}},
$$

allows us to rewrite eq. (21) as

$$
\dot S^m - \dot S^{\text{abs}} - \dot S^{\text{ins}} \equiv \dot S^{\text{rst}} \geq 0.
$$

This is the second law of thermodynamics for a discrete system with resetting.

We finally note that for time-dependent dynamics, for example when relaxing towards the stationary state [27], $\partial_t p(x, t) \neq 0$ in the continuous case and $\partial_t p_n(t) \neq 0$ in the discrete case, respectively, leading to $\dot S^{\text{sys}} \neq 0$ in contrast to eq. (20). However, the derivation of the second law can be written analogously and it hence reads

$$
\dot S^{\text{sys}} + \dot S^m - \dot S^{\text{rst}} \geq 0,
$$

for both continuous and discrete dynamics.

**Resetting entropy rate and Landauer’s principle.** Resetting can decrease or increase the Shannon entropy compared to steady states without resetting, if they exist, depending on whether the distribution $p(x)$ is compressed or broadened. For example, resetting a freely diffusing particle to some point $x_0$ concentrates the particle to that region and hence reduces the uncertainty about its position, resulting in a resetting entropy rate $\dot S^{\text{rst}} < 0$. Specifically, for discrete dynamics we have $p_n \leq 1$ and hence $\dot S^{\text{abs}} < 0$, which reduces the system entropy due to the flux of probability out of each state $n$. For the continuous case, no general statement about $\dot S^{\text{abs}}$ can be made. On the other hand, for discrete dynamics, $\dot S^{\text{ins}} > 0$, while it can be both positive or negative for continuous dynamics, depending on the stochastic entropy of the designated reset state, $-\ln p(x_0)$. So inserting into a state with low steady-state probability increases the system entropy, $\dot S^{\text{rst}} > 0$, while inserting into a state with high probability effectively erases information from the system, $\dot S^{\text{rst}} < 0$.

Let us expand on this last point by analysing the following toy model for erasure. We consider a two-state system with equal energy levels and a reset rate $r$ from state 2 to state 1, but no thermal transition rates $w_{mn}$. Starting from the equilibrium state with $p_1 = p_2 = 1/2$ and Shannon entropy $S^{\text{sys}} = \ln 2$, the probability $p_2$ decreases exponentially, flowing to state 1 and leaving the system with $S^{\text{sys}} = 0$, implementing the erasure of a single bit. There is no thermodynamic entropy production since $w_{mn} = 0$, but the rate of change of the system’s entropy $\dot S^{\text{sys}} \neq 0$ and the second law (26) reduces to $\dot S^{\text{sys}} = \dot S^{\text{rst}}$, with equality due to the vanishing rates $w_{mn}$, cf. (21). Integrating leads to

$$
\int_0^\infty dt \dot S^{\text{rst}} = \Delta S^{\text{rst}} = \Delta S^{\text{sys}} = -\ln 2
$$

which is exactly minus the minimal amount of work needed to erase one bit originally derived by Landauer [14]. A natural question is now how the resetting entropy production is related to Landauer’s principle more generally.

On the single trajectory level, stochastic entropy is defined as $s(t) \equiv -\ln p_n(t)$ [28] in the discrete case and
analogy for continuous dynamics. Each time a reset from state \( n \) takes place, the system’s entropy changes by \( \Delta s_n = -\ln p_{n_0} + \ln p_n \) and the energy of the system changes by \( \Delta E_n = E_{n_0} - E_n \), corresponding to minus the extracted work. The difference in non-equilibrium free energy then reads

\[
\Delta F_n = \Delta E_n - \Delta s_n = E_{n_0} - E_n - \ln \frac{p_n}{p_{n_0}}.
\]  

(28)

Moving on to the ensemble level, we average using \( r_n p_n \) and use the second law of non-equilibrium thermodynamics [17, 29] to find the following lower bound for the amount of work necessary to maintain the resetting:

\[
\dot{W}_\text{ext} \geq \sum_n r_n p_n \Delta F_n
\]

\[
= \sum_n r_n p_n (E_{n_0} - E_n) - \sum_n r_n p_n \ln \frac{p_n}{p_{n_0}}
\]  

(29)

We can now identify the extracted work \( \dot{W}\text{out} \) (19) and the resetting entropy production (24), such that

\[
\dot{W}_\text{ext} \geq -\dot{W}_\text{out} - \dot{S}_\text{ext}.
\]  

(30)

We note that \( \dot{W}_\text{ext} \) needs to be provided by the resetter, but is not performed on the system and thus has no influence on the first law (6). This result gives a lower bound for the work \( \dot{W}_\text{ext} \) which must be performed by an external mechanism to maintain the resetting process. Comparing (30) to the non-equilibrium Landauer principle [17, 29], we thus find that the resetting entropy production equals the rate with which information is erased from or created in the system. Hence, (30) implements Landauer’s principle in a steady state, providing a scheme complementary to autonomous demons involving a tape as information reservoir [30–32]. Two regimes emerge from this result: to continuously reduce the information content of the system in the steady state, we must apply work. However, we can also extract work by increasing the system entropy, leading to a Maxwell demon.

**Examples.** — We illustrate crucial properties of the entropy rates that we have introduced with simple examples. First we revisit the freely diffusing particle with resetting and find that \( \dot{S}_\text{ext} < 0 \), as we do for diffusion with drift. However, this inequality is not a general result as we show in our third example. We also demonstrate that we can extract heat from the bath using a reset mechanism and give a full discussion of the different regimes that emerge in our final example.

**Free diffusion with resetting.** First, we choose \( r(x) = r \) for \( |x| \geq a \) and \( r = 0 \) otherwise. The ensuing steady state has been solved by Evans and Majumdar [23] and is shown in fig. 2(a). Since \( F = 0 \), there is no thermodynamic entropy production \( \dot{S}_\text{th} = 0 \). The entropy rates related to resetting, eqs. (10) and (11), are all non-zero and given by

\[
\dot{S}_\text{abs} = \frac{-2r}{2 + a^2 r + 2a\sqrt{r}} \left( 1 - \ln \frac{1 + \sqrt{r}}{2 + a^2 r + 2a\sqrt{r}} \right),
\]  

(31)

\[
\dot{S}_\text{ins} = \frac{-2r}{2 + a^2 r + 2a\sqrt{r}} \ln \frac{(1 + \sqrt{r})\sqrt{r}}{2 + a^2 r + 2a\sqrt{r}}.
\]  

(32)

The key point here is that the resetting counteracts the free diffusion by confining the particle to a region around the origin. This reduction in the uncertainty of the particle position is reflected by the rate of resetting entropy production (12) being strictly smaller than zero, \( \dot{S}_\text{ext} < 0 \).

**Diffusion in a V-shaped potential.** We now apply a force \( F(x) = \text{sgn}(x) f \) to the colloidal particle which
Fig. 3: (Colour online) Resetting entropy production rate $\dot{S}^{\text{rst}}$ (blue, dashed line) and thermodynamic entropy production rate $\dot{S}^{\text{m}}$ (yellow, dashed line) as a function of $f$ for the setup shown in fig. 2(c) with $r = 1$.

The resetting entropy production rate $\dot{S}^{\text{rst}}$ corresponds to a rooftop potential ($f > 0$) or a V-shaped potential ($f < 0$) as shown in fig. 2(b). The particle is reset to $x_0 = 0$ at a constant rate $r(x) = r$, such that the Fokker-Planck equation (1) simplifies to

$$\partial_x^2 p(x) - \text{sgn}(x) r \partial_x p(x) - r p(x) + r \delta(x) = 0. \quad (33)$$

We first solve eq. (33) for $x \neq 0$. Using the continuity of $p(x)$ at $x = 0$ and imposing natural boundary conditions leads to $p(x) = c \exp(-(f + \sqrt{f^2 + 4r})/2|x|)$ with $c$ determined by the normalisation. This distribution and the corresponding probability current $j(x)$ are plotted in fig. 2(b). The entropy rates then follow as

$$\dot{S}^{\text{abs}} = r \ln \frac{f + \sqrt{f^2 + 4r}}{4} - r, \quad (34)$$
$$\dot{S}^{\text{ins}} = -r \ln \frac{f + \sqrt{f^2 + 4r}}{4}, \quad (35)$$
$$\dot{S}^{\text{m}} = \frac{-2fr}{f + \sqrt{f^2 + 4r}}. \quad (36)$$

The resetting entropy production rate obeys $\dot{S}^{\text{rst}} = -r < 0$, so again resetting reduces the uncertainty of the system. The thermodynamic entropy production depends on the potential shape, determined by the sign of $f$. For the rooftop potential, heat is dissipated into the medium ($\dot{S}^{\text{m}} > 0$) and work needs to be performed on the system, $W^{\text{out}} < 0$. For the V-shaped potential, heat is absorbed from the surrounding medium, $\dot{S}^{\text{m}} < 0$, while work is extracted. Thus, in the latter case, the external reset mechanism operates like a Maxwell demon, extracting heat from a bath at constant temperature. From the perspective of the second law, however, this is compensated by the work $W^{\text{ext}}$ required to maintain the resetting in the steady state (30). Hence the resetter has to apply more work than can be extracted as $W^{\text{out}}$ from the particle, ensuring thermodynamic consistency.

**Resetting with $\dot{S}^{\text{rst}} > 0$.** So far, the resetting entropy production has been negative in all the examples. However, inspection of the rate, eq. (12), reveals that it can also be positive if $p(x) > p(x_0)$ in regions of large $r(x)$, which can be realized in a setup like the one shown in fig. 2(c). Here, we have a reflecting boundary at $x = a$, hence $j(a) = 0$, and apply a potential $V(x) = -fx$ for $x \leq a$. We reset the particle to $x_0 = 0$ at a rate $r(x) = r$ for $x \geq 0$ and $r(x) = 0$ otherwise. The steady-state distribution and current, calculated analogously to the second example, are also shown in fig. 2(c). The entropy production rates cannot be calculated in closed form for such a system, but numerical results are shown as a function of the force parameter $f$ in fig. 3. For $f$ larger than a critical value $f_c$, the resetting entropy production rate $\dot{S}^{\text{rst}}$ becomes indeed negative, which corresponds to increasing the information content of the system.

**Operation diagram for the discrete random walk.** In our last example, we expand on the idea of different signs for the thermodynamic and resetting entropy production rates by considering the three-state system shown in fig. 4. The transition rates between neighbouring states are $k_+$ from left to right and $k_-$ in the opposite direction. The random walker is reset from states 2 and 3 to the initial state $n_0 = 1$ with constant rate $r$. The master equation (15) is then easily solved for the steady state. In this system, three regimes emerge with regard to the sign of the rates of entropy production in the medium (18) and due to resetting (24). We sketch them in the operation...
diagram, fig. 5, as a function of the parameters $k = k_{+}/k_{-}$ and $r' = r/k_{-}$. For $k < 1$, as the random walker jumps to the right, it absorbs heat $-k > 0$ from the reservoir which is extracted as work $W_{\text{out}}$ when it is reset to $n_{0} = 1$, yielding an average thermodynamic entropy production of $\dot{S} = 0$, thus implementing a Maxwell demon. On the other hand, for $k > 1$, the random walker produces entropy in the medium at a rate $\dot{S} > 0$ and work has to be put in to reset the particle, yielding $\dot{W}_{\text{out}} < 0$, effectively lifting it to a higher state. Meanwhile, the resetting entropy rate depends on the ratio of probabilities in the initial state $n_{0} = 1$ and in states $n = 2, 3$, respectively, either erasing or increasing the information content of the system. However, for all $r'$ and $k$, $\dot{S} > 0$.

Conclusion and perspectives. – We have derived the first and second law of thermodynamics for continuous and discrete stochastic dynamics with resetting and identified the contribution of resetting to the total entropy production. This resetting entropy rate $\dot{S}_{\text{reset}}$ quantifies the rate of information creation or erasure in the steady state. Using Landauer's principle, we were able to derive a lower bound on the work input required to implement a given resetting mechanism. We note that work can also be extracted from the heat bath at constant temperature in a Maxwell-demon–type setup at the expense of increased external work input.

We expect that our results can be verified experimentally, for example using colloids in an optical trap, which have a history of successful validation of concepts from stochastic thermodynamics [18,33–35]. Among the avenues for further theoretical work, it should be worthwhile to apply our framework to biological systems featuring resetting, such as the detection of pathogens by the immune system [36]. An appreciation of the thermodynamic costs involved could guide the search for the fundamental principles underlying these highly efficient processes.

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REFERENCES