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TOPICAL REVIEW

Certification and applications of quantum nonlocal correlations

A Elena Piceno-Martínez, Laura E C Rosales-Zárate* and P Ornelas-Cruces

Centro de Investigaciones en Óptica A.C., León, Guanajuato 37150, Mexico
* Author to whom any correspondence should be addressed.

E-mail: laura@cio.mx

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Abstract

Entanglement and Einstein–Podolsky–Rosen (EPR) steering are nonlocal quantum correlations, which are relevant resources for quantum information protocols. EPR steering, or quantum steering, refers to the correlation where a party might 'steer', or modify, the state of another, which is spatially separated. Entanglement is a symmetric resource while steering is asymmetrical, since it depends on the direction of the effect. Due to these different characteristics and the therefore different possible applications, there has been both theoretical and experimental research on forms to certify the distinct quantum nonlocal correlations. In recent years, alongside the investigation on quantum correlations between two systems, there has been a great interest in investigating multipartite/multimode entanglement as well as steering, since they include a high dimension and it may be possible to store more information than in a single qubit. In this review, we will summarize the different criteria and measures that have been developed for the characterization of these two kinds of correlations. We first focus on bipartite entanglement and steering. We then review the progress that has been made in the investigation of multipartite quantum correlations. We revise the theoretical work in quantum nonlocal correlation witnesses and measures, which respectively allow one to certify that the system is entangled or presents EPR steering, and give a quantification of the content of these correlations in the system. Then, we briefly review the experiments that have been designed and that demonstrate multipartite quantum correlations. We also include applications in quantum information protocols, in particular in quantum teleportation and quantum cryptography.

1. Introduction

In the last decades, great progress has been made in both quantum information and quantum technologies [1–4] even with focus on possible realizations and applications to be carried out in space [5]. A very important resource for applications in quantum information are nonlocal quantum correlations [6–9]. In this review, we summarize the state of the art of entanglement and Einstein–Podolsky–Rosen (EPR) steering.

Nonlocal quantum correlations are strong correlations which cannot be described by classical physics. In the seminal paper by Einstein *et al* [10], two spatially separated particles with perfect correlations were considered. Then, if a measurement is performed on either momentum or position in one of the particles, the other particle will have a defined momentum or position. This is what is nowadays known as the EPR paradox. In the EPR paradox, the premise of *local realism*, which implies that one particle cannot influence another spatially separated particle, is assumed for a quantum state, finally leading to the conclusion that quantum mechanics is incomplete. In a series of articles starting in the same year, Schrödinger [11–15], as a reply to the ongoing discussion, brought forward the concepts of steering and entanglement.

In 1964 Bell [16, 17] established the Bell inequality, which showed the incompatibility between *local hidden variables*, which are the proposal to maintain the local properties of classical physics, and quantum mechanics. Later, other Bell-type inequalities, some of them experimentally accessible, were proposed. For instance, two that are amply used in the present are the Clauser–Horne [18] and the Clauser–Horne–

Shimony–Holt inequalities [19], which are experimentally accessible and thus have had a great relevance in the investigation of quantum correlations since they were proposed. In the decades of the 1970–1980's the first experiments of the Bell inequality were carried out [20–22], proving that nonlocal quantum correlations are present in nature and therefore showing that entanglement is an important resource for processes and tasks in quantum information [23].

Moreover, the EPR paradox shows the existence of the nonlocal quantum correlations [7, 8, 22], pioneering the three nonlocal quantum correlations, which are quantum entanglement, quantum steering and Bell nonlocality. Mathematically, an entangled state is a composite state that cannot be described in a separable or factorizable form, this means that one party is strongly correlated with the other, even if they are spatially separated [6, 11, 12]. Bell nonlocality and quantum steering are stronger forms of quantum correlations. Quantum steering is an asymmetric correlation, since for two spatially separated parties local measurements performed by one of the parties can *steer* the state of the other, and is the manifestation of the EPR paradox. This property is what had been long known as 'spooky action at a distance'. Steering can also be expressed in terms of violations of the so named *local hidden state* (LHS) models or defined in terms of a quantum task [8, 24, 25]. Due to the asymmetry property, steering has applications in one-sided device independent quantum key distribution (QKD), where only one of the measurement apparatuses of the parties is trusted, which in turn requires the demonstration of steering [26]. Steering is also important in quantifying the randomness that can be extracted in an scenario where one of the parties does not trust their device [27].

Bipartite entanglement refers to the type of entanglement that manifests/occurs between two systems or particles. This form of entanglement has been thoroughly investigated [2, 6, 28–36] from a quantitatively and qualitatively point of view, as well as both in a theoretical and in an experimental way. However, more recently there have been described and studied other forms of entanglement, which include high dimensional entanglement [37, 38], hybrid entanglement [39, 40] and multipartite entanglement [41–45], in which several systems are involved. In particular, hybrid entanglement is that between different types of subsystems, including at least one continuous variable (CV) subsystem and one discrete variable (DV) subsystem, in contrast with continuous entanglement which is entanglement between CVs only, and discrete entanglement, which is only between DV subsystems. The inclusion of the two different types of variables allows for novel applications of entanglement. For instance, this has been implemented in QKD protocols using the spin and orbital angular momentum [46], or polarization and orbital angular momentum [47]. In the sense of high dimensional entanglement, a higher dimensional Hilbert space might allow to increase the capacity of and generally improve quantum communication protocols [48–50], since more bits can be encoded in the system [51–53]. On the other hand, multipartite entanglement can be implemented in integrated optics [54], and can be used to enhance sensitivity [55] or for quantum metrology [56].

Besides the applications, it is important to investigate how quantum correlations can be generated. This can be done through nonlinear processes. In the bipartite case, entanglement and steering can be produced by parametric processes, as harmonic cascades with sum-frequency or by a direct third-harmonic generation [57]. In [58], a nonlinear optical apparatus where the steering can be controlled and even set as one-way steering is described, which shows the asymmetry of quantum steering. For the tripartite case, entanglement can be generated by an optical parametric oscillator [59]. Both genuine tripartite entanglement and steering have been investigated for a system of optical modes in a combined down conversion and sum of frequency process [60], while other proposals concerning only genuine tripartite entanglement are given by spontaneous parametric down conversion processes of spatial degrees of freedom of the photons [61]. There are also recent studies considering triple photon generation [62–65] or a nonlinear process of third order [66]. The generation of higher order multipartite steering and entanglement is a subject of current investigation [67, 68] which may require multiple nonlinear cascaded processes [69, 70].

In this review, we first give an overview of quantum correlations, in section 2, with focus on their definition and relationship. Next, the different measures to certify either entanglement and steering for both bipartite and multipartite entanglement are described in section 3. Section 4 shows the applications of multipartite quantum correlations. Concluding remarks are given in section 5, the last section of this review.

2. Overview

In quantum theory there exist quantum correlations in what is observed between two systems that do not correspond to classical causality. This important fact is known and was developed from the beginnings of quantum theory [71–76]. Currently, our understanding is that there exist three different types of quantum correlations and they represent an appropriately complete panorama.

2.1. Quantum correlations

2.1.1. Bell nonlocality

In the EPR paper [10], local hidden variable theories (LHVTs) were proposed to complete the description that quantum mechanics gives of physical systems, in such a way that classical realism is recovered in the predictions we make of the measurement results. Bell identifies the *local causality principle* as the component of LHVTs that disagrees with the correlations predicted in quantum theory. The kind of quantum correlations that, by Bell's theorem, do not satisfy local causality are named *nonlocality*.

Additionally, nonlocality and Bell's inequalities experimental proofs have allowed us in the past decade to observe the existence of quantum correlations in quantum mechanical systems [20].

Nonlocality, Bell inequalities and their applications were reviewed more in depth by Brunner *et al* [22], while Bell inequalities for mesoscopic and macroscopic systems were reviewed by Teh *et al* [77]. As this is outside of the scope of this work, the interested reader is directed to these references.

2.1.2. Entanglement

The concept of entanglement surges from the idea that was early observed by Schrödinger [11, 12, 78–80], that quantum systems do not obey the independence of correlated physical systems, like classical systems do. In this way, entanglement is defined as the proper *non-separability* of quantum systems, in the sense that in order to describe a quantum system composed of two or more subsystems, it can not be described by only the description of the subsystems individually, but must be described in full.

The conceptual and mathematical sense of the phenomenon of entanglement will be revised in more detail in the following section discussing the detection of quantum correlations, section 3.1.

2.1.3. EPR steering

In discussing the situation presented by Einstein *et al* [10], Schrödinger [13–15] proposed the term *steering* to refer to the influence of a part of the system over the other, in search of a more general discussion than that of the conjugate variables position and momentum [72, 74]. In this way, steering can be described as the ability to affect a system that we do not have access to, through performing local measurements in a different system.

A formal definition of steering in terms of quantum operations and information theory was given in 2007 by Wiseman *et al* and Jones *et al* [24, 25], through the attentive revision of Schrödinger's works. To reflect the fact that the concept of steering carries the essence of the correlations in the case considered by EPR, it was named *EPR steering*. In the information operational sense, steering is defined as the impossibility of a description through a LHS model. In steering, the EPR effect is present, but the correlations are not strong enough to discard *all* hidden variable models. Steering is a quantum correlation that, in strength, is located between entanglement and Bell nonlocality [24, 25].

A remarkable work about steering's growth since its conception is given by Uola *et al* [7]. There, the theoretical basis and historical milestones are collected, giving a general steering framework as guidance for future research. A recent review of the state of the art of quantum steering was made by Xiang *et al* [81], where the main collaborations that had arisen through the years about the cited topic are recognized. Also a series of open questions and research paths to follow are presented. This review [81], gathers together theory, experiments and the scope of quantum steering for the years to come. It is important to comment that this work gives a wide outlook about steering detection with entropic criteria. In addition, different criteria are compared in different references, for instance [82–84].

2.1.3.1. Steering asymmetry

In a different manner as for the other quantum correlations, steering possesses a peculiar characteristic, that of asymmetry. Given that it is so rare in the context of quantum correlations, this asymmetry is an important property and promises interesting applications [85]. The asymmetry has been realized in different scenarios, in which there is also evidence of manipulation of this asymmetry [58, 86–93]. In this way, the asymmetry of steering provides important advantages for applications, such as in QKD, where now protocols in which we only require one party to be trustworthy are realizable [94], as well as permitting different characteristics that improve the security of quantum teleportation [95].

The steering asymmetry also implies that in some states it can be certified *only* in one direction between the two parties, this is known as one way steering [24, 96]. This means that for an entangled state, an observer Alice can steer Bob's system, but no the other way around.

Different measures to characterize this asymmetry have been proposed. For example by defining an *steering radius* it is possible to quantify the steerability of the system and also demonstrate the asymmetry of the system [97], other measures include the use of entropic measures [98] (see section 3.2.2.4). The

asymmetry of steering for Gaussian measurements has been theoretically shown in an intracavity nonlinear Kerr coupler [99]. Gaussian one way steering was first experimentally observed by Händchen *et al* [100].

Later experiments demonstrate genuine one way steering, but with the added important properties of being for a general type of measurements and with no detection loopholes, which include the positive operator value measurements (POVMs) and the assumption of the quantum state [101]. A rigorous experimental detection loophole-free test for a two qubit system was performed in [102]. For two-qubit states, a theoretical and experimental certification of one-way steering with no assumptions on state fidelity or measurement settings have been performed [103]. Since this asymmetry can be detected in several systems, a proposal for generating asymmetric steering in macroscopic steering is given in [104].

Due to the asymmetry property of steering being important for potential applications, there has been research in ways to generate this asymmetry. For example, it was studied in microwave photons by using a superconducting circuit system [105], a quantum interference from the incoherent pumping of an atomic system [106], or through atomic coherent effects in a resonant four-level system [107].

Related to the direction of steering, is the investigation of the decoherence of steering when an entangled system is coupled to a reservoir. In this case, steering can change depending if system A is steered by B or otherwise [108]. For multipartite systems the asymmetry property has been experimentally performed for optical networks [109], and theoretically for Gaussian states by using a modulating scheme [89].

2.2. The relationship between different quantum correlations

From the work of [24, 25], we know that every nonlocal state possesses steering and any steerable state is entangled, however, the converse relationships are not true. Moreover, it was found that the concepts of steering and the uncertainty principle are the two elemental components of nonlocality [110].

2.2.1. Trust and quantum information processes

Trust in the measuring devices is an important characteristic of the tasks that are carried out in quantum information, and corresponds to the characterization of the measuring processes. Here also we can see the hierarchy of the quantum correlations. In entanglement all of the measurements are well-characterized and we have a full description of the quantum system. Then, for Bell-nonlocality we do not need a characterization of the measurements, just to know the resulting probabilities. Steering is found between these two extremes: in the bipartite case only one system is completely characterized, conventionally the measurements of Bob. This allows for *one-sided device-independent* protocols [26, 111, 112]. Even more, in the steering case presented by [113], Bob's system is not completely characterized and only its dimension is known, with which the certification of steering can be seen as a separability problem.

3. Certification of nonlocal quantum correlations

To date there are a great variety of different criteria to certify bipartite entanglement, with important contributions in the cases of multipartite and continuous entanglement. As well, there exist diverse manners to certify steering in different kinds of systems. A recent review with focus on theoretical and experimental certification of entanglement is given in [114]. Other reviews of entanglement include [6, 33, 34, 36, 115, 116]. For steering, the recent reviews of [7, 8, 81, 117] discuss aspects of certification. Below we review the different measures or witnesses used to certify quantum entanglement and steering.

3.1. Entanglement detection measures

3.1.1. Bipartite entanglement

An entangled state is a state of a quantum system for which its parties are strongly correlated; entanglement is a very important phenomenon on quantum mechanics. Nowadays, it is a fundamental question to know whether a state presents nonlocal correlations. These can be defined for bipartite and multipartite systems, being the bipartite systems the most studied. For these, entanglement can be both certified and quantified.

Let us consider a bipartite pure state $|\psi\rangle_{AB}$, in the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The system is entangled if the state cannot be expressed as a factorized form of product of the states of each of the systems, which correspond to $|\phi\rangle_A$ and $|\varphi\rangle_B$, respectively, for system A and B [28, 33]

$$|\psi\rangle_{AB} \neq |\phi\rangle_A |\varphi\rangle_B. \tag{1}$$

A separable state is a state that can be written as a factorized form of product of the states of each of the systems. In this case, the state is uncorrelated, and measurements made in system A are independent of measurements of B.

For mixed states, the states of each system are described by a density matrix, ρ_A for system A and ρ_B for B, respectively, which are defined in the corresponding Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. A bipartite state ρ_{AB} is separable if it can be written as a product of the form

$$\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i. \tag{2}$$

Depending on the system under consideration there are different criteria that one can consider in order to certify whether the system is entangled. The quantum systems that have an infinite-dimensional Hilbert space, and a continuous spectrum for the observables, are called CV systems, while the quantum systems with a finite-dimensional Hilbert space and discrete spectrum are DV systems. Examples of CV systems are harmonic oscillators and electromagnetic modes. They are characterized by the quadrature operators

$$\hat{X}_1 = \frac{c}{2}\left(\hat{a} + \hat{a}^\dagger\right), \qquad \hat{X}_2 = \frac{c}{2i}\left(\hat{a} - \hat{a}^\dagger\right). \tag{3}$$

Here $\hat{a}$ and $\hat{a}^\dagger$ are bosonic operators, while c is a constant which values are usually 1, 2 or $\sqrt{2}$. Gaussian states are examples of CV systems. These systems present entanglement and steering in their quadrature operators. Thus, as an example, entanglement witnesses are given in terms of variances of these operators.

3.1.2. Entanglement witnesses for bipartite systems

3.1.2.1. Schmidt decomposition

The simplest case of entanglement is seen in a bipartite quantum state. In this occurrence, the Schmidt decomposition of the state seen as a vector in the product state space, allows to determine unequivocally the separability of the state from its *Schmidt rank*.

The Schmidt decomposition corresponds to the construction of coefficients that can be used to describe the compound state trough tensor products between the subsystem states [36]. For an state $|\psi\rangle_{AB}$ the Schmidt Theorem assures the existence of bases $\{|u_i\rangle_A\}_i$ and $\{|v_j\rangle_B\}_j$ of A and B, respectively, such that

$$|\psi\rangle_{AB} = \sum_{j=1}^{\min(d_A,d_B)} \alpha_j |u_j\rangle_A \otimes |v_j\rangle_B. \tag{4}$$

The Schmidt rank is defined as the number of non-zero coefficients in equation (4). Then, an state is entangled *iff* its Schmidt rank is greater than one [36].

Several measures can also be written using the *Schmidt coefficients*, $\lambda_j = \alpha_j^2$, for instance, the concurrence (section 3.1.3.8) and *entanglement of formation* (EoF) (section 3.1.3.3), further details can be found in [36]. In the case of multipartite systems, the *Schmidt rank vector*, constructed from the Schmidt rank of the possible bipartitions, can be used to characterize the entanglement [118, 119], see also [37].

3.1.2.2. Positivity criteria

The two main results of positivity criteria, that of Peres–Horodecki [28, 120] for DV and of Simon [121] for CV, are of great importance due to the simplicity of the determination of entanglement they represent, specially when compared to the most straightforward approach of resolving the non-separability of a quantum state, and have been amply studied as a result. Here we briefly present some of the results in this direction.

3.1.2.3. The positive partial transpose (PPT) criterion

The PPT criterion was developed by Peres [120] and Horodecki *et al* [28] and is an operational criterion to detect entanglement based upon the fact that a given state is separable if its matrix representation has a PPT. It is noteworthy that this condition is just necessary, and not sufficient, for the separability in discrete systems of general dimensions [28].

3.1.2.4. The Simon positivity criterion

There exists a generalization of the criterion for CV that was proposed in 2000 by Simon [121], such generalization utilizes second order momenta of the canonical operators.

3.1.2.5. Momentum matrices positivity criteria

The positivity criteria can be further extended in terms of the momentum matrices [122–124]. The criteria of [123, 124] also allows for the detection of nonseparability for DV systems of general dimensions. For the case of non-Gaussian continuous states, the Simon criterion [121], becomes a hierarchy of positivity conditions on the minor matrices obtained from the quadrature momentum matrices in the approach of [122].

Even further, there is a generalization of the work of [122], which puts constrictions on the CV systems in order to generalize the criterion for hybrid systems [125].

3.1.2.6. Entanglement witnesses

An important disadvantage of the PPT criterion and its extensions is that they just offer necessary conditions to detect entanglement in a given system in most cases. Fortunately, there are *entanglement witnesses* [6], which are another type of tests that in turn use quantum observables to detect quantum entanglement.

It is said that an observable is an entanglement witness if at least it is capable of distinguishing one entangled state from the set of all separable states. The detection provided by these witnesses is defined thanks to a general operational condition [126]. The entanglement witnesses are Hermitian operators W acting over the compounded Hilbert space of a system, and they always have positive mean values for all pure separable states $\langle W\rangle_{\rho_{\text{sep}}} \geqslant 0$, even though they are not positive definite, then pointing out the presence of entanglement with the existence of negative mean values. This is, if $\langle W\rangle_{\rho} < 0$, then ρ is entangled.

Clearly, a downside of the witnesses is that they are not global, because one single operator is not capable of detecting entanglement in every general state. Conversely, for each entangled state, there exist multiple witnesses capable of distinguishing it from all separable states. Even more, there are entangled states that can not be detected utilizing any witnesses [36]. Regarding the mathematical formulation, entanglement witnesses are defined in a geometrical manner starting from the convexity of the set of separable states. For two qubits, when the two parties perform Pauli measurements σ_x and σ_z, there exist closed descriptions for every witness [127]. More introductory details on this important topic can be found in [6, 36].

3.1.2.7. Information theory

Entanglement, as all quantum correlations, is closely related to the transmission of information from one system to another. This can be seen both in the initial discussions of correlations and intuitive understandings, and the upsurging fields of quantum information and quantum communications, in which these correlations are fundamental concepts and building blocks.

Since it results that many of the concepts developed in this current of research provide closed expressions for the amount of entanglement in bipartite DV states, a more detailed basic review appears in section 3.1.3.1.

3.1.2.8. Entropic entanglement criteria

Entropic criteria and quantities have been able to be defined in order to detect the presence and give a measure of quantum correlations. Entanglement criteria from entropic uncertainty relations have been obtained in references such as [128–131]. In an interesting analysis, a general formalism to obtain criteria from local uncertainty relations, which bound the variances, is proposed in [132]. This is then upgraded to a formalism to obtain both entanglement and steering criteria from entropic uncertainty relations in [131], see section 3.2.2.4.

In order to certify entanglement, we require to know the measurements of both systems. For separable states, it follows from the existence of generalized entropic formulations of the uncertainty principle, that there is an entropic limit on the probabilities of both subsystems [132]

$$\sum_k S^{\lambda}(A_k) \geqslant C_A; \qquad \sum_k S^{\lambda}(B_k) \geqslant C_B. \tag{5}$$

Therefore, we can find for the generalized conditional entropies the following bound for separable states

$$\sum_k S(B_k|A_k) \geqslant C_A + C_B. \tag{6}$$

3.1.2.9. Entanglement criteria from local uncertainty relations

The previous limit based on the uncertainty principle can be also obtained for the regular uncertainty relations, where the uncertainty is given in terms of variances instead of entropies. In this case, equation (6) becomes a bound on the variances of the global measurements $M_k = A_k \otimes \mathbb{I} + \mathbb{I} \otimes B_k$ [132]

$$\sum_k \delta^2(M_k) \geqslant C_A + C_B. \tag{7}$$

There also exists a formulation using covariance matrices [133]. This criterion is studied in detail in several references, such as [134–136].

3.1.3. Entanglement measures for bipartite systems

3.1.3.1. Entropic measures

The transmission of information is one of the quantum tasks that is possible thanks to entanglement. It is found that, referring to the Bohm experiment [137], the perfect correlation apparent from Bohm's state is the basic unit of entanglement. Following the last statement we can value the origin of the entanglement entropy as a measure of entanglement from a prevailingly informational operational point of view.

3.1.3.2. Entanglement entropy

The seminal work of von Neumann [138] about the formalism of quantum mechanics gave an ideal theoretical framework to link this field with that of information theory. In particular, this entailment helped applying some informational ideas to quantum mechanics, for instance the mutual information and Shannon entropy [139], that converge towards the main idea of the *von Neumann entropy*, which uses marginal probability distributions (related with traces of density matrices of given systems), to measure the discrepancy between the joint probability distribution and the marginal probabilities related with each one of the subsystems, this is the entanglement entropy. It is such the importance of this quantity because, historically, in its use during the first investigations related with entanglement, the entanglement entropy was considered as a synonymous of the proper entanglement. The von Neumann Entropy of a quantum system in the ρ state is

$$S = -Tr(\rho \ln \rho). \tag{8}$$

Writing ρ in terms of its eigenvectors, $\{|i\rangle\}_i$, and eigenvalues, $\{\lambda_i\}_i$ we have,

$$\rho = \sum_i \lambda_i |i\rangle\langle i|, \tag{9}$$

then

$$S = -\sum_i \lambda_i \ln \lambda_i, \tag{10}$$

is the so called Shannon entropy for the quantum state ρ.

Now, consider a discrete composite system whose density matrix is ρ_{AB}, ergo the subsystems under consideration are A and B. The entanglement entropy is defined as the von Neumann entropy of the reduced states, which are defined through the partial trace. Let us take the case of subsystem A,

$$\rho_A^{(B)} = Tr_B \rho_{AB}, \tag{11}$$

thus the entanglement entropy is

$$E = S(\rho_A^{(B)}) = -Tr\left(\rho_A^{(B)} \ln \rho_A^{(B)}\right). \tag{12}$$

3.1.3.3. Other entropic measures

Although the entanglement entropy entirely characterizes the entanglement in bipartite DV states, this does not happen for other types of states. The lack of entanglement measures for multipartite DV and CV has stimulated the development of other entanglement measures, that are reduced to the entanglement entropy in the limiting case of pure states.

In the case of mixed states, it is possible to define two measures that are related with the amount of entanglement necessary as a resource within preparation of states. The first one is the EoF and it can be understood as the minimum necessary amount of entanglement to prepare an entangled state utilizing local operations and classical communication (LOCC) (see the following section 3.1.3.11). The second one is the *distillable entanglement* [140], that can be defined as the amount of entanglement that one can obtain from a certain state trough LOCC.

3.1.3.4. Rényi-α entropies

Another generalization of the entanglement entropy are the Rényi-α entropies [141]. These measures are α power generalizations of the von Neumann entropy and consist as well of the Rényi entropies applied over the discrete bipartite reduced state. In the limit when $\alpha \to 1$, the Rényi-α entropies reduce to the entanglement entropy.

3.1.3.5. q-Entropies (unified entropies)

There are other generalized entropic measures that are useful since they reflect the information in not ideal scenarios [142]. Under a framework of informational axioms developed in [143], they are obtainable by means of relaxation of one or more of such axioms.

For example, in this group we have Rényi entropies that do not satisfy the convexity property, as well as the Tsallis entropies [144, 145], which do not satisfy the positivity axiom. With both families the entanglement entropy is recovered in the limit of the q parameter when $q \to 1$.

3.1.3.6. Mixed states

The definition of entanglement measures is more complicated for the case of mixed states. While in the case of pure bipartite qubit systems, the entanglement entropy characterizes completely the entanglement and quantum correlations, many measures have been found in the search to characterize the properties of mixed states. In this section, we will highlight the case of convex roofs.

3.1.3.7. Convex roofs

The convex roof entanglement measures [146] are based on the fact that it is possible to write a mixed state as a statistical mixture of pure states, as follows

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \tag{13}$$

with p_i elements of a probability distribution. In this way, a natural extension of a given measure, M, which quantifies the entanglement of pure states, can be seen as the mean value when taking into consideration each state of the decomposition of equation (13). However this decomposition is not unique. Considering this, the infimum mean value over all possible decompositions,

$$M(\rho) = \inf_{p_i;|\psi_i\rangle} \sum_i p_i M(\psi_i), \tag{14}$$

is a quantity monotonically decreasing under LOCC, and, therefore, a measure for the mixed state ρ, called the convex roof of M.

3.1.3.8. Concurrence

The concurrence was proposed in 1997 by Hill and Wooters [147] for discrete bipartite mixed states with two levels. It is defined by means of the unitary transformation θ [6, 148],

$$\theta|\psi\rangle = \sigma_y \otimes \sigma_y |\psi^*\rangle, \tag{15}$$

with σ_y the Pauli matrix. The concurrence is then [147]

$$C = \langle\psi| \overleftarrow{\theta} |\psi\rangle. \tag{16}$$

This definition permits to calculate the convex roof of the concurrence as

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{17}$$

with λ_i the square roots of the eigenvalues of the matrix $R = \sqrt{\rho}\sqrt{\tilde{\rho}}$ ordered in a descending order, $\tilde{\rho} = \theta\rho\theta$.

3.1.3.9. Negativity

The negativity is an entanglement measure derived from the PPT criterion. According to this criterion, an state is an entangled state if its partial transpose is negative, namely, if it has at least one negative eigenvalue. Therefore, we can quantify the entanglement of the given state through the magnitude of its negative eigenvalues. The negativity is defined as [149]

$$N = \sum_{\lambda_i < 0} |\lambda_i|, \tag{18}$$

where λ_i are the eigenvalues of the partial transpose of ρ. The negativity is a monotone under LOCC.

3.1.3.10. Logarithmic negativity

The logarithmic negativity [150–152] is a version of the negativity given by the trace norm (Schatten 1-norm). In a way related to the entropic measures, it corresponds to the entanglement cost for creating an entangled state using operations that preserve the positivity of the partial transpose [153].

3.1.3.11. Quantum operations

Quantum correlation characterization provides a different kind of definition for quantum entanglement, which is consistent with separability. In this way, it is said that if a correlation between variables of a system can be created only by making *local operations* and using *classical communications*, within schemes known as LOCC, then such correlations do not correspond to quantum entanglement. By definition, these operations maintain the desired classical properties.

Hence, by using solely LOCCs we can create only classical correlations. In addition, when considering all possible LOCCs we have that they will in general decrease the entanglement of a certain state. Therefore, a quantity which under action of LOCCs is always reduced, corresponds to quantum entanglement [36, 154]. Noticeably, the definition of entanglement by means of LOCC allows the description of quantities that determine entanglement in a quantitative manner, such quantities are known as *entanglement monotones* [151, 155]. LOCC theory provides a definite way of constructing such monotones and to corroborate if a certain quantity is an appropriate entanglement quantifier.

3.1.3.12. Entanglement measures for CV systems

The biggest obstacle in the study of continuous systems is the infinite dimensionality of the involved systems. However, Gaussian continuous states can be described in a simple way due to their structure. Additionally, these states are vastly utilized in quantum information and quantum optics applications. For the previous stated reasons, many of the developed continuous measures had been created for Gaussian states. A more comprehensive review than the one presented here can be found in [156].

It is common to use the covariance matrix of a Gaussian system to determine the separability of such system. The covariance matrix is directly associated with the second moment of a probability distribution.

A covariance matrix, σ, corresponds to a separable state *iff* the covariance matrices of the subsystems σ_A and σ_B exist such that [157]

$$\sigma \geqslant \sigma_A \oplus \sigma_B. \tag{19}$$

The description of Gaussian CV states is then finite thanks to the covariance matrix, in this way, it is possible to generalize for these types of systems results found for DV systems, such as the Schmidt decomposition [158–160], entropic measures [161, 162], negativity [163] and tangle [164, 165] (see section 3.1.6).

It is important to mention that one of the main results in the study of non-Gaussian entanglement is the Shchukin–Vogel separability criterion [122]. This is a PPT-like criterion (see section 3.1.2.2), in the form of a generalization of the Simon criterion, that uses a hierarchy of conditions from the no-null higher order components of the momentum matrix and its partial transposition.

3.1.4. Multipartite entanglement

In recent years it has been possible to experimentally create multipartite entangled states for different types of systems, for example, in optical systems, it has been achieved for three photon states [63, 166–168], four photons [169], six photon states [170, 171] or even eight photons [172]. In order to detect that the system is entangled it is necessary to formulate either conditions or criteria that certify multipartite entanglement.

We can mathematically formulate the separability of a composite quantum system, and therefore, of non-separability, in a simple manner, by stating that any composed system whose state can not be described as a product of the states of the subsystems as follows

$$|\psi_{1,2,\ldots}\rangle = |\psi_1\rangle\,|\psi_2\rangle\ldots, \tag{20}$$

is non-separable, and therefore has multipartite entanglement [173]. However, in most cases it is very difficult to determine the entanglement of a quantum state by simple inspection, and this is even more so in the case in which we have more than two systems involved. Therefore, quantities that allow to determine if there is entanglement are very valuable, these are called witnesses.

3.1.5. Entanglement witnesses for multipartite systems

Entanglement witnesses depend on the system under consideration, either discrete or continuous, or pure or mixed states. Here we review different entanglement witnesses for multipartite systems.

Sperling and Vogel [174] defined arbitrary multipartite witnesses, for a given partition of the complete system, in terms of an Hermitian operator $\hat{W}$, that satisfy:

- For all $\hat{\sigma}$ separable:

$$\langle \hat{W} \rangle = \mathrm{Tr}\left[\hat{\sigma}\hat{W}\right] \geqslant 0.$$

- For at least one $\hat{\rho}$:

$$\langle \hat{W} \rangle = \mathrm{Tr}\left[\hat{\rho}\hat{W}\right] < 0.$$

Thus, it is possible to construct an operator of the form:

$$\hat{W} = f_{\mathcal{I}_1:\ldots:\mathcal{I}_k}(\hat{L})\hat{1} - \hat{L}, \tag{21}$$

here $\hat{L}$ is a general Hermitian operator, $\mathcal{I}_1 : \ldots : \mathcal{I}_k$ are partitions, and $f_{\mathcal{I}_1:\ldots:\mathcal{I}_k}(\hat{L})$ is the maximum expectation value for separable states. In terms of this witness, a quantum state $\hat{\rho}$ is entangled with respect to the partition $\mathcal{I}_1 : \ldots : \mathcal{I}_k$, if and only if

$$\mathrm{Tr}\left[\hat{\rho}\hat{L}\right] > f_{\mathcal{I}_1:\ldots:\mathcal{I}_k}(\hat{L}), \tag{22}$$

for a Hermitian operator $\hat{L}$.

3.1.5.1. Multipartite separable eigenvalue (MSEvalue) witnesses
Sperling and Vogel [174] also defined a general entanglement witness constructed from an algebraic set of equations, known as the MSEvalue equations

$$\hat{W} = \sup\{g\}\hat{1} - \hat{L}, \tag{23}$$

with

$$g = \left\langle a_1 \ldots a_k | \hat{L} | a_1 \ldots a_k \right\rangle.$$

3.1.5.2. CV tripartite and multipartite entanglement conditions
For discrete multipartite systems genuine entanglement measures have been established in [55]; while for CV systems, measures for detecting tripartite entanglement were first given by van Loock and Furusawa [175]. For this case, linear combinations of the quadratures, $\hat{x}_i$ and $\hat{p}_i$, of the electromagnetic modes under consideration, were defined as follows

$$\hat{u} = h_1\hat{x}_1 + h_2\hat{x}_2 + h_3\hat{x}_3 \tag{24}$$

$$\hat{v} = g_1\hat{p}_1 + g_2\hat{p}_2 + g_3\hat{p}_3.$$

Here h_i and g_i are real arbitrary parameters. A necessary condition for separability in a tripartite three mode states is then given in terms of the variances of $\hat{u}$ and $\hat{v}$, which is:

$$\langle(\Delta\hat{u})^2\rangle + \langle(\Delta\hat{v})^2\rangle \geqslant \frac{1}{2}(|h_1g_1| + |h_2g_2| + |h_3g_3|). \tag{25}$$

Thus, a violation of equation (25) implies tripartite entanglement.

A particular form is given once the real parameters are chosen, for example, on taking $h_1 = g_1 = 1$, $g_2 = g_3 = -h_2 = -h_3 = \frac{1}{\sqrt{2}}$. In this case, the condition equation (25) is expressed as:

$$\langle[\Delta(\hat{x}_1 - (\hat{x}_2 + \hat{x}_3)/\sqrt{2})]^2\rangle + \langle[\Delta(\hat{p}_1 + (\hat{p}_2 + \hat{p}_3)/\sqrt{2})^2]\rangle \geqslant \frac{1}{2}. \tag{26}$$

The violation of the inequality given in equation (26) is a sufficient condition for tripartite entanglement.

These inequalities can be generalized for the multipartite case. For N-partite systems, linear combinations of the quadrature operators are similarly defined

$$\hat{u} = h_1\hat{x}_1 + h_2\hat{x}_2 + \ldots h_N\hat{x}_N,$$
$$\hat{v} = g_1\hat{p}_1 + g_2\hat{p}_2 + \ldots g_N\hat{p}_N.$$

In order to express a sufficient condition for entanglement it is necessary to first choose a pair of modes (m, n), next, to consider an appropriate linear combination of the quadratures such as those in equation (27),

and finally consider pairs (m, n) to negate all partial separabilities with the other modes, k_r and k_s. Thus, a violation of the following inequality

$$\langle(\Delta\hat{u})^2\rangle + \langle(\Delta\hat{v})^2\rangle \geqslant \frac{1}{2}(|h_m g_m + \sum_r h_{k_r} g_{k_r}| + |h_n g_n + \sum_s h_{k_s} g_{k_s}|), \tag{27}$$

implies that the multipartite system is entangled.

3.1.5.3. Full N-partite entanglement

For multipartite CV entanglement, two types of entanglement have to be defined [176]. These are the *genuine N-partite entanglement*, and the *full N-partite inseparability* [177]. The former refers to the falsification of all mixtures of the bipartitions of the form $\rho_{m,n}\rho_k$. Such a bipartition means that the systems m and n can be entangled, but there is no entanglement between either m or n, and k. On the other hand, for full N-partite inseparability it is not necessary to negate all mixtures of the bipartitions.

Conditions for fully inseparable entangled states are given in terms of the sum of variances as well as product criteria. For the tripartite case, using the linear combinations of the quadrature operators defined in equation (24) [176]:

$$\begin{aligned}(\Delta\hat{u})^2 + (\Delta\hat{v})^2 &\geqslant 2(|h_1 g_1| + |h_2 g_2 + h_3 g_3|),\\ \Delta\hat{u}\Delta\hat{v} &\geqslant |h_1 g_1| + |h_2 g_2 + h_3 g_3|.\end{aligned} \tag{28}$$

For tripartite systems, conditions for fully inseparable tripartite entangled states, were derived by van Loock and Furusawa [175] as the following inequalities

$$B_I \equiv [\Delta(\hat{x}_1 - \hat{x}_2)]^2 + [\Delta(\hat{p}_1 + \hat{p}_2 + g_3\hat{p}_3)]^2 \geqslant 4, \tag{29}$$

$$B_{II} \equiv [\Delta(\hat{x}_2 - \hat{x}_3)]^2 + [\Delta(g_1\hat{p}_1 + \hat{p}_2 + \hat{p}_3)]^2 \geqslant 4,$$

$$B_{III} \equiv [\Delta(\hat{x}_1 - \hat{x}_3)]^2 + [\Delta(\hat{p}_1 + g_2\hat{p}_2 + \hat{p}_3)]^2 \geqslant 4.$$

3.1.5.4. Genuine N-partite entanglement

Criteria for genuine N-partite entanglement can be defined in analogy to the tripartite case, given in the set of equations of equation (28) [176]. In order to define this, a set of X_N bipartitions I_k are defined, and A_k and B_k are two distinct sets of parties for each bipartition. An inequality is defined in terms of the sum of variances for each bipartition; a sufficient condition for N-partite genuine entanglement is the violation of

$$\sum_{k=1}^{X_N} I_k \geqslant 4. \tag{30}$$

Different criteria for genuine tripartite entanglement where also defined in [176]. Here, we show the sum of variances and product of variances for the linear combinations of the quadratures operators defined in equation (24)

$$\begin{aligned}(\Delta\hat{u})^2 + (\Delta\hat{v})^2 \geqslant &2\min(|h_3 g_3| + |h_1 g_1 + h_2 g_2|, |h_2 g_2| + |h_1 g_1 + h_3 g_3|,\\ &|h_1 g_1| + |h_2 g_2 + h_3 g_3|),\end{aligned} \tag{31}$$

$$\begin{aligned}\Delta\hat{u}\Delta\hat{v} \geqslant &\min(|h_3 g_3| + |h_1 g_1 + h_2 g_2|, |h_2 g_2| + |h_1 g_1 + h_3 g_3|,\\ &|h_1 g_1| + |h_2 g_2 + h_3 g_3|).\end{aligned} \tag{32}$$

Other methods to detect multipartite entanglement are based in a semi-device-independent scenario, where, in a quantum network, one of the parties use untrusted measurements [178].

3.1.6. Entanglement measures for multipartite systems

While an entanglement witness answers the question: 'is a quantum system entangled?', an entanglement measure quantifies the amount of entanglement in a quantum system. Research work on multipartite entanglement is recent and is ongoing. In order to quantify the entanglement for multipartite systems, measures are based on the *residual tangle* or *residual entanglement*, which is a measure for a three qubit system [179].

For the qubit case, genuine tripartite measures were defined, taking into account that the measure should be zero for separable states and product states [180]. Xie and Eberly [180] defined a triangle tripartite measure based on the concurrence of one qubit and the other remaining two. Generalized m-partite entanglement measurements where derived as well in [55].

Multipartite measurements related to the entropy were reviewed in [45].

3.2. EPR steering detection measures

Given the fact that steering is found between entanglement and nonlocality, there is ample variety and flexibility within the approaches for its detection. In a similar way as is done in the case of Bell nonlocality, one can proceed by simply analyzing the correlations in the probability distributions obtained in the measurement processes. In this sense, we have linear criteria, that is closely related to entanglement witnessing, as well as criteria based on entropic or variance uncertainty relations, similar to existing entanglement criteria, and detecting inequalities analogous to Bell inequalities.

In steering examples, we have the following basic setup: a quantum state is shared between two experimenters, Alice and Bob, who carry out their experiments in a spatially separated way; these experiments correspond to measures of an observable on their half of the state. In this way, Alice makes a measurement on her system, and claims that with this she can affect the state inside Bob's setup. However, confidence in Alice's asseveration does not exist. In the case that Alice can carry out this effect in Bob's state, there is steering present from the state of the subsystem of Alice, A, to that of Bob, B.

Steering arises from probabilities that can not be explained using an LHS model. That is, that Bob's probability distribution can not be inferred from prior existence of a given state in his part of the array [24, 25]. This means that steering appears on the failure of the description of the modification on the state of Bob from Alice via sending and then measuring on a local hidden shared state that she knows. However, it is not straightforward to determine if there exists some state that permits such a description, in the general case, and thus, to decide on the steerability of our system.

According to Schrödinger's analyses on steering [12–15] there is not transmission of information with steering, since Bob's reduced state (the part of the system that Bob has) is independent on the measurement choice of Alice. Instead, she can decide with this choice, whether Bob's part of the system will be found on an eigenstate of one or the other observable. Therefore, Bob *believes* that Alice can influence his particle.

3.2.1. Bipartite steering

In order to certify EPR steering, there are many different criteria that can be used depending on the system. For a review of different steering witnesses the reader is referred to the reviews [7–9, 81, 181]. The first practical criteria for steering was given by Reid [182]. This is given in terms of inferred variances of the quadrature operators. This was first experimentally verified by [183]. A definition of steering as a task in quantum information was introduced by Wiseman, Jones and Doherty in 2007 [24, 25].

3.2.1.1. Information theory

Steering can be thought in terms of information as the knowledge we have of one part of the system affecting the information that can be obtained about the other part of the system [15].

With informational criteria, the quantification of steering, previously a question of considerable difficulty, becomes possible [184]. The definition of steering through informational properties, allows its analysis in theories involving quantum operators and quantum protocols, for instance by way of POVMs [96, 185]. Even more, steering was proved completely equivalent to the capacity to carry out joint generalized measurements [186–191]; this means that the measurements that cannot be realized at the same time are the ones that reveal steering.

3.2.2. Steering witnesses for bipartite systems

3.2.2.1. Linear criteria

For linear steering criteria, several results bounding summations of squared mean values have been known for many years [192–194] for different anticommuting operators on Bob's measurements and mutually unbiased bases. Then, a generalization to other classes of observables is carried out in [195]. Optimal

measurements for this type of criteria are studied in different references, as [196–198]. Interestingly, it is possible to define inequalities that present unbounded violations [199, 200].

An interesting case is that presented in [201], from which the following general linear criteria is derived. In this case, we have N measurements A_k of Alice; the results of Alice's measurements are a and Bob realizes Pauli measurements. Defining

$$T_x^{(k)} = \sum_a p(a|k)(\langle\sigma_x\rangle|_a)^2, \tag{33}$$

then, unsteerable states satisfy the following relation

$$T_x^{(1)} + T_y^{(2)} + T_z^{(3)} \leqslant 1. \tag{34}$$

Another form of general linear criteria for witnessing steering in bipartite systems is given in [202, 203]. For N dichotomous measurements A_k of Alice, and arbitrary B_k for Bob, an state is unsteerable if

$$\sum_{k=1}^{N} |\langle A \otimes B\rangle| \leqslant \max_{\{a_k\}} \left[\lambda_{\max}\left(\sum_{k=1}^{N} a_k B_k\right)\right], \tag{35}$$

where $a_k = \pm 1$ and $\lambda_{\max}(B)$ is the maximum eigenvalue of B_k.

Some of these type of criteria were studied experimentally as well in [201–207].

3.2.2.2. Information theory

As we have seen above, concepts of information theory have been transferred to describe the EPR steering present in a system. In this section we will focus on the uncertainty-derived criteria. Different criteria have also been found from other notions, for instance, the criterion of [208] is obtained using the Fisher information, a measure of the information content, while the average fidelity is used in [209, 210]. In [211] criteria is derived using correlation matrices, and in [212, 213] with magic squares, as well as in [214] for POVMs and mutually unbiased measurements. Other informational quantities are used in [215].

3.2.2.3. Uncertainty

Criteria to detect correlations on quantum systems can be obtained from the satisfaction of generalized uncertainty relations. In the case of entanglement, both measures satisfy an entropic limit for separable states, then, in an entanglement criterion, both limits would appear [132]. For non-steerable states, only one system, that of Bob, would satisfy the entropic limit, and the steering criterion resulting would only have this limit. For Bell nonlocality, neither entropic limit exist and the limit for the criteria is 0, equivalent to Bell's theorem.

Uncertainty can be understood as a description of the amount of information we can obtain from a system. In steering, the information Alice can obtain about Bob's system has a lower uncertainty than that allowed by the uncertainty principle.

One of the first steering criteria was developed by [182], considering measurements of position or momentum and uncertainty given by variances. Bob's minimal uncertainty is

$$\delta^2_{\min}(X_B) = \int \mathrm{d}x_A p(x_A)\delta^2(x_B|x_A). \tag{36}$$

Then, unsteerable states satisfy

$$\delta^2_{\min}(X_B)\delta^2_{\min}(P_B) \geqslant \frac{1}{4}. \tag{37}$$

3.2.2.4. Entropic steering criteria

Walborn *et al* [216] developed a steering inequality based on the generalized entropic uncertainty relation of [217] for CVs. An inequality for the certification of steering is obtained in [218] by analyzing the EPR paradox and the part that the uncertainty principle plays. Even though the criterion obtained here is not the strongest one, the informational approach and ease of calculation pose important advantages. Another comparison of the strength of entropic criteria is carried out in [219]. It is then proven that from any entropic uncertainty relation, steering criteria can be obtained [219]. In this way, a criterion using Shannon entropies is found [219], this also allows for the detection of steering on only one direction. Entropic quantities to witness entanglement can be also extended to obtain steering criteria [219]. Steering criteria found from entropic uncertainty relations include those of [131, 184, 219–223].

In [219] also a general criteria is found in terms of the Tsallis entropies. Kriváchy *et al* [224] developed an inequality for Rényi entropies when there are only two measurements at both sides.

A generalized formalism to witnessing correlations via entropic criteria can be understood as follows [131, 219]. In the usual case of bipartite steering, only the measurements of Bob are characterized, and the probabilities of a LHS description are bounded by an entropic uncertainty relation

$$\sum_k S^\lambda(B_k) \geqslant C_B, \tag{38}$$

where B_k are the different measurement operators of Bob, $S^\lambda(B_k)$ are the corresponding entropies, and λ represents the LHS. Then, we can obtain an entropic criteria based on the corresponding relative entropy

$$\sum_k S(B_k|A_k) \geqslant C_B. \tag{39}$$

This is closely related to entropic witnessing criteria for entanglement, section (3.1.2.8). The above inequality can be found for any generalized entropies when it holds that

- the entropy is at least pseudo-additive for independent probability distributions,
- the entropic uncertainty relation is independent on the state,
- the relative entropy is jointly convex.

Different kinds of entropic uncertainty relations are used to find steering criteria, such as noisy uncertainty relations [225], fine-grained [226]. A relationship to joint measurability was explored in [227]. Further, in [222, 223] the proposed criteria were analyzed experimentally, and in [228] this was done for hybrid systems. Other experimental proofs of criteria are found in [83, 229–231].

3.2.2.5. Steering criteria based on local uncertainty criteria

A similar construction as the previous one can be carried out for local uncertainty relations, the difference being that the uncertainty is now characterized by variances [131, 232, 233]

$$\sum_k \delta^2(B_k) \geqslant C_B \tag{40}$$

and the criterion results in terms of the variances of the global observables $M_k = A_k \otimes \mathbb{I} + \mathbb{I} \otimes B_k$

$$\sum_k \delta^2(M_k) \geqslant C_B. \tag{41}$$

Again, this can be formulated in terms of covariance matrices [232, 233].

3.2.2.6. Positivity criteria

Inherited from the PPT and moment matrix positivity criteria for entanglement [234] (see section 3.1.2.2), we have the following situation for steering [235]. If there exist a possible choice for the unknown entries on the moment matrices corresponding to Alice's measurements, that will make the matrix positive, then the state is unsteerable. This is for conditions on commutativity of Alice's measurements and structure of Bob's characterized measurements [113, 235]. Kogias *et al* [235] also defined the equivalence of this approach with a semidefinite programming problem (see section 3.2.2.8).

Steering detection can be carried out in a device-independent way using matrix positivity criteria, by analyzing the positivity of the moment matrices for each element of the conditional probability distribution [234].

Additionally, the realignment method, originally derived for the quantification of entanglement, was studied in [131] for steering, this is related to the positivity of states containing steering in the sense of the PPT criterion [236, 237].

3.2.2.7. LHS model inequalities

When Bob realizes two mutually unbiased measurements, the following inequality determines if the correlations, represented by $\langle A_i B_j \rangle$, can be described by an LHS model and the system is thus unsteerable [178]

$$\sqrt{\langle (A_1 + A_2)B_1 \rangle^2 + \langle (A_1 + A_2)B_2 \rangle^2} + \sqrt{\langle (A_1 - A_2)B_1 \rangle^2 + \langle (A_1 - A_2)B_2 \rangle^2} \leqslant 2. \tag{42}$$

Different criteria are obtained for the case where the LHS model only describes the correlations $\langle A_i B_j \rangle$ [238], or also the marginal distributions $\langle A_i \rangle$ and $\langle B_j \rangle$ [239]. The last equation (42) is also a criterion for three uncharacterized projective measurements on Bob's system [238], as well as for POVMs [238]. An important result of this paper [238] is that, when the LHS model is only required to describe the correlations, states that are steerable according to this criterion are nonlocal via the Bell-CHSH inequality. However, in the case where we want the LHS model to describe the full distributions, this result is no longer true [239].

Another interesting criterion in this category is the *all versus nothing proof* [240, 241], which is similar to the GHZ proof for Bell nonlocality without inequalities [41], and is obtained through bounds on the conditional probabilities of Bob. This proof can be also used to determine the presence of one-direction steering.

3.2.2.8. Semidefinite programming

There exist different inequalities that can be used to corroborate the presence of steering; many of them depend on *semidefinite programming* in order to carry out the required calculations [181, 242, 243], this is a numerical method that requires Alice's measurements to be known, which poses important difficulties. With semidefinite programming, the determination of steering is a notorious problem of convex optimization and is equivalent to a dual program, and a linear steering inequality is obtained [244].

A review on the computation involved on this approach is found in [181]. A review of a general framework of quantum steering focusing on semidefinite programming is given in [181] only for Gaussian states. Important results are summarized in [7].

In these criteria, in order to determine the steerability, the LHS model problem is reduced to the problem of finding a finite ensemble of LHS states for a finite assemblage depending on the measurement settings of Alice [242, 245], an assemblage being a set of conditional states for Bob. The LHS states are then the operators that define all of the conditional states given a *deterministic* result $a = \lambda(x)$. If the conditional state is given as $\rho_{a|x}$ and the assemblage $\{\rho_{a|x}\}$, with λ the local hidden variables and the setting x, then the state is unsteerable if it can be written in terms of the LHS states operators $\sigma_\lambda \geqslant 0$ as

$$\rho_{a|x} = \sum_\lambda \delta_{a,\lambda(x)} \sigma_\lambda. \tag{43}$$

3.2.3. Steering measures for bipartite systems

3.2.3.1. Entropic measures

Entropic measures can be used to approach the quantification question. An step in this direction is found in [184], where the violation of the entropic steering inequalities (see section 3.2.2.4) puts lower bounds on the entanglement monotones found in resource theory (see section 3.1.3.11). In this work [184], steering detecting inequalities are then found through the bounds on entanglement measures using the uncertainty principle in presence of quantum memory and the relationship between entanglement monotones and the negative quantum conditional entropy [246, 247] to optimize the bounds. The resulting criteria allow for scalability to higher dimensions and to witness steering for hybrid systems also.

3.2.3.2. Semidefinite programming

With the concepts of semidefinite programming and assemblages representing the possible probability distributions that Bob can obtain (see section 3.2.2.8), we can propose a quantification of the steering from the following principle [181]. The space of all assemblages is a convex set, in this, the subset of unsteerable assemblages is also convex; then, we can quantify steering as some kind of distance in this convex structure from our steerable state to the set of unsteerable assemblages.

3.2.3.3. Steering weight

The steering weight was the first measure proposed under the semidefinite programming concepts, developed in 2014 in [185]. In this measure, the assemblage under investigation is written as a convex combination of an unsteerable assemblage $\rho^{\mathrm{LHS}}_{a|x}$ and a general assemblage $\rho^{G}_{a|x}$, both with the same parameters

$$\rho_{a|x} = p\rho^{G}_{a|x} + (1-p)\rho^{\mathrm{LHS}}_{a|x}. \tag{44}$$

The minimum weight possible p in this combination is the steering weight. We seek out to minimize the relative distance of the distance between the unsteerable assemblage and our assemblage, and the distance between the general assemblage and $\rho_{a|x}$, this is called the unsteerable fraction. This optimization problem can be solved via semidefinite programming naturally.

3.2.3.4. Steering robustness

The steering robustness [215] is linked to the steering weight, since it is obtained by considering the weight that accompanies the general assemblage in the convex combination of equation (44) when taking the general assemblage as noise that needs to be added to the unsteerable assemblage to obtain the desired steerable assemblage. In this case, the desired minimization is of the relative distance of the distance between the unsteerable assemblage and the assemblage under investigation, and the distance between the unsteerable assemblage and the general assemblage. The relative distance measures the noise tolerance. This is also solvable via semidefinite programming [181, 215].

Steering robustness was first defined in the quantum information and communication task of subchannel discrimination [215], while, on the other hand, it was then found out that steering weight is associated to the task of subchannel exclusion [248]. Both of these measures can be connected to the maximal violation of every potential steering inequality [249].

A related quantification of steering, the generalized steering robustness, is proposed in [250], in a similar way as what is done for the generalized entanglement robustness [251, 252]. This quantification is proven to have desirable characteristics for a proper steering measure.

3.2.3.5. Geometric trace distance

In [253] a geometric measure for the steering is found based on the trace distance between the studied assemblage and the closest LHS assemblage. For qubit states, the volume of the geometric surfaces, with the related concepts of *steering ellipsoid* and *LHS surface*, correspondingly, gives a witness for steering. The measure is found through a maximization on all mutually unbiased bases. In resource theory, it is found that this measure is a convex monotone of steering.

3.2.3.6. Other semidefinite programming measures

Defining the moment matrices assemblage as the collection of moment matrices corresponding to the conditional states, a device independent measure is constructed in [254]. Different measures have also been proposed based on the conceptually similar *critical steering radius* [255–257].

3.2.3.7. Other informational measures

In [189] a steering measure is found for the case of two-qubit, two-settings, from the correspondence of steering with joint measurability. In the sense of the distance to the unsteerable assemblages (see section 3.2.3.2), a class of steerable states is defined, for which upper and lower bounds are found that are related to the nonlocality given by maximal violation of the CHSH inequality. As an interesting result, a class and therefore a qualification for asymmetric steering can both be found in a similar way.

A similar quantification is derived in [258]. From the criteria of [178, 259], the quantification of the steering of a state is given by the amount in which a steering inequality is maximally violated. The intuition behind this proposal is given as follows: the more inequalities a state violates, the most robust under noise the steering is, and the steerability is said to be greater. This is therefore related to steering robustness [215] (see section 3.2.3.4) and usual approaches to nonlocality quantification [22, 260].

Kogias *et al* [243] propose a measure called the *Gaussian steerability* for arbitrary bipartite Gaussian CV systems. Conditions are found on the steerability of a covariance matrix created by Gaussian measurements. Then, a quantity is defined based on conditions on the diagonalization of the matrices and symplectic operations. The measure is designed depending on the direction of the steering, therefore a construction can be made that determines the degree and direction of asymmetry in the steering. For two modes, the Gaussian steerability reduces to coherent information. This measure is related to the key rate in one-sided device-independent QKD.

3.2.4. Multipartite steering

3.2.4.1. Steering witnesses for multipartite systems

Similar to multipartite entanglement, it is possible to define multipartite steering witnesses. It is also important to differentiate the concepts of full N-partite steering and genuine N-partite steering. Full N-partite steering inseparability refers to the certification of steering for each one of all the bipartitions of the system in at least one direction. While genuine N-partite steering refers to bilocality along different bipartitions, therefore steering is confirmed for all the different bipartitions [261]. For pure states, genuine N partite steering is demonstrated if all bipartitions of the model are negated.

For tripartite states, a condition for genuine tripartite steering is given in terms of the steering of party i by the other parties, denoted by $S_{i|jk}$. This condition is as follows:

$$S_{1|23} + S_{2|13} + S_{3|12} < 1. \tag{45}$$

3.2.4.2. CV tripartite steering criteria

For CV systems, steering is certified in the quadrature operators. For tripartite states, linear combinations of the quadrature operators are defined as in equation (24). Genuine tripartite steering is verified if one of the following inequalities is violated [176]:

$$(\langle\Delta\hat{u}\rangle)^2\langle(\Delta\hat{v}\rangle)^2 \geqslant 2\min(|g_1h_1|,|g_2h_2|,|g_3h_3|). \tag{46}$$

$$\Delta\hat{u}\Delta\hat{v} \geqslant \min(|g_1h_1|,|g_2h_2|,|g_3h_3|). \tag{47}$$

Different criteria can be obtained depending on the values of the parameters g_i and h_i. Important steering relations where each of the possible bipartitions are consider are given below [176]:

$$B_I \equiv [\Delta(\hat{x}_1-\hat{x}_2)]^2+[\Delta(\hat{p}_1+\hat{p}_2+g_3\hat{p}_3)]^2 \geqslant 2, \tag{48}$$

$$B_{II} \equiv [\Delta(\hat{x}_2-\hat{x}_3)]^2+[\Delta(g_1\hat{p}_1+\hat{p}_2+\hat{p}_3)]^2 \geqslant 2,$$

$$B_{III} \equiv [\Delta(\hat{x}_1-\hat{x}_3)]^2+[\Delta(\hat{p}_1+g_2\hat{p}_2+\hat{p}_3)]^2 \geqslant 2.$$

Stronger steering inequalities in terms of products of the variances can also be defined:

$$S_I \equiv \Delta(\hat{x}_1-\hat{x}_2)\Delta(\hat{p}_1+\hat{p}_2+g_3\hat{p}_3) \geqslant 1, \tag{49}$$

$$S_{II} \equiv \Delta(\hat{x}_2-\hat{x}_3)\Delta(g_1\hat{p}_1+\hat{p}_2+\hat{p}_3) \geqslant 1,$$

$$S_{III} \equiv \Delta(\hat{x}_1-\hat{x}_3)\Delta(\hat{p}_1+g_2\hat{p}_2+\hat{p}_3) \geqslant 1.$$

These inequalities together with the ones for tripartite entanglement were used to detect genuine tripartite entanglement and steering in an experiment implemented with optical quantum networks [109]. Similar witnesses can be obtained for multipartite spin systems [262].

3.2.4.3. DV tripartite and four-partite steering criteria

For discrete systems, in particular for three qubit states, genuine multipartite steering can be detected for W states and GHZ states by a sequence of multipartite observers [263] or by using entropic uncertainty relations [221] or linear linear steering inequalities [264], which leads to two way steering. Genuine four-partite steering was demonstrated by constructing a witness of kernel of quantum steering, denoted by W_4, which is expressed in terms of joint probabilities of experimental outcomes [265]. If this witness satisfies

$$W_4 > 1+\frac{1}{\sqrt{2}}, \tag{50}$$

then the system exhibits genuine four-partite EPR steerability. In the experiment carried out by Li *et al* [265], in a four photon entangled system, genuine four-partite steerability was verified since $W_4 = 1.8829 \pm 0.0049$.

4. Applications of multipartite quantum correlations

4.1. Quantum steering applications

Steering has been found already in a variety of quantum systems; for instance, in Bose–Einstein condensates is studied in [266–268], where the multipartite case is considered in [263, 269, 270], and in [269] for atomic coherence systems. Recently, in 2022 an steering distillation protocol was proposed [271], with this, we can envisage an improvement of the use of steering as a resource in quantum information.

The Peres conjecture, and the difference between nonlocality and the concepts of bound entanglement and entanglement distillation, was proven false through counterexamples found in systems that present steering [69, 272, 273].

In relation to the experimental proofs of quantum correlations, steering allows to setup experiments without loopholes easier than when this kind of setup is attempted based on Bell nonlocality [201, 206].

Advances in telecommunication wavelengths have been made [274] thanks to steering, since recently it has been possible to certify EPR steering of photon pairs at these wavelengths, by using the process of parametric down conversion. For this kind of position-momentum entanglement, it is possible to calculate

the dimensionality of entanglement, however, this depends on the length of the crystal. Other methods that can be used to calculate the dimensionality in position and momentum correlations rely in imaging techniques, for example the use of a charge-coupled device camera. Using this, spatial entanglement and EPR correlations have been observed in [275]. It is worth to mention than steering is more robust to experimental noise than nonlocality [202–205, 276].

4.2. Quantum teleportation

Quantum teleportation is an important protocol in quantum information and quantum communication based on entanglement as a resource. In the quantum teleportation protocol [277, 278], Alice, the sender, transfers an unknown quantum state to the receiver, Bob. They share an entangled state and are spatially separated. Alice measures her entangled state and the unknown state, and sends the results of the measurement to Bob, through a classical communication. Then, Bob reconstructs the state by using Bell measurements. There have been several experimental implementations for different systems for the bipartite scenario, refer to [278, 279] and references therein.

At the same time, for the multipartite scenario, proposals of an experimental implementation were performed for CV systems [48], and by using tripartite entanglement. In this scenario, besides Alice and Bob, there is a third party, usually named Claire or Charlie. The three of them share a tripartite quantum state, and Claire can control the transfer of the state form the sender to the receiver. This protocol, that considers a controller, is known as controlled quantum teleportation (CQT) [280], and is the generalization of the teleportation protocol to a multipartite system, in this case the tripartite one. An experimental demonstration of the CQT protocol for discrete particles was performed in [281], by using photonic GHZ states. In contrast to the CV case, tripartite entanglement is not a necessary resource. For the experimental implementation in CVs a fidelity of 0.64 ± 0.02 was obtained [48], while for discrete systems a fidelity of 0.83 was obtained [281].

Implementation of high dimensional quantum teleportation was first realized for 12 qutrit states [282], obtaining an average fidelity of 0.75.

Once a multipartite system is considered for a quantum teleportation protocol, it is important to address whether there are multiple senders or receivers. This implies how to share the information. A protocol of the teleportation of a quantum secret between multiple senders and receivers was proposed by Lee *et al* [283] and also implemented in an experiment.

4.3. QKD

QKD is a protocol for secure communication between two distant parties that involves a cryptographic scheme using secret keys. The sender and the receiver are usually called Alice and Bob respectively, and are connected via a classical channel (CC) and a quantum channel (QC). The CC is used to send classical messages back and forth, while the QC allows the parties to share quantum states.

In the QKD protocol, it is relevant to authenticate the CC to avoid an eavesdropper, usually called Eve, in the communication between Alice and Bob. If the QC presents a leak of information, Alice and Bob abort the protocol. Thus, the security of the QKD protocol is essential and can be achieved with the asymmetric property of steering [26]. It is useful the fact that in steering, it is only required trust on one party of the configuration [284, 285]. For a review of the security of QKD the reader is referred to reviews [286–289].

QKD protocols can be implemented depending on the way that Alice encodes and Bob decodes the classical information. QKD protocols can be separated in discrete variable QKD (DV-QKD) [290] and continuous variable QKD (CV-QKD) [111, 112, 291]. In the DV-QKD, the secret keys are encoded in qubits. That is, Alice encodes her qubit, for the bipartite case, or her qudit, for the multipartite case, into the quantum states, while Bob decodes the encoded quantum information. In CV-QKD, the secret keys are encoded in quadrature operators and are decoded using coherent detection. On the other hand, it is possible to implement a QKD protocol that shares the properties of both DV-QKD and CV-QKD protocols as in [292] where the proposal uses the quantum state preparation in DV-QKD and the homodyne detectors used in CV-QKD. From the security analysis, they conclude that the proposal allows to work in the trusted device scenario since it is possible to isolate the quantum noise of the homodyne detector.

Further practical challenges and security of QKD protocols have been reviewed in [286–288, 293–295]. Also, we refer the reader to the implementation of QKD in [111, 112, 291], as well as via satellite in [171, 296]. Works on high dimensional QKD protocols are such as [297–300], where the most recent experimental research uses biphotons (pairs of entangled photons) to encode information [300].

Steering improves the security of QKD [26]. Steering inequalities have been used to establish a link with the secret key rate in one-sided device-independent quantum key distribution (1SDI-QKD) protocols. In [26], the condition to achieve a positive key in their 1SDI-QKD protocol is equivalent to demonstrating that quantum steering is present between Alice and Bob. In [112], the EPR inequality [182] was used to establish

a direct link with the secret key rate. This connection gives an interpretation of steering as a quantifier of the number of secure 1SDI bits that can be extracted.

4.4. Further applications and experiments in quantum information

Quantum networks [301] are important for the transmission of information in the field of quantum information. Moreover, it is important to investigate how entanglement can be distributed in these quantum networks [302]. Two basic elements of a quantum network are the nodes, where qubits are located, and links, which represent entanglement between the qubits in the nodes. Therefore, in a quantum network it is necessary to verify that there is entanglement present. A protocol for verifying multipartite entanglement in a network and its experimental implementation was done by McCutcheon *et al* [303], by using polarized tripartite and four-partite entangled photons.

5. Concluding remarks

This review gives an overview of the certification of nonlocal quantum correlations. We focus on two of these correlations, namely entanglement and quantum steering. Definitions, and different forms to certify and quantify entanglement and steering were reviewed for different types of systems. In order to generalize the results for multipartite systems it is important to understand the bipartite systems, since generalizations are performed considering entanglement or steering between one party and the other ones.

Great advances have been made both from a theoretical point of view and experimental in multipartite entanglement and steering, mainly for tripartite and quadripartite states. However this is not an easy task and there is ongoing work, since the understanding of these correlations is important for potential applications in quantum information and quantum communications.

Data availability statement

This is a review, therefore there is no data. The data that support the findings of this study are available upon reasonable request from the authors.

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ORCID iDs

A Elena Piceno-Martínez https://orcid.org/0000-0003-4786-3746
Laura E C Rosales-Zárate https://orcid.org/0000-0003-0137-8999

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