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Long-lived entanglement of two multilevel atoms in a waveguide

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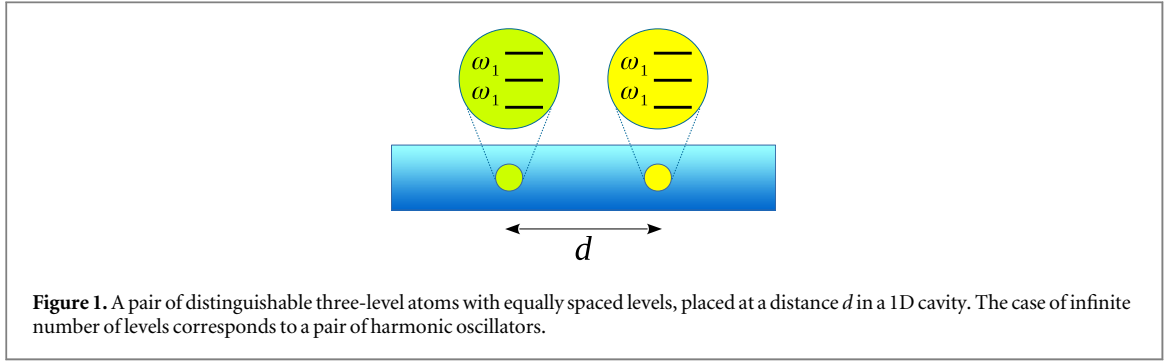
Abstract

We study the presence of nontrivial bound states of two multilevel quantum emitters and the photons propagating in a linear waveguide, when the emitters can be modeled as harmonic oscillators. We characterize the conditions for the existence of such states and determine their general properties, focusing in particular on the entanglement between the two emitters, that increases with the number of excitations. We discuss the relevance of our results for entanglement preservation and generation by spontaneous relaxation processes. We finally introduce a perturbative method for determining the existence of long-lived states, for general level spacings and couplings.

1. Introduction

The physics of quantum systems confined in one-dimensional (1D) geometries has recently attracted a lot of attention [1, 2], and is motivating interesting theoretical and experimental research. The behavior of an excited atom coupled to a field is among the peculiarities of such systems: although decay by spontaneous emission occurs in free (three-dimensional) space, boundary conditions and artificial dimensional reduction alter the picture, enhancing or inhibiting (and sometimes hindering) decay. These effects have been extensively studied and observed in cavity-QED settings [3–14], where the spectrum of the electromagnetic field is discrete. It is much less trivial that similar phenomena occur in effectively 1D unbound systems, in which the field spectrum is continuous and photons are free to propagate in 1D space. Dimensional reduction can be implemented in a range of experimental platforms, that include cold atoms in tightly focused fields [15–17], photonic crystals [18–21], optical fibers [22, 23], quantum dots in photonic nanowires [24, 25], and superconducting qubits in integrated circuit waveguides [26–29]. Theoretical studies focused on the interplay between the spectral features of the field and the structure of the emitters [30–37].

The vacuum of quasi-1D fields and their coupling with quantum emitters (real or artificial atoms) can be engineered by properly adjusting the distance between the emitters and a perfect mirror at one end of the system [28, 38, 39]. However, the interplay between absorption, stimulated and spontaneous emission provides a quantum emitter with mirror-like properties [29, 40, 41]. Hence, a pair of emitters can confine the field in the region between them, yielding nontrivial bound states above the threshold for photon propagation, that can be exploited for their robust entanglement features [42–47].

The objective of this article is to study the possible stable configurations of two multilevel atoms placed in a 1D cavity. See figure 1. In analogy with the 2-level case, each atom behaves both as an emitter and a mirror, confining the photon field and giving rise to a bound state endowed with highly nontrivial entanglement between the two atoms and between the atoms and the field. The discussed effects are non-perturbative and enable entanglement generation by relaxation. We will first adopt a method of resolution that applies to the case of two identical atoms that can be modeled as truncated harmonic oscillators. The technique consists in solving the problem for full-fledged harmonic oscillators, endowed with infinite number of levels. Due to the rotating-wave form of the Hamiltonian and the ensuing conservation law, the evolution in a given sector will involve only

Figure 1. A pair of distinguishable three-level atoms with equally spaced levels, placed at a distance d in a 1D cavity. The case of infinite number of levels corresponds to a pair of harmonic oscillators.

a finite number of atomic excitations. Finally, we will consider a perturbative method that enables to extend the analysis to the case of general level spacings and couplings, yielding the conditions for which the decay rate vanishes at second order in the atom-field coupling constant.

2. Harmonic-oscillator model

We consider a pair of emitters consisting of distinguishable harmonic oscillators A and B, with the same characteristic frequency ω_1, placed in a linear waveguide at a distance d. We assume that the photons coupled to the oscillators belong to a single nondegenerate transverse mode of the waveguide, with dispersion relation $\omega(k)$. Hence, at the zeroth order in the coupling constant, the oscillator frequency must be larger than the low-energy cutoff

$$M := \min_k \omega(k) \geqslant 0 \tag{1}$$

to enable propagation along the guide, and smaller than the other mode cutoffs to justify the assumption of coupling to a single mode. These conditions can be typically realized in a linear rectangular waveguide, where the dispersion relation of the lowest-energy mode reads $\omega(k) = (k^2 + M^2)^{1/2}$, with M inversely proportional to the longer side of the guide cross-section. However, we will keep the discussion as general as possible. In the dipolar and rotating-wave approximations, the Hamiltonian reads

$$H = \omega_1(b_A^\dagger b_A + b_B^\dagger b_B) + \int \mathrm{d}k \, \omega(k) b^\dagger(k) b(k) + \int \mathrm{d}k \, g(k)[(b_A^\dagger + b_B^\dagger \mathrm{e}^{\mathrm{i}kd}) b(k) + \mathrm{H.c.}], \tag{2}$$

where $b(k)$ and $b^\dagger(k)$ are the photon field operators in longitudinal momentum space, satisfying the canonical commutation relation $[b(k), b^\dagger(k')] = \delta(k - k')$, while b_J and $b_J^\dagger$ ($J = A, B$) are the canonical harmonic oscillator operators, satisfying $[b_J, b_K^\dagger] = \delta_{JK}$. The real coupling function $g(k)$ naturally decouples at high frequencies, such that

$$\int \frac{g^2(k)}{1 + \omega(k)} \mathrm{d}k < +\infty \tag{3}$$

(see e.g. [47] for photon waveguides).

The excited (number) states are created by acting on the vacuum $|0\rangle = |0_A, 0_B\rangle \otimes |\mathrm{vac}\rangle$ with the creation operators $b_A^\dagger$, $b_B^\dagger$, and $b^\dagger(k)$. The Hamiltonian (2) commutes with the total number of excitations

$$\mathcal{N} = \mathcal{N}_{\mathrm{at}} + \mathcal{N}_{\mathrm{field}} = b_A^\dagger b_A + b_B^\dagger b_B + \int \mathrm{d}k \, b^\dagger(k) b(k) \tag{4}$$

and does not mix different sectors, belonging to different values of $\mathcal{N}$. Due to this conservation law and the robustness of our approximations [47], our analysis applies equally well to a pair of harmonic oscillators and to a pair of $(N + 1)$-level atoms (with equally spaced levels) in a waveguide. See figure 1.

3. Bound states

The $\mathcal{N} = 0$ sector contains only the vacuum $|0\rangle$. The existence of nontrivial atom-photon bound states in the $\mathcal{N} = 1$ sector, occurring for discrete values of the interatomic distance and having no counterpart in more than one dimension, was proven in the simpler case of two-level atoms [47]. We shall prove that the presence of these bound states is fundamental in determining the properties of highly-excited sectors.

Since the Hamiltonian is quadratic in the field operators, it can be diagonalized by a proper linear combination of the bosonic field operators. In particular, consider the generic combination

$$b_\phi = \phi_A b_A + \phi_B b_B + \int dk\, \phi(k) b(k), \tag{5}$$

with

$$|\phi_A|^2 + |\phi_B|^2 + \int dk\, |\phi(k)|^2 = 1. \tag{6}$$

b_ϕ is a bosonic annihilation operator and satisfies the canonical commutation relation $[b_\phi, b_\phi^\dagger] = 1$. Suppose it satisfies the equation

$$[b_\phi, H] = E b_\phi, \tag{7}$$

for some real E. Then it is immediate to see that $b_\phi^\dagger$ creates *bound* eigenstates of H from the vacuum. Moreover, applying it N times to the vacuum will create eigenstates of H, belonging to the sector $\mathcal{N} = N$, with eigenvalues NE.

By (5), equation (7) is equivalent to

$$E\phi_A = \omega_1 \phi_A + \int dk\, g(k) \phi(k), \tag{8}$$

$$E\phi_B = \omega_1 \phi_B + \int dk\, g(k) \phi(k) e^{ikd}, \tag{9}$$

$$\phi(k) = g(k) \frac{\phi_A + \phi_B e^{-ikd}}{E - \omega(k)}. \tag{10}$$

If the system admits a solution with $E < M \leqslant \omega(k)$, then the amplitude $\phi(k)$ is square integrable and the state is also normalizable, provided condition (3) holds. These solutions correspond to bound states below the threshold for photon propagation: therefore, they occur even in the case of a single excited emitter coupled to the waveguide.

The solutions above threshold ($E > M$) are more interesting and nontrivial, since in such conditions a single excited emitter would spontaneously decay by photon emission. In fact, this happens in most situations also for a pair of emitters. The relevant properties of these solutions can be determined by a number of non-perturbative and general arguments, that can be eventually specialized to specific dispersion relations and to the case of small coupling. Normalizability of $\phi(k)$ requires that the poles k_i in (10), solutions to $E = \omega(k)$, are compensated by zeros in the numerator: thus, $g(k)$ or $\phi_A + \phi_B e^{-ikd}$ must vanish at all poles. Let us exclude the former possibility, that does not depend on the atomic state, and focus on the latter: since

$$\phi_A + \phi_B e^{-ik_i d} = 0, \tag{11}$$

for all i, in order to obtain a nontrivial solution, the poles must be constrained by the conditions $(k_i - k_j)d = 2\pi n$, with n an integer [47, 48]. Plugging these results into equations (8)–(9) must yield a real solution E, which does not depend on the choice of the pole k_i. This sets a strong limitation to the possibility of bound states above threshold. A solution can in principle be found when both $\omega(k)$ and $g^2(k)$ are symmetric in k, and ω is increasing in $|k|$, in which case $k_1 = -k_2 = \bar{k}$ and the energy $E = \omega(\bar{k})$ of the bound state must satisfy the coupled equations

$$\omega(\bar{k}) = \omega_1 + \int dk\, g^2(k) \frac{1 - e^{i(\bar{k}-k)d}}{\omega(\bar{k}) - \omega(k)}, \tag{12}$$

$$\bar{k}d = n\pi, \quad \text{with } n \in \mathbb{N}. \tag{13}$$

The conditions on the existence of a bound state in this case imply $\phi_A = (-1)^{n+1}\phi_B$, so that the probability $p_{at} = 2|\phi_A|^2$ associated to the emitter component of b_ϕ reads
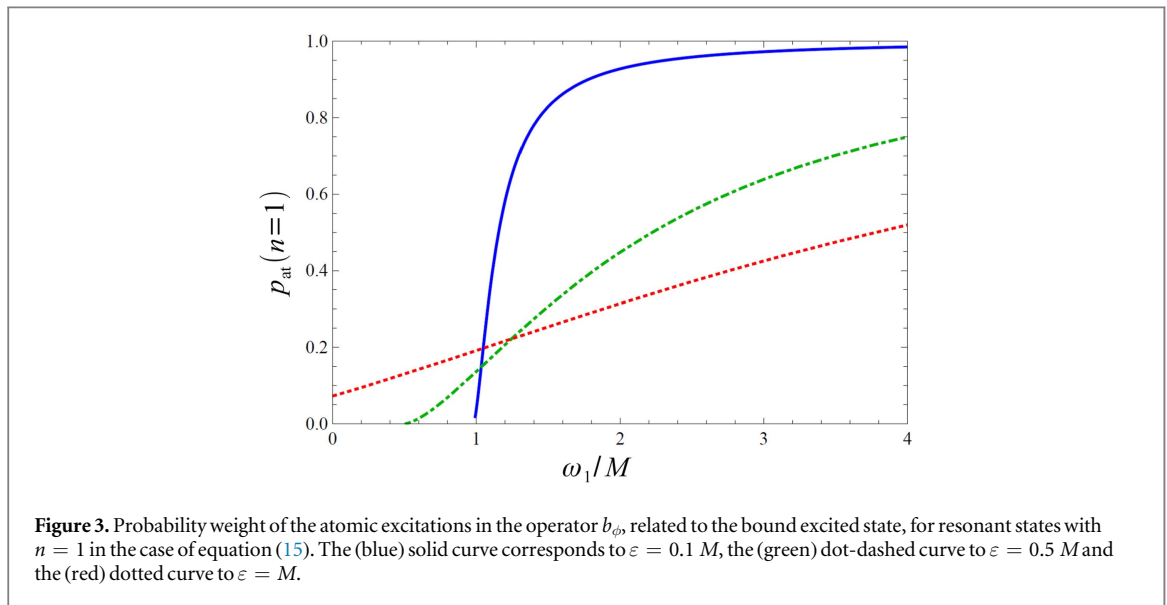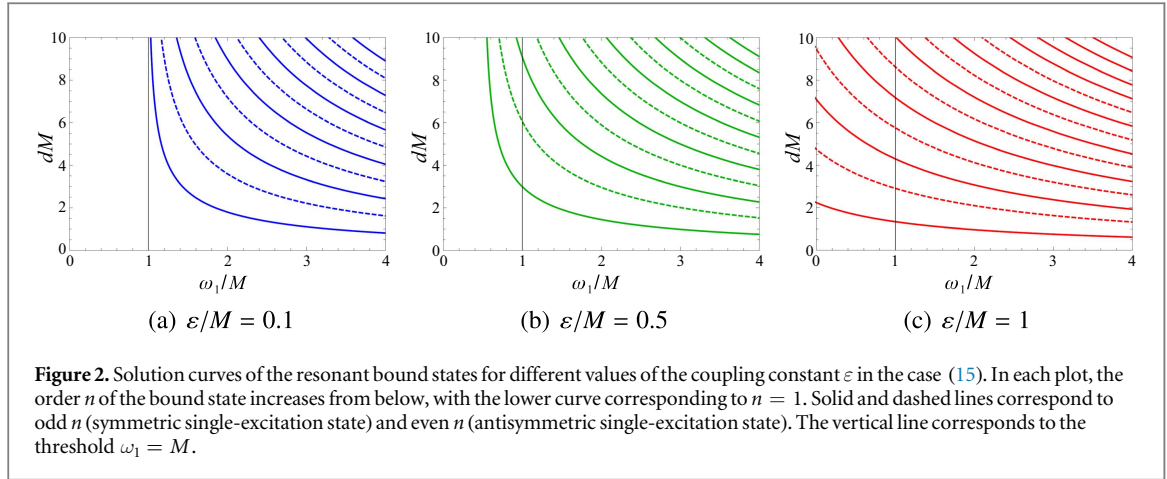
$$p_{at} = \left(1 + \int dk\, g^2(k) \frac{1 - \cos[(\bar{k}-k)d]}{[\omega(\bar{k}) - \omega(k)]^2}\right)^{-1}. \tag{14}$$

This quantity, which is an exact result as well as (12)–(13), differs from unity by an order $\lambda^2 \equiv \|g\|^2$ in the perturbative regime, but can become small for $\omega(\bar{k})$ close to the threshold M [47].

In order to clarify the effect of a finite threshold for propagation, let us consider the case of dipolar coupling to a massive photon (e.g. a guided mode), with the form factor derived from electrodynamics:

$$\omega(k) = \sqrt{k^2 + M^2}, \qquad g(k) = \frac{\varepsilon}{\sqrt{\omega(k)}}, \tag{15}$$

with ε being a constant with dimensions of energy. The relevant parameters are ω_1/M, ε/M and dM. In figure 2, the curves along which stable states exist are represented in the $(\omega_1/M, dM)$ plane for different values of the coupling. In the case $\varepsilon = 0.1\,M$, the curves are very well approximated by the positive solutions of $\omega(n\pi/d) = \omega_1$, and no resonant bound state corresponds to $\omega_1 < M$ in the range $dM \in (0, 10)$. Bound states above threshold ($E = \omega(\bar{k}) > M$) can generally exist even in the case $\omega_1 < M$, if the distance d is large enough

Figure 2. Solution curves of the resonant bound states for different values of the coupling constant ε in the case (15). In each plot, the order n of the bound state increases from below, with the lower curve corresponding to $n = 1$. Solid and dashed lines correspond to odd n (symmetric single-excitation state) and even n (antisymmetric single-excitation state). The vertical line corresponds to the threshold $\omega_1 = M$.

Figure 3. Probability weight of the atomic excitations in the operator b_ϕ, related to the bound excited state, for resonant states with $n = 1$ in the case of equation (15). The (blue) solid curve corresponds to $\varepsilon = 0.1\,M$, the (green) dot-dashed curve to $\varepsilon = 0.5\,M$ and the (red) dotted curve to $\varepsilon = M$.

to make $\omega(\bar{k})$ of the same order as the interaction integral in equation (12). Actually, in the plot in panel (b), corresponding to the intermediate value $\varepsilon = 0.5\,M$, resonant bound states with $\omega_1 < M$, corresponding to $n = 1, 2, 3$, can be observed for sufficiently large d. Finally, the case $\varepsilon = M$ shows the possibility that the curves extend to very small values of ω_1, at which the rotating-wave approximation is no longer valid, also crossing the line $\omega_1 = 0$.

Once the solution curves, that relate characteristic frequencies and distances for each n, have been determined, it is possible to compute the atomic probability p_{at} by equation (14). In figure 3, this quantity has been represented, as a function of ω_1/M, in the case $n = 1$ for three different coupling constants. It is worth observing that the value of p_{at} is close to one only in the perturbative regime, namely for a sufficiently small coupling constant and a large enough characteristic frequency, which requires smaller interatomic distances to form the bound state.

4. Entanglement

We can now study the entanglement properties of the bound states, that occur, at fixed characteristic frequency ω_1, whenever the distance d between A and B is such that the system (12)–(13) is satisfied for some n. The bound state can be expanded as

$$|N\rangle = \frac{(b_\phi^\dagger)^N}{\sqrt{N!}}|0\rangle = \sum_{m=0}^{N} \binom{N}{m}^{\frac{1}{2}} p_{at}^{\frac{m}{2}} (1 - p_{at})^{\frac{N-m}{2}} |\psi_{AB}^{(m)}\rangle \otimes |\phi^{(N-m)}\rangle, \tag{16}$$

where we have introduced the normalized states of the emitters

$$|\psi_{AB}^{(m)}\rangle := \frac{(b_{\text{res}}^\dagger)^m}{\sqrt{m!}}|0_A, 0_B\rangle = 2^{-\frac{m}{2}} \sum_{\ell=0}^{m} \binom{m}{\ell}^{\frac{1}{2}} (-1)^{\ell(n+1)} |\ell_A, (m-\ell)_B\rangle, \tag{17}$$

with

$$b_{\text{res}} = \frac{b_A + (-1)^{n+1} b_B}{\sqrt{2}}, \tag{18}$$

and of the photon field

$$|\phi^{(m)}\rangle := \frac{1}{\sqrt{m!}} \left[\sqrt{\frac{p_{\text{at}}}{2(1-p_{\text{at}})}} \int dk \, g(k) \frac{1 + (-1)^{n+1} e^{ikd}}{\omega(n\pi/d) - \omega(k)} b^\dagger(k) \right]^m |\text{vac}\rangle. \tag{19}$$

It is also convenient to introduce the antiresonant combination

$$b_{\text{anti}} = \frac{b_A + (-1)^n b_B}{\sqrt{2}}, \tag{20}$$

associated with a state that is orthogonal to both the states associated with b_{res} and b_ϕ. The state $|N\rangle$, characterized by a fixed number of total excitations, has components in the emitter excitation sectors ranging from $\mathcal{N}_{\text{at}} = 0$, correlated with the presence of N photons, to $\mathcal{N}_{\text{at}} = N$, with no photon in the guide. The state, defined by the above expressions, can be exactly determined for any value of the parameters, after numerical computation of the atomic probability as in (14) for a given resonance. An interesting and relevant case is represented by the perturbative regime, in which $p_{\text{at}} \simeq 1$ and the bound state is dominated by the contribution from $\mathcal{N}_{\text{at}} = N$, namely

$$|N\rangle = p_{\text{at}}^{\frac{N}{2}} |\psi_{AB}^{(N)}\rangle \otimes |\text{vac}\rangle + O(N\lambda^2). \tag{21}$$

In this regime, the bound state is thus characterized by a large projection on the state with maximal entanglement between atoms. Notice that the state $|\psi_{AB}^{(N-1)}\rangle$, that is dominant in $|N-1\rangle$, appears in the $O(N\lambda^2)$ term of equation (21).

Two comments are in order. First, the state belongs to the $\mathcal{N} = N$ sector and never leaves it, under the action of the Hamiltonian (2). Thus, although the preceding analysis has been done for two harmonic oscillators, it is still valid for two $(N+1)$-level systems in the waveguide. This point will be formalized in the final part of the article. Second, the reduced density matrix of the two emitters

$$\rho_{AB}^{(N)} = \text{tr}_{\text{field}} |N\rangle \langle N| = \sum_{m=0}^{N} \binom{N}{m} p_{\text{at}}^m (1 - p_{\text{at}})^{N-m} |\psi_{AB}^{(m)}\rangle \langle \psi_{AB}^{(m)}| \tag{22}$$

is a mixture of entangled states, dominated by the term $m = N$, whose entanglement increases with the number of excitations. Before quantitatively clarifying the last point, let us comment on the consequences and possible applications of entanglement in a bound state.

The interesting properties related to entanglement persistence stem from the decomposition of the Hamiltonian (2) into a 'stable' and a 'decaying' part, as

$$H = E b_\phi^\dagger b_\phi + H_c, \tag{23}$$

where H_c, that commutes with b_ϕ, accounts for the continuous spectrum and the spontaneous decay of excited states orthogonal to $|N\rangle$. Here we have assumed that there is only one resonant bound state in the spectrum, which is valid in the small coupling regime. In the one-excitation sector (namely two 2-level atoms), it has been demonstrated [47] that the presence of the bound state enhances the decay rate of the orthogonal states towards the configuration in which the emitters are both in their ground states and a photon propagates in the waveguide.

If the emitters are placed at infinite distance from one another, an initial state close to $|\psi_{AB}^{(N)}\rangle \otimes |\text{vac}\rangle$ with $N > 0$ would rapidly decay to an orthogonal state with a smaller number of emitter excitations. By contrast, in a waveguide, with d and ω_1 close to the resonance conditions, such a state will be left almost invariant by the Hamiltonian evolution, with a dressing (slight, in the perturbative regime) due to the imperfect superposition of the initial and the bound state (i.e. $p_{\text{at}} < 1$). Therefore, on timescales smaller than the waveguide losses, entanglement is preserved *without* imposing constraints or external control.

Another interesting application is related to the decay of the unstable component of an *arbitrary* initial state. An initial state $\rho_0 = \rho_{\text{in}} \otimes |\text{vac}\rangle \langle \text{vac}|$, with ρ_{in} in the $\mathcal{N}_{\text{at}} = N$ sector, will generally evolve into $\rho(t) = e^{-iHt} \rho_0 e^{iHt}$. After a transient, the N-excitation states will relax towards the atomic ground state by emission of N photons if no bound state is present in the sector. If, instead, one of the resonance conditions is fulfilled, the state $|\psi_{AB}^{(N)}\rangle \otimes |\text{vac}\rangle$, which has a nonvanishing projection on $|N\rangle$, will relax towards an excited atomic state. The probability of having maximally entangled atoms is given by

$$p_{\text{in}} = \lim_{t \to \infty} \text{Tr}[\rho(t)(\mathbb{I}_{\text{at}} \otimes |\text{vac}\rangle \langle \text{vac}|)] = p_{\text{at}}^{2N} \langle \psi_{AB}^{(N)} | \rho_{\text{in}} | \psi_{AB}^{(N)} \rangle. \tag{24}$$

Therefore, the atoms, starting from a general density matrix, which can be e.g. factorized, eventually reach a highly entangled state with N excitation with finite probability. The typical relaxation time of the reduced atomic density matrix towards its asymptotic state is determined by the lifetime of the component $b_{\text{anti}}^{\dagger} | \psi_{AB}^{(N-1)} \rangle$, and coincides with the lifetime of a single atomic excitation created in the antiresonant state [47]. The described entanglement generation strategy is similar to the Hamiltonian generation of entanglement, in which an initial (factorized) state is let to coherently evolve until it reaches an entangled state. Such a procedure can be applied in the case of the bound states well below the threshold for photon propagation. However, the Hamiltonian nature of the evolution in the AB Hilbert space yields oscillations, which implies that the evolution must be stopped at a proper time to obtain the desired state. This drawback is absent in entanglement generation by relaxation, in which the final state is approached asymptotically. This technique does not require energy pumping into the system, since a constant entanglement is reached after an initial transient [47].

Let us go back to the discussion of the entanglement properties of $|\psi_{AB}^{(N)}\rangle$. The emitter state in the sector $\mathcal{N}_{\text{at}} = 1$ (two 2-level atoms) has been extensively studied, and is particularly interesting since it corresponds to one of the two Bell states, according to the sign of $(-1)^n$ (see equations (13)–(17)). It is thus maximally entangled in the $\mathcal{N}_{\text{at}} = 1$ subspace. The bound state $|N = 2\rangle$, relative to the pair of 3-level atoms in figure 1, reads

$$\begin{aligned}
|2\rangle = &\frac{p_{\text{at}}}{\sqrt{6}}(|0_A, 2_B\rangle - 2|1_A, 1_B\rangle + |2_A, 0_B\rangle) \otimes |\text{vac}\rangle \\
&+ \sqrt{2p_{\text{at}}(1 - p_{\text{at}})}(|0_A, 1_B\rangle - |1_A, 0_B\rangle) \otimes |\phi^{(1)}\rangle \\
&+ (1 - p_{\text{at}})|0_A, 0_B\rangle \otimes |\phi^{(2)}\rangle.
\end{aligned} \tag{25}$$

By projecting onto a suitable photonic state, one can select the desired (long-lived) atomic entangled state $|\psi_{AB}^{(N)}\rangle$, for $N = 1$ and 2. To extend the analysis of entanglement to large-N states, we can use the fact that the reduced one-emitter density matrices obtained from $|\psi_{AB}^{(m)}\rangle$ have a particularly simple binomial form, that leads to

$$\rho_A^{(N)} = \text{tr}_B \rho_{AB}^{(N)} = \sum_{\ell=0}^{N} C_\ell^{(N)}(p_{\text{at}}) |\ell_A\rangle \langle \ell_A|, \tag{26}$$

where the coefficients

$$C_\ell^{(N)}(p_{\text{at}}) := \sum_{m=0}^{N} \frac{1}{2^m} \binom{N}{m} \binom{m}{\ell} p_{\text{at}}^m (1 - p_{\text{at}})^{N-m} \tag{27}$$

are dominated by the terms $m = N$ at small coupling. These states appear in the analysis of coherently illuminated beam splitters, when the input states are very imbalanced [49], and their interesting entanglement features generalize those of NOON states. Bipartite entanglement can be quantified through any measure based on the eigenvalues of the reduced state $\rho_A^{(N)}$. Let us adopt purity of A:

$$\pi_A^{(N)} = \sum_{\ell=0}^{N} (C_\ell^{(N)}(p_{\text{at}}))^2 = \frac{\Gamma\left(N + \frac{1}{2}\right)}{\sqrt{\pi} N!}(1 + O[(N\lambda^2)^2]) \sim \left[\frac{1}{\sqrt{\pi N}} + O(N^{-3/2})\right](1 + O[(N\lambda^2)^2]), \tag{28}$$

as $N \to \infty$, where Γ is the Euler gamma function. Strictly speaking, this quantity measures entanglement between A and its environment (B + field). However, since the state of the field is quasi factorized at small coupling, it is also an approximate measure of entanglement between the two emitters A and B. On one hand, purity (28) scales more slowly than the minimal value $(N + 1)^{-1}$ in the sector, corresponding to maximally mixed reduced density matrices. On the other hand, this result is consistent with the minimal purity for states whose reduced density matrices are effectively approximated by the superposition of $O(N^{1/2})$ states.

It is also possible to determine the entanglement properties of coherent and incoherent superpositions of the bound states $|N\rangle$. For example, one can consider the 'pseudothermal' state

$$\rho_{AB}^{\text{th}} = (1 - e^{-\beta E}) e^{-\beta E b_\phi^{\dagger} b_\phi}, \tag{29}$$

whose reduced density matrix is the thermal average of (26), yielding the purity

$$\pi_A^{\text{th}} = \sum_{\ell=0}^{\infty} \left(\sum_{N=\ell}^{\infty} (1 - e^{-\beta E}) e^{-\beta E N} C_\ell^{(N)}(p_{\text{at}})\right)^2 = (1 - e^{-\beta E}) + O(N_{\text{th}}\lambda^2), \tag{30}$$

with Nth the average excitation number. Another interesting case is the coherent state $|\alpha\rangle = e^{\alpha b_\phi^{\dagger} - \alpha^* b_\phi}|0\rangle$. Since it is a product of coherent states, $|\alpha\rangle = e^{\alpha \phi_A^* b_A^{\dagger} - \alpha^* \phi_A b_A} e^{\alpha \phi_B^* b_B^{\dagger} - \alpha^* \phi_B b_B} e^{\int dk \, [\alpha \phi^*(k) b^{\dagger}(k) - \alpha^* \phi(k) b(k)]}|0\rangle$, there is no entanglement between the two atoms. In particular, in the small-coupling limit, the atom density matrix ρ_{AB}^{α} is dominated by the projection on

$$|\tilde{\alpha}\rangle = \mathrm{e}^{-|\alpha|^2} \sum_{\ell=0}^{\infty} \frac{(-\alpha)^\ell}{\sqrt{2^\ell \ell!}} \sum_{m=0}^{\infty} \frac{(-\alpha)^m}{\sqrt{2^m m!}} |\ell_A, m_B\rangle, \tag{31}$$

which yields a factorized reduced density matrix.

Let us finally comment on the truncation of the oscillator Hilbert spaces. The bound state $|N\rangle$ contains up to the Nth power of the oscillator creation operators $b_{A,B}^\dagger$. Therefore, truncating both oscillators at the $(N+1)$th level, namely formally adding the constraint $(b_{A,B}^\dagger)^{N+1} = 0$, has no effect on the existence and properties of state $|N\rangle$. On the other hand, the bosonic commutation relations satisfied by *all* operators, which is reflected in the bosonic character of the b_ϕ operator, are fundamental in the derivation of the commutator equation (7), and in the subsequent reduction of the problem to the determination of ϕ_A, ϕ_B, $\phi(k)$, and E. If the oscillators are truncated at $\bar{N}$, then the existence of a bound state in the sector $\mathcal{N} = N > \bar{N}$ does not follow automatically from the presence of a bound state in the lowest-excited $\mathcal{N} = 1$ sector. Summarizing, in bound states, the excited levels of each oscillator must be in a sufficient number to absorb all the photons in the state. And, of course, considerations on the pseudothermal and coherent states are valid, only approximately, if the average occupation number is not close to $\bar{N}$.

5. Stable states in the general case

We now relax the assumptions of equally-spaced levels and oscillator-like interaction. Unlike in the previous sections, where the analysis could be perfomed non-perturbatively, we will work at second order in the atom-field coupling.

Let us consider a Hamiltonian describing two identical $(N+1)$-level atoms A and B with the same spectra $\vec{\omega} = (\omega_0, \omega_1, \ldots, \omega_N)$ (with $\omega_0 = 0$), placed at a distance d from each other and coupled with the guided field. We assume that the interaction with the field can induce transitions between neighboring levels $|j_I\rangle$ ($j = 0, 1, \ldots,$ N with $I = A, B$), with an arbitrary couplings:

$$H = H_0 + H_{\mathrm{int}} = \sum_{j=1}^{N} \omega_j |j_A\rangle\langle j_A| + \int \mathrm{d}k\, \omega(k) b^\dagger(k) b(k)$$
$$+ \sum_{j=1}^{N} \int \mathrm{d}k\, g_j(k)[(|j_A\rangle\langle(j-1)_A| + \mathrm{e}^{ikd}|j_B\rangle\langle(j-1)_B|)b(k) + \mathrm{H.c.}]. \tag{32}$$

A state

$$|\psi_N^{(0)}\rangle = \sum_{j=0}^{N} c_j |(N-j)_A; j_B\rangle \tag{33}$$

with $\mathcal{N}_{\mathrm{at}} = N$ and bare energy

$$E_N^{(0)}(\vec{c}) = \langle\psi_N^{(0)}|H_0|\psi_N^{(0)}\rangle = \sum_{j=0}^{N} |c_j|^2 (\omega_{N-j} + \omega_j), \tag{34}$$

where $\vec{c} = (c_0, c_1, \ldots, c_N)$, generally decays into N photons. We have shown in the previous sections that, in the harmonic-oscillator-like case ($\omega_j = j\omega_1$ and $g_j(k) = \sqrt{j}\, g_1(k)$) an exact eigenstate with nonvanishing amplitude in the $\mathcal{N} = N$ sector can exist. In all other cases, one can nonetheless identify the most stable combinations (33) by analyzing the decay rate at second order in the atom-photon coupling [50]. This quantity can be immediately computed from the action of the interaction Hamiltonian on the general state, yielding

$$\Gamma_N^{(0)}(\vec{c}) = 2\pi \int \mathrm{d}k \sum_{j=0}^{N-1} |c_j g_{N-j}(k) + c_{j+1} g_{j+1}(k) \mathrm{e}^{-ikd}|^2 \delta(\omega(k) + \omega_{N-j-1} + \omega_j - E_N^{(0)}(\vec{c})). \tag{35}$$

The integral can be computed by determining the set of momenta $\{\bar{k}_\ell\}_{\ell=1,\ldots,M}$ at which the argument of the delta function vanishes. Such a set depends on $\vec{c}$ if the levels are not evenly spaced. The resulting $\Gamma_N^{(0)}(\vec{c})$ can be eventually minimized with respect to the coefficients, to find the most stable state in the N-excitation sector.

Let us look explicitly to the case $N = 2$, where $\vec{c} = (c_0, c_1, c_2)$. The on-shell condition reads

$$\omega(\bar{k}) = \omega_2 - \omega_1 - (\omega_2 - 2\omega_1)|c_1|^2. \tag{36}$$

Let us assume that dispersion relation and form factors are symmetric, and that the argument of the delta function in equation (35) has only two symmetric solutions $\pm\bar{k}(|c_1|^2)$. Then, the decay rate reads

$$\Gamma_3^{(0)}(\vec{c}) = \frac{2\pi}{\nu(|c_1|^2)} \sum_{s=\pm} (|c_0 g_2(\bar{k}(|c_1|^2)) + c_1 g_1(\bar{k}(|c_1|^2)) e^{-is\,\bar{k}(|c_1|^2)d}|^2$$
$$+ |c_2 g_2(\bar{k}(|c_1|^2)) + c_1 g_1(\bar{k}(|c_1|^2)) e^{is\,\bar{k}(|c_1|^2)d}|^2) \tag{37}$$

with $\nu(|c_1|^2) = \mathrm{d}\omega/\mathrm{d}k|_{k=\bar{k}}$. The stationarity conditions with respect to c_0 and c_2 require that the minimizing sequence of coefficients $\vec{c}_*$ satisfy

$$c_{0*} = c_{2*} = -c_{1*} \frac{g_1(\bar{k}(|c_{1*}|^2))}{g_2(\bar{k}(|c_{1*}|^2))} \cos(\bar{k}(|c_{1*}|^2)d). \tag{38}$$

Choosing real phases for c_{0*} and combining equation (38) with the normalization condition, we obtain an equation for c_{1*}, that, in the case of equal spacings, in which $\bar{k}$ does not depend on the state, leads to the solution

$$c_{1*} = -\mathrm{sgn}(\mathcal{G}(\bar{k}))(1 + 2\mathcal{G}^2(\bar{k}))^{-\frac{1}{2}}, \quad \text{with } \mathcal{G}(k) = \frac{g_1(k)}{g_2(k)} \cos(kd). \tag{39}$$

Let us consider two resonance cases, with $\cos(\bar{k}d) = \pm 1$. If $g_1/g_2 = 1/\sqrt{2}$, one recovers the truncated harmonic oscillators analyzed in sections 3–4, with $c_{1*} = \mp 1/\sqrt{2}$. In the interesting case of equal couplings $g_1/g_2 = 1$, stationarity and normalization lead to the solution $c_{1*} = \mp 1/\sqrt{3}$, yielding a uniform minimizer. In both cases, the decay rate is also vanishing at the minimum.

We finally comment on the robustness with respect to small variations of the level spacings. The minimal decay rate can be written as a function of the bare energy levels as $\bar{\Gamma}_N(\vec{\omega}) = \Gamma_N^{(0)}(\vec{c}_*)$, with $\vec{c}_*$ the minimizing sequence of coefficients. Generally, the minimal rate varies linearly with small displacements of the energy levels. However, a relevant exception is represented by the cases, like those mentioned before for $N = 2$, in which $\bar{\Gamma}_N(\vec{\omega}) = 0$. Since $\bar{\Gamma}_N$ is nonnegative, if it is also a smooth function of $\vec{\omega}$, then its gradient with respect to $\vec{\omega}$ must vanish as well. In such cases, the decay rate increases quadratically in the variation of the bare energy levels.

6. Conclusion

We have investigated the existence and properties of the stable states of a pair of N-level atoms in a waveguide, discussing the possibility to generate and preserve robust entanglement between the two atoms. As a case study, we have considered identical distinguishable emitters with uniform level spacing. The generalization to the presence of (small or large) asymmetries in the excitation energy and in the coupling, or uneven levels, has been considered in section 5. Among possible interesting applications, we mention the use of a single two-level emitter as a dynamical probe of the state of the field, along the guidelines discussed in [28]. These ideas can be generalized, by using (entangled) atomic *multilevel pairs*, paving the way to unprecedented possibilities and possibly super-resolution.

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