Positive impact of decoherence on spin squeezing in GHZ and W states

To cite this article: Kapil K Sharma and Swaroop Ganguly 2018 J. Phys. Commun. 2 015012

View the article online for updates and enhancements.
Positive impact of decoherence on spin squeezing in GHZ and W states

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Keywords: spin squeezing, spin squeezing sudden death, decoherence, quantum information

Abstract

We study spin squeezing behaviour in tripartite unsqueezed maximally entangled Greenberger–Horne–Zeilinger (GHZ) and W states under various decoherence channels with Kitagawa–Ueda criteria. In order to search spin squeezing sudden death (SSSD) and perform the analysis of spin squeezing production we use bit flip, phase flip, bit-phase-flip, amplitude damping, phase damping and depolarization channels in the present study. In literature, the influence of decoherence has been studied usually as a destroying element. On the contrary here we investigate the positive impact of decoherence, which produces spin squeezing in unsqueezed GHZ and W states under certain channels. Our careful study shows that GHZ state remains unsqueezed under aforementioned channels except bit-phase-flip and depolarization channels, on the other hand all the decoherence channels produce spin squeezing in W state. So we find that, GHZ is more robust in comparison to W state in the sense of spin squeezing production under decoherence. Most importantly we find, none of the decoherence channels produce SSSD in either of the states.

1. Introduction

The phenomenon of squeezing exist in light from a long time in literature [1, 2]. Further the concept has been investigated in the context of spin systems by Kitagawa–Ueda (KU) in 1993, so called spin squeezing [3–5]. Spin squeezing attracted much attention of quantum information community as it has its lucid connection with entanglement [6]. Many features of spin squeezing have been investigated to detect the multipartite and pairwise entanglement [7, 8], even though negative pairwise entanglement [9] in quantum systems. A class of spin squeezing inequalities [10–13] also established for the same purpose. Enhancement of sensitivity and precision in quantum metrology [14, 15] is an important aspect which has been achieved by using many techniques, spin squeezing is one of these. Squeezing of spins play an important role for quantum metrology and magnetometry to improve the sensitivity and precision because of its ability to reduce the quantum noise [16–20]. Early experimental manifestations of spin squeezing is demonstrated in varieties of physical systems such as, in entangled ion trap systems [21], in Bose–Einstein condensate (BEC) through repulsive interactions [22], measurements by partial projection [23], squeezing of huge ensemble of atoms through light–matter interaction to produce atomic clocks [24–30], gravitational interferometers [31–33] and in many qubit systems [34–36]. There are many types of squeezing like dipolar, phase and planar squeezing including spin squeezing with various definitions [37] and mathematical criteria. The definition depends on the type of the system and each has its own significance. Here we deal with spin squeezing, which further demand any mathematical criteria to know its degree. There are many criteria present for spin squeezing [38], KU is one of the highly studied criteria [3] which also have its experimental manifestations. KU criteria has major drawback as it is mostly suitable for symmetric states and fail to check the spin squeezing under local unitary operations on the symmetric and non-symmetric states. However generalised version of KU criteria is developed by Usha Devi et al for the states which are invariant under unitary transformations for non-symmetric as well as for symmetric
states \cite{39}. A well known example of symmetric states which are invariant under particle exchange is coherent spin state (CSS) \cite{40, 41}. Primarily spin squeezing with KU criteria has been studied in CSS with one and two axis-twisting Hamiltonians \cite{42}, which are nonlinear Hamiltonians and used to produce spin squeezing in separable CSS. The best squeezing scales achieved with N number of particles by these Hamiltonians are $\epsilon_{\text{2DTH}} \propto 1/N^{2/3}$ and $\epsilon_{\text{2TH}} \propto 1/N^2$ respectively without decoherence affects. The one axis twisting Hamiltonian under decoherence assumed in the form of particle loss, has been studied in two mode BEC and author have shown the spin squeezing production in the state \cite{7}. They have been observed the scaling of spin squeezing with one particle loss is $\epsilon^2 \propto N^{-4/15}$, for two particle loss the scaling is independent of N and for three particle loss it is $\epsilon^2 \propto N^{-4/15}$. Further the spin squeezing production with two axis twisting model with decoherence in the form of particle loss has been studied by A Andre et al with Raman scattering based approach \cite{43}. Early studies of decoherence in spin squeezing with CSS have been investigated under amplitude damping, pure dephasing and depolarizing channels by Wang et al \cite{44}. This study have shown that CSS goes under spin squeezing sudden death (SSSD) under decoherence except amplitude damping channel. The phenomenon of SSSD is inspired by entanglement sudden death which is widely studied in many physical settings \cite{45–53}. So investigations of spin squeezing behaviour under decoherence in CSS motivate us to carry out the study in another symmetric states. Here in the present study we consider tripartite maximally entangled Greenberger–Horne–Zeilinger (GHZ) and W states \cite{54, 55} which are symmetric under particle exchange. The aim of the present study is two fold, we be diligent to search the phenomenon SSSD and some interesting results of spin squeezing production in these states under decoherence.

The structure of the paper is as follows, in section 2, we give the brief formulation of spin squeezing, which is used to reckon the behaviour of spin squeezing throughout the paper. In section 3 and its subsequent sections from 3.1 to 3.6, we proceed the study on spin squeezing in tripartite GHZ and W states under decoherence channels like bit-flip, phase flip, bit-phase flip, amplitude damping, phase damping and depolarization. In section 4, we provide the conclusion of the paper. In the last, we have added an appendix for detailed calculations of variance $(\Delta J_z)^2$.

2. Spin squeezing formulation

In this section we cover basic definition of spin squeezing and its brief mathematical formulation bases on KU criteria. In the treatment of spin squeezing formulation covered in literature \cite{3, 38, 42}, here we apply efforts for simplified and straight forward calculations of spin squeezing expression along with the calculations of variance $(\Delta J_z)^2$ added in appendix. In many body physics the collective behaviour of observables play an important role, because the individual particles can not be addressed like in BEC \cite{56}, in such systems spin squeezing play an important role to reduce the noise. Apart from it, spin squeezing also have strong connection with multipartite entanglement. To proceed the formulation of spin squeezing in N body spin-$\frac{1}{2}$ systems, we begin with the collective angular momentum $\vec{J}$. The collective angular momentum $\vec{J}$ can be represented in three dimensional Cartesian coordinate system as below,

$$\vec{J}_{\text{tot}} = (J_x, J_y, J_z),$$

(1)

where $J_x$, $J_y$ and $J_z$ are angular momentum components along $x$, $y$ and $z$ respectively in Euclidean space. Further we transform the axis of Euclidean space i.e. $(x, y, z)$ to spherical coordinate system using the orthonormal vectors ($\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$). This transformation is needed to remove the dipolar squeezing and to generate the spin squeezing in the system. The unit orthonormal vectors used for transformation are given below.

$$\vec{n}_1 = (-\sin \phi, \cos \phi, 0)$$

(2)

$$\vec{n}_2 = (-\cos \theta \cos \phi, -\cos \theta \sin \phi, \sin \theta)$$

(3)

$$\vec{n}_3 = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

(4)

The collective angular momentum vector $\vec{J}_{\text{tot}}$ in old coordinate system can be transformed in new coordinate system by using the rotation matrix as,

$$\begin{bmatrix}
J_{\text{tot}} \\
J_{\text{tot}} \\
J_{\text{tot}}
\end{bmatrix} =
\begin{bmatrix}
-\sin \phi & \cos \phi & 0 \\
-\cos \theta \cos \phi & -\cos \theta \sin \phi & \sin \theta \\
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta
\end{bmatrix}
\begin{bmatrix}
J_x \\
J_y \\
J_z
\end{bmatrix}.
$$

(5)

In L.H.S. of the above equation, $(J_{nx}, J_{ny}, J_{nz})$ are the components of the collective angular momentum $\vec{J}$ in new spherical coordinate system along the bases ($\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$). Further simplification of equation (5) leads as,

$$J_{nx} = -J_x \sin \phi + J_y \cos \phi$$

(6)

$$J_{ny} = -J_x \cos \theta \cos \phi - J_y \cos \theta \sin \phi + J_z \sin \theta$$

(7)
The above equations (6)–(8) give the projection of the vector \( \vec{J}_{xy} \) along the unit vectors \( \hat{n}_1, \hat{n}_2, \hat{n}_3 \). Knowing the components \( J_x, J_y, J_z \) and the set of angles \( (\theta, \phi) \) gives the clue to obtain the components \( (J_n, J_{n+1}, J_{n+2}) \) of the vector \( \vec{J}_n \). In an N spin-1/2 system, the angular momentum vectors of individual spins may be in different directions, so we need to calculate the mean values of the angular momentum components in \( x, y \) and \( z \) directions, these components form a mean spin vector \( \vec{J}_{\text{mean}} \) as obtained below,

\[
\vec{J}_{\text{mean}} = \langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle.
\]  

The mean spin vector depends on the state of the system. Here it is important to mention that, we rotate the Cartesian coordinate system with the angles \( (\theta, \phi) \) such that the mean spin vector \( \vec{J}_{\text{mean}} \) align along the vector \( \vec{n}_3 \) in new spherical coordinate system, this makes calculations simple and assist the removal of dipolar spin squeezing also. Further, the mean spin vector in new spherical coordinate system can be expressed as

\[
\vec{J}_{\text{mean}} = (J_n, J_{n+1}, J_{n+2})
\]

with,

\[
\langle J_n \rangle = -\langle J_z \rangle \sin \phi + \langle J_y \rangle \cos \phi
\]

\[
\langle J_{n+1} \rangle = \langle -J_x \rangle \cos \theta \cos \phi - \langle J_y \rangle \cos \theta \sin \phi + \langle J_z \rangle \sin \theta
\]

\[
\langle J_{n+2} \rangle = \langle J_x \rangle \sin \theta \cos \phi + \langle J_y \rangle \sin \theta \sin \phi + \langle J_z \rangle \cos \theta.
\]

Equations (11)–(13) play an important role in calculating the spin squeezing, which basically reveal the connections of mean components of the vector \( \vec{J}_{\text{mean}} \) in spherical coordinate system to the mean components of the vector \( \vec{J}_{\text{mean}} \) in Cartesian coordinate system. To go towards the mathematical definition of spin squeezing first we give the light on the arrangement of the vectors \( \vec{n}_1, \vec{n}_2, \vec{n}_3 \) which are orthonormal. Let assume the orthonormal vectors \( \vec{n}_1 \) and \( \vec{n}_2 \) are in the same plane denoted as \( n_{12} \), obviously this plane is perpendicular to the vector \( \vec{n}_3 \) i.e. \( (\perp n_{12}) \). We can choose any arbitrary vector lying in the plane \( n_{12} \) and making an arbitrary angle \( \varphi \) with the vector \( \vec{n}_1 \). This vector is called \( \vec{n}_\varphi \), which is automatically perpendicular to the vector \( \vec{n}_3 \). The arrangement of the vectors \( \vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_\varphi, \vec{J}_{\text{mean}} \) and \( \vec{O}\vec{B} \) is shown in the figure 1. \( \vec{O}\vec{B} \) is the projection vector of \( \vec{J}_{\text{mean}} \) in \( xy \) plane. As we know the vectors \( \vec{n}_1 \) and \( \vec{n}_2 \) are mutually perpendicular unit vectors which are staying in two dimensional plane i.e. \( (\perp n_{12}) \) and can be written as \( \vec{n}_1 = (1, 0)^T \) and \( \vec{n}_2 = (0, 1)^T \). Here we mention that rotation of the vector \( \vec{n}_1 \) with an angle \( \varphi \) produce the vector \( \vec{n}_\varphi \). So \( \vec{n}_\varphi \) can be obtained as,

\[
\vec{n}_\varphi = S\vec{n}_1,
\]

where \( S \) is the orthogonal unitary rotation matrix expressed as,

\[
S = \begin{bmatrix}
\cos \varphi & -\sin \varphi \\
\sin \varphi & \cos \varphi
\end{bmatrix}.
\]

Simplification of equation (14) produce the vector \( \vec{n}_\varphi \), which is obtained below,

\[
\vec{n}_\varphi = \vec{n}_1 \cos \varphi + \vec{n}_2 \sin \varphi.
\]
The vector \( \mathbf{n}_n \) lies in the plane \( n_{12} \) in spherical coordinate system and play an important role for the definition of spin squeezing. The definition of spin squeezing is given as \( \text{The minimum value of the variance of the total angular momentum } J_n \text{ along the vector } \mathbf{n}_n \text{ is less than or equal to the stranded quantum limit} \) i.e.

\[
[(\Delta J_n)^2]_{\text{min}} \leq J^2 / 2,
\]

where \( J \) is the spin quantum number and given as \( J = \frac{n}{2} \). Rearranging the equation (17) we get

\[
\epsilon = \frac{4[(\Delta J_n)^2]_{\text{min}}}{N} \leq 1,
\]

where \( N \) is the number of spins in the system and \( \epsilon \) is the spin squeezing parameter. If \( \epsilon = 1 \), the state is unsqueezed and there is no quantum correlation present in the state. For pure CSS uncorrelated state \( \epsilon = 1 \). If there are certain type of quantum correlations present in the state than \( \epsilon < 1 \). We can easily obtain the projection of the vector \( \mathbf{J}_n \) along the vector \( \mathbf{n}_n \), which produce the vector \( \mathbf{J}_n' \). This vector is obtained as,

\[
\mathbf{J}_n' = \mathbf{n}_n \mathbf{n}_n = J_n \cos \varphi + J_n \sin \varphi.
\]

The variance of the vector \( \mathbf{J}_n' \), and its optimization is obtained in the appendix, which is expressed as,

\[
(\Delta J_n')^2_{\text{min}} = \frac{1}{2} \left[ O - \sqrt{M^2 + N^2} \right],
\]

where the coefficients \( M, N \) and \( O \) are given below,

\[
M = (J_n^2 - J_n'^2)
\]

\[
N = \langle J_n J_n' + J_n J_n' \rangle
\]

\[
O = \langle J_n'^2 + J_n'^2 \rangle.
\]

Putting the value from the equation (20) in (18) we get the spin squeezing as,

\[
\epsilon = \frac{2}{N} \left( O - \sqrt{M^2 + N^2} \right) \leq 1.
\]

This is called KU criteria for spin squeezing\([3–5]\). The condition \( \epsilon < 1 \) represents the state is spin squeezed and with \( \epsilon = 1 \) the state is unsqueezed. The criteria also give the indications about the separability and entanglement of a state\([3]\). The parameter \( \epsilon < 1 \) exhibits that the state is entangled and with \( \epsilon = 1 \), the state is separable. But this is true only for symmetric states, the criteria fails for non symmetric states and under local operations, to overcome this difficulty a generalised KU criteria is developed in \([39]\).

### 3. Spin squeezing under decoherence in GHZ and W states

In this section we study the spin squeezing behaviour in tripartite maximally entangled GHZ and W states\([34, 35]\) under decoherence. We here consider that individual qubits face the decoherence, this situation is modelled by Kraus operators formalism\([57]\). To start with, the tripartite maximally entangled GHZ and W states can be written as,

\[
|\psi\rangle_{\text{GHZ}} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).
\]

\[
|\psi\rangle_{\text{W}} = \frac{1}{\sqrt{2}}(|100\rangle + |010\rangle + |001\rangle).
\]

Both the states are symmetric under particle exchange. The corresponding initial density matrices of these states can be obtained as \( \rho_{\text{GHZ}}(0) \) and \( \rho_{\text{W}}(0) \). In subsequent subsections we proceed the study under bit flip, phase flip, bit-phase flip, amplitude damping, phase damping and depolarization channels. Let initial density matrix passes through any one of the quantum channel, than by using Kraus operators the decoherence prone density matrix can be written as,

\[
\rho_{\text{dp}} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} E_{i,j,k} \otimes E_{j} \otimes E_{k} |\rho(0)E_{i}^\dagger \otimes E_{j}^\dagger \otimes E_{k}^\dagger
\]

where \( n \) is the number of Kraus operators for a particular channel and \( E_{i,j,k} \) are the Kraus operators for three qubits \( i, j \) and \( k \), which are operating independently. The state \( \rho(0) \) represents the initial state either \( \rho_{\text{GHZ}}(0) \) or \( \rho_{\text{W}}(0) \). The equation (27) will be used for the decoherence analysis in spin squeezing throughout the paper for various quantum channels.
3.1. Bit flip channel

In this subsection we study the spin squeezing behaviour of GHZ and W states under bit flip channel. Bit flip error is a common error produced in many physical systems, a common example is the stray magnetic field which may flip the bit and produce this error. To begin with we write the Kraus operators of bit flip channel, which are given as,

\[
E_1 = \begin{bmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{p} \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & \sqrt{(1-p)} \\ \sqrt{(1-p)} & 0 \end{bmatrix}.
\] (28)

The bit flip channel has two Kraus operators, we plug-in these operators with \( n = 2 \), in equation (27). Putting \( \rho(0) = \rho_{\text{GHZ}}(0) \) for GHZ state and \( \rho(0) = \rho_{\text{W}}(0) \) for W state in equation (27), we get the decoherence prone density matrix \( \rho_{\text{bf}} \) for bit flip channel, which is used to reckon the spin squeezing in GHZ and W states by using equation (24). The expressions of degree of spin squeezing in GHZ and W states are obtained as,

\[
\epsilon_{\text{GHZ}} = \frac{1}{2} \left[ -2(2p - 1)^2 \sin^2(\theta) - (1 - 2p^2) \cos(2\theta) + 4(p - 1)p + 3 \right].
\] (29)

\[
\epsilon_{\text{W}} = \frac{2}{3} \left[ -\sin(\theta)\cos(\theta)\cos(\phi) + \frac{3\cos^2(\theta)}{8} + \left( 2(p - 1)p + \frac{7}{8} \right)\sin^2(\theta) + p\sin(2\theta)\cos(\phi) \right]
- \left( 2p - 1 \right)\sin(\theta) \sqrt{\sin^2(\phi) + \frac{1}{4} \left( 2\cos(\theta)\cos(\phi) + (2p - 1)\sin(\theta) \right)^2} + \frac{9}{8}.
\] (30)

First, we reckon the behaviour of spin squeezing in GHZ state with the equation (29). In the absence of bit flip error i.e. \( p = 0 \), the equation (29) become free from the angle \( \theta \) and we get \( \epsilon_{\text{GHZ}} = 1 \), which reveals the state is initially unsqueezed. Taking the other side of the discussion we can also look at the length of mean spin vector in GHZ state, which is zero in the absence of decoherence [58, 59]. We calculate the length of the mean spin vector \( \langle r \rangle \) for GHZ state in the decoherence prone matrix \( \rho_{\text{bf}} \) by using the equation (A.17), which is still found as zero and independent from the parameter \( p \). So, the result \( \langle r \rangle = 0 \) in the state \( \rho_{\text{bf}} \) represents the origin ‘O’ in the figure 1, which reveals that \( (\theta, \phi) = (0, 0) \) in GHZ state. So by putting \( \theta = 0 \) in equation (29) we get,

\[
\epsilon_{\text{GHZ}} = \frac{1}{2} \left[ -(1 - 2p)^2 + 4(p - 1)p + 3 \right] = 1.
\] (31)

The parameter \( p \) vanish from the equation and we get \( \epsilon_{\text{GHZ}} = 1 \). This concludes that the GHZ state remains unsqueezed and do not feel the influence of bit flip channel.

Next we give the look at equation (30), in which the spin squeezing parameter \( \epsilon_{\text{W}} \) is the function of three parameters \( (\theta, \phi, p) \). We plot \( \epsilon_{\text{W}} \) versus parameter \( p \) with different values of \( (\theta, \phi) \) in figure 2. We have calculated the length of mean spin vector for W state in the absence of bit flip error i.e. \( p = 0 \), it is obtained as \( \langle r \rangle = 0.372678 \approx 0 \). Which is reverse case to the GHZ state. So there is a possibility for spin squeezing and its production in W state. Giving the look at the subfigure of figure 2 with \( \theta = 0^\circ \), we find with \( p \in [0, 1] \) and \( \phi \in [0^\circ, 180^\circ] \), the parameter \( \epsilon_{\text{W}} \) is always equal to 1. Which concludes that as long as the mean spin vector is along the \( z \) axis, the state remain unsqueezed and the state remain unaffected by bit flip channel. But as the mean spin vector rotates with \( \theta \in (0^\circ, 90^\circ) \) and \( \phi \in [0^\circ, 180^\circ] \) the spin squeezing produces in the state. Direction of the mean spin vector has important significance which play the role to determine the plane of reduced variances, which lie normal to the direction of mean spin vector. By giving the look at the figure 2 with \( \theta = 10^\circ, 60^\circ, 90^\circ \), we find the behaviour of spin squeezing parameter \( \epsilon_{\text{W}} \), such that the bit flip parameter \( p \) and rotation angle \( \phi \) produce the spin squeezing in the state. But it is important to note that with \( p = 0.5 \), the parameter \( \epsilon_{\text{W}} = 1 \), and there is no spin squeezing produced. Here we mention that the movement of mean spin direction represents the movement of projection vector \( \hat{OB} \) in \( xy \) plane with an angle \( \phi \) and vice versa. With \( p \in [0, 0.5] \), \( \theta \in (0, 90^\circ) \) as the projection vector \( \hat{OB} \) rotates in the \( xy \) plane with an angle \( \phi \) the spin squeezing parameter achieves the value as \( \epsilon_{\text{W}} < 1 \), so spin squeezing production takes place. The condition \( \phi = 0^\circ \), \( \theta \in (0^\circ, 90^\circ) \) represents that the mean spin lie in \( xz \) plane and \( [\phi = 180^\circ, \theta \in (0^\circ, 90^\circ)] \) represents that the mean spin lie in the plane \( yz \). As mean spin or projection vector \( \hat{OB} \) rotates with \( \phi \in (0^\circ, 180^\circ) \) in \( xy \) plane, the degree of spin squeezing rises with \( p < 0.5 \) and decreases with \( p > 0.5 \). Bases on discussion for the results obtained in figure 2, we conclude that the W state is fragile under bit flip channel and channel produces spin squeezing in the state.

3.2. Phase flip channel

In this section we study the spin squeezing behaviour of GHZ and W states under phase flip channel. The Kraus operators of phase flip channel are given below.
By using the equation (27) with \( n = 2 \), we can obtain the decoherence prone matrix corresponding to the phase-flip channel as \( \rho_{\text{dp}}^{\text{fp}} \). Further calculating the spin squeezing parameter in GHZ and W states by using the equation (24) we get,

\[
\epsilon_{\text{GHZ}} = \frac{1}{2}(-2 \sin^2(\theta) - \cos(2\theta) + 3) = 1.
\]

\[
\epsilon_{\text{W}} = \frac{2}{3} \left[ \frac{1}{4} (-\cos(2\theta) + 2(2p - 1)\sin(2\theta)\cos(\phi) + 7) \right.
\]

\[
- \left. \frac{1}{4} (1 - 2p)\sin(2\theta)\cos(\phi) - \sin^2(\theta) \right]^2.
\]

Equation (33) reveals, the equation is free from the decoherence parameter \( p \) and simplification of the equation (33) leads as \( \epsilon_{\text{GHZ}} = 1 \). In fact this equation is similar to equation (29) with \( (p = 0) \). With \( \epsilon_{\text{GHZ}} = 1 \), we conclude that GHZ state do not feel decoherence by phase flip channel and avoid spin squeezing production.

Now we concentrate at the squeezing parameter obtained for W state in equation (34). The squeezing parameter \( \epsilon_{\text{W}} \) is the function of three parameters (\( \theta, \phi, p \)). With varying values of these parameters the results are plotted in figure 3. At \( (\theta = 0^\circ) \) under phase flip channel, we obtain the similar result as obtained with \( (\theta = 0^\circ) \) for bit flip channel as shown in figure 2. This result shows that as long as the mean spin vector is along the \( z \) axis, the W state remains initially unsqueezed and unaffected by phase flip channel. Here in figure 3 with \( \theta \in [10^\circ, 90^\circ] \) except \( (\phi = 0^\circ) \), spin squeezing plots have round behaviour in the vicinity of \( (p = 0.5) \). However for bit flip channel the spin squeezing was sharply meeting at \( (p = 0.5) \). At \( (p = 0) \) and \( (p = 1) \), the degree of spin squeezing obtained under this channel is same as obtained for bit flip channel with varying values of \( (\theta, \phi) \). When mean spin vector is in \( xz \) plane with \( (\theta = 60^\circ, \phi = 0^\circ) \), the spin squeezing is produced only with \( p \in [0.05, \ 0.1] \). After \( (p > 0.1) \), the state is unsqueezed. Once the mean spin vector switch to \( yz \) plane with \( (\theta = 60^\circ, \phi = 180^\circ) \), the state is squeezed only with \( p \in [0.95, \ 1.0] \). As the projection vector \( \vec{OB} \) rotates in \( xy \) plane with \( \phi \in [0^\circ, 180^\circ] \), there are good features of spin squeezing production. Here we mention that phase flip channel has the capability to produce spin squeezing in W state.

Figure 2. Plot of \( \epsilon_{\text{W}} \) versus parameter \( p \) with \( \phi \in [0^\circ, 180^\circ] \) for bit flip channel.
3.3. Bit-phase-flip channel

In this section we study the behaviour of spin squeezing under bit-phase-flip channel. This channel flip the bit along with the emergence of relative phase factor in the state. The Kraus operators of bit-phase-flip channel are given below.

\[
E_1 = \begin{bmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & -i\sqrt{1-p} \\ i\sqrt{1-p} & 0 \end{bmatrix}.
\]

We use the equation (27) with \( n = 2 \), and obtain the decoherence prone density matrix after passing through the channel as \( \rho_{\text{bfip}}^{\phi} \). By using \( \rho_{\text{bfip}}^{\phi} \), the spin squeezing parameter for GHZ and W states are obtained as,

\[
\epsilon_{\text{GHZ}} = \frac{1}{2} \left\{ (2p - 1)(-\cos(2\theta) + 8(p - 1)p + 3) - 2(1 - 2p)\sin^2(\theta) \right\},
\]

\[
\epsilon_{\text{W}} = \frac{2}{3} \left\{ \frac{1}{4} (2p - 1) \{ 2\sin(2\theta)\cos(\phi) - \cos(2\theta) + 24(p - 1)p + 7 \} \right. \\
\left. - (2p - 1) \sqrt{\sin^2(\theta)\sin^2(\phi) + \frac{1}{4} \{ \sin(2\theta)\cos(\phi) + \sin^2(\theta) \}^2} \right\}.
\]

Looking at the equation (36), the parameter \( \epsilon_{\text{GHZ}} \) is the function of \((p, \theta)\). In the absence of decoherence i.e. \((p = 0)\), we get \( \epsilon_{\text{GHZ}} = 1 \), hence GHZ state is initially unsqueezed. As the direction of mean spin vector i.e. \((\theta)\) and the value of parameter \( p \) increases, the GHZ state become squeezed. The result is shown in figure 4. We also observe in the figure 4 with \((p < 0.5)\), the squeezing parameter achieve negative values, which is an indicator that the channel induces negative correlations in the state [9]. With \((p > 0.5)\), as the value of parameter \( p \) increases the degree of spin squeezing exponentially increases. Here we conclude, the bit-phase-flip channel has the capability to produce spin squeezing in GHZ state.

Further for W state, giving the glance to equation (37), we find the squeezing parameter is the function of three parameters (\(\theta, \phi, p\)). For \((\theta = 0^\circ)\), the equation becomes free from the angle (\(\phi\)) and remains the function of parameter \((p)\) only, but it is not true with \((\theta = 90^\circ)\). With \((\theta = 0^\circ)\), the mean spin vector in this case is along the \( z \) axis and result is plotted in figure 5. Which reveals that as the value of probability increases with \((p > 0.5)\), the degree of spin squeezing exponentially increases. But there is no spin squeezing produced in the state with \((p < 0.5)\). Further observing the figure 5 with \((\theta = 25^\circ)\) reveals, as the direction of mean spin vector changes, the degree of spin squeezing exponentially grows with varying values of \((\phi)\). As the value of the angle \((\phi)\) tends to \(180^\circ\), the squeezing is produced with higher values of \((p)\) beyond the range \((p > 0.5)\). Here we find, the bit-phase-flip channel produces spin squeezing with \((p > 0.5)\) in both the GHZ and W states. So both the GHZ and W states exhibit fragile behaviour w.r.t. bit-phase-flip channel and channel has better positive impact on spin squeezing production.

3.4. Amplitude damping channel

In this section, we study the spin squeezing under amplitude damping channel. This channel is used to describe the dissipation of the interaction between quantum systems and its environment, for example the spontaneous emission of the photon from a quantum system. In actual the amplitude damping channel describe the energy loss in the system. The Kraus operators of amplitude damping channel are given below,

\[
E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{e^{-\gamma t}} \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & \sqrt{1 - e^{-\gamma t}} \\ \sqrt{1 - e^{-\gamma t}} & 0 \end{bmatrix},
\]

where \(\gamma\) is the damping rate for the channel. We obtain the decoherence prone density matrix \( \rho_{\text{ad}}^{\phi} \) corresponding to the amplitude damping channel by using the equation (27) with \( n = 2 \). Further, the spin squeezing

![Figure 3. Plot of parameter \( \epsilon_W \) versus parameter \( p \) with \( \phi \in [0^\circ, 180^\circ] \) for phase flip channel.](image-url)
parameters for GHZ and W states are calculated as below,
\[
\epsilon_{\text{GHZ}} = \frac{1}{2}[-2e^{-2\gamma t}(e^{i\phi} - 2) + 2\sin^2(\theta) - 4e^{-2\gamma t}(e^{i\phi} - 1)\sin^2(\theta) - \cos(2\theta) + 3] = 1.
\] (39)
\[
\epsilon_{\text{W}} = \frac{2}{3} \left[ \frac{1}{4} (8e^{-3\gamma t} \sin(\theta) \cos(\phi)) \cos(\phi) - 2 \sqrt{e^{-3\gamma t}} \sin(2\theta) \cos(\phi) - 4e^{-2\gamma t}(3e^{i\phi} - 2) \sin^2(\theta) - 3\cos(2\theta) + 9 \right] - \sqrt{A_1}
\] (40)

with
\[
A_1 = e^{-3\gamma t}(e^{i\phi} - 2)^2 \sin^2(\theta) \sin^2(\phi) + \frac{1}{4} e^{-3\gamma t} \sin^2(\theta) (\sqrt{e^{-3\gamma t}} (3e^{i\phi} - 2) - 4) \sin(\theta) - 2(e^{i\phi} - 2) \cos(\theta) \cos(\phi))^2
\] (41)

Simplification of the equation (39) leads as ($\epsilon_{\text{GHZ}} = 1.$). We find, the squeezing parameter $\epsilon_{\text{GHZ}}$ is independent from decoherence parameter ($\gamma t$) and angle ($\theta$). So it implies that amplitude damping channel do not exhibit the spin squeezing production in GHZ state. Further, equation (40) is used to explore the results for W state. These results are shown in figure 6. When mean spin vector is along the z axis with ($\theta = 0^\circ$), the equation (40) leads the value ($\epsilon_{\text{W}} = 1$), which represents the state is unsqueezed. As the value of the parameter ($\gamma t, \theta$) increases with variations in the angle ($\phi$), the spin squeezing is produced in the state. Further we find there are
good features of spin squeezing production with \((\gamma t < 0.6)\). It is found, with \((\phi = 0^\circ, \forall \theta)\), the state is always unsqueezed. We notice, as the mean spin vector rotates over the plane \(xy\) with \(\phi \in (0^\circ, 180^\circ)\), there are nice plots of spin squeezing production, which are clearly shown in figure 6 with increasing values of parameter \((\theta)\). Here we find, W state shows fragile character under the specified channel and channel has good capability to produce spin squeezing.

3.5. Phase damping channel

In this section we study the spin squeezing behaviour under phase damping channel. Phase damping channel is the model to represent the information loss in quantum system because of the relative phase produced in the system with system–environment interaction. This channel do not involve the energy loss in the system as it is done in the case of amplitude damping channel. Recently it is observed that spin squeezing can be produced with phase damping channel in the system by using quantum non demolition interaction \([60–67]\), so it is important to study the affect of this channel on spin squeezing. The Kraus operators are given for this channels as below.

\[
E_1 = \begin{bmatrix} \sqrt{e^{-\gamma t}} & 0 \\ 0 & \sqrt{e^{-\gamma t}} \end{bmatrix}, \\
E_2 = \begin{bmatrix} \sqrt{1 - e^{-\gamma t}} & 0 \\ 0 & \sqrt{1 - e^{-\gamma t}} \end{bmatrix}, \\
E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

We obtained decoherence prone density matrix \(\rho_{\text{opc}}\) by putting \(n = 3\) in equation (27). Further we have calculated the spin squeezing parameters for phase damping channel for both the GHZ and W states, these are obtained below,

\[
\epsilon_{\text{GHZ}} = \frac{1}{2} \left[ -2 \sin^2(\theta) - \cos(2\theta) + 3 \right] = 1 \tag{43}
\]

\[
\epsilon_W = \frac{2}{3} \left[ e^{-\gamma t} \sin(\theta) \cos(\theta) \cos(\phi) - \sqrt{e^{-2\gamma t}} \sin^2(\theta) \sin^2(\phi) - (A_2)^2 - \frac{1}{4} \cos(2\theta) + \frac{7}{4} \right] \tag{44}
\]

with

\[
A_2 = \sin(\theta) \cos(\theta) \cos(\phi) (e^{-\gamma t}) + \frac{\sin^2(\theta)}{2} \tag{45}
\]

Simplification of equation (43) shows that for GHZ state the spin squeezing parameter is obtained as \((\epsilon_{\text{GHZ}} = 1)\), it represents the state is unsqueezed and channel do not produce spin squeezing in the state.

Further for W state, with the equation (44) we find at \((\theta = 0^\circ)\), the squeezing parameter is \((\epsilon_W = 1)\). So it implies as long as the mean spin vector is along the \(z\) axis with \((\theta = 0^\circ)\), the W state is unsqueezed. For higher values of the parameter \((\theta)\), the results are plotted in figure 7. We found, as the mean spin vector is in \(xz\) plane with \((\phi = 0^\circ, \forall \theta)\), the spin squeezing has not been produced in the state. While the rotation of mean spin vector with \((\phi \in (0^\circ, 180^\circ))\) produces spin squeezing with the increasing values of decoherence parameter \((\gamma t)\).

3.6. Depolarization channel

Under this section we study the spin squeezing behaviour under depolarization channel \([68]\). This channel is widely studied in polarization encoding in quantum information, the map of depolarization is described as it lives the system in fully mixed state with the probability \((p)\) and the systems is unchanged with the probability \((1 - p)\). The Kraus operators for depolarization channels are given below.
We obtained the density matrix $\rho_{pdc}$ by putting $n = 4$ in equation (27). Further, we obtained the spin squeezing parameters for GHZ and W states with $\rho_{pdc}$. These are obtained below,

$$
E_1 = \begin{bmatrix} \sqrt{e^{-\gamma t}} & 0 \\ 0 & \sqrt{e^{-\gamma t}} \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & \frac{i}{\sqrt{3}}(1 - e^{-\gamma t}) \\ \frac{i}{\sqrt{3}}(1 - e^{-\gamma t}) & 0 \end{bmatrix}, 
E_3 = \begin{bmatrix} 0 & -i \frac{1}{\sqrt{3}}(1 - e^{-\gamma t}) \\ \frac{1}{\sqrt{3}}(1 - e^{-\gamma t}) & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} \frac{i}{\sqrt{3}}(1 - e^{-\gamma t}) & 0 \\ 0 & -\frac{i}{\sqrt{3}}(1 - e^{-\gamma t}) \end{bmatrix}.
$$

We obtained the density matrix $\rho_{pdc}$ by putting $n = 4$ in equation (27). Further, we obtained the spin squeezing parameters for GHZ and W states with $\rho_{pdc}$. These are obtained below,

$$
\epsilon_{GHZ} = \frac{1}{54}(-2e^{-3\gamma t}(e^{i\theta} + 2)^3\sin^2(\theta) - e^{-3\gamma t}(e^{i\theta} + 2)^3(\cos(2\theta) - 3))
= \frac{1}{27}e^{-3\gamma t}(e^{i\theta} + 2)^3.
$$

$$
\epsilon_W = \frac{2}{3} \left[ -\frac{1}{729}e^{-6\gamma t}(e^{i\theta} + 2)^6\sin^2(\theta)\sin^2(\phi) + \frac{A_3}{2916} \\
- \frac{1}{108}e^{-3\gamma t}(e^{i\theta} + 2)^3(-2\sin(2\theta)\cos(\phi) + \cos(2\theta) - 7) \right].
$$

Looking at the equation (48), we find the squeezing parameter for GHZ state is the function of the damping rate $(\gamma t)$ and independent from the parameters $(\theta, \phi)$. It implies that, the mean spin can be in any direction in the space. We have plotted this function in figure 8. At $(\gamma t = 0)$ we have $\epsilon_{GHZ} = 1$, the state is initially unsqueezed, but as the depolarization rate increases the parameter $\epsilon_{GHZ}$ achieve the value less than one and spin squeezing is produced in the state. The degree of spin squeezing decay exponentially and stay at $\epsilon_{GHZ} = 0.2$. This channel shows lucid behaviour for spin squeezing production in GHZ state and the state is very much fragile under this channel.

For W state, we study the equation (49), we find at $(\theta = 0^\circ)$, the equation convert into the equation (48) and spin squeezing in W state exhibit the same behaviour as found for GHZ state. This result is shown in figure 9.
with $\theta = 0^\circ$. Looking at subfigures of figure 9, we find as the values of $\theta$ increases the spin squeezing produces in W state and decreases exponentially as the depolarization rate $\gamma t$ increases. Most importantly we have found, as the mean spin vector lies in $xz$ or in $yz$ plane with $\phi = 0^\circ$, $\phi = 180^\circ$, $\forall \theta$, still the spin squeezing is produced in W state under depolarization channel. We have found the depolarization channel has great impact to produce spin squeezing in both the GHZ and W states.

4. Conclusion

In this article we investigate the positive impact of decoherence on spin squeezing in tripartite maximally entangled GHZ and W states under bit flip, phase flip, bit-phase flip, amplitude damping, phase damping and depolarization channels. Initially GHZ state is unsqueezed and W state is also unsqueezed as long as its mean spin vector is along the $z$ axis. When decoherence is applied, we have found the spin squeezing production in these states. However GHZ state remain unsqueezed under all the decoherence channels except bit-phase-flip and depolarization channels. The W state shows fragile behaviour under decoherence and it permits all the channels to produce spin squeezing in it and exhibit less robust character than GHZ state. More specifically we have found depolarization channel has lucid characteristic to produce good degree of spin squeezing in both the GHZ and W states. We also investigated that none of the state exhibit spin squeezing sudden death under any one of the decoherence channels. There are extensive experimental efforts to generate spin squeezing in varieties of physical systems such as BEC, Ultra cold atoms, NV diamond centres, gravitational interferometers, atomic clocks etc. As quantum systems are too evasive and the phenomenon of decoherence is unavoidable which has its negative impact on quantum systems, but investigating the positive aspects for the same may lead to new outcomes. So, in this direction the present study theoretically explore the impact of decoherence on spin squeezing as a positive factor in GHZ and W states, which may be useful for quantum information community.

Acknowledgments

The authors acknowledge support from the Ministry of Electronics & Information Technology, Government of India, through the Centre of Excellence in Nanoelectronics, IIT Bombay.

Appendix. Calculations of variance $(\Delta J^2)$

Under this section we give the calculations of the variance $(\Delta J^2)$. To proceed, we define the variance as,

$$(\Delta J^2)^2 = (J_n^2) - (J_n^2).$$  \hspace{2cm} (A.1)

By using the equation (19), we obtain the terms $(J_n^2)$ and $(J_n^2)$ as follows,

$$J_n^2 = (J_n \cos \varphi + J_n \sin \varphi)(J_n \cos \varphi + J_n \sin \varphi) = J_n^2 \cos^2 \varphi + J_n^2 \sin^2 \varphi + \frac{1}{2} (J_n J_n + J_n J_n) \sin 2 \varphi$$

$$= \frac{1}{2} (J_n^2 - J_n^2) \cos 2 \varphi + \frac{1}{2} (J_n J_n + J_n J_n) \sin 2 \varphi + (J_n^2 + J_n^2).$$  \hspace{2cm} (A.3)
Taking the averages on both the sides we get,

\[
\langle J_x^2 \rangle = \frac{1}{2} \langle J_m^2 - J_n^2 \rangle \cos 2\varphi + \langle J_n J_n + J_m J_m \rangle \sin 2\varphi + \langle J_m^2 + J_n^2 \rangle.
\] (A.4)

Here we assume,

\[
M = \langle J_m^2 - J_n^2 \rangle
\] (A.5)

\[
N = \langle J_n J_n + J_m J_m \rangle
\] (A.6)

\[
O = \langle J_m^2 + J_n^2 \rangle.
\] (A.7)

Hence the equation (A.4) can be re written as,

\[
\langle J_x^2 \rangle = \frac{1}{2} [M \cos 2\varphi + N \sin \varphi + O].
\] (A.8)

Now focusing on the factor \( \langle J_x \rangle \), by using the equation (19) we get,

\[
\langle J_x \rangle = \langle J_m \rangle \cos \varphi + \langle J_n \rangle \sin \varphi.
\] (A.9)

Putting the values of the factors \( \langle J_m \rangle \) and \( \langle J_n \rangle \) from the equations (11) and (12) we further obtain,

\[
\langle J_x \rangle = (-\langle J_x \rangle) \sin \phi + (\langle J_y \rangle \cos \phi) \cos \varphi + ((-\langle J_n \rangle) \cos \theta \cos \phi - (\langle J_x \rangle \cos \theta \sin \phi + (\langle J_y \rangle \sin \theta) \sin \varphi).
\] (A.10)

Here we use geometric description to find out the values of the factors \( \langle J_x \rangle \), \( \langle J_y \rangle \) and \( \langle J_z \rangle \) used in the above equation. These are the components of mean vector along the x, y and z axis respectively. We refer the figure 1 and redraw two figures A1 and A2. Giving the look to the geometry in these figures, from the figure A1, we find...
\[
\cos \phi = \frac{\langle J_z \rangle}{R} \Rightarrow \langle J_z \rangle = R \cos \phi \quad (A.11)
\]
\[
\sin \phi = \frac{\langle J_y \rangle}{R} \Rightarrow \langle J_y \rangle = R \sin \phi \quad (A.12)
\]
\[
\tan \phi = \frac{\langle J_y \rangle}{\langle J_x \rangle} \quad (A.13)
\]
\[
R = \sqrt{\langle J_z \rangle^2 + \langle J_y \rangle^2}. \quad (A.14)
\]

From figure A2 we get the following results
\[
\sin \theta = \frac{R}{r} \Rightarrow R = r \sin \theta \quad (A.15)
\]
\[
\cos \theta = \frac{\langle J_z \rangle}{r} \Rightarrow \langle J_z \rangle = r \cos \theta \quad (A.16)
\]
\[
r = \sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2 + \langle J_z \rangle^2} = R, \quad (A.17)
\]

where \(r\) is the length of mean vector \(\vec{R} \text{mean}\), which is represented in the figures as a ray OA. The above expression obtained from the geometry shown in both the figures are independent from the state of the system and true for any state. We plug-in the values of the factors obtained in the equations (A.11–A.17), in equation (A.10), we get
\[
\langle J_z \rangle = 0 \Rightarrow \langle J_y \rangle^2 = 0. \quad (A.18)
\]

This is very interesting result and beauty of this result is that, it is true for any state of the system. We plug-in the values from equations (A.8) and (A.18) in equation (A.1), we obtain the variance of the vector \(\vec{J}_z\) as below,
\[
(\Delta J_z^2) = \frac{1}{2} [M \cos 2\varphi + N \sin 2\varphi + O]. \quad (A.19)
\]

The variance \((\Delta J_z^2)\) is the function of angle \(\varphi\), we find the minimum and maximum value of the function over the angle \(\varphi\), so doing first derivative of the function \((\Delta J_z^2)\) w.r.t. the angle \(\varphi\), we get,
\[
\frac{d(\Delta J_z^2)}{d\varphi} = \frac{1}{2} \left[ 0 + \frac{M}{2} (-\sin 2\varphi) + \frac{N}{2} \cos 2\varphi \right]. \quad (A.20)
\]

For maximization and minimization we put \(\frac{d}{d\varphi} = 0\), which leads.
\[
\tan 2\varphi = \frac{N}{M}. \quad (A.21)
\]

The equation (A.21) further leads the conclusion as,
\[
\sin 2\varphi = \pm \frac{N}{\sqrt{M^2 + N^2}}, \quad \cos 2\varphi = \pm \frac{M}{\sqrt{M^2 + N^2}}. \quad (A.22)
\]

By putting the values from the equation (A.22) in equation (A.19), we obtain,
\[
(\Delta J_z)^2 = \frac{1}{2} \left[ O \pm \frac{M^2}{\sqrt{M^2 + N^2}} + \frac{N^2}{\sqrt{M^2 + N^2}} \right] \quad (A.23)
\]
\[
= \frac{1}{2} [O \pm \sqrt{M^2 + N^2}]. \quad (A.24)
\]

As per the definition of spin squeezing we consider the minimum value of the variance along the vector \(\vec{n}\), So the final expression for the variance of \(J_z\) is obtained as,
\[
(\Delta J_z)^2 = \frac{1}{2} [O - \sqrt{M^2 + N^2}]. \quad (A.25)
\]

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