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Parametric instabilities in resonantly-driven Bose–Einstein condensates

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Abstract

Shaking optical lattices in a resonant manner offers an efficient and versatile method to devise artificial gauge fields and topological band structures for ultracold atomic gases. This was recently demonstrated through the experimental realization of the Harper–Hofstadter model, which combined optical superlattices and resonant time-modulations. Adding inter-particle interactions to these engineered band systems is expected to lead to strongly-correlated states with topological features, such as fractional Chern insulators. However, the interplay between interactions and external time-periodic drives typically triggers violent instabilities and uncontrollable heating, hence potentially ruling out the possibility of accessing such intriguing states of matter in experiments. In this work, we study the early-stage parametric instabilities that occur in systems of resonantly-driven Bose–Einstein condensates in optical lattices. We apply and extend an approach based on Bogoliubov theory (Lellouch et al 2017 Phys. Rev. X 7 021015) to a variety of resonantly-driven band models, from a simple shaken Wannier–Stark ladder to the more intriguing driven-induced Harper–Hofstadter model. In particular, we provide \textit{ab initio} numerical and analytical predictions for the stability properties of these topological models. This work sheds light on general features that could guide current experiments to stable regimes of operation.

1. Introduction

Driving quantum systems periodically in time has been proposed as a versatile tool to generate unusual quantum phases of matter [1–6]. In the context of ultracold quantum gases, it was shown that subjecting neutral atoms to an external time-periodic drive could be used to design artificial gauge fields in these systems [6–10], hence opening promising perspectives in the quantum simulation of topological states of matter [11–13] and quantum magnetism [10, 14, 15]; see also the recent work [16] on the control of magnetic correlations in driven cold gases. The underlying concept of Floquet engineering [4, 8–10, 13] builds on the fact that the dynamics of periodically-driven systems can be well described by a static effective Hamiltonian, whose properties can be suitably designed by tailoring the driving protocol. On the experimental side, this idea has been extensively used to design artificial gauge fields for neutral atoms [11, 12, 17–22], some of which lead to non-trivial geometrical and topological properties [13].

A particularly interesting class of periodically-driven setups is that featuring \textit{resonant} time-modulations [23–28], in which the driving frequency \( \omega \) resonates with an energy separation \( \Delta \approx h \omega \) that is inherent to the underlying static system. In particular, such schemes can be exploited to finely control the tunneling matrix elements connecting neighboring sites of a lattice, and can be simply implemented by resonantly modulating a superlattice or a Wannier–Stark ladder; see [29, 30] for experimental realizations and [10] for a review. This so-called photon-assisted tunneling effect constitutes a natural ingredient for the generation of artificial fluxes within cold-atom systems [23, 25–27], as exemplified by the recent experimental realizations of the Harper–Hofstadter
model \cite{17-21, 31}, a 2D lattice penetrated by a uniform magnetic flux \cite{32}; the latter band model is of particular interest, due to its rich topological band structure \cite{33, 34}.

However, a major challenge remains in this context, namely, the addition of inter-particle interactions in view of creating novel strongly-correlated states of matter, e.g. fractional Chern insulators \cite{34-38}. Indeed, the interplay between inter-particle interactions and an external drive, such as lattice shaking, has been shown to lead to significant heating and losses in experiments involving Bose–Einstein condensates (BEC) \cite{31, 39}. The complexity of this issue lies on the fact that several underlying mechanisms are believed to be responsible for those undesired effects, and these possibly interplay in a complex manner, as we now explain.

On the one hand, at very short times, time-modulated BECs are believed to be mostly affected by strong parametric instabilities, which are characterized by an exponential growth of collective (Bogoliubov) excitations and are accompanied with a fast decay of the BEC; such processes were thoroughly characterized in our previous work \cite{40}, where instability rates, stability diagrams and robust physical signatures of such processes were obtained for simple non-resonant shaken systems. Other approaches have been adopted to characterize dynamical instabilities in shaken BECs \cite{41-45}, and a recent study even pointed out the possibility of dynamically stabilizing a modulated BEC, inspired by the Kapitza pendulum \cite{46}; the experimental evidence of staggered-states in time-modulated BECs, whose formation also stems from an instability involving the external drive and collective excitations, was recently reported in \cite{47}; we note that time-modulating the trapping potential can also be exploited to create correlated excitations in BECs \cite{48}.

On the other hand, at longer times, dissipative processes are expected to be dominated by scattering events, which are typically associated with two-body processes and are captured by the so-called Floquet–Fermi golden rule \cite{11, 49, 50}. Finally, the role of (resonant) inter-band transitions \cite{39, 51-53}, and the formation of collective emissions of matter–wave jets upon driving \cite{54}, have been analyzed in very recent ultracold-atom experiments.

The aim of this paper is to analyze the onset of parametric instabilities in resonantly-modulated BECs. Building on the tools developed in \cite{40}, we identify the general features of those instabilities that occur in the resonant-modulations context, and provide useful results that could be readily applied to topical systems, such as the driven-induced Harper–Hofstadter model \cite{17-21, 31}.

The rest of the paper is organized as follows: in section 2, we recall the general method of \cite{40} to treat parametric instabilities. In section 3, we apply this method to a simple resonantly-shaken Wannier–Stark ladder, highlighting the main features of these driven-induced instabilities. We then address the topical case of an optical lattice that is modulated by a secondary moving lattice, and which includes a space-dependent phase \cite{18-20}; we first study the 1D case in section 4, and then discuss the 2D configuration (leading to the Harper–Hofstadter Hamiltonian \cite{18-20}) in section 5. Final remarks are provided in the concluding section 6.

\section{2. General method}

We first briefly summarize the general method of \cite{40} to study parametric instabilities in periodically-driven BEC lattice systems; we point out that similar or complementary approaches have been proposed to characterize dynamical instabilities in \cite{41-45, 55, 56}.

\subsection{2.1. Linear stability analysis}

Consider a weakly-interacting Bose gas, in a generic periodically-driven system of period $T = 2\pi/\omega_T$; this discussion disregards the system dimension, and/or the presence of a lattice. To study the stability properties of a potential BEC, the general idea is to assume that the system is initially fully-condensed in some well-defined state and to perform a linear stability analysis around this specific state. Let us denote by $a_n^{(0)}(t = 0)$ the condensate wavefunction at initial time $t = 0$, where $n$ is a generic (possibly multidimensional, discrete or continuous) index for spatial position. This state could be set by hand (based on analytical arguments), or it could be more precisely estimated for a given experimental protocol, by numerically implementing the full preparation sequence or using Floquet adiabatic perturbation theory \cite{57, 58}.

Given this state $a_n^{(0)}(t = 0)$ as an input, we are interested in the stability of the full time-dependent solution $a_n^{(t)}(t)$. This property can be evaluated by following the guideline below:

(i) In the weakly-interacting regime, this solution is governed by the time-dependent Gross–Pitaevskii equation (tGPE), whose exact form depends on the precise model under consideration. Thus, we first determine the time-evolution of the condensate wavefunction $a_n^{(0)}(t)$, by solving the (tGPE) with the initial condition $a_n^{(0)}(t = 0)$. This can be performed numerically using direct real-time propagation in real space. We stress that this calculation is based on the full time-dependent equations, so that the dynamics of the BEC is exactly computed (including all micromotion effects \cite{8}).
(ii) Given the time-dependent solution for the condensate wave function $a_n^{(0)}(t)$, we analyze its stability by considering a small perturbation

$$ a_n(t) = a_n^{(0)}(t)[1 + \delta a_n(t)] $$

and linearizing the (GPE) in $\delta a_n$. This yields the time-dependent Bogoliubov–de Gennes equations, which take the general form

$$ \begin{pmatrix} \delta a_n \\ \delta \Phi_n \end{pmatrix} = \mathcal{L}(t) \begin{pmatrix} \delta a_n \\ \delta \Phi_n \end{pmatrix}, $$

where $\mathcal{L}(t)$ is a $T$-periodic operator (we set $\hbar = 1$ here and in the following). Based on this time-periodicity, it is convenient to exploit the Floquet theorem and to focus the stability analysis on the ‘time–evolution’ (propagator) matrix $\Phi(T)$, which is obtained by time-evolving equation (2) over a single period $T$. From the knowledge of $\Phi(T)$, we extract the ‘Lyapunov’ exponents $\epsilon_q$, which are related to the eigenvalues $\lambda_q$ of $\Phi(T)$ through the relation $\lambda_q = e^{-i\epsilon_q T}$; here we explicitly introduced the momentum $q$, which will be used to index the corresponding excitation modes. The appearance of Lyapunov exponents with positive imaginary parts indicates a dynamical instability [43, 44], i.e. an exponential growth of the corresponding modes, given by the rate $s_q = \text{Im} \ \epsilon_q$.

### 2.2. Observables

As previously discussed in [40], a first quantitative indicator of the instability is the maximum growth rate of the spectrum,

$$ \Gamma \equiv \max_q s_q, $$

which, in the following, will be referred to as the instability rate $\Gamma$. This instability rate is independent of the reference frame (or gauge). It quantifies the parametric instabilities occurring in the system in the sense that it governs the stroboscopic dynamics ($t = T \times \text{integer}$) of physical observables in the system. For instance, $\Gamma > 0$ indicates that physical observables (e.g. the total energy, the depleted fraction, ...) will exponentially grow up with the rate $2\Gamma$.

Another relevant indicator is the most unstable mode

$$ q^{\text{num}} \equiv \arg\max_q s_q. $$

The excitations momentum distribution indeed shows a pronounced contribution around $q^{\text{num}}$ and structures of momentum $q^{\text{num}}$ develop in real/momentum space, producing clear signatures of those parametric instabilities. We stress that, due to the factorization of the BEC wavefunction in equation (1), all momentum modes $q$ (and hence, the most unstable mode) are always defined relatively to the ground-state (which may not always be associated with a vanishing momentum).

As pointed out in [40], saturation effects generically alter the instability rates ($\Gamma$) predicted by Bogoliubov theory at longer times. These effects can be attributed to different mechanisms, which include the coupling between the condensate and the Bogoliubov modes, but also the interaction between the various Bogoliubov modes; while the latter effect is captured by the full nonlinear Gross–Pitaevskii equation (and can be explored numerically through a Truncated–Wigner approach [59]), we analyzed the former effect in [40] using a weak-coupling-conserving-approximation method. As a consequence, the parametric instabilities investigated in this framework are truly associated with short-time dynamics. While the instability rate must therefore be treated as a dynamical quantity (affected by saturation effects and other, possibly incoherent, mechanisms [49, 50]), the most unstable mode $q^{\text{num}}$ and the associated structures developing in real and momentum space (e.g. in the momentum distribution) are found to be very robust; this could, therefore, provide clear signatures of parametric instabilities in realistic experimental configurations.

We point out that the instabilities analyzed in this work are obtained from the classical Gross–Pitaevskii equation, and in this sense, the following results apply to a wide class of systems, including classical systems. However, when applied to the situation of a BEC trapped in a time-dependent optical lattice, the predicted instabilities are genuinely of quantum origin in the sense that they are triggered by a non-zero initial population of collective modes (i.e. a feature of the Bogoliubov ground-state [60]).

\footnote{The factor of 2 stems from the fact that $\Gamma$ is defined in amplitude while physical observables involve squared moduli of wavefunctions.}
3. A first example: the resonantly-shaken Wannier–Stark ladder

We first apply this method to a simple one-dimensional (1D) resonantly-driven Wannier–Stark ladder; we note that this toy-model is a direct extension of the shaken 1D lattice studied in [40]. Beyond its physical interest, this section aims at demonstrating how the (numerical and analytical) tools that were previously developed can indeed be successfully applied in the context of resonantly-driven models.

3.1. The model

We consider a system of weakly-interacting bosons, trapped in a shaken 1D Wannier–Stark ladder, as described by the periodically-driven Bose–Hubbard Hamiltonian [10]

$$\hat{H}(t) = -J \sum_n (\hat{a}_{n+1}^{\dagger} \hat{a}_n + \text{h.c.}) + \frac{U}{2} \sum_n \hat{a}_n^{\dagger} \hat{a}_n \hat{a}_n^{\dagger} \hat{a}_n + \sum_n [\Delta \hat{n}_n \hat{a}_n + K \cos(\omega t) \hat{n}_n \hat{a}_n^{\dagger}],$$

where $\hat{a}_n$ annihilates a particle at lattice site $n$, $J > 0$ denotes the tunneling amplitude of nearest-neighbor hopping, and $U > 0$ is the repulsive on-site interaction strength. The second line in equation (5) captures the on-site potential term, which contains two effects: the Wannier–Stark-ladder potential, which introduces an energy shift $\Delta > 0$ between consecutive sites, and a time-periodic modulation of amplitude $K$ and frequency $\omega = \frac{2\pi}{T}$; this time-modulation simply corresponds to an external shaking of the 1D optical lattice, as viewed from the moving frame [10, 27]. The frequency modulation $\omega$ is chosen to be resonant with the offset $\Delta$, i.e. $\Delta = \omega \pi$ with $\pi$ denoting some integer; see [29, 30].

It is well known that the main effect of the time-modulation is to restore the tunneling, which is suppressed by the strong offset $\Delta$; this is reflected in the (non-interacting) Floquet effective Hamiltonian associated with equation (5), and which reads [10, 26, 29, 61]

$$\hat{H}_{\text{eff}}^{(0)} = -J_{\text{eff}} \sum_n [\hat{a}_{n+1}^{\dagger} \hat{a}_n + \text{h.c.}],$$

where the effective tunneling is $J_{\text{eff}} = J \mathcal{J}_l(K/\omega)$, with $\mathcal{J}_l$ the $l$th order Bessel function, and where we set $U = 0$.

3.2. Numerical results

To compute the instability rates of the system, we first numerically implement the procedure detailed in section 2. The initial state $\hat{a}_n^{(0)}(t = 0)$ is chosen to be the ground state of the naive ‘interacting effective Hamiltonian’

$$\hat{H}_{\text{eff}}^{(0)} + \frac{U}{2} \sum_n \hat{a}_n^{\dagger} \hat{a}_n \hat{a}_n^{\dagger} \hat{a}_n,$$

which is a reasonable approximate for an experimentally-prepared ground-state; in general, we note that this ground-state can be numerically determined using imaginary time propagation. For the specific model under consideration, we find that $\hat{a}_n^{(0)}(t = 0)$ is the Bloch state $e^{i\mathbf{p}_0 \cdot \mathbf{r}}$ of momentum $\mathbf{p}_0 = 0$ if $J_{\text{eff}} > 0$ (homogeneous condensate), and $\mathbf{p}_0 = \pi$ for $J_{\text{eff}} < 0$; see analytical details in appendix. This fact is reminiscent of the non-resonant case; see [40, 44].

Figure 1 displays the behavior of the instability rate $\Gamma$ as a function of the interaction strength $g = U \rho$ (with $\rho$ the condensate density, which enters the normalization of the initial state) and modulation amplitude $K/\omega$, in the purely resonant case ($l = 1$) and for a driving frequency $\omega = 5J$. We note that the general features are very similar to what was observed in [40, 44] in the non-resonant case: the stability diagram displays lobes, which are separated by stable regions corresponding to cancellation points of the effective tunneling $J_{\text{eff}}$ (here the zeros of the function $\mathcal{J}_l(K/\omega)$). For large enough driving frequencies (such as in figure 1), the system is stable at low values of $g$, and a transition to instability appears at finite $g$; close to the transition, these instabilities are found to be dominated by the Bogoliubov mode $q_{\text{max}}^{\text{max}} = \pi$, which is the most unstable one; for smaller values of $\omega$ (smaller than the effective free-particle bandwidth, i.e. $\omega < 4|J_{\text{eff}}|$ here), the system would be unstable for any non-zero interaction strength, with the most unstable mode corresponding to $q_{\text{max}}^{\text{max}} < \pi$ [40].

We find that this situation is very general: for other values of $l$, we find similar stability diagrams, except that the positions of the lobes are now governed by the Bessel function $\mathcal{J}_l(K/\omega)$ (instead of $\mathcal{J}_{l-1}$); besides, similar conclusions hold for the nature of the most unstable mode.

3.3. Analytical approach

The numerical results described in the previous section can be understood using the analytical method developed in [40], which can indeed be readily transposed to the present model; see appendix for the full calculations.
The main idea is that the Bogoliubov equations of motion (2) can be mapped onto a parametric oscillator model [45, 62], a seminal model of periodically-driven harmonic oscillator known to display dynamical instabilities as soon as the drive frequency approaches twice its own (intrinsic) frequency. Similarly here, we find (see appendix) that each Bogoliubov mode \( q \) will display a dynamical instability (characterized by an exponential growth of its population) whenever the drive frequency \( \omega \) approaches its time-averaged Bogoliubov energy, namely, \( \omega \approx E_{av}(q) \), with

\[
E_{av}(q) = \sqrt{4|J_{eff}| \sin^2(q/2)(4|J_{eff}| \sin^2(q/2) + 2g)}.
\]

Note that the latter represents the Bogoliubov dispersion associated with the linearized GPE, based on the naive ‘interacting effective Hamiltonian’ in equation (7). As detailed in appendix, the growth rate \( s_q \) associated with this instability can be computed analytically using a perturbative method [40, 62]. From the knowledge of those individual rates, it is then straightforward to infer the total instability rate \( \Gamma \) and the most unstable mode (see equations (3) and (4)).

Figure 1 (bottom panel) shows the analytical stability diagram associated with the present model, as obtained from the aforementioned analytical perturbative method; as a technical remark, we point out that the latter calculation was performed up to second order with respect to the perturbation’s amplitude \( \alpha_q \) defined in equation (A11); see appendix and [40, 62] for details. It shows very good agreement with the numerical diagram of figure 1, and we attribute the small discrepancies to the perturbative nature of the method (higher order terms are generically expected to provide small corrections).

Importantly, the analytical approach explains the existence of stable regions in the vicinity of the cancellation points \( J_{eff} \approx 0 \), as identified in [44]. Indeed, when \( J_{eff} \) vanishes, the time-averaged Bogoliubov dispersion \( E_{av}(q) \) becomes trivially flat, so that no excitation mode fulfills the resonance criterion \( E_{av}(q) \approx \omega \) associated with the existence of parametric instability.

Besides, the analytical approach also offers a simple view on the boundaries separating stable and unstable regions; in particular, it predicts the nature of the most unstable mode responsible for the onset of instabilities in the vicinity of these boundaries. In the regime where \( \omega > 4|J_{eff}| \), no resonance can occur at \( g = 0 \), so that the system is stable; upon increasing \( g \), a first mode can become unstable, which is the one of maximum \( E_{av}(q) \),

\[
\Gamma = \sum s_q
\]

Figure 1. Stability of the resonantly-driven Wannier–Stark ladder. Numerical (top) and analytical (bottom) instability rate \( \Gamma \) as a function of the interaction strength \( g = U/\rho \) and the modulation amplitude \( K/\omega \). Here we set \( \Delta = 5J \).
namely $q = \pi$; see equation (8). Conversely, for $\omega < 4|J_{\text{eff}}|$, there always exists a particular mode fulfilling the resonance condition: the system is unstable at any interaction strength, and the most unstable mode is located at a certain momentum $q < \pi$ (at lowest order, $q_{\text{num}}$ is the mode at resonance; see equation (8)). We stress that such conclusions are similar to those found in the non-resonant case [40].

3.4. Extension to higher dimensional model

The present analysis can be straightforwardly extended to models featuring transverse directions (be it lattice or continuous directions).

For instance, in the case where a continuous transverse degree of freedom is present, we find the stability diagram of figure 2. As observed in [40], the presence of transverse modes enhance the instabilities by opening new instability channels. More precisely, in the case of an unbounded bandwidth, there always exists Bogoliubov modes that are resonant with the drive frequency $\omega$, as a consequence, instabilities occur at any finite interaction strength $g$. We note that, in contrast with the purely 1D case treated above, the cancellation of the effective tunneling does not result in a vanishing of $\Gamma$; hence, in this case, the stability regions located near the zeros of $J_{l}(K/\omega)$ are rather imputable to the cancellation of the perturbation’s amplitude $\alpha_q$ that is associated with the underlying (effective) parametric oscillator (equation (A11) in appendix), and which also scales as $J_{l}(K/\omega)$.

Finally, in that case, simple analytical formulas are obtained for both $\Gamma$ and $q_{\text{num}}$ (see appendix for details; here the index $x$ denotes the lattice direction):

(i) If $\omega > \sqrt{4|J_{\text{eff}}|(4|J_{\text{eff}}| + 2g)}$, one finds

$$q_{x_{\text{num}}} = \pi; \quad (q_{l_{\text{num}}}^2)^2/2m = \sqrt{g^2 + \omega^2} - g - 4|J_{\text{eff}}|.$$  \hspace{1cm} (9)

$$\Gamma = 2|J_{l}(K/\omega)| \frac{g}{\omega}. \hspace{1cm} (10)$$

(ii) If $\omega < \sqrt{4|J_{\text{eff}}|(4|J_{\text{eff}}| + 2g)}$, we find

$$q_{x_{\text{num}}} = 2 \arcsin \sqrt{\frac{g^2 + \omega^2 - g}{4|J_{\text{eff}}|}}; \quad q_{l_{\text{num}}}^2 = 0.$$  \hspace{1cm} (11)

$$\Gamma = (\sqrt{g^2 + \omega^2} - g) \frac{g}{\omega}.$$  \hspace{1cm} (12)

We note a strong similarity with the non-resonant case [40]; a notable difference, though, is the fact that the rate $\Gamma$ does not depend on the shaking amplitude $K$ in the low-frequency regime (equation (12)). These predictions, in particular the existence of two different regimes characterized by very specific dependences of $\Gamma$ and $q_{\text{num}}$ on the model parameters, could be readily tested by present-day experiments, providing a clear and unambiguous signature of parametric instabilities in resonantly-modulated systems.

![Figure 2. Stability of the resonantly-driven Wannier–Stark ladder in the presence of a transverse continuous dimension. Shown is the instability rate $\Gamma$ as a function of the interaction strength $g = U \rho$ and the modulation amplitude $K/\omega$. We set $\omega = \Delta = 5f$.](image)
4. Moving lattices with a space-dependent phase: the 1D case

We now consider another type of modulation scheme, which involves a main (primary) optical lattice that is perturbed by a (secondary) moving-lattice potential [26, 63]. Introducing a space-dependent phase in this moving-lattice potential has been shown to generate non-trivial effective gauge fields and topological band structures in the context of 2D neutral gases [17–21, 31]. Before addressing the case of 2D systems (section 5), where such gauge structures appear, we first investigate the properties of a simpler 1D toy model.

4.1. The model

We consider a 1D model described by the Hamiltonian

\[
\hat{H} = \sum_n \left[ -J (\hat{a}_{n+1}^\dagger \hat{a}_n + \text{h.c.}) + \frac{U}{2} \hat{a}_n^\dagger \hat{a}_n^\dagger \hat{a}_n \hat{a}_n + \Delta \hat{n}_n \right] + K \cos (\omega t + n\theta) \hat{a}_n^\dagger \hat{a}_n, \quad \theta = \pi,
\]

(13)

where \( \hat{a}_n \) annihilates a particle at lattice site \( n \), \( J > 0 \) denotes the tunneling amplitude of nearest-neighbor hopping, \( U > 0 \) is the repulsive on-site interaction strength, and \( \Delta > 0 \) is the energy difference between two consecutive sites. The second line of equation (13) captures the effects of the secondary (moving-lattice) potential; the latter is characterized by the amplitude \( K \), the frequency \( \omega \), and a phase difference of \( \theta = \pi \) between two consecutive sites. In the following, we consider the resonant case where \( \omega = \Delta \).

Before analyzing the existence of parametric instabilities in this model, we point out that such moving-lattice potentials generically produce momentum kicks, which can be directly revealed in momentum distributions [26]. However, these effects have a zero average over one period of the drive [7]; in particular, such momentum kicks do not influence the parametric instabilities explored in the present work.

It is convenient to perform our analysis in the rotating frame, defined by the unitary transformation

\[
R(t) = e^{i \sum_n \left[ \alpha \sin (\omega t + n\theta) \hat{a}_n^\dagger \hat{a}_n + \text{h.c.} \right]}
\]

with \( \alpha = K/\omega \). In this frame, the Hamiltonian reads

\[
\hat{H} = -J \sum_n \left[ e^{i(2\alpha(-1)^{n+1}) \sin (\omega t - \theta)} \hat{a}_n^\dagger \hat{a}_{n+1} + \text{h.c.} \right] + \frac{U}{2} \sum_n \hat{a}_n^\dagger \hat{a}_n^\dagger \hat{a}_n \hat{a}_n
\]

(14)

so that translational invariance (with a periodicity of two lattice sites) is restored.

In the absence of interactions \( (U = 0) \), we recall that the Floquet effective Hamiltonian associated with equation (14) simply reads [26]

\[
\hat{H}_\text{eff}^{(0)} = -J_{\text{eff}} \sum_n \left[ (-1)^{n+1} \hat{a}_{n+1}^\dagger \hat{a}_n + \text{h.c.} \right],
\]

(15)

where \( J_{\text{eff}} = J \mathcal{J}_-(2\alpha) \). The Hamiltonian in equation (15) has a periodicity of two lattice sites, so that its spectrum (shown in figure 3(a)) displays two energy bands,

\[
E_p^\pm = \pm 2|J_{\text{eff}} \sin p|.
\]

(16)

We note that the corresponding eigenstates are labeled by the quasi-momentum \( p \), which is defined in a reduced Brillouin zone \( p \in [-\pi/2; \pi/2] \). Here, the ground state corresponds to the state with \( p_0 = \pi/2 \) in the ‘–’ band; see figure 3(a). Thus, this state has alternate on-site amplitudes from one unit cell to the consecutive one; furthermore, depending on the sign of \( J_{\text{eff}} \), we find that its one-site coefficients are either equal or opposite within each cell; therefore, the ground state is found to be the same, globally, whatever the sign of \( J_{\text{eff}} \) (see figure 3(c)); this is in striking contrast with the models analyzed in the previous section 3 and in [40].

4.2. Numerical results

As previously in section 3, the initial state is again taken to be the ground state of the naive ‘interacting effective Hamiltonian’ (i.e. equation (7) with (15)), which is obtained numerically through imaginary-time propagation. Interestingly, we find that this state is still characterized by the quasi-momentum \( p_0 = \pi/2 \), even in the presence of interactions.

The stability diagram obtained from this initial state is displayed in figure 4, which shows the instability rate \( \Gamma \) as a function of the modulation amplitude and the interaction strength.

The stability diagram in figure 4 is dominated by a quasi ‘periodic’ structure, as a function of \( \alpha = K/\omega \), which arises from the dependence of the effective tunneling along the lattice direction, \( J \mathcal{J}_-(2\alpha) \); similarly to what was observed for other modulated band models (see [40, 44] and section 3.3), we note that stable regions
are indeed privileged when the effective tunneling $J_{\text{eff}} \approx 0$, which is consistent with the parametric resonance criterion $E_{\text{av}}(q) = \omega$ introduced in section 3.3.

This important observation allows one to anticipate the presence of stable regions in other time-modulation schemes. An immediate extension is the case where the phase difference $\theta$ between consecutive sites (which we took equal to $\theta = \pi$ in equation (13)) takes another value. Effective Hamiltonians for such models can be simply obtained using the formulas of [7]; for instance, considering a moving lattice with a phase difference of $\theta = \pi / 2$, we find an effective tunneling $J_{\text{eff}} = J_{\text{eff}}(4\alpha)$ (see [7] for details), which implies a stability diagram associated with narrower instability lobes. More generally, for models of the form given in equation (13) with arbitrary phase $\theta$, stable zones are obtained whenever $J_{\text{eff}}(\alpha p) \approx 0$, where $\alpha$ is the order of the resonance ($\Delta = \omega$) and where $p$ is the spatial periodicity of the phase $n\theta$ entering the time-modulation (i.e. the moving lattice).

Besides, our numerical calculations reveal that the onset of instability (i.e. the boundaries of the stability diagram in figure 4) is dominated by a most unstable mode, which in this case corresponds to the $q = 0$ mode; we remind that this momentum value is evaluated with respect to the ground-state, as always tacitly implied within our formalism. To understand this, let us compute the time-averaged Bogoliubov dispersion $E_{\text{av}}(q)$ for the present model, which yields

$$E_{\text{av}}(q) = \sqrt{4|J_{\text{eff}}|^2(1 + \cos^2 q) + 4|J_{\text{eff}}|g \pm \sqrt{16|J_{\text{eff}}|^2g^2\cos^2 q + 64|J_{\text{eff}}|^4\cos^2 q(|J_{\text{eff}}| + g)}}. \quad (17)$$

Figure 3. (a) Single-particle spectrum of the effective Hamiltonian in equation (15) for $J = 1$ and $\alpha = 0.5$; the ground state corresponds to the state $p_0 = \pi / 2$ in the first band. (b) Time-averaged Bogoliubov dispersion for the same parameters and $g = 3J$. The two branches are reminiscent of those of the single-particle spectrum (the global shift of $\pi / 2$ comes from the fact that excitation momenta are defined with respect to the ground-state, which is a $\pi / 2$ state), but these are modified by the interactions; in particular, Goldstone modes arise at $q = 0$ in the lowest band, as generically expected. (c) Qualitative picture of the ground-state in which condensation occur; this state does not depend on the sign of $J_{\text{eff}}$.

Figure 4. Stability in the presence of a 1D moving lattice with a spatially-dependent phase. Shown is the numerical instability rate of the model described by equation (13) as a function of the interaction strength $g = U/\rho$ and the modulation amplitude $K/\omega$; here we set $\omega = \Delta = 5J$. The quasi-repetition of patterns, as a function of $\alpha = K/\omega$, follows the dependence of the effective tunneling, $J_{\text{eff}}(2K/\omega)$. 
As shown in figure 3(b), this dispersion is made of two branches, which are reminiscent of the two-branch single particle spectrum, but are modified by the interactions. In particular, Goldstone modes arise at \( q \approx 0 \) in the lowest band, as generically expected as a result from the broken \( U(1) \) symmetry [64]. The absolute maximum is reached for \( q = 0 \) in the upper branch, which indeed precisely corresponds to the most unstable mode identified in our numerics: similarly to the previous model discussed in section 3, we thus find that the onset of parametric instability is governed by the mode of highest time-averaged Bogoliubov energy, which is consistent with the fact that this mode will be the first one to resonate [\( E_\nu(q = 0) = 0 \)] as one increases the interaction strength \( g \) [40]. We emphasize that this most unstable mode differs from the one identified for the resonantly-shaken Wannier–Stark ladder of section 3.

It is remarkable that the conclusions emanating from our analysis of the moving lattice are qualitatively similar to those related to other shaken-lattice models; see section 3 and [40]. This suggests a reliable and intuitive guideline to predict stable zones of the stability diagram and to identify the most unstable mode, based on the simple knowledge of the effective band structure (i.e. the effective tunneling and the time-averaged Bogoliubov dispersion).

5. The driven-induced Harper–Hofstadter model

In this section, we extend our previous analysis (section 4) to a two-dimensional setting, which has been exploited to realize the Harper–Hofstadter Hamiltonian in cold atoms [17–21, 31].

5.1. The model

We consider a two-dimensional extension of the previous model (equation (13)), which we define by the Hamiltonian

\[
\hat{H} = \sum_{m,n} \left\{ -j_x (\hat{a}_{m+1,n}^\dagger \hat{a}_{m,n} + \text{h.c.}) - j_y (\hat{a}_{m,n+1}^\dagger \hat{a}_{m,n} + \text{h.c.}) + [K \cos(\omega t + m\theta_x + n\theta_y) \\
+ \Delta u] \hat{a}_{m,n}^\dagger \hat{a}_{m,n} + \frac{U}{2} \hat{a}_{m,n}^\dagger \hat{a}_{m,n} \hat{a}_{m,n} \hat{a}_{m,n} \right\}, \quad \theta_x = \theta_y = \pi, \tag{18}
\]

where \( \hat{a}_{m,n} \) annihilates a particle at lattice site \((m, n)\), \( j_x \) (resp. \( j_y \)) is the tunneling along the \( x \) (resp. \( y \)) direction, and \( U \) models on-site repulsive interactions. A linear (Wannier–Stark) potential is introduced along the \( x \)-direction; besides the model features a moving lattice of amplitude \( K \), frequency \( \omega \) and a \( \pi \)-phase dependence along both directions (\( \theta_x, \theta_y = \pi \)). In the following, we set the resonance condition \( \omega = \Delta \). This model, and other variants, have been realized in several recent ultracold-atom experiments [17–21, 31]. Here, we focus on the \( \pi \)-flux configuration [19, 20], which corresponds to setting \( \theta_x = \pi \) in equation (18), but we note that several of our results hold for other values of the synthetic flux (see below).

In the absence of interactions \((U = 0)\), the effective Hamiltonian associated with equation (18) reads [7]

\[
\hat{H}_{\text{eff}} = - j_x^\text{eff} \sum_{m,n} \left\{ (-1)^{m+1} \hat{a}_{m+1,n}^\dagger \hat{a}_{m,n} + \text{h.c.} \right\} - j_y^\text{eff} \sum_{m,n} \left\{ \hat{a}_{m,n+1}^\dagger \hat{a}_{m,n} + \text{h.c.} \right\}, \tag{19}
\]

with \( \alpha = K/\omega \), \( j_{x,\text{eff}} = j_x \mathcal{J}_x(2\alpha) \), and \( j_{y,\text{eff}} = j_y \mathcal{J}_y(2\alpha) \). This corresponds to the Harper–Hofstadter model [32], with a magnetic flux \( \Phi = \pi \) in each unit cell, and with different effective tunnelings along the \( x \) and \( y \) directions. As stated above, other choices for the moving-lattice phase (\( \theta_x \neq \pi \)) lead to different fluxes per unit cell [7, 18, 31]. We recall that the Harper–Hofstadter model displays the well-known ‘Hofstadter’s butterfly’ spectrum [32], a rich fractal structure that hosts Dirac semimetals and Chern insulating phases [33]. In this sense, at the single-particle level, the time-dependent model under consideration (i.e. equation (18) with \( U = 0 \)) is one of the simplest systems realizing artificial gauge fields and topological band structures for neutral atoms in 2D optical lattices [17–21, 31]; we point out that a BEC was recently realized in the \( \pi \)-flux configuration of this model, which further motivates the present study [20].

5.2. Numerical results

In order to extract the instability properties of the 2D model in equation (18), we now apply the same procedure as for the 1D model of section 4. The initial state is again chosen as the ground state of the naive ‘interacting

\[5\] The global shift of \( \pi/2 \) comes from the fact that excitation momenta are defined with respect to the ground-state, which is a \( \pi/2 \) state in the present case.
effective Hamiltonian; however, we point out that a complexity arises here from the degeneracy of this condensation state [32, 65].

To see this, one observes that the single-particle effective Hamiltonian in equation (19) is periodic over a $2 \times 2$ cell, so that its eigenstates are labeled by the quasi-momentum $p$ defined in the reduced Brillouin zone $p_x, p_y \in [-\pi/2, \pi/2]$. The spectrum features two energy bands (labeled ±)

$$E^\pm_p = \pm 2 \sqrt{J_{\text{eff}}^x \sin^2 p_x + J_{\text{eff}}^y \cos^2 p_y},$$

which are both twofold degenerate (see figures 5(a), (b)). The ground state corresponds to the state with $p = (\pi/2, 0)$ in the ‘−’ band, and is twofold degenerate.

In the presence of interactions, the ground state still features this twofold degeneracy [65], as well as the momentum characteristic ($p_x = \pi/2, p_y = 0$). Therefore, several choices can a priori be made for the initial state of our analysis, within the whole degenerate ground space, as we now explore.

5.2.1. Stability diagram and sensitivity to the ground state

Figure 6 shows the instability diagrams obtained for two different orthogonal ground-states in this subspace. At first sight, they are found to be very similar in the sense that the rates are of the same order of magnitude, and the diagrams feature very similar structures as a function of the model parameters; in particular, the same ‘periodicity’ as in figure 4 is observed as a function of modulation amplitude $\alpha = K/\omega$. Yet, the exact positions of the stable and unstable regions are shifted, reflecting that the later are very sensitive to the ground-state in which the system is prepared.

The later observation that stable regions cannot be simply obtained by analyzing the behavior of the effective tunneling $I_{\text{eff}}^{x0}$ (section 3.3) can be related to the fact that these quantities never cancel simultaneously; indeed, in this specific 2D model, the time-averaged Bogoliubov dispersion never vanishes, meaning that the simple ‘flat-band’ criterion of section 3.3 cannot be applied (a priori, there might always exist some modes fulfilling the resonance condition $E_{\text{av}} = \omega$). Consequently, we conjecture that the stable zones in figure 4 should rather be governed by the amplitude of the perturbation entering the underlying effective parametric-oscillator model (i.e. the analog of the quantity $q_0$ that was defined in equation (A11), in appendix, for the model of section 3); although its full analytical derivation cannot be obtained in the present case, it is natural to believe that this amplitude is indeed ground-state dependent, since the Bogoliubov treatment leading to this effective parametric-oscillator model relies on the actual condensation state that is chosen within the degenerate ground manifold.

5.2.2. Most unstable mode

In contrast, we find that the most unstable mode associated with the onset of the instability is robust with respect to the ground-state. Indeed, independently of the ground state, and within the whole parameter range of the diagram, we find that this most unstable mode always corresponds to the $q = 0$ mode (with respect to the ground-state). This can again be accounted for by the fact that it is the mode of highest time-averaged Bogoliubov energy. To verify this, we calculate the time-averaged Bogoliubov spectrum $E_{\text{av}}$, which is obtained by numerically diagonalizing the Bogoliubov Hamiltonian derived from the GPE combined with the static effective Hamiltonian in equation (19); we note that analytics exist in the symmetric case where $I_{\text{eff}}^x = I_{\text{eff}}^y$, see [65]. The resulting Bogoliubov spectrum $E_{\text{av}}$ is plotted in figure 5(c). Consistently with the general features reported in [65], this spectrum is made of four branches (the degeneracy observed at the single-particle level being lifted) and indeed has its absolute maximum at $q = 0$ in the upper branch.

Interestingly, this behavior is in fact expected to be very generic. For a model configuration leading to a flux $\Phi = 1/3$, i.e. $\theta_y = \pi/3$, we also find a most unstable mode at $q = 0$, consistently with the fact that it is the mode of highest time-averaged Bogoliubov energy (see [65] for the Bogoliubov spectrum in that case).

We conclude this section by noting that the stability diagram of the driven-induced Harper–Hofstadter model (figure 6) displays relatively large stability regions, which extend up to significant values of the interaction strength $g$, hence suggesting potentially favorable regimes of operation (as far as parametric instabilities are concerned). However, one should point out that current experiments typically feature a transverse (‘tube’) direction, which is generically expected to trigger or increase instabilities [40]. Besides, our result that the stability diagram strongly depends on the prepared ground state appears as an important feature of these time-modulated systems, which should be taken in account in experiments.

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6. Let us stress that the effective Hamiltonian in equation (19) obtained from equation (18), does not correspond to the Landau gauge; hence the difference compared to the more common version of the Harper–Hofstadter model whose spectrum is $E^\pm_p = \pm 2 \sqrt{J_{\text{eff}}^x \cos^2 p_x + J_{\text{eff}}^y \cos^2 p_y}$ and ground-state in $p_x = p_y = 0$. 

6. Conclusion

In this work, we analyzed the parametric instabilities that occur in a variety of resonantly-driven band models. While the general method used to identify and characterize these instabilities had already been applied to non-resonant models [40, 44], we have hereby demonstrated its usefulness and flexibility to address the relevant case of resonant time-modulations. *Ab initio* predictions for instability rates and for the most unstable mode (which appears to be the most robust and directly accessible signature of parametric instabilities) were obtained; these could be directly tested in current ultracold-atom experiments [17–21, 31], for instance, in view of optimizing their operating regimes.

Figure 5. (a) Single-particle spectrum of the effective Hamiltonian in equation (19) for $J_{x,y} = 1$ and $\alpha = 0.5$; each branch is twofold degenerate, and the two ground-states correspond to states at $p_x = \pi/2, p_y = 0$ in the $1^{st}$ band; (b) cut of the single-particle spectrum at $p_y = 0$; (c) time-averaged Bogoliubov dispersion for the same parameters and $g = 3 f$, plotted as a function of $q_x$ at fixed $q_y = 0$. The four branches are reminiscent of the two branches of the single-particle spectrum, with a lifting of their twofold degeneracy through interactions; the global shift of $\pi/2$ in the $x$-direction comes from the fact that excitation momenta are defined with respect to the ground-state, which is a $p_x = \pi/2$ state; in particular, Goldstone modes arise at $q \approx 0$ in the lowest band, as generically expected.
Our study has confirmed the generic role played by parametric resonances in the instabilities that appear in these driven systems, and which directly involve the drive frequency and the dispersion of the Floquet–Bogoliubov modes. In particular, while genuine analytical results can only be obtained in simple cases, many qualitative features of these instabilities appear to be generic, hence providing an intuitive picture that could guide experiments to stable regimes of operation.

On the one hand, instability rates exhibit a dependence on the modulation amplitude that is, in most cases, governed by the effective tunneling, and favorable regions are generally found whenever this effective tunneling is weak (based on the resonance criterion $E_{av} = \omega$ underlying parametric instabilities and the fact that $E_{av} \approx 0$ in these regions); while this conclusion is justified in models where all effective tunneling amplitudes can simultaneously vanish (as in the simple 1D models discussed in this work), it should however be treated with care in higher dimensions, where stability regions seem to be rather governed by the amplitude of the perturbation entering the underlying effective parametric–oscillator model (i.e. the analog of the quantity $\alpha_q$ in equation (A11) in appendix); in particular, in the presence of degenerate ground states, as in the ‘Harper–Hofstadter’ case treated in section 5, the instability rates were found to be very sensitive to the actual ground-state in which the system was prepared. Besides, we note that the energy scales of the (effective) system mostly rely on the effective tunneling, which indicates that a compromise must be found in actual implementations (in the context of fractional Chern insulators, it is crucial to maximize the size of topological gaps in view of generating strongly-correlated states in the presence of finite temperature).

On the other hand, for models exhibiting a finite bandwidth, the most unstable mode responsible for the onset of instability is always found to be that of highest effective Bogoliubov energy $E_{av}$ (for experimental values of the drive frequency [40]). When a continuum is present and the dispersion is unbounded, the most unstable mode is found (at lowest order) to fulfill the resonance condition $E_{av}(q_{mun}) = \omega$. Therefore, the simple knowledge of the effective band structure ($J_{eff}$ and $E_{av}$) allows one to anticipate such behaviors, which are expected to be directly relevant to experiments.

Altogether, this work constitutes a first step in view of controlling and exploiting the potentialities of interacting modulated BECs. In particular, achieving stable BECs in the context of time-modulated optical lattices, with reduced instabilities and losses, will constitute a first step towards the cold-atom realization of
strongly-correlated states of matter with topological features, such as fractional Chern insulators [34–38]. However, many open problems remain. First of all, this study was performed in the mean-field interaction regime, which is expected to give access to the short-term dynamics. Longer-time dynamics, including the possibility for thermalization, are dominated by nonlinear couplings between the excitation modes and the condensate, which go beyond the present Bogoliubov treatment. Moreover, the analysis presented in this work neglects the effects of higher bands, which are indeed expected to be weak when setting the drive frequency within a gap of the (effective) spectrum; however, higher-band effects are also expected to become important at longer times. Then, independently from the driving scheme itself and the obtained stability properties, a central question is that of the preparation of the initial state, as loading the system into a desired eigenstate with highest fidelity is far from being trivial: one solution could be to apply adiabatic perturbation theory in the presence of the periodic drive [57, 58], but there might exist other alternatives and their impact on the stability properties of the prepared state remains uncharacterized. Finally, the interplay between parametric instabilities and other instability mechanisms neglected in our approach, especially inter-band transitions [39, 51, 52], is expected to lead to rich behaviors that still remain to be studied.

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Appendix. Analytical treatment of the resonantly-driven Wannier–Stark ladder

We present the analytical derivation of instability rates for the resonantly-driven Wannier–Stark ladder of section 3. For the sake of generality, we consider an even more general situation where the modulation has a non-trivial uniform phase, and with possible additional transverse directions in the model. We thus deal with the Hamiltonian

\[ \hat{H}(t) = -\int \sum_{n,r} \left( \hat{a}_{n+1,r}^\dagger \hat{a}_{n,r} + \text{h.c.} \right) + \hat{H}_L + \frac{U}{2} \sum_{n,r} \left( \hat{a}_{n+1,r}^\dagger \hat{a}_{n+1,r} \hat{a}_{n,r} \hat{a}_{n,r}^\dagger + \Delta \hat{a}_{n+1,r}^\dagger \hat{a}_{n+1,r} \right) + K \cos(\omega t + \phi) n \hat{a}_{n+1,r}^\dagger \hat{a}_{n+1,r}, \]  \[ (A1) \]

where \( \hat{a}_{n,r} \) annihilates a particle at lattice site \( n \) and transverse position \( r \), \( J > 0 \) denotes the tunneling amplitude of nearest-neighbor hopping along the \( x \)-direction, \( \hat{H}_L \) describes a kinetic part along transverse directions (be it a lattice or a continuous one), and \( U > 0 \) is the repulsive on-site interaction strength. The on-site potential term is made of a Wannier–Stark ladder along the \( x \)-direction introducing an energy shift \( \Delta > 0 \) between consecutive sites, and a time-periodic modulation along the \( x \)-direction of amplitude \( K \), phase \( \phi \), and frequency \( \omega = 2\pi/T \). The frequency modulation \( \omega \) is chosen to be resonant with the offset \( \Delta (\Delta = k\omega) \) with \( l \) denoting some integer.

We will perform all the theoretical analysis in the rotating frame (where translational invariance is restored), which is defined by the unitary transformation \( \hat{R}(t) = e^{-i(\alpha \sin(\omega t + \phi) + \Delta t)\hat{X}} \) (with \( \hat{X} \equiv \sum_n n \hat{a}_{n+1,r}^\dagger \hat{a}_{n,r} \) the position operator on the lattice).

A.1. Condensate dynamics and Bogoliubov equations

The single-particle quasienergy spectrum of this model reads

\[ \epsilon_q = -2J_{\text{eff}} \cos(q_x + l\phi) + \epsilon^{\parallel}(q_x), \]  \[ (A2) \]

where \( J_{\text{eff}} = J_-(\alpha) \) is the effective tunneling (with \( \alpha = K/\omega \)) and \( \epsilon^{\parallel} \) is the dispersion associated with transverse directions. The effective ground-state, which is taken as the initial state in which the system is supposed to be prepared, is a Bloch state \( e^{i0_{\text{eff}}(\hat{p}_x + e^{\parallel}(q_x)\hat{p}_y)\theta} \) of momentum \( p_x = 0 \), and \( p_y = -\hbar\omega \) if \( J_{\text{eff}} > 0 \) or \( p_y = \pi - \hbar\omega \) if \( J_{\text{eff}} < 0 \).

Taking this state, normalized to the condensate density \( \rho_0 \), as our initial condition \( \hat{a}_{n,r}^{0\parallel}(t = 0) \) for the propagation of the tGPE, we find that the solution of the (tGPE) reads

\[ \hat{a}_{n+1,r}^{0\parallel}(t) = \sqrt{\rho_0} e^{i0_{\text{eff}}(\Delta t - i\cos(\omega t + \phi)\theta)n_0 2J(\cos(\omega t) - \sin(\omega t))\epsilon^{\parallel}(0) e^{-i0_{\text{eff}}(0)} e^{-it}}, \]  \[ (A3) \]
where we have introduced the functions

\[ S(t) = \sum_{m=-\infty}^{\infty} \frac{\sin[(m\omega + \Delta)t + m\phi] - \sin(m\phi)}{m\omega + \Delta} J_m(\alpha) \]

and

\[ C(t) = \sum_{m=-\infty}^{\infty} \frac{\cos[(m\omega + \Delta)t + m\phi] - \cos(m\phi)}{m\omega + \Delta} J_m(\alpha). \]

Plugging it into the Bogoliubov–de Gennes equations, and taking advantage of the translational invariance to rewrite these in momentum space, we find that the dynamics of a momentum mode \( q \) is governed by the equation

\[
i\hbar \partial_t \begin{pmatrix} u_q \\ v_q \end{pmatrix} = \begin{pmatrix} \varepsilon(q, t) + g & g \\ -g & -\varepsilon(-q, t) - g \end{pmatrix} \begin{pmatrix} u_q \\ v_q \end{pmatrix}, \tag{A4} \]

where

\[
\varepsilon(q, t) = 4J \sin \frac{q}{2} \sin \left( \frac{q}{2} + p_x - \alpha \sin(\omega t + \phi) - \Delta t \right) + \varepsilon^+_{q, c}.
\]

**A.2. Mapping on a parametric oscillator model**

Proceeding as in [40], we then introduce the two following changes of basis: first,

\[
\begin{pmatrix} u_q \\ v_q \end{pmatrix} = \begin{pmatrix} \cosh(\theta_q) & \sinh(\theta_q) \\ \sinh(\theta_q) & \cosh(\theta_q) \end{pmatrix} \begin{pmatrix} u_q' \\ v_q' \end{pmatrix}, \tag{A5} \]

where \( \cosh(2\theta_q) \equiv (\varepsilon_{av}(q) + g)/E_{av}(q) \) and \( \sinh(2\theta_q) \equiv g/E_{av}(q) \) with

\[
E_{av}(q) = \sqrt{\varepsilon_{av}(q)(\varepsilon_{av}(q) + 2g)}, \tag{A6} \]

the time-averaged Bogoliubov dispersion and

\[
\varepsilon_{av}(q) = 4J[\alpha\sin^2(q_x/2) + \varepsilon^+_{q, c}],
\]

and then,

\[
\begin{pmatrix} \hat{u}_q' \\ \hat{v}_q' \end{pmatrix} = \begin{pmatrix} e^{2iE_{av}(q)t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_q \\ v_q \end{pmatrix}. \tag{A7} \]

This allows us to write the Bogoliubov equations under the form

\[
i\hbar \partial_t \begin{pmatrix} \hat{u}_q' \\ \hat{v}_q' \end{pmatrix} = E_{av}(q) \hat{1} + \hat{W}_q(t) + \sinh(2\theta_q) \begin{pmatrix} 0 & h_q(t)e^{-2iE_{av}(q)t} \\ -h_q(t)e^{2iE_{av}(q)t} & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_q' \\ \hat{v}_q' \end{pmatrix}, \tag{A8} \]

where \( E_{av}(q) \) is the time-averaged Bogoliubov dispersion associated with the effective GPE, within the Bogoliubov approximation (see equation (A6)), \( \hat{W}_q(t) \) is a diagonal matrix of zero average over one driving period,

\[
\hat{W}_q(t) = 4J\sin \frac{q}{2} \sum_{m=0}^{\infty} J_m(\alpha) \times \begin{pmatrix} \cosh 2\theta_q \sin \frac{q}{2} \cos [(m + l)(\omega t + \phi)] - \cos \frac{q}{2} \sin[(m + l)(\omega t + \phi)] & 0 \\ 0 & \cosh 2\theta_q \cos \frac{q}{2} \sin[(m + l)(\omega t + \phi)] + \sin \frac{q}{2} \cos [(m + l)(\omega t + \phi)] \end{pmatrix},
\]

and \( h_q(t) \) is a (real-valued) function,

\[
h_q(t) = 4J \sin^2(q_x/2) \sum_{m=0}^{\infty} J_m(\alpha) \cos[(m + l)(\omega t + \phi)]. \tag{A9} \]

As shown in [40], equation (A8) are formally equivalent to a set of independent parametric oscillators (one for each mode \( q \)), since they describe harmonic oscillators of eigenfrequencies \( E_{av}(q) \) driven by a weak time-periodic modulation of frequency \( \omega \). As can be anticipated from a RWA treatment of equation (A8), a parametric instability will thus appear in the system as soon as one of the harmonics of the modulation is close to twice the energy of any of the (effective, time-averaged) Bogoliubov modes, \( 2E_{av}(q) \), i.e. \( m\omega \approx 2E_{av}(q) \) (resonance condition). The later will result in a long-term explosion in the (stroboscopic) dynamics of the corresponding mode. Assuming that the harmonics can be treated independently, the instability rate of a given mode \( q \) and a
given harmonics can be analytically computed following the perturbative method developed in [40, 62], and the total instability rate and most unstable mode straightforwardly inferred from equations (3) and (4).

A.3. Explicit analytics in the strict resonant case $\omega = \Delta$

As an illustration, we now consider the simple case $l = 1$ (i.e. $\omega = \Delta$) and $\phi = 0$ addressed in the main text. For the typical parameters used in figure 1, one immediately sees that only the smaller harmonics of the modulation will allow the resonance condition to be fulfilled; this corresponds to the terms $m = \pm 1$ in equation (A9). We can thus disregard other harmonics, as well as the term $W_q(t)$ which has no long-term stroboscopic contribution, and we get the effective model

$$i\hbar \partial_t \begin{pmatrix} \hat{a}_q' \\ \hat{a}_q'' \end{pmatrix} = \begin{pmatrix} E_{\omega}(q) \mathbf{i} + \frac{\alpha_q}{E_{\omega}(q)} \begin{pmatrix} 0 \\ -\cos(2\omega t) e^{2iE_{\omega}(q)t} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \hat{a}_q' \\ \hat{a}_q'' \end{pmatrix}. \tag{A10}$$

with

$$\alpha_q = 8|\mathcal{J}_{-1}(\alpha)|\sin^2(qx/2) g E_{\omega}(q)^2. \tag{A11}$$

As shown in [40, 45], equation (A10) is indeed formally equivalent to the equations of motion describing a parametric oscillator of eigenfrequency $E_{\omega}(q)$, driven by a perturbation of amplitude $\alpha_q$ and frequency $2\omega$. The quantity $\alpha_q$ will thus be referred to as the perturbation’s amplitude in the effective parametric oscillator model; it is the small parameter of the perturbative method used to compute the instability rates [40, 62]. At lowest order in $\alpha_q$, we find in this case

$$\Gamma = \max_q \left[ G \frac{2|\mathcal{J}_{-1}(\alpha)|\sin^2(qx/2)}{E_{\omega}(q)} \sqrt{1 - \left| \frac{\omega - E_{\omega}(q)}{2|\mathcal{J}_{-1}(\alpha)|\sin^2(qx/2) g E_{\omega}(q)} \right|^2} \right]. \tag{A12}$$

In the purely 1D case (no transverse direction), this yields the analytical stability diagram on the bottom panel of figure 1 (although the latter is actually computed using the analytical expression at second order, which is an implicit equation). In that case, the most unstable mode is $q_{x,\text{num}} = \pi$ at the onset of instability.

In the case where a continuous degree of freedom is present, we find two regimes:

(i) If $\omega > \sqrt{4|\mathcal{J}_{\text{eff}}||4|\mathcal{J}_{\text{eff}}| + 2G}$, one finds

$$q_{x,\text{num}} = \pi; \quad (q_{x,\text{num}}^2/2m = \sqrt{G^2 + \omega^2} - g - 4|\mathcal{J}_{\text{eff}}|). \tag{A13}$$

$$\Gamma = 2|\mathcal{J}_{\text{eff}}(K/\omega)| \frac{G}{\omega}. \tag{A14}$$

(ii) If $\omega < \sqrt{4|\mathcal{J}_{\text{eff}}||4|\mathcal{J}_{\text{eff}}| + 2G}$, we find

$$q_{x,\text{num}} = 2\arcsin \sqrt{\frac{\sqrt{G^2 + \omega^2} - g}{4|\mathcal{J}_{\text{eff}}|}}; \quad q_{x,\text{num}}^2 = 0. \tag{A15}$$

$$\Gamma = (\sqrt{G^2 + \omega^2} - g) \frac{G}{\omega}. \tag{A16}$$

References


7 Interestingly however, we find that this term can play a role for higher values of $l$, with non trivial effects arising from the non-commutation between the diagonal $W_q(t)$ and the off-diagonal matrix in equations (A8).
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