ON THE PARALLEL SPECTRUM IN MAGNETOHYDRODYNAMIC TURBULENCE

ANDREY BERESNYAK

Los Alamos National Laboratory, Los Alamos, NM 87545, USA Nordita, KTH Royal Institute of Technology and Stockholm University, SE-10691 Stockholm, Sweden Received 2014 December 24; accepted 2015 January 24; published 2015 February 27

ABSTRACT

Anisotropy of MHD turbulence has been studied extensively for many years, most prominently by measurements in the solar wind and high-resolution simulations. The spectrum parallel to the local magnetic field was observed to be steeper than the perpendicular spectrum, typically k^{-2} , consistent with the widely accepted Goldreich & Sridhar model. In this Letter, I looked deeper into the nature of the relation between parallel and perpendicular spectra and argue that this k^{-2} scaling has the same origin as the ω^{-2} scaling of the Lagrangian frequency spectrum in strong hydrodynamic turbulence. This follows from the fact that Alfvén waves propagate along magnetic field lines. It has now became clear that the observed anisotropy can be argued without invocation of the "critical balance" argument and is more robust that was previously thought. The relation between parallel (Lagrangian) and perpendicular (Eulerian) spectra is an inevitable consequence of strong turbulence of Alfvén waves, rather than a conjecture based on the uncertainty relation. I tested this using high-resolution simulations of MHD turbulence, in particular, I verified that the cutoff of the parallel spectrum scales as a Kolmogorov timescale, not lengthscale.

Key words: magnetohydrodynamics (MHD) – turbulence

1. INTRODUCTION

Astrophysical and space plasmas are well conductive and can be described as a magnetohydrodynamic fluid, which is usually turbulent (Armstrong et al. 1995; Biskamp 2003; Beresnyak & Lazarian 2014). At the same time, high-quality measurements in the solar wind have been available for more than two decades (Goldstein et al. 1995). The presence of a large-scale magnetic field is expected to change the dynamics dramatically. Analytic weak turbulence theory (Galtier et al. 2000) found that a turbulent cascade proceeds in the direction perpendicular to the mean field, resulting in stronger turbulence, which should eventually break down weak turbulence on sufficiently small scales. Earlier similar qualitative arguments lead Goldreich & Sridhar (1995) to suggest that the inertial range of MHD turbulence will be strong turbulence. They also argued that such turbulence will be "critically balanced," or marginally strong, with the linear propagation term always contributing comparably with the nonlinear interaction term, predicting the $k_{\parallel} \sim k_{\perp}^{2/3}$ anisotropic cascade, which has found support in numerics (e.g., Cho & Vishniac 2000; Maron & Goldreich 2001). The $k_{\perp}^{-5/3}$ perpendicular dicular spectrum and the $k_{\parallel} \sim k_{\perp}^{2/3}$ anisotropy results in a k_{\parallel}^{-2} parallel spectrum. One observation is paramount for our understanding of this parallel spectrum. While Goldreich & Sridhar (1995) suggested a closure model predicting the $k_{\parallel} \sim k_{\perp}^{2/3}$ anisotropy in the frame associated with the global mean field, it was not observed in Cho & Vishniac (2000); rather, this anisotropy was observed in the structure function measurement performed in the frame associated with the local magnetic field. Similarly, Horbury et al. (2008) observed the k_{\parallel}^{-2} parallel scaling using the wavelet technique and associating parallel direction to the direction of the local field.

Given the importance of the parallel spectrum for a variety of phenomena, e.g., resonant scattering of solar energetic particles, the measurement of the parallel spectrum in the solar wind attracted considerable attention (see, e.g., Horbury et al. 2008; Osman & Horbury 2009; Podesta 2009; Luo & Wu 2010; Wicks et al. 2010, 2011). These measurements followed the prescription of the local field direction and generally confirmed the k_{\parallel}^{-2} scaling; however, the debate surrounding the critical balance argument and the nature of anisotropy continued in Grappin & Müller (2010) and Grappin et al. (2013).

In this Letter, I will argue that there is a conceptually simpler way to look at the MHD anisotropy, namely, as a relation between Lagrangian and Eulerian spectra. I will also introduce the statistically averaged one-dimensional spectrum along the field line and show that high-resolution numerics support steep parallel spectra, consistent with k_{\parallel}^{-2} , just like in the solar wind measurements.

2. STRONG TURBULENCE AND THE LAGRANGIAN SPECTRUM

Strong turbulence was suggested to be scale-local and selfsimilar in Kolmogorov (1941), which led to his $k^{-5/3}$ Eulerian power spectrum of velocity perturbations. Another basic spectrum of hydrodynamic turbulence is called the Lagrangian frequency spectrum, which statistically evaluates how the velocity of the fluid element changes with time. Assuming that the dot product of the total time derivative of the velocity and the velocity vector itself is work done upon a fluid element, one could estimate $\delta v_{\tau} \cdot \delta v_{\tau}/\tau$ as the turbulence energy cascade rate per unit mass ϵ , measured in cm² s⁻³, also known as the dissipation rate. More precisely, in stationary turbulence, the second-order structure function of velocity should satisfy

$$SF(\tau) = \left\langle (\mathbf{v}(t+\tau) - \mathbf{v}(t))^2 \right\rangle \approx \epsilon \tau \tag{1}$$

in the inertial range, where v(t) is a velocity of a given fluid element. Such a time structure function is dual to the frequency spectrum of $E(\omega) = \epsilon \omega^{-2}$ (Landau & Lifshitz 1944; Corrsin 1963; Tennekes & Lumley 1972). The cutoff of this spectrum is associated with the timescale of critically viscously damped eddies—the Kolmogorov timescale $\tau_{\eta} = (\nu/\epsilon)^{1/2}$, which has a different dependence on the Reynolds number Re = vL/ν compared to the Kolmogorov lengthscale $\eta = (\nu^3/\epsilon)^{1/4}$ —the cutoff of the Eulerian spectrum.

Magnetized dynamics is qualitatively different from hydrodynamics in that locally there is always a propagating wave characteristic. In particular, following a fluid element, we may find oscillations associated with the wave train that propagates though this fluid element in the direction of the local mean magnetic field, which makes the classic Lagrangian measurement of limited value. Therefore, the Lagrangian evolution in MHD takes on a different meaning. The Alfvén perturbations decomposed into can be Elsässer components $w^{\pm} = v \pm B/\sqrt{4\pi\rho}$, each of which propagate either along or against the local field direction, i.e., along the magnetic field line. The functional dependence of such perturbations will take the form $f(s \mp v_A t)$ in the absence of interaction, where s is a distance along the field line. If the nonlinear interaction is present, the trajectory $s = \pm v_A t$ would act as an analogy to hydrodynamic fluid element trajectory if we want to study Lagrangian dynamics in MHD.

The above argument suggests that following the evolution of w^+ and w^- along the field line in fixed time and in the positive direction in *s* would be equivalent to following the evolution of w^+ backward in time and w^- forward in time. This simple argument already has been fruitful in explaining the asymmetric Richardson diffusion of magnetic field lines (Beresnyak 2013). As far as the frequency spectrum goes, the sign of time is unimportant, and any measurement of the power spectrum along the field line of either v, B or w^{\pm} will be analogous to the Lagrangian frequency spectrum with frequency f replaced by wavenumber f/v_A : $E(k_{\parallel}) \sim ev_A^{-1}k_{\parallel}^{-2}$. The spatial structure function will be expressed correspondingly as $SF_{\parallel}(l) = elv_A^{-1}$.

Another way to argue the k_{\parallel}^{-2} parallel scaling is the dimensional argument using the Alfvén symmetry of reduced MHD in Beresnyak (2012). Indeed, this symmetry dictates that changing v_A while keeping $k_{\parallel}v_A$ constant leaves equations unchanged. Therefore, one must keep energy $E(k_{\parallel})dk_{\parallel}$ constant under such transformation, which requires that $E(k_{\parallel}) \sim v_A^{-1}$. Using scale-locality, i.e., assuming that the spectrum can only depend on v_A , ϵ and k_{\parallel} , we arrive at

$$E\left(k_{\parallel}\right) = C_{\parallel} \epsilon v_A^{-1} k_{\parallel}^{-2}, \qquad (2)$$

where C_{\parallel} is dimensionless constant. Logically, this dimensional argument is a restatement of the Lagrangian spectrum argument. Note that the parallel second-order spectrum scales linearly with the dissipation rate ϵ , similar to the *third-order* Eulerian scaling and not to $\epsilon^{2/3}$ scaling of the second-order Eulerian spectrum.

Unlike reduced MHD, full MHD has no exact Alfvén symmetry. The arguments in favor of using it in the inertial range are still quite compelling (Beresnyak 2012). It is interesting to check if the parallel spectrum still follows Equation (2) not only in Alfvénic MHD but in the general MHD case. Especially interesting is the case with zero mean magnetic field where the v_A will be determined only by local fluctuations.

3. NUMERICS

The first half of the numerical data is from my DNS of strong reduced MHD turbulence (Beresnyak 2014), which are well-

 Table 1

 Three-dimensional MHD and RMHD Simulations

Run	N^3	Dissipation	v_A	ϵ	η	$v_A \tau_\eta$
MHD1 MHD2	1536 ³ 1536 ³	$-5 \times 10^{-10} k^4 -6.2 \times 10^{-10} k^4$	0.73 1.53	0.091 0.728	0.0021 0.0018	0.026 0.025
M1 M2 M3	$ 1024^3 \\ 2048^3 \\ 4096^3 $	$\begin{array}{c} -1.75 \times 10^{-4} k^2 \\ -7 \times 10^{-5} k^2 \\ -2.78 \times 10^{-5} k^2 \end{array}$	1 1 1	0.06 0.06 0.06	0.0031 0.00155 0.00077	0.044 0.028 0.017
M1H M2H M3H	$ 1024^3 \\ 2048^3 \\ 4096^3 $	$\begin{array}{c} -1.6\times 10^{-9}k^4\\ -1.6\times 10^{-10}k^4\\ -1.6\times 10^{-11}k^4\end{array}$	1 1 1	0.06 0.06 0.06	0.0030 0.00152 0.00076	0.045 0.029 0.018

resolved, statistically stationary driven simulations intended to precisely calculate averaged quantities. Note that reduced MHD, i.e., Alfvén dynamics, does not depend on plasma pressure and can be applied to situations with different values of plasma β , from zero to infinity. I list the most important parameters of these simulations in code units in Table 1 under rows M1-3 and M1H-3H. The only difference between rows M1-3 and M1H-3H was that the latter were performed with higher-order diffusivities. Additionally, I performed simulations of statistically isotropic driven incompressible MHD turbulence with zero mean field with parameters presented in Table 1, rows MHD1-2. For all cases, I have calculated the spectra along the magnetic field line, and for the reduced MHD cases, I additionally have calculated the one-dimensional spectra along the x direction, which was the global mean field direction.

Three dimensional numerics have modest Re and are are always affected by the finite Re effects. I used a rigorous scaling study method, fairly common in the analysis of experimental data and DNS (Sreenivasan 1995; Gotoh et al. 2002; Kaneda et al. 2003; Beresnyak 2012, 2014), which compares spectra from simulations with several different Re values on the same plot with dimensionless axes. The parallel spectrum was plotted versus dimensionless wavenumber $kv_A \tau_{\eta}$ and compensated by $k^2 \epsilon^{-1} v_A$ to see how the scaling is consistent with (2). This measurement is presented in Figure 1. For the reduced MHD case, the spectra collapsed on the dissipation scale, corresponding to an overall scaling of k^{-2} .

Given that reduced MHD has precise Alfvén symmetry and the requirement of turbulence to be strong on the outer scale assumes a certain value of ϵ , it does not allow us to check the linear scaling with ϵ in Equation (2), as I could not vary ϵ in M1-3. I used statistically isotropic MHD simulations with zero mean field MHD1-2, for which Alfvén symmetry is absent and the inertial range scaling (2) can not be rigorously argued based on units. Despite that, the standard argumentation, introduced by Iroshnikov (1964) and Kraichnan (1965) is that the rms magnetic field can play the role of the local mean field and this could still be regarded as the strong mean field limit. I conjecture that the parallel spectrum will still follow Equation (2) in the inertial range in this case as well. In the MHD case, I used simulations with different ϵ and substituted the rms field instead of v_A in Equation (2). Figure 1 demonstrates that there is an inertial range convergence to k^{-2} even in this zero mean field case. The linear scaling with ϵ , not $\epsilon^{2/3}$, is also confirmed.



Figure 1. Energy spectrum along the magnetic field line compensated by the theoretical scaling ek_{\parallel}^{-2} (2). Solid, dashed, and dashed–dotted lines are spectra from 4096³, 2048³, and 1024³ simulations, respectively, on the upper plot. The M1-3H has been multiplied by a factor of two to separate the curves. On the lower plot, the dashed and solid lines are MHD1 and MHD2, respectively.

Another possible spectral measurement is with respect to the global mean field. We do not expect such scalings to deviate significantly from the perpendicular scalings for the following reason: Alfvén waves propagate along the local field direction that deviates by an angle of $\delta B_L/B_0$ from B_0 , while the angular anisotropy in this frame is $\delta B_l/B_0$, with inertial range values of δB_l much smaller than the outer scale value of δB_L . It follows that the anisotropy will be washed out. Figure 2 presents a measurement of the spectrum along the *x*-global mean field direction. It is grossly consistent with -5/3, i.e., the perpendicular spectral scaling observed in Beresnyak (2014).

4. DISCUSSION

Critical balance refers to the interaction parameter $\xi = \delta v \lambda_{\parallel} / v_A \lambda_{\perp}$ being around unity in strong MHD turbulence. It was first argued based on the uncertainty relation between the wave frequency and the cascade timescale in Goldreich & Sridhar (1995) and had been restated in various forms, including the decorrelation argument by Gruzinov (Maron & Goldreich 2001). While plausibility arguments like this are certainly useful in qualitative understanding, their apparent generality is problematic. For example, the decorrelation argument does not explicitly refer to nonlinear interaction; however, it could not be generally valid, as pure propagating solutions with $\xi \gg 1$, strong Alfvénic waves, do exist. The naive application of the uncertainty relation argument fails, e.g., in imbalanced turbulence, where it predicts that the anisotropy of the stronger Elsässer component should be higher than the anisotropy of the weaker component, while in reality the opposite is true (Beresnyak & Lazarian 2008, 2009b). The new argument, presented in this Letter, circumvents this problem by noticing that the energy cascade is manifested both in space and time domains, also the parallel direction is equivalent to the time domain. Therefore, the well-known anisotropy relation $k_{\parallel} \sim k_{\perp}^{2/3}$ is simply the correspondence between space domain (Eulerian) and frequency domain (Lagrangian) spectra. The old arguments required that the



Figure 2. Spectra along the global mean field in M1-3 and M1-3H. The M1-3H spectra have been multiplied by a factor of two. This plot demonstrates that this energy spectrum scaling is consistent with -5/3, i.e., the same as the perpendicular scaling.

average ξ must be close to unity, while the new argument only requires that the average ξ is a dimensionless, scaleindependent quantity, i.e., a constant similar to the Kolmogorov constant.

Most observational data from the solar wind have been pointing to the k^{-2} parallel spectrum. For example, Horbury et al. (2008) used a wavelet technique to follow the local field direction and obtained k^{-2} . This has been further improved in Wicks et al. (2010, 2011) and compared with the global Fourier spectra. Podesta (2009) obtained similar results with wavelets and demonstrated scale-dependent anisotropy. The structure function measurement in Luo & Wu (2010) again confirmed the same scaling. Multi-spacecraft measurements allowing better coverage of k space (Osman & Horbury 2009) also confirmed k^{-2} . Earlier measurements in the *global* frame (e.g., Matthaeus et al. 1990) reported scale-independent anisotropy, which, as I argued above, is consistent with theory and numerics as well. As far as numerics go, the measurements along the local field direction gave the k^{-2} slope (see, e.g., Cho & Vishniac 2000; Maron & Goldreich 2001; Beresnyak & Lazarian 2009a, 2009b; Chen et al. 2011; Beresnyak 2012), while the measurements in the global frame gave scaleindependent anisotropy (see, e.g., Grappin & Müller 2010 or my Figure 2). The robustness of the critical balance with a properly defined nonlinear timescale has recently been discussed in Mallet et al. 2014.

Recently, the debate on the parallel spectrum has been revived, in particular, Turner et al. (2012) measured the quasiisotropic spectrum in the solar wind after filtering out discontinuities, while Grappin et al. (2012, 2013) suggested a new model with a "ricochet" cascade that effectively fills parallel direction and results in the same slope, as in perpendicular direction, citing Grappin & Müller (2010) and Turner et al. (2012) as motivation. My numerical data strongly disfavor this model, as the observed $-2.0 \div -1.9$ parallel spectral slope is much steeper than either -5/3 or the -1.5suggested in Grappin et al. (2012, 2013). It is also not clear why global measurements (Grappin & Müller 2010) should support the alternative model or whether or not filtering in Turner et al. (2012) interferes with the local field direction enough to destroy the weaker k^{-2} parallel spectrum.

Measurements of the Lagrangian frequency spectrum in hydrodynamics has been performed by many authors (see, e.g., Yeung et al. 2006 and references therein) and has showed correspondence with the theoretical ω^{-2} . The first direct measurement of the Lagrangian frequency spectrum in statistically isotropic MHD turbulence has been performed in Busse et al. (2010) and tentatively has confirmed the ω^{-2}



Figure 3. Scaling study of alignment measures $DA = \langle | \delta v \times \delta b | \rangle / \langle | \delta v \delta b | \rangle$ and IM = $\langle | \delta (w^+)^2 - \delta (w^-)^2 | \rangle / \langle \delta (w^+)^2 + \delta (w^-)^2 \rangle$ from M1-3H (top) and M1-3 (bottom). The alignment slopes converge to relatively small values, e.g., 0.06 for *DA*, which is smaller than 0.25, predicted by Boldyrev (2006). See also Beresnyak & Lazarian (2009a) and Beresnyak (2011, 2012).

scaling, but the results from simulations with a strong mean field were less clear. The connection between the Lagrangian frequency spectrum and the parallel spatial spectrum has not been made in Busse et al. (2010); also, as I argued above, the classic Lagrangian measurement could be meaningless in MHD as high-frequency perturbations cross the fluid element causing oscillatory velocity changes not associated with the energy cascade. In this Letter, I argued that the measurement of the spectrum *along the magnetic field line* is similar to the measurement of the Lagrangian frequency spectrum; therefore, the Goldreich & Sridhar (1995) scale-dependent anisotropy is a simple relation between the Eulerian and Lagrangian spectra.

Using the scaling study argument, I found that the best convergence corresponds to k^{-2} ; however, the deviation around 0.1 in the scaling exponent on medium scales is evident in Figure 1 and is somewhat interesting from a theoretical viewpoint as a long-range finite-Re effect. It is clear that the -1.9 slope on the medium scales in Figure 1 is still much steeper than the Eulerian $-5/3 \approx -1.7$, but what is the nature of this deviation? First, the prediction of the models with so-called dynamic alignment modifies only the perpendicular spectrum, leaving the parallel spectrum unchanged at the expense of higher anisotropy (Boldyrev 2005, 2006). Second, even if such modification was suggested by some inertial-range theory, it would be inconsistent with my numerics, as the 0.1 correction is not universal and disappears with higher Re measurements.

In the past several years, various spectral scalings, deviations from theoretically predicted scalings, and alignment measures have been studied in some detail (Beresnvak & Lazarian 2009a; Beresnyak 2011, 2012). The overall picture seems to be that while moderate Re shows a scale dependency of several alignment measures, normally in the range of 0.1-0.2, in the higher Re measurements, these alignment measures flatten out and their slopes are fairly close to zero (see, e.g., Figure 3). Similarly, the deviation from the expected perpendicular -1.7 slope is around ~ 0.2 in the medium scales, but disappeared when higher-resolution data became available. The ~ 0.1 deviation of the parallel slope fits nicely into this tendency. We see that the deviations from theoretical scalings had been observed so far only within around an order of magnitude in scale from the driving scale, and modifications of theory of the inertial range, such as in Chandran et al. (2014), are probably excessive at this point. Further solar wind measurements with better statistics and/or larger-scale numerics will help to shed light on this problem.

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