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FLUCTUATION DYNAMO AT FINITE CORRELATION TIMES AND THE KAZANTSEV SPECTRUM

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ABSTRACT

Fluctuation dynamos are generic to astrophysical systems. The only analytical model of the fluctuation dynamo is the Kazantsev model which assumes a velocity field that is delta-correlated in time. We derive a generalized model of fluctuation dynamos with finite correlation time, τ, using renovating flows. For $\tau \to 0$, we recover the standard Kazantsev equation for the evolution of longitudinal magnetic correlation, M_L. To the next order in τ, the generalized equation involves third and fourth spatial derivatives of M_L. It can be recast to one with at most second derivatives of M_L using the Landau–Lifschitz approach. Remarkably, we then find that the magnetic power spectrum remains the Kazantsev spectrum of $M(k) \propto k^{3/2}$, in the large k limit, independent of τ.

Key words: dynamo – galaxies: magnetic fields – magnetic fields – magnetohydrodynamics (MHD) – turbulence

1. INTRODUCTION

Magnetic fields are ubiquitously present in most astrophysical systems from stars to galaxies and galaxy clusters. They could be generated by dynamo amplification of weak seed fields. A particularly generic dynamo is the fluctuation or small-scale dynamo (Kazantsev 1967; Molchanov et al. 1985; Zeldovich et al. 1990; Kulsrud & Anderson 1992; Subramanian 1997, 1999; Rogachevskii & Kleeorin 1997; Brandenburg & Subramanian 2005; Cho et al. 2009; Federrath et al. 2011; Tobias et al. 2011; Sur et al. 2012; Brandenburg et al. 2012; Bhat & Subramanian 2013). Here, turbulence in a conducting plasma, with even a modest magnetic Reynolds number ($\mathrm{R}_M > R_{\mathrm{crit}} \sim 30$–$500$), leads to the amplification of magnetic fields on the fast eddy turnover timescale, usually much smaller than the age of the astrophysical system (Haugen et al. 2004; Schekochihin et al. 2004, 2005; Malyshkin & Boldyrev 2010; Schober et al. 2012). (Here $\mathrm{R}_M = u/(q\eta)$ with u and q, respectively, indicating the characteristic velocity and wavenumber of the flow and η representing the resistivity.) The R_{crit} depends on $\mathrm{P}_M = \nu/\eta$, where ν is the viscosity and the R_{crit} upper limit corresponds to $\mathrm{P}_M \ll 1$. The fast growth rate implies that fluctuation dynamos are crucial for the early generation of magnetic fields in primordial stars, galaxies, and galaxy clusters. It is therefore important to have a clear understanding of the fluctuation dynamo.

The only analytical treatment of the fluctuation dynamo is that due to Kazantsev (1967), where the velocity field is assumed to be delta-correlated in time (correlation time, $\tau \to 0$). In this case, one derives a partial differential equation describing the evolution of the longitudinal magnetic correlation function, $M_L(r, t)$. From its solutions, Kazantsev also predicted that the magnetic power spectrum for a single scale or a large P_M turbulent flow scales asymptotically as $M(k) \propto k^{3/2}$, for $q \ll k \ll k_\eta$, with k_η, the wavenumber where resistive dissipation becomes important. This spectrum is known as the Kazantsev spectrum. Also, in the same limit, Chertkov et al. (1999) extended analytic considerations to multi-point correlators, in a random smooth (linear) flow.

Finite-τ effects have been derived for the magnetic energy growth (Chandran 1997), and single point PDF in the ideal limit (Schekochihin & Kulsrud 2001). Kleeorin et al. (2002) considered a finite-τ correction to the two-point correlator evolution, but seem to have kept only a subset of the terms we derive here. Mason et al. (2011) show that solutions to the Kazantsev equation can be made to agree with simulations involving finite-τ velocity flows if the diffusivity spectrum is appropriately filtered out at small scales. However, an analytic understanding of the magnetic spectrum at finite-τ is still lacking.

In this Letter, we give an analytic generalization of the results of Kazantsev (1967) to flows with a finite correlation time, τ, by modeling the velocity as a renovating flow. We recover the Kazantsev evolution equation for M_L in the $\tau \to 0$ limit and derive the complete evolution equation for M_L to the next order in τ. We show for the first time an intriguing result that the Kazantsev spectrum is in fact preserved even for such finite-τ.

2. FLUCTUATION DYNAMO IN RENOVATING FLOWS

Consider the induction equation for magnetic field ($\boldsymbol{B}$) evolution in a conducting fluid with velocity $\boldsymbol{u}$,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B} - \eta \nabla \times \boldsymbol{B}). \qquad (1)$$

We assume $\boldsymbol{u}$ to have zero mean and a random component, which renovates every time interval τ (Dittrich et al. 1984; Gilbert & Bayly 1992). It is given in the form assumed by Gilbert & Bayly (1992, GB),

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a} \sin(\boldsymbol{q} \cdot \boldsymbol{x} + \psi), \qquad (2)$$

with $\boldsymbol{a} \cdot \boldsymbol{q} = 0$ for an incompressible flow. In each time interval $[(n-1)\tau, n\tau]$, (1) ψ is chosen to be uniformly random between 0 to 2π, (2) $\boldsymbol{q}$ is uniformly distributed on a sphere of radius $q = |\boldsymbol{q}|$, and (3) for every fixed $\hat{\boldsymbol{q}} = \boldsymbol{q}/q$, the direction of $\boldsymbol{a}$ is uniformly distributed in the plane perpendicular to $\boldsymbol{q}$. Specifically, for computational ease, we modify the GB ensemble by choosing $a_i = P_{ij} A_j$, where $\boldsymbol{A}$ is uniformly distributed on a sphere of radius A, and $P_{ij}(\hat{\boldsymbol{q}}) = \delta_{ij} - \hat{q}_i \hat{q}_j$ projects $\boldsymbol{A}$ to the plane perpendicular to $\boldsymbol{q}$. Then $\langle a^2 \rangle = 2A^2/3$. This modification in ensemble does not affect any result using the renovating flows. Condition (1) on ψ ensures statistical homogeneity, while (2) and (3) ensure statistical isotropy of the flow.

The magnetic field evolution in any time interval $[(n-1)\tau, n\tau]$ is

$$B_i(\boldsymbol{x}, n\tau) = \int \mathcal{G}_{ij}(\boldsymbol{x}, \boldsymbol{x}_0) B_j(\mathbf{x_0}, (n-1)\tau)\, d^3\boldsymbol{x}_0, \qquad (3)$$

where $\mathcal{G}_{ij}(\boldsymbol{x}, \boldsymbol{x}_0)$ is the Green's function of Equation (1). To obtain $\mathcal{G}_{ij}(\boldsymbol{x}, \boldsymbol{x}_0)$ in the renovating flow, we use the method introduced by GB. The renovation time, τ, is split into two equal sub-intervals. In the first sub-interval $\tau/2$, resistivity is neglected and the frozen field is advected with twice the original velocity. In the second sub-interval, $\boldsymbol{u}$ is neglected and the field diffuses with twice the resistivity. This method, plausible in the $\tau \to 0$ limit, has been used to recover the standard mean field dynamo equations in renovating flows (Gilbert & Bayly 1992; Kolekar et al. 2012).

In the first sub-interval $\tau/2 = t_1 - t_0$, from the advective part of Equation (1), we obtain the standard Cauchy solution,

$$B_i(\boldsymbol{x}, t_1) = \frac{\partial x_i}{\partial x_{0j}} B_j(\boldsymbol{x}_0, t_0) \equiv J_{ij}(\boldsymbol{x}(\boldsymbol{x}_0)) B_j(\boldsymbol{x}_0, t_0). \quad (4)$$

Here $B_j(\mathbf{x}_0, t_0)$ is the initial field, which is propagated from $\boldsymbol{x}_0$ at time t_0, to $\boldsymbol{x}$ at time $t_1 = t_0 + \tau/2$. The phase $\Phi = \boldsymbol{q} \cdot \boldsymbol{x} + \psi$ in Equation (2) is constant in time as $d\Phi/dt = \boldsymbol{q} \cdot \boldsymbol{u} = 0$, from incompressibility. Thus $d\boldsymbol{x}/dt = 2\boldsymbol{u}$ can be integrated to give at time $t_1 = t_0 + \tau/2$,

$$\boldsymbol{x} = \boldsymbol{x}_0 + \tau \boldsymbol{u} = \boldsymbol{x}_0 + \tau \mathbf{a} \sin(\boldsymbol{q} \cdot \boldsymbol{x}_0 + \psi), \quad (5)$$

with the Jacobian

$$J_{ij}(\boldsymbol{x}(\boldsymbol{x}_0)) = \delta_{ij} + \tau a_i q_j \cos(\boldsymbol{q} \cdot \boldsymbol{x}_0 + \psi). \quad (6)$$

It will be more convenient to work with the field in Fourier space,

$$\hat{B}_i(\mathbf{k}, t_1) = \int J_{ij}(\mathbf{x}(\mathbf{x_0})) B_j(\mathbf{x_0}, t_0) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} d^3\boldsymbol{x}. \quad (7)$$

In the second sub-interval $(t_1, t = t_1 + \tau/2)$, where only diffusion operates with resistivity 2η,

$$\hat{B}_i(\boldsymbol{k}, t) = G^{\eta}(\boldsymbol{k}, \tau) \hat{B}_i(\boldsymbol{k}, t_1) = e^{-(\eta\tau\boldsymbol{k}^2)} \hat{B}_i(\boldsymbol{k}, t_1), \quad (8)$$

where G^{η} is the resistive Greens function. To derive the evolution equation for the magnetic two-point correlation function, we combine Equation (7) and Equation (8) to get

$$\langle \hat{B}_i(\boldsymbol{k}, t) \hat{B}_h^*(\boldsymbol{p}, t) \rangle = e^{-\eta\tau(\boldsymbol{k}^2 + \boldsymbol{p}^2)} \int \langle J_{ij}(\boldsymbol{x}_0) J_{hl}(\boldsymbol{y}_0) \times e^{-i(\boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{p}\cdot\boldsymbol{y})} \rangle \langle B_j(\boldsymbol{x}_0, t_0) B_l(\boldsymbol{y}_0, t_0) \rangle d^3\boldsymbol{x} d^3\boldsymbol{y}. \quad (9)$$

Here $\langle\cdot\rangle$ denotes an ensemble average over the random velocity field and $*$ a complex conjugate. We have split the averaging between the initial two-point correlation of the magnetic field and the rest of the integral, as the initial field at t_0 is uncorrelated with renovating flow in the next interval $t_1 - t_0 = \tau/2$.

We use Equation (5) to transform from $(\boldsymbol{x}, \boldsymbol{y})$ to $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ in Equation (9). The Jacobian of this transformation is unity, due to incompressibility of the flow. Also, the initial statistical homogeneity and isotropy of the magnetic field are preserved at any time step. Thus $\langle B_j(\boldsymbol{x}_0, t_0) B_l(\boldsymbol{y}_0, t_0) \rangle = M_{jl}(|\boldsymbol{r}_0|, t_0)$, where $\boldsymbol{r}_0 = \boldsymbol{x}_0 - \boldsymbol{y}_0$. Let us also write $\boldsymbol{k} \cdot \boldsymbol{x}_0 - \boldsymbol{p} \cdot \boldsymbol{y}_0 = \boldsymbol{k} \cdot \boldsymbol{r}_0 + \boldsymbol{y}_0 \cdot (\boldsymbol{k} - \boldsymbol{p})$ in Equation (9), transform now from $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ to a new set of variables $(\boldsymbol{r}_0, \boldsymbol{y}_0{}' = \boldsymbol{y}_0)$, and integrate over $\boldsymbol{y}_0{}'$. This leads to a delta function in $(\boldsymbol{k} - \boldsymbol{p})$ and Equation (9) becomes

$$\langle \hat{B}_i(\boldsymbol{k}, t) \hat{B}_h^*(\boldsymbol{p}, t) \rangle = (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{p}) \hat{M}_{ih}(\boldsymbol{p}, t),$$

$$\hat{M}_{ih}(\boldsymbol{p}, t) = e^{-2\eta\tau\boldsymbol{p}^2} \int \langle R_{ijhl} \rangle M_{jl}(\boldsymbol{r}_0, t_0) e^{-i\boldsymbol{p}\cdot\boldsymbol{r}_0} d^3\boldsymbol{r}_0$$

$$\langle R_{ijhl} \rangle = \left\langle J_{ij}(\boldsymbol{x}_0) J_{hl}(\boldsymbol{y}_0) e^{-i\tau(\boldsymbol{a}\cdot\boldsymbol{p})(\sin A - \sin B)} \right\rangle, \quad (10)$$

where $A = (\boldsymbol{x}_0 \cdot \boldsymbol{q} + \psi)$ and $B = (\boldsymbol{y}_0 \cdot \boldsymbol{q} + \psi)$. We will see explicitly that $\langle R_{ijhl} \rangle$ is only a function of $\boldsymbol{r}_0$ as it should be from statistical homogeneity.

3. THE GENERALIZED KAZANTSEV EQUATION

It is difficult to evaluate $\langle R_{ijhl} \rangle$ exactly. However, we motivate a Taylor series expansion of the exponential in $\langle R_{ijhl} \rangle$ for a small Strouhl number $St = q|\boldsymbol{a}|\tau = qa\tau$ as follows. First, $(\sin A - \sin B) = \sin(\boldsymbol{q} \cdot \boldsymbol{r}_0/2) \cos(\psi + \boldsymbol{q} \cdot \boldsymbol{R}_0)$, where $\boldsymbol{R}_0 = (\boldsymbol{x}_0 + \boldsymbol{y}_0)/2$. Also for the kinematic fluctuation dynamo, the magnetic correlation function peaks around the resistive scale $r_0 = |\boldsymbol{r}_0| \sim 1/(q\mathrm{R}_M^{1/2})$, or the spectrum peaks around $p \sim (q\mathrm{R}_M^{1/2})$. (Here $p = |\boldsymbol{p}|$ and $\mathrm{R}_M \sim a/(q\eta) \gg 1$.) Thus, $qr_0 \ll 1$ and hence $\sin(\boldsymbol{q} \cdot \boldsymbol{r}_0) \sim \boldsymbol{q} \cdot \boldsymbol{r}_0$. Subsequently, the phase of the exponential in Equation (10) is of the order of $(pa\tau qr_0) \sim qa\tau = St$. Thus, for $St \ll 1$, one can expand the exponential in Equation (10) in τ. We do this retaining terms up to τ^4 order; keeping up to τ^2 terms in Equation (10) gives the Kazantsev equation, while the τ^4 terms give finite-τ corrections. We get

$$\langle R_{ijhl} \rangle = \left\langle H_{ijhl} \left[1 - i\tau\sigma - \frac{\tau^2\sigma^2}{2!} + \frac{i\tau^3\sigma^3}{3!} + \frac{\tau^4\sigma^4}{4!} \right] \right\rangle, \quad (11)$$

where $\sigma = (\boldsymbol{a} \cdot \boldsymbol{p})(\sin A - \sin B)$ and $H_{ijhl} = J_{ij}(\boldsymbol{x}_0) J_{hl}(\boldsymbol{y}_0)$ contains terms up to order τ^2. (We note that Kleeorin et al. (2002) seem to have kept only up to p^2 terms in Equation (11).) To calculate $\langle R_{ijhl} \rangle$, we average over ψ, $\hat{\mathbf{a}}$, and $\hat{\mathbf{q}}$. Terms that are proportional to $\sin(\cdots + n\psi)$ or $\cos(\cdots + n\psi)$ become zero upon averaging over ψ. Survival of such terms that depend explicitly on $\boldsymbol{x}_0$, $\boldsymbol{y}_0$, or $\boldsymbol{R}_0$ would break the statistical homogeneity. Naturally, surviving terms are those that depend on the relative coordinate $\boldsymbol{r}_0$ or are constant. For example, $\langle \sin A \cos A \rangle = \langle \sin(2\boldsymbol{q}\cdot\boldsymbol{x}_0 + 2\psi) \rangle/2 = 0$, while $\langle \sin A \cos B \rangle = \langle \sin(\boldsymbol{x}_0 + \boldsymbol{y}_0 + 2\psi) \rangle/2 + \langle \sin(\boldsymbol{q} \cdot \boldsymbol{r}_0) \rangle/2 = \langle \sin(\boldsymbol{q} \cdot \boldsymbol{r}_0) \rangle/2$. Next, we average over $\hat{\mathbf{a}}$ by using $a_i = P_{ij}(\boldsymbol{q}) A_j$ and averaging independently over $\mathbf{A}$. The remaining q_i dependent terms can be written in terms of either $\langle \cos(\boldsymbol{q} \cdot \boldsymbol{r}_0) \rangle$, $\langle \cos(2\boldsymbol{q} \cdot \boldsymbol{r}_0) \rangle$ or its spatial derivatives. Consider a simple example of the turbulent diffusion tensor, $T_{ij} = (\tau/2) \langle u_i(\boldsymbol{x}_0) u_j(\boldsymbol{y}_0) \rangle = (\tau/2) \langle a_i a_j \sin(A) \sin(B) \rangle$, which arises upon averaging terms proportional to τ^2. Note that in the $\tau \to 0$ limit, τ in T_{ij} is kept finite to recover the Kazantsev equation. This is the reason for multiplying the velocity two-point correlator by τ. We have

$$T_{ij} = \frac{\tau}{4} \langle A_l A_m P_{il} P_{jm} \cos(\boldsymbol{q} \cdot \boldsymbol{r}_0) \rangle = \frac{A^2\tau}{12} \langle P_{ij} \cos(\boldsymbol{q} \cdot \boldsymbol{r}_0) \rangle$$
$$= \frac{a^2\tau}{8} \left[\delta_{ij} + \frac{1}{q^2} \frac{\partial^2}{\partial r_{0i} r_{0j}} \right] j_0(qr_0). \quad (12)$$

Here we have used the fact that for the isotropically distributed vector $\mathbf{A}$, $\langle A_i A_j \rangle = A^2 \delta_{ij}/3$ and the average over directions of $\boldsymbol{q}$ gives $\langle \cos(\boldsymbol{q} \cdot \boldsymbol{r}_0) \rangle = j_0(qr_0)$.

The averages of terms that are of the order of τ^4 also introduce the fourth-order velocity correlators,

$$T_{mnih}^{x^2y^2} = \tau^2 \langle u_m(\boldsymbol{x}) u_n(\boldsymbol{y}) u_i(\boldsymbol{x}) u_h(\boldsymbol{y}) \rangle,$$
$$T_{mnih}^{x^3y} = \tau^2 \langle u_m(\boldsymbol{x}) u_n(\boldsymbol{x}) u_i(\boldsymbol{x}) u_h(\boldsymbol{y}) \rangle,$$
$$T_{mnih}^{x^4} = \tau^2 \langle u_m(\boldsymbol{x}) u_n(\boldsymbol{x}) u_i(\boldsymbol{x}) u_h(\boldsymbol{x}) \rangle. \quad (13)$$

Again, we multiply the fourth-order velocity correlators by τ^2, as we envisage that T_{ijkl} will be finite even in the $\tau \to 0$ limit, behaving like products of turbulent diffusion. Note that the renovating flow is not Gaussian random, and hence higher order correlators of $\boldsymbol{u}$ are not the product of two-point correlators. Interestingly, we find that the terms from Equation (11) of the order of τ^3 become 0 when averaging.

Similarly, we expand the exponential in the resistive Greens function in Equation (10), $e^{-2\eta\tau \boldsymbol{p}^2} = 1-2\eta\tau \boldsymbol{p}^2 \ldots$ and consider only the leading order term in η, relevant in the independent small η (or $\mathrm{R}_M \gg 1$) limit.

On combining these steps, we find that the integrand determining the magnetic spectral tensor $\hat{M}_{ih}(\boldsymbol{p}, t)$, is of the form $G(\boldsymbol{p})F_{ih}(\boldsymbol{r}_0, t_0)$, where $G(\boldsymbol{p})$ is a polynomial up to fourth order in p_i. This allows for a simple inverse Fourier transform of $\hat{M}_{ih}(\boldsymbol{p}, t)$, in Equation (10) back to configuration space and then magnetic field correlation function is

$$M_{ih}(\boldsymbol{r}, t) = \int G(\boldsymbol{p})F_{ih}(\boldsymbol{r}_0, t_0)e^{i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)}d^3\boldsymbol{r}_0\frac{d^3\boldsymbol{p}}{(2\pi)^3}. \quad (14)$$

The various powers of p_i in $G(\boldsymbol{p})$ above can be written as derivatives with respect to r_i. The integral over $\boldsymbol{p}$ then simply gives a delta function $\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)$ and this makes the integral over $\boldsymbol{r}_0$ trivial. Carrying out these steps, the magnetic correlation function can be written in the form

$$M_{ih}(\boldsymbol{r}, t) = M_{ih}(\boldsymbol{r}, t_0) + \tau^2 f_{ih}(\boldsymbol{r}, t_0) + \tau^4 g_{ih}(\boldsymbol{r}, t_0). \quad (15)$$

We then divide Equation (15) by τ, take the limit of $\tau \to 0$, and write $(M_{ih}(\boldsymbol{r}, t) - M_{ih}(\boldsymbol{r}, t_0))/\tau = \partial M_{ih}/\partial t$. The remaining τ multiplying the term f_{ih}, is absorbed into keeping T_{ij} finite, while τ^2 multiplying the term g_{ih}, is absorbed into T_{ijkl}, leaving one remaining τ as a small effective finite time parameter. The resulting equation for M_{ih} is given by

$$\begin{aligned}\frac{\partial M_{ih}(\boldsymbol{r}, t)}{\partial t} &= 2(-[T_{ih}M_{jl}]_{,jl} + [T_{jh}M_{il}]_{,jl} + [T_{il}M_{jh}]_{,jl} \\ &\quad - [T_{jl}M_{ih}]_{,jl}) + (2T_L(0) + 2\eta)\nabla^2 M_{ih} + \tau \\ &\quad \times \Bigg([\tilde{T}_{mnih}M_{jl}]_{,mnjl} - 2[\tilde{T}_{mnrh}M_{il}]_{,mnrl} \\ &\quad + \left[\left(\tilde{T}_{mnrs} + \frac{T^{x^4}_{mnrs}}{12}\right)M_{ih}\right]_{,mnrs}\Bigg), \end{aligned} \quad (16)$$

where $\tilde{T}_{mnih} = T^{x^2y^2}_{mnih}/4 - T^{x^3y}_{mnih}/3$, $T_L(r) = \hat{r}_i\hat{r}_jT_{ij}$ with $\hat{r}_i = r_i/r$. The first two lines in Equation (16) contain exactly the terms which give the Kazantsev equation, while the last two lines contain the finite-τ corrections. We write these latter terms as the fourth derivative of the combined velocity and magnetic correlators; however, as both the velocity and magnetic fields are divergence free, each spatial derivative only acts on one or the other.

Note that for a statistically homogeneous, isotropic, and nonhelical magnetic field, the correlation function $M_{ih} = (\delta_{ih} - \hat{r}_i\hat{r}_h)M_{\mathrm{N}}(r, t) + \hat{r}_i\hat{r}_hM_{\mathrm{L}}(r, t)$. Here $M_L(r, t) = \hat{r}_i\hat{r}_hM_{ih}$ and $M_N(r, t) = (1/2r)[\partial(r^2M_L)/\partial r]$ are, respectively, the longitudinal and transversal correlation functions of the magnetic field. Upon contracting Equation (16) with $\hat{r}_i\hat{r}_h$ we obtain the dynamical equation for $M_L(r, t)$, the generalized Kazantsev equation,

$$\begin{aligned}\frac{\partial M_L(r, t)}{\partial t} &= \frac{2}{r^4}\frac{\partial}{\partial r}\left(r^4\eta_{tot}\frac{\partial M_L}{\partial r}\right) + GM_L \\ &\quad + \tau M_L^{''''}\left(\overline{T}_L + \frac{\overline{T}_L(0)}{12}\right) + \tau M_L^{'''}\left(2\overline{T}_L^{'} + \frac{8\overline{T}_L}{r} + \frac{2\overline{T}_L(0)}{3r}\right) \\ &\quad + \tau M_L^{''}\left(\frac{5\overline{T}_L^{''}}{3} + \frac{11\overline{T}_L^{'}}{r} + \frac{8\overline{T}_L}{r^2} + \frac{2\overline{T}_L(0)}{3r^2}\right) \\ &\quad + \tau M_L^{'}\left(\frac{2\overline{T}_L^{'''}}{3} + \frac{17\overline{T}_L^{''}}{3r} + \frac{5\overline{T}_L^{'}}{r^2} - \frac{8\overline{T}_L}{r^3} - \frac{2\overline{T}_L(0)}{3r^3}\right). \end{aligned} \quad (17)$$

Here, $\eta_{tot} = \eta + T_L(0) - T_L(r)$ and $G = -2(T_L^{''} + 4T_L^{'}/r)$. Also, a prime denotes $\partial/\partial r$. Furthermore, $\overline{T}_L(r) = (\overline{T}_L^{x^2y^2}/4 - \overline{T}_L^{x^3y}/3)$, with

$$\begin{aligned}\overline{T}_L^{x^2y^2} &= \hat{r}_m\hat{r}_n\hat{r}_i\hat{r}_hT^{x^2y^2}_{mnih} = -24\left(\frac{3\partial_{2z}j_0(2z)}{(2z)^3} + \frac{j_0(2z)}{(2z)^2}\right) \\ \overline{T}_L^{x^3y} &= \hat{r}_m\hat{r}_n\hat{r}_i\hat{r}_hT^{x^3y}_{mnih} = -24\left(\frac{3\partial_z j_0(z)}{z^3} + \frac{j_0(z)}{z^2}\right), \end{aligned} \quad (18)$$

where $z = qr$ and the derivatives ∂_z and ∂_{2z} are derivatives with respect to z and $2z$, respectively. These latter equalities give the explicit expressions of these fourth-order correlators for the renovating flow. Again, in the limit $\tau \to 0$, we recover exactly the Kazantsev equation for M_L. Equation (17) allows eigen solutions of the form $M_L(z, t) = \tilde{M}_L(z)e^{\gamma\tilde{t}}$, where $\tilde{t} = t\eta_tq^2$, with $\eta_t = T_L(0) = a^2\tau/12 = A^2\tau/18$, and γ is the growth rate. Boundary conditions are given as $M_L'(0, t) = 0$, $M_L \to 0$ as $r \to \infty$. Implications of the higher spatial derivative terms are discussed below.

4. KAZANTSEV SPECTRUM AT FINITE CORRELATION TIME

We will solve Equation (17) numerically in our follow up paper (P. Bhat & K. Subramanian, in preparation). However, to derive both the standard Kazantsev spectrum in the large k limit, and its finite-τ modifications, it suffices to go to the limit of small $z = qr \ll 1$. Expanding the Bessel functions in Equations (12) and (18) in this limit, and substituting $M_L(z, t) = \tilde{M}_L(z)e^{\gamma\tilde{t}}$, Equation (17) becomes

$$\begin{aligned}\gamma\tilde{M}_L(z) &= \left(\frac{2\eta}{\eta_t} + \frac{z^2}{5}\right)\tilde{M}_L^{''} + \left(\frac{8\eta}{\eta_t} + \frac{6z^2}{5}\right)\frac{\tilde{M}_L^{'}}{z} + 2\tilde{M}_L \\ &\quad + \frac{9\bar{\tau}}{175}\left(\frac{z^4}{2}\tilde{M}_L^{''''} + 8z^3\tilde{M}_L^{'''} + 36z^2\tilde{M}_L^{''} + 48z\tilde{M}_L^{'}\right), \end{aligned} \quad (19)$$

where $\bar{\tau} = \tau\eta_tq^2 = (St)^2/12$ and prime is now z-derivative.

Close to the origin, where $z \ll \sqrt{\eta/\eta_t}$, we can write $\tilde{M}_L(z) = M_0(1 - z^2/z_\eta^2)$. Using Equation (19), $z_\eta = qr_\eta = [240/(2-\gamma)]^{1/2}[\mathrm{R}_M(St)]^{-1/2}$. The $\bar{\tau}$ dependent terms, which are small because both z and $\bar{\tau}$ are small, do not affect this result. Thus, for $\mathrm{R}_M \gg 1$, the resistive scale $r_\eta \ll 1/q$, although one has to go to sufficiently large $\mathrm{R}_M \gg 240/((2-\gamma)St)$ for this conclusion to obtain.

Now consider the solution for $z_\eta \ll z \ll 1$. In this limit, Equation (19) is scale free, as scaling $z \to cz$ leaves it invariant. Thus, Equation (19) has power-law solutions of the form $\tilde{M}(z) = \tilde{M}_0z^{-\lambda}$. The appearance of higher order (third and fourth) spatial derivatives in Equation (19) (or in Equation (17)), when going to finite-τ, implies that in this case, M_L evolution becomes nonlocal, determined by an integral type equation whose leading approximation for small $\bar{\tau}$ is Equation (19). However, for small $\bar{\tau}$ or St, these higher derivative terms only appear as perturbative terms multiplied by the small parameter $\bar{\tau}$. Thus it is possible to make the Landau–Lifshitz-type approximation, used in treating the effect of radiation reaction force in electrodynamics (see Landau & Lifshitz 1975 Section 75). In this treatment, one first ignores the perturbative terms proportional to $\bar{\tau}$, which gives basically the Kazantsev equation for $\tilde{M}_L$, and uses this to express $\tilde{M}_L'''$ and $\tilde{M}_L''''$ in terms of the lower order derivatives $\tilde{M}_L''$ and $\tilde{M}_L'$. This gives for $z \gg z_\eta$, $z^3\tilde{M}_L''' = -8z^2\tilde{M}_L'' - z(16 - 5\gamma_0)\tilde{M}_L'$ and $z^4\tilde{M}_L'''' = (56 + 5\gamma_0)z^2\tilde{M}_L'' + 10(16 - 5\gamma_0)z\tilde{M}_L'$. Here γ_0 is the growth rate obtained for the Kazantsev equation in the $\tau \to 0$ limit. Substituting these expressions back into the full Equation (19) we get

$$\tilde{M}_L'' z^2\left(\bar{\tau}\gamma_0\frac{9}{70} + \frac{1}{5}\right) + \tilde{M}_L' z\left(\bar{\tau}\gamma_0\frac{27}{35} + \frac{6}{5}\right) + (2 - \gamma)\tilde{M}_L = 0. \tag{20}$$

Remarkably, the coefficients of the perturbative terms in Equation (19) are such that all perturbative terms that do not depend on γ_0 cancel out in Equation (20)! Also interesting is the nature of the power-law solution $\tilde{M}_L(z) = \bar{M}_0 z^{-\lambda}$ to Equation (20). One gets for λ,

$$\lambda^2 - 5\lambda + \frac{5(2-\gamma)}{1 + \frac{9}{14}\gamma_0\bar{\tau}} = 0; \quad \text{so } \lambda = \frac{5}{2} \pm i\lambda_I, \tag{21}$$

where $\lambda_I = [20(2 - \gamma)/(1 + 9\gamma_0\bar{\tau}/14) - 25]^{1/2}/2$, and importantly, the real part of λ is $\lambda_R = 5/2$, independent of the value of $\bar{\tau}$! We can also get the approximate growth rate, assuming $\mathrm{R}_M \gg 1$, following the argument from Gruzinov et al. (1996); that one evaluates γ by substituting into Equation (21), the value of $\lambda = \lambda_m$ where $d\gamma/d\lambda = 0$. This gives $\gamma_0 \approx 3/4$ and $\gamma \approx (3/4)(1 - (45/56)\bar{\tau})$, which also implies $\lambda_I \approx 0$. (Including the effects of resistivity gives λ_I, a small positive non-zero value $\propto 1/(\ln(\mathrm{R}_M))$ as will be shown in our detailed follow up paper (P. Bhat & K. Subramanian, in preparation)). The γ_0 we obtain agrees with that of Kulsrud & Anderson (1992), which is obtained from looking at the evolution of $M(k, t)$. We also note that the growth rate is reduced for a finite $\bar{\tau}$. Such a reduction is found in simulations that directly compare with an equivalent Kazantsev model (Mason et al. 2011).

From Equation (21), for $z_\eta \ll z \ll 1$, M_L is then given by

$$M_L(z, t) = e^{\gamma\bar{t}}\tilde{M}_0 z^{-5/2}\cos(\lambda_I \ln(z) + \phi), \tag{22}$$

where $\tilde{M}_0$ and ϕ are constants. Thus, in this range, M_L varies dominantly as $z^{-5/2}$, modulated by the weakly varying cosine factor (as λ_I is small). Note that the magnetic power spectrum is related to M_L by

$$M(k, t) = \int dr(kr)^3 M_L(r, t) j_1(kr). \tag{23}$$

The spherical Bessel function $j_1(kr)$ peaks around $k \sim 1/r$, and a power-law behavior of $M_L \propto z^{-\lambda_R}$ for a range of $z_\eta \ll z = qr \ll 1$, translates into a power law for the spectrum $M(k) \propto k^{\lambda_R - 1}$ in the corresponding wavenumber range $q \ll k \ll q/z_\eta$. From the solution given in Equation (22), we see that in the range $z_\eta \ll z \ll 1$, M_L dominantly varies as a power law with $\lambda_R = 5/2$, independent of τ. This implies remarkably that the magnetic spectrum is of the Kazantsev form with $M(k) \propto k^{3/2}$ in k space, independent of τ! This is the main result of this Letter.

5. DISCUSSION AND CONCLUSIONS

Fluctuation dynamos are important as they ubiquitously lead to a rapid generation of magnetic fields in astrophysical systems. However, their only analytical treatment, the Kazantsev model, assumes a delta-correlated velocity field. Here, we have generalized the Kazantsev model to finite correlation time, τ, using a velocity field that renovates every time period τ. We have shown that the Kazantsev equation for M_L is recovered when $\tau \to 0$ and have extended it to the next order in τ. In order to treat the resulting higher order (third and fourth) spatial derivatives of M_L perturbatively, we use the Landau–Lifshitz approach, which was earlier used to treat the effect of the radiation reaction force. An asymptotic treatment shows first that the fluctuation dynamo growth rate is reduced due to finite $\bar{\tau}$. More important is the novel and remarkable result that the Kazantsev spectrum of $M(k) \propto k^{3/2}$, is preserved even at finite-τ.

The finite-τ evolution equation for M_{ih} (Equation (16)) or M_L (Equation (17)). is cast in terms of the general velocity correlators T_{ij} and T_{ijkl} and matches exactly with the Kazantsev equation for the $\tau \to 0$ case. Moreover, the forms of T_{ij} and T_{ijkl} at $r \ll (1/q)$ are expected to be universal due to their symmetries and divergence-free properties. These features indicate that our result on the spectrum could have a more general validity than the context (of a renovating velocity) in which it is derived. It would be very interesting to see if such a result also holds for $St \sim 1$ and to extend the finite-τ result to helical renovating flows, issues which we hope to address in the future.

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