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NUMERICALLY DETERMINED TRANSPORT LAWS FOR FINGERING ("THERMOHALINE") CONVECTION IN ASTROPHYSICS

A. Traxler[1], P. Garaud[1,3], and S. Stellmach[2]
[1] Department of Applied Mathematics and Statistics, Baskin School of Engineering, University of California, Santa Cruz, 1156 High Street, Santa Cruz, CA 95064, USA
[2] Institut für Geophysik, Westfälische Wilhelms-Universität Münster, D-48149 Münster, Germany
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ABSTRACT

We present the first three-dimensional simulations of fingering convection performed at parameter values approaching those relevant for astrophysics. Our simulations reveal the existence of simple asymptotic scaling laws for turbulent heat and compositional transport, which can be straightforwardly extrapolated from our numerically tractable values to the true astrophysical regime. Our investigation also indicates that thermo-compositional "staircases," a key consequence of fingering convection in the ocean, cannot form spontaneously in the fingering regime in stellar interiors. Our proposed empirically determined transport laws thus provide simple prescriptions for mixing by fingering convection in a variety of astrophysical situations, and should, from here on, be used preferentially over older and less accurate parameterizations. They also establish that fingering convection does not provide sufficient extra-mixing to explain observed chemical abundances in red giant branch stars.

Key words: hydrodynamics – instabilities – turbulence

1. INTRODUCTION

Recent years have seen rapidly growing interest in astrophysical applications of fingering convection (often referred to as "thermohaline convection"), a process by which the unequal diffusion rates of thermal and compositional fields can drive a double-diffusive instability and lead to significant turbulent transport across stably stratified regions. In Earth's oceans this process has been well studied as "salt fingering" (see Stern 1960, or a recent review in Kunze 2003). In stellar contexts, by contrast, its exact form and contribution to vertical mixing is much less clear. Nevertheless, fingering convection has been invoked in various scenarios. It is thought to explain the lack of observed metallicity signatures from the infall of planets onto their host star (Vauclair 2004; Garaud 2011). It may also explain the presence of extra-mixing in the radiative zones of red giant branch (RGB) stars needed to fit abundance observations (Charbonnel & Zahn 2007; Stancliffe 2010; Denissenkov 2010).

While field, laboratory, and numerical measurements abound for salt fingering, data are much more scarce for the astrophysical regime. Transport by fingering convection in astrophysics has until now been modeled through mixing-length theory, commonly used parameterizations being those of Ulrich (1972) and Kippenhahn et al. (1980). In these works, turbulent compositional transport depends sensitively on the finger aspect ratio. As recently shown by Denissenkov (2010), however, there is a fundamental contradiction between the large aspect ratio required to explain abundance observations of RGB stars using these prescriptions (Charbonnel & Zahn 2007; Cantiello & Langer 2010), and the near-unit aspect ratio of the "fingers" actually measured in his own two-dimensional simulations.

Three possibilities emerge to resolve this problem. First, the Kippenhahn et al. (1980) and Ulrich (1972) parameterizations may not adequately model transport by fingering convection. Second, two-dimensional simulations may not correctly capture the dynamics of this inherently three-dimensional system. Finally, thermo-compositional layers may form in the objects studied, in which case transport could be strongly enhanced above the level of homogeneous fingering convection. Indeed, "staircases" of convectively mixed layers separated by thin fingering interfaces are ubiquitous in the ocean. In their presence, vertical mixing increases by as much as an order of magnitude above the already enhanced mixing due to fingering (Schmitt et al. 2005; Veronis 2007). Whether such staircases form in astrophysics has never been established.

In this Letter, we consider all three possibilities and present the first three-dimensional simulations to address directly the question of thermal and compositional transport by fingering convection in astrophysics. Using a high-performance algorithm designed to study fingering convection, we are able to run simulations at moderately low values of the Prandtl number ($\mathrm{Pr} = \nu/\kappa_T$) and diffusivity ratio ($\tau = \kappa_\mu/\kappa_T$), where ν is viscosity, and κ_T and κ_μ are the thermal and compositional diffusivities, respectively. We find, in Section 3.1, asymptotic scaling laws for transport which are applicable to the stellar parameter regime ($\mathrm{Pr} \ll 1$, $\tau \ll 1$). These provide a parameter-free, empirically motivated prescription for mixing by homogeneous fingering convection. The compositional turbulent diffusivity derived is, as expected (Stern et al. 2001), a few times larger than that obtained by Denissenkov (2010) for the same parameters, but remains too small to account for RGB abundance observations.

Since the presence of thermo-compositional staircases is known to increase transport in the ocean, we next address the question of whether such staircases are likely to form spontaneously in the astrophysical context in Section 3.2. Applying a recently validated theory for staircase formation (Radko 2003; Stellmach et al. 2010), we find that they are in fact not expected to appear in this case. We conclude in Section 4 that our small-scale flux laws can reliably be used to estimate heat and compositional transport for a variety of astrophysical fingering systems and concur with Denissenkov's view that fingering convection is not sufficient to explain the required extra-mixing in RGB stars.

[3] On sabbatical leave at: Institute for Astronomy, 34 ʻOhiʻa Ku Street, Pukalani, HI 96768-8288, USA.

2. MODEL DESCRIPTION AND NUMERICAL ALGORITHM

The dynamics of double-diffusive systems depend on the fluid properties ($\nu, \kappa_T, \kappa_\mu$), as well as its local stratification, measured by $\nabla - \nabla_{\rm ad}$ (where $\nabla = d\ln T/d\ln p$ and $\nabla_{\rm ad} = (d\ln T/d\ln p)_{\rm ad}$ is the adiabatic gradient) and ∇_μ (where $\nabla_\mu = d\ln\mu/d\ln p$). Here, T is temperature, p is pressure, and μ is the mean molecular weight. The fingering instability occurs in thermally stably stratified systems ($\nabla - \nabla_{\rm ad} < 0$) destabilized by an adverse compositional gradient ($\nabla_\mu < 0$). In most regimes of interest the main governing parameter is the density ratio, defined in astrophysics as (Ulrich 1972)

$$R_0^* = \frac{\nabla - \nabla_{\rm ad}}{\nabla_\mu}. \tag{1}$$

This linear instability is well understood, thanks to early work in the oceanic context (Stern 1960; Baines & Gill 1969), later extended in astrophysics by Ulrich (1972) and Schmitt (1983). A system is Ledoux-unstable when $R_0^* < 1$ and fingering-unstable when $R_0^* \in [1, 1/\tau]$. Unfortunately, as discussed in Section 1, the nonlinear saturation of this instability, or in other words its turbulent properties, remains poorly understood.

We approach the problem from a numerical point of view and model a fingering-unstable region using a local Cartesian frame (x, y, z) with gravity $\mathbf{g} = -g\mathbf{e}_z$. We run our numerical experiments using a high-performance spectral code, which was recently used to model three-dimensional fingering convection at oceanic parameters (Traxler et al. 2010), as well as thermohaline staircase formation (Stellmach et al. 2010). As a consequence of its original purpose, our code uses the Boussinesq approximation, which relates the density perturbations $\tilde{\rho}$ to the temperature and compositional perturbations $\tilde{T}$ and $\tilde{\mu}$ via

$$\frac{\tilde{\rho}}{\rho_0} = -\alpha\tilde{T} + \beta\tilde{\mu}, \tag{2}$$

where ρ_0 is a reference density, $\alpha = -\rho_0^{-1}\partial\rho/\partial T$, and $\beta = \rho_0^{-1}\partial\rho/\partial\mu$. Fingering convection is driven by constant large-scale temperature and compositional gradients $T_{0z} > 0$ and $\mu_{0z} > 0$, with a stable density gradient $\rho_{0z} = -\rho_0\alpha T_{0z} + \rho_0\beta\mu_{0z} < 0$. These approximations, which greatly simplify the governing equations, are nevertheless justified since fingers are typically much smaller than a pressure or density scale height (see below), and velocities are small compared with the sound speed. As first shown by Ulrich (1972), results in the Boussinesq case can straightforwardly be extended to the astrophysical case by replacing the Boussinesq density ratio $R_0 = \alpha T_{0z}/\beta\mu_{0z}$ with the true density ratio R_0^* defined in Equation (1).

We now describe the governing equations in more detail. We express the velocity, temperature, and composition fields as the linear background stratification plus perturbations,

$$T(x, y, z, t) = T_0(z) + \tilde{T}(x, y, z, t), \tag{3}$$

$$\mu(x, y, z, t) = \mu_0(z) + \tilde{\mu}(x, y, z, t), \tag{4}$$

$$\mathbf{u}(x, y, z, t) = \tilde{\mathbf{u}}(x, y, z, t), \tag{5}$$

where $T_0(z) = T_{0z}z$ and $\mu_0(z) = \mu_{0z}z$. We non-dimensionalize the system noting that an appropriate length scale is provided by the anticipated finger width (Stern 1960), $[l] = d = (\kappa_T\nu/g\alpha T_{0z})^{1/4}$. At typical RGB parameters (Denissenkov 2010), $d \sim 40$ m. With structures thus many orders of magnitude smaller than the density scale height, the Boussinesq approximation discussed by Spiegel & Veronis (1960) provides excellent accuracy.

We scale time using d^2/κ_T, temperature with $T_{0z}d$, and composition with $(\alpha/\beta)T_{0z}d$. The non-dimensional governing equations are

$$\frac{1}{\rm Pr}\left(\frac{\partial\tilde{\mathbf{u}}}{\partial t} + \tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}\right) = -\nabla\tilde{p} + (\tilde{T} - \tilde{\mu})\mathbf{e}_z + \nabla^2\tilde{\mathbf{u}}, \tag{6}$$

$$\nabla\cdot\tilde{\mathbf{u}} = 0, \tag{7}$$

$$\frac{\partial\tilde{T}}{\partial t} + \tilde{w} + \tilde{\mathbf{u}}\cdot\nabla\tilde{T} = \nabla^2\tilde{T}, \tag{8}$$

$$\frac{\partial\tilde{\mu}}{\partial t} + \frac{1}{R_0}\tilde{w} + \tilde{\mathbf{u}}\cdot\nabla\tilde{\mu} = \tau\nabla^2\tilde{\mu}. \tag{9}$$

Note that the Rayleigh number Ra then depends only on the computational domain height L_z:

$$\mathrm{Ra} = \frac{\alpha g T_{0z}L_z^4}{\kappa_T\nu} = \left(\frac{L_z}{d}\right)^4. \tag{10}$$

We solve Equations (6–9) in a triply periodic box of size (L_x, L_y, L_z), so that

$$\begin{aligned}\tilde{T}(x, y, z, t) &= \tilde{T}(x + L_x, y, z, t) = \tilde{T}(x, y + L_y, z, t)\\ &= \tilde{T}(x, y, z + L_z, t),\end{aligned} \tag{11}$$

and similarly for $\tilde{p}$, $\tilde{\mu}$, and $\tilde{\mathbf{u}}$.[4]

3. RESULTS

3.1. Small-scale Fluxes

Six sets of simulations were conducted at decreasing values of Pr and τ, for a range of density ratios listed in Table 1 spanning each instability range $R_0 \in [1, 1/\tau]$. The computational domains were chosen to allow enough fingers for accurate statistics (for details on the protocol for box sizes, see the Appendix of Traxler et al. 2010). Near $R_0 = 1$, where the system dynamics are most turbulent, one additional simulation with doubled vertical resolution (192 or 384 points vertically) was run for each set. The measured fluxes in the more fully resolved runs differed by no more than a few percent from the lower-resolution runs, confirming that the spatial resolution used was sufficient to extract accurate fluxes. Sample snapshots of two simulations are shown in Figure 1. As an important note, we also find that, except when $R_0 \to 1/\tau$, the finger aspect ratio is close to unity (Denissenkov 2010).

We define non-dimensional heat and compositional fluxes through the Nusselt numbers Nu_T and Nu_μ, ratios of the total flux to the diffusive flux of the field considered. Equivalently, $\mathrm{Nu}_{T,\mu} - 1$ represents the ratio of the turbulent diffusivity to the microscopic diffusivity. We measure Nu_T and Nu_μ from each simulation once the system has settled into a statistically steady state. The results are summarized in Figure 2.

[4] Because the length scale of the convective motions is set by the diffusive scales in the fingering problem, it does not suffer from the known pathology of triply periodic thermal convection, i.e., the homogeneous Rayleigh–Bénard problem (Calzavarini et al. 2006).

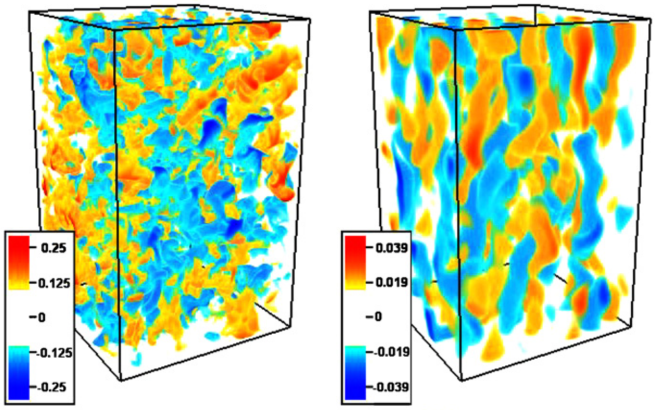

Figure 1. Compositional perturbations, in units of the compositional contrast across the height of the domain, of two simulations at $\mathrm{Pr} = \tau = 1/10$, at density ratio $R_0 = 1.45$ (left) and $R_0 = 9.1$ (right). Fingering convection is more laminar, and the fingers are taller, as $R_0 \to 1/\tau$ (as the background stratification becomes more stable). However, as Pr and τ decrease, secondary instabilities enforce shorter fingers even at high density ratio.

Table 1
Summary of Governing Parameters for all Simulation Sets

Set	Pr	τ	R_0
1	1/3	1/3	1.025, 1.075, 1.1, 1.1[a], 1.1[b], 1.125, 1.2, 1.3, 1.5, 2, 2.4, 2.8, 2.9[a,c]
2	1/3	1/10	1.5, 1.5[b], 4, 7, 9.5[a,c]
3	1/10	1/3	1.1, 1.1[a], 1.7, 2.3, 2.8[a,c] 2.9[a,c]
4	1/10	1/10	1.1, 1.45, 1.45[a], 2, 2.8, 3, 3.3, 5, 7, 9.1[a,c]
5	1/10	1/30	11[d,e], 11[d,f], 20[d,e]
6	1/30	1/10	4[d,e], 7[d,e]

Notes. Relevant parameters include values of Pr and τ, of the density ratio $R_0 \in [1, 1/\tau]$, domain size, and resolution. Unless otherwise specified (see footnotes), simulations were run at domain sizes of $67d \times 67d \times 107.2d$ and resolution of 96^3 grid points.
[a] resolution: $96 \times 96 \times 192$.
[b] resolution: $192 \times 192 \times 384$.
[c] domain size: $83.75d \times 83.75d \times 268d$.
[d] domain size: $67d \times 67d \times 67d$.
[e] resolution: $192 \times 192 \times 192$.
[f] resolution: $384 \times 384 \times 384$.

For ease of comparison between different simulation sets, we define the rescaled density ratio as $r = (R_0 - 1)/(\tau^{-1} - 1)$ so that the instability range is $r \in [0, 1]$ in all cases. We find that all simulation sets collapse onto a single universal profile for the turbulent fluxes $\mathrm{Nu}_T(r) - 1$ and $\mathrm{Nu}_\mu(r) - 1$ as Pr and τ decrease, profiles which can be expressed as

$$\mathrm{Nu}_T(r) - 1 = \tau^{3/2}\sqrt{\mathrm{Pr}}\, f(r), \tag{12}$$

$$\mathrm{Nu}_\mu(r) - 1 = \sqrt{\frac{\mathrm{Pr}}{\tau}}\, g(r), \tag{13}$$

where the functions $f(r)$ and $g(r)$ are adequately fitted to the data with a Levenberg–Marquardt algorithm using

$$g(r),\ f(r) \sim a e^{-br}(1 - r)^c. \tag{14}$$

This functional form was chosen as the simplest that captures both the near-exponential decay of fluxes as r increases from the

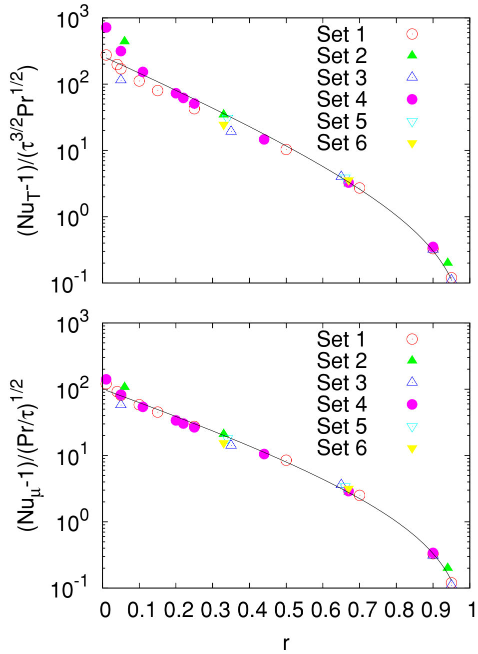

Figure 2. Turbulent heat and compositional fluxes as a function of rescaled density ratio r. Dividing the turbulent heat flux by $\tau^{3/2}\mathrm{Pr}^{1/2}$ and compositional flux by $\sqrt{\mathrm{Pr}/\tau}$ reveals the universal scaling laws (Equations (12) and (13)). Also shown are $f(r)$ and $g(r)$ fitted to the data (solid lines).

point of marginal convective instability, and the sharp drop off of transport as r approaches 1. Additionally, the non-dimensional turbulent flux ratio $\gamma = \langle \tilde{w}\tilde{T}\rangle / \langle \tilde{w}\tilde{\mu}\rangle$ approaches unity as $r \to 1$ (Radko 2003), and so turbulent transport of heat and composition must drop to zero at the same rate in this limit. To ensure that this condition is met, we first fit c in $g(r)$, then use that coefficient for $f(r)$.

For $f(r)$ the data suggest that $a = 264 \pm 1$, $b = 4.7 \pm 0.2$, $c = 1.1 \pm 0.1$, and for $g(r)$ we find $a = 101 \pm 1$, $b = 3.6 \pm 0.3$, $c = 1.1 \pm 0.1$. Figure 2 shows these fits with the rescaled Nusselt numbers. The physical interpretation of these scalings with τ and Pr is the following: for $\nu, \kappa_\mu \ll \kappa_T$, the instability is driven by the compositional field and the dependence of compositional transport on κ_T must drop out. The turbulent compositional diffusivity $D_\mu = (\mathrm{Nu}_\mu - 1)\kappa_\mu$ is then proportional to the geometric mean of ν and κ_μ. The scaling for the temperature Nusselt number follows by assuming γ is always $O(1)$ (Radko 2003; Rosenblum et al. 2010).

Our simulations were conducted at Pr, $\tau \sim O(10^{-2})$, while the true astrophysical parameters are $O(10^{-6})$. However, the clear asymptotic progression of our results provides the best empirical estimate to date of fingering fluxes at more extreme parameters. The simplicity of the system considered (in particular, the lack of boundary layers) also suggests that these laws can reliably be extrapolated to the astrophysical regime (while Pr is of order τ). From this we draw two critical observations. As found by Denissenkov (2010), the contribution of fingering convection to heat transport at stellar parameter values (Pr, $\tau \sim 10^{-6}$) is negligible. It therefore does not affect the thermal structure of the object in any way. Compositional transport, on the other hand, is significantly enhanced above molecular diffusion, by about two orders of magnitude depending on the ratio $\mathrm{Pr}/\tau = \nu/\kappa_\mu$.

We now compare our flux laws with the parameterization of Ulrich (1972) and Kippenhahn et al. (1980) as summarized in

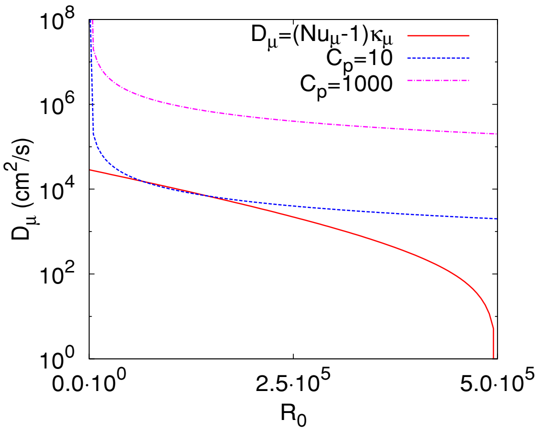

Figure 3. Comparison between our empirically determined compositional turbulent diffusivity and the parameterizations of Kippenhahn et al. (1980) and Ulrich (1972; where the coefficient C_p serves as a free parameter). Since we find $L/W \sim 1$, $C_p \sim O(10)$ for both prescriptions, and they overestimate D_μ as $R_0 \to 1$ and $R_0 \to 1/\tau$. Our results show that turbulent mixing by fingering convection is not sufficient to explain observations, for which $C_p \sim 1000$ is required (see the text for detail). The compositional diffusivity is $\kappa_\mu = 2 \times 10^2$ as in Denissenkov (2010).

Charbonnel & Zahn (2007). Both have $\mathrm{Nu}_\mu - 1 = C_p/\tau R_0^*$, with model constant $C_p = 12$ for Kippenhahn et al. (1980) and $C_p = 8\pi^2\alpha^2/3$ for Ulrich (1972), where $\alpha = L/W$ is the aspect ratio of the fingers (length divided by width). Figure 3 compares D_μ as a function of density ratio, at the fluid parameters used by Denissenkov (2010), $\mathrm{Pr} = 4 \times 10^{-6}$ and $\tau = 2 \times 10^{-6}$. Given that the numerically determined finger aspect ratio is close to one for all simulations as $\tau \ll 1$, $\mathrm{Pr} \ll 1$, both prescriptions vastly overestimate mixing near $R_0^* = 1$ (where the system is closest to Ledoux instability) and $R_0^* = 1/\tau$, close to linear stability. Crucially, our results show that the diffusivity obtained is not sufficient to explain RGB abundance observations for which a value of $C_p \approx 1000$ is needed (Charbonnel & Zahn 2007; Cantiello & Langer 2010; Denissenkov 2010).

3.2. Layer Formation

The small-scale flux laws presented above are accurate measurements of transport by homogeneous fingering convection. However, they may not appropriately describe transport in the presence of the kind of large-scale thermo-compositional staircases which are known to form in oceanic thermohaline convection (e.g., Schmitt et al. 2005). The origin of such staircases has recently been discovered by Radko (2003), who demonstrated the existence of a positive feedback mechanism that can drive the growth of horizontally invariant perturbations in the temperature and salinity fields; large-scale variations of the density ratio lead to convergences and divergences of fingering fluxes that reinforce the original perturbation, producing a growing disturbance that ultimately overturns into regular "steps." This mechanism relies fundamentally on the variation of the flux ratio $\gamma = F_T/F_S$ (where F_T is the turbulent heat flux and F_S is the turbulent salt flux) with density ratio and was therefore called the γ-instability. The γ-instability mechanism accurately predicts the growth rate of large-scale perturbations, and subsequent overturning into layers, in both two-dimensional simulations (Radko 2003) and three-dimensional simulations of staircase formation (Traxler et al. 2010; Stellmach et al. 2010). The instability requires that γ decreases as R_0 increases, a condition which must be experimentally determined from the fluxes for each set of fluid parameters Pr and τ.

In Radko's original formulation for oceanic staircases, the systems of interest were dominated by turbulent fluxes. Diffusive contributions to the total fluxes were neglected for convenience and simplicity. However, as shown in Section 3.1, turbulent heat transport is negligible in the astrophysical case, so the γ-instability theory is re-derived here including all diffusive terms for accuracy and completeness. We begin with the equations of Section 2 and average them over many fingers:

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + (T-\mu)\mathbf{e}_z + \nabla^2\mathbf{u} - \frac{1}{\mathrm{Pr}}\nabla\cdot\mathbf{R}, \tag{15}$$

$$\frac{\partial T}{\partial t} + w + \mathbf{u}\cdot\nabla T = \nabla^2 T - \nabla\cdot\mathbf{F}_T, \tag{16}$$

$$\frac{\partial \mu}{\partial t} + \frac{1}{R_0}w + \mathbf{u}\cdot\nabla\mu = \tau\nabla^2\mu - \nabla\cdot\mathbf{F}_\mu, \tag{17}$$

where $\mathbf{u}$, T, and μ now represent averaged large-scale fields. On the right-hand sides appear the usual Reynolds stress term, $R_{ij} = \overline{\tilde{\mathbf{u}}_i\tilde{\mathbf{u}}_j}$, and the turbulent fluxes, $\mathbf{F}_T = \overline{\tilde{\mathbf{u}}\tilde{T}}$, $\mathbf{F}_\mu = \overline{\tilde{\mathbf{u}}\tilde{\mu}}$.

The γ-instability involves only the temperature and compositional fields, so we consider only zero-velocity perturbations and discard the momentum equation. We then define the total fluxes as well as their ratio:

$$F_T^{\mathrm{tot}} = F_T - (1 + \partial T/\partial z), \tag{18}$$

$$F_\mu^{\mathrm{tot}} = F_\mu - \tau\,(1/R_0 + \partial\mu/\partial z), \tag{19}$$

$$\gamma^{\mathrm{tot}} = \frac{F_T^{\mathrm{tot}}}{F_\mu^{\mathrm{tot}}}, \tag{20}$$

so that the thermal and compositional Nusselt numbers are

$$\mathrm{Nu}_T = \frac{F_T^{\mathrm{tot}}}{-(1+\partial T/\partial z)}\quad \mathrm{Nu}_\mu = \frac{F_\mu^{\mathrm{tot}}}{-\tau\left(R_0^{-1} + \partial\mu/\partial z\right)}.$$

The final assumption is that Nu_T, Nu_μ, and γ^{tot} depend only on the local value of the density ratio R_ρ. Note that $R_\rho \neq R_0$ and may vary with z. We now linearize Equations (16) and (17) around a state of homogeneous turbulent convection in which $T = 0 + T'$, $\mu = 0 + \mu'$, and $R_\rho = R_0 + R'$. For example, linearizing the density ratio R_ρ, we have

$$\begin{aligned} R_\rho &= \frac{\alpha T_{0z}(1+\partial T'/\partial z)}{\beta\mu_{0z}\left[1 + \left(\frac{\alpha T_{0z}}{\beta\mu_{0z}}\right)\partial\mu'/\partial z\right]}, \\ &\approx R_0(1 + \partial T'/\partial z - R_0\partial\mu'/\partial z). \end{aligned} \tag{21}$$

The temperature equation yields

$$\begin{aligned} \frac{\partial T'}{\partial t} &= -\frac{\partial F_T^{\mathrm{tot}}}{\partial z}, \\ &= \frac{\partial \mathrm{Nu}_T}{\partial z} + \mathrm{Nu}_T\frac{\partial^2 T'}{\partial z^2}, \\ &= \left.\frac{\partial \mathrm{Nu}_T}{\partial R_\rho}\right|_{R_0}\frac{\partial R_\rho}{\partial z} + \mathrm{Nu}_T(R_0)\frac{\partial^2 T'}{\partial z^2}, \\ &= \left.\frac{\partial \mathrm{Nu}_T}{\partial R_\rho}\right|_{R_0} R_0\left(\frac{\partial^2 T'}{\partial z^2} - R_0\frac{\partial^2\mu'}{\partial z^2}\right) + \mathrm{Nu}_T(R_0)\frac{\partial^2 T'}{\partial z^2}, \end{aligned} \tag{22}$$

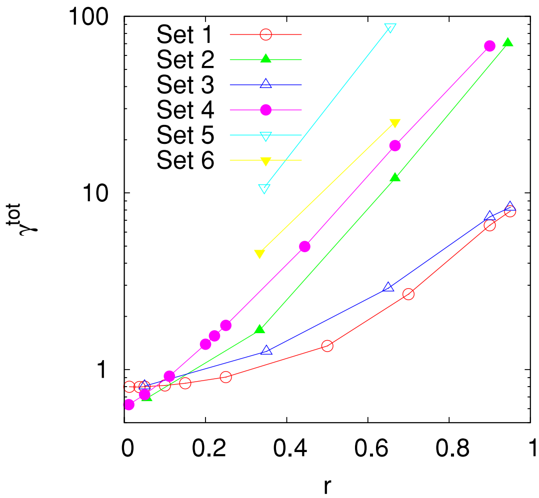

Figure 4. Total flux ratio $\gamma^{\rm tot} = F_T^{\rm tot}/F_\mu^{\rm tot}$ for various systems. As the density stratification becomes more stable ($r \to 1$), the fluxes of heat and composition are dominated by diffusive contributions and their ratio approaches the limiting value R_0/τ. The flux ratio increases monotonically with r, indicating that staircases will not form spontaneously in this parameter regime ($\Pr \ll 1, \tau \ll 1$).

and similarly for Equation (17). Assuming normal modes of the form $T', \mu' \sim e^{ikz+\lambda t}$, Equations (22) and the equivalent linearization of Equation (17) combine into the quadratic

$$\lambda^2 + \lambda k^2 \left[\mathrm{Nu}_0(1 - A_1 R_0) + A_2 \left(1 - \frac{R_0}{\gamma_0^{\rm tot}}\right)\right] - k^4 A_1 \mathrm{Nu}_0^2 R_0 = 0, \quad (23)$$

where we use the following notation for simplicity:

$$A_1 = R_0 \left.\frac{d(1/\gamma^{\rm tot})}{dR_\rho}\right|_{R_0}, \quad A_2 = R_0 \left.\frac{d\mathrm{Nu}_T}{dR_\rho}\right|_{R_0},$$
$$\mathrm{Nu}_0 = \mathrm{Nu}_T(R_0), \quad \gamma_0^{\rm tot} = \gamma^{\rm tot}(R_0).$$

Note that this is identical to the quadratic of Radko (2003) simply replacing the turbulent flux ratio γ by the total flux ratio $\gamma^{\rm tot}$.

As discussed by Radko (2003), a sufficient condition for the instability is that $\gamma^{\rm tot}$ is a decreasing function of density ratio. Indeed, when this is the case, the coefficient A_1 is positive, and the Equation (23) has two real roots, one positive and one negative. In order to determine whether the γ-instability occurs and therefore whether staircases may spontaneously form at low Pr, low τ, we simply need to calculate $\gamma^{\rm tot}$ from our previous measurements. The results are shown in Figure 4. We find that the flux ratio always increases with density ratio. Crucially, this suggests that thermo-compositional layers are not expected to form spontaneously in astrophysical fingering convection. In the absence of such "staircases," the fluxes are accurately supplied by our flux laws (Equations (12) and (13)).

4. CONCLUSION

Our results can be summarized in a few key points. Using high-performance three-dimensional numerical simulations we find that turbulent compositional transport by fingering convection follows the simple law:

$$D_\mu = \kappa_\mu(\mathrm{Nu}_\mu - 1) = 101\sqrt{\kappa_\mu \nu}\, e^{-3.6r}(1-r)^{1.1}, \quad (24)$$

where $r = (R_0^* - 1)/(\tau^{-1} - 1)$. Meanwhile, the turbulent heat flux is negligibly small at stellar parameters (see Denissenkov 2010). We also find that large-scale thermo-compositional staircases are not expected to form spontaneously for Pr, $\tau \ll 1$ in fingering convection, so that the turbulent diffusivity proposed above may and should be used as given. It is interesting to note that by contrast, staircases do form in the opposite double-diffusive regime (semi-convection) (see Rosenblum et al. 2010).

Applying these findings to the problem of RGB stars, we concur with Denissenkov's conclusion that mixing by fingering convection alone cannot explain the observed RGB abundances, and that additional mechanisms should be investigated instead (e.g., gyroscopic pumping; see Garaud & Bodenheimer 2010). A second example of application to the planetary pollution problem is presented in a companion paper (Garaud 2011).

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Figure 1 was rendered using VAPOR (Clyne et al. 2007; Clyne & Rast 2005), a product of the National Center for Atmospheric Research.

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