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Unambiguous ultrashort pulse reconstruction from double spectrograms alone

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Abstract
We report a spectrographic technique for amplitude and phase measurements of ultrashort laser pulses (above 10 fs). Pulse information is obtained directly from two different spectrograms, using the mathematical relations between Wigner–Ville function projections. Pulses are reconstructed rapidly and unambiguously without stagnation. This non-interferometric method is demonstrated experimentally for the successful characterization of 100 fs pulses.

Keywords: ultrafast optical fields, phase retrieval, numerical optimization

(Some figures may appear in colour only in the online journal)

1. Introduction
Research into the retrieval of accurate phase and intensity information of ultrashort pulses has aided the advancement of various research fields. Examples include time-resolved chemical analysis (femto-chemistry) [1], femto-second spectroscopy [2], and potential telecommunications systems based on soliton molecules [3]. Because these examples involve complex time-resolved dynamics accurate pulse information diagnostics are required. In situations where complex pulse structures arise, such as the characterization of three-pulse soliton molecules [4], effective diagnostic methods must be able to retrieve intensity and phase information as a function of time in order to generate a precise description of the ultrashort process. There are several methods which can meet these specifications. Interferometric techniques determine the phase difference between waves using interferograms. This is commonly exemplified by spectral phase interferometry for direct electric field reconstruction of ultrashort optical pulses [5]. Tomographic methods use Wigner–Ville function projections in cronocyclic space to obtain pulse information [6]. Spectrographic techniques exemplified primarily by frequency resolved optical gating (FROG) or second-harmonic generation FROG (SHG FROG) obtain phase and intensity of a single pulse from a single spectrogram [7].

In further FROG-based methods, cross-correlation has been used to characterize two independent pulses [8]. This approach has provided the means to characterize weak pulses by cross-correlating them with synchronized intense pulses. In addition, sum-frequency generation (SFG) or difference-frequency generation provides scope for versatility in terms of center wavelengths of the pulses measured. Furthermore, in order to remove the necessity of fully characterized reference pulses, as defined for cross-correlation FROG (XFROG) [8] self-referenced techniques have been reported which are also capable of characterizing two independent pulses [9]. The measurement of two independent pulses via cross-correlation using a single spectrogram can lead to solution ambiguity [10] which can be solved by using dual spectrogram methods [11, 12]. Unfortunately, these dual spectrogram methods remain subject to the phase-retrieval problem as described in detail in [13]. A method which bypasses the phase-retrieval problem, by combining interferometric with spectrographic methods, is blind-MEFISTO [14]. However, there is a constraint that blind-MEFISTO requires close to exact matching
of spectral domains of the two independent pulses for successful characterization [14].

In this work an approach to bypassing the phase-retrieval problem is presented for a dual spectrogram method known as ‘very advanced method for phase and intensity retrieval of e-fields’ (VAMPIRE), which can be used to characterize two pulses of different center wavelengths and spectral ranges [3, 11]. Specifically, we replace the pre-existing phase-retrieval problem with a reconstruction problem based on a partial differential equation relating phase and spectrogram information derived from the theory of Wigner–Ville functions. Although Wigner–Ville function projections are widely used in tomographic methods, to our knowledge this is the first report documenting their use to characterize ultrashort pulses using spectrographic methods. From this method we report two noteworthy outcomes; firstly, accurate phase information retrieval can be achieved without using an iterative Fourier transform algorithm, constituting direct analytical pulse reconstruction using purely spectrographic data. Secondly, pulse spectra can be obtained directly without complementary spectral measurement, which was previously a practical constraint with respect to VAMPIRE. Overall, we report a practical and convenient, self-referenced method for femtosecond pulse reconstruction.

2. Theory

2.1. Simplified VAMPIRE

VAMPIRE, a pulse characterization technique for two independent pulses, was the first method used to generate dual spectrograms [3, 11]. While the two pulses $E_{1,\text{in}}$ and $E_2$ (illustrated in figure 1) may be independent of each other, their corresponding spectrograms hold mathematical dependence. This is achieved by using a Mach–Zehnder interferometer (MZI) in conjunction with a dispersive element to create two temporally separated subpulses from $E_{1,\text{in}}$ (denoted by $E_{1,\text{disp}}$ and $E_1$) which are both temporally cross-correlated with $E_2$. The overall purpose of the MZI is to obtain these collinear subpulses, the static delay line is only used as an optimization feature for avoiding interference between the emerging subpulses by providing the necessary temporal separation. The input pulse $E_{1,\text{in}}$ and the subpulse $E_1$ are related simply by the known dispersion functions of the 50:50 beam splitters of the MZI. The dispersion generated by the dispersive element is considered to be the unique difference between the subpulses as the dispersion generated by the 50:50 beam splitters is considered to be equal for both paths of the MZI. The two subpulses are related by

$$E_{1,\text{disp}}(\omega) = E_1(\omega)\exp[i\phi_{\text{DE}}(\omega)],$$

(1)

where $\phi_{\text{DE}}(\omega)$ is the spectral phase function of the dispersive element.

The pulses emerging from the MZI are focused by a cylindrical lens upon the nonlinear crystal (BBO). At the position of the nonlinear crystal the pulse fronts of the incident beams coincide, i.e., there is a spatial and temporal overlap between them. The beam diameters and the angle between the incident pulse fronts define the range of delay which can be obtained in a given setup. The delay $(\tau)$ is a function of the transverse coordinate of the SFG beam. The SFG signal emanating from the crystal is then mapped onto the entrance slit of the spectrometer. Time resolution is achieved using the delay information contained in the transverse coordinate of the SFG beam. This well established method for the generation of a delay axis is described in detail in [15] and exemplified for VAMPIRE in [16].

The delay between the subpulses emerging from the MZI should be optimized using the static delay line so that the pulses have minimal time separation without temporal overlap; which may lead to interference effects. However, if the temporal pulse separation is too large then there will be no spatial and temporal overlap of both pulses emerging from the

Figure 1. Scheme for simplified VAMPIRE: $E_{1,\text{in}}$ and $E_2$ are synchronized input pulses; NPBS, non polarizing 50:50 beam splitter; DE, dispersive element (2 mm thick SF10 glass); L, cylindrical lens; BBO, 300 $\mu$m thick type I BBO crystal, in which the nonlinear optical process of sum-frequency generation was used. DL (static) is an adjustable but constant delay line. (a) The single-beam simplified VAMPIRE apparatus. (b) and (c) The two previous non-simplified VAMPIRE schemes which required the measurement of three beams.
MZI with $E_2$ in the BBO crystal; in this case a ($\tau$) axis cannot be obtained for both spectrograms. Optimal separation can be found in situ using the camera to observe the relative positions of the spectrograms.

One should note that figure 1 is a scheme for a simplified version of VAMPIRE, where part (A) represents the advancement to single beam measurement from the previous multibeam measurement systems shown in (B) and (C). This is possible due to the pulse reconstruction method proposed here. This is a considerable advantage as one no longer has to measure three non-collinear beams in different spectral domains, as was necessary in previous cases.

2.2. Spectrogram equations

For VAMPIRE both spectrograms are expressed in the form of

$$I_\Omega(\tau) = I(\Omega, \tau) \propto \int d\omega \hat{E}_1(\omega)\hat{E}_2(\Omega - \omega)\exp(i\omega\tau)\, d\omega,$$

(2)

$$I_\Phi(\Omega) = I_\Phi(\Omega, \tau) \propto \int d\omega \hat{E}_{1,\text{disp}}(\omega)\hat{E}_2(\Omega - \omega)\exp(i\omega\tau)\, d\omega,$$

(3)

where the angular frequency $\Omega$ emerges from the SFG process. With reference to figure 1, $\hat{E}_1$, $\hat{E}_{1,\text{disp}}$, and $\hat{E}_2$ are the complex electric fields, expressed in the spectral domain, of the respective pulses shown. Equations (2) and (3) are valid for situations where $E_1$ and $E_{1,\text{disp}}$ are temporally separated and non-overlapping. The use of the spectral domain is convenient as it facilitates the inclusion of the spectral phase function $\varphi_{\text{disp}}(\omega)$ of the dispersive element in one of the arms of the MZI. Examples of both spectrograms $I_\Omega(\tau)$ and $I_\Phi(\Omega, \tau)$ are given in figure 2.

2.3. Pulse reconstruction method

According to developments published in [17] a partial differential equation was derived from corresponding Wigner functions and used to obtain the spectral phase function derivative of an ultrashort signal. Mathematically, the same partial differential equation must be true for conversely interchanged spectral and time domains [18]. Hence, if the electric field $E(t)$ of an ultrashort pulse (expressed in the spectral domain as $\hat{E}(\omega)$) is subjected to a quadratic spectral phase modulation as described by

$$\hat{E}_\Psi(\omega) = \hat{E}(\omega)\exp\left(i\frac{\Psi}{2}\omega^2\right),$$

(4)

then the time dependent phase function derivative $\partial \varphi_E(t)/\partial t$ of the electric field $E(t)$ can be obtained from

$$\left\{ \frac{\partial I(t)}{\partial \Psi} \right\}_{\Psi=0} = \frac{\partial}{\partial \Psi} \left[ I(t) \frac{\partial \varphi_E(t)}{\partial t} \right].$$

(5)

$I(t)$ and $I_\Psi(t)$ are the intensities corresponding to the electric fields $E(t)$ and $E_\Psi(t)$, respectively.

It follows that equation (5) can be used in conjunction with spectrographic information for the reconstruction of the pulses $E_1$ and $E_2$ as defined in figure 1, this outcome can be shown with the following steps:

For each $\Omega$ (treated as rows) of the first spectrogram equation (2), a function $\tilde{A}_\Omega(\omega)$ is defined in the spectral domain by

$$\tilde{A}_\Omega(\omega) = \hat{E}_1(\omega)\hat{E}_2(\Omega - \omega),$$

(6)

where the product $\hat{E}_1(\omega)\hat{E}_2(\Omega - \omega)$ is part of the integrand in both spectrogram equations (2) and (3), this can be seen by considering equation (1).

$\tilde{A}_\Omega(\omega)$ can be Fourier transformed into the time domain producing

$$A_\Omega(\tau) = |A_\Omega(\tau)|\exp[i\varphi_{A_\Omega}(\tau)] \propto \int d\omega \tilde{A}_\Omega(\omega)\exp(i\omega\tau),$$

(7)
where \( \varphi_{A_0}(\tau) \) is the phase function of \( A_0(\tau) \). The square modulus of the complex function \( A_1(\tau) \) is directly proportional to the first spectrogram intensity \( I_1(\tau) \) in equation (2).

In an ideal case, as defined in equation (4), the spectral phase function (the phase change imposed by a dispersive element) should take the form of

\[
\phi_{\text{DE,ideal}}(\omega) = \frac{\Psi}{2} \omega^2
\]

(8)

and the spectral phase modulation of \( \tilde{A}_1(\omega) \) can be written as

\[
\tilde{A}_{1,\Psi}(\omega) = \tilde{A}_1(\omega) \exp[i\phi_{\text{DE,ideal}}(\omega)],
\]

(9)

which in the time domain gives

\[
A_{1,\Psi}(\tau) \propto \int \text{d}\omega \tilde{A}_{1,\Psi}(\omega) \exp(i\omega\tau).
\]

(10)

The spectrogram intensities hold the following proportionality

\[
I_{1,\Psi}(\tau) \propto |A_1(\tau)|^2 \quad \text{and} \quad I_{0,\Psi}(\tau) \propto |A_0,\Psi(\tau)|^2,
\]

and thus using equation (5) are related by

\[
\frac{\partial I_{0,\Psi}(\tau)}{\partial \Psi} \bigg|_{\delta=0} = \frac{\partial}{\partial \Psi} \left[ I_{1,\Psi}(\tau) \frac{\partial \varphi_{A_0}(\tau)}{\partial \tau} \right],
\]

(11)

where \( \partial \varphi_{A_0}(\tau)/\partial \tau \) is the phase function derivative of \( A_0(\tau) \).

Equation (11) is then integrated to obtain the phase function derivative

\[
\frac{\partial \varphi_{A_0}(\tau)}{\partial \tau} \approx \frac{1}{I_1(\tau)\Psi} \int_{-\infty}^{\tau} [I_{0,\Psi}(\tau') - I_{1,\Psi}(\tau')] \text{d}\tau'.
\]

(13)

By integrating \( \partial \varphi_{A_0}(\tau)/\partial \tau \) one gets the phase function itself up to the constant of integration \( F_{01} \).

By integrating \( \partial \varphi_{A_0}(\tau)/\partial \tau \) one gets the phase function itself up to the constant of integration \( F_{01} \).

Having modulus \( |A_1(\tau)| \propto \sqrt{I_1(\tau)} \) and phase \( \varphi_{A_0}(\tau) + F_{01} \), the complex function \( A_1(\tau) \) is transformed back into the spectral domain to give

\[
\tilde{A}_1(\omega) \exp[i F_{01}]
\]

\[
\propto \int \text{d}\tau |A_1(\tau)| \exp[i(\varphi_{A_0}(\tau) + F_{01})] \exp(-i\omega\tau).
\]

(14)

The modulus of the complex function \( \tilde{A}_1(\omega) \) is shown in figure 4.

2.3.1. Retrieval of pulse spectra. The modulus term

\[
|\tilde{A}_1(\omega)\exp[i F_{01}]| = |\tilde{A}_1(\omega)| = |\tilde{E}_1(\omega)||\tilde{E}_2(\Omega - \omega)|
\]

(15)

can be applied to retrieve the spectral intensities of both pulses as follows.

By defining the variable

\[
\Omega' = \Omega - \omega,
\]

(16)

one can define the function

\[
M(\omega, \Omega') = |\tilde{A}(\omega, \Omega')| = |\tilde{E}_1(\omega)|^2|\tilde{E}_2(\Omega')|^2
\]

(17)

and obtains

\[
I_1(\omega) \propto |\tilde{E}_1(\omega)|^2 = \frac{\int M(\omega, \Omega') \text{d}\Omega'}{\int |\tilde{E}_2(\Omega')|^2 \text{d}\Omega'} \propto \int M(\omega, \Omega') \text{d}\Omega',
\]

(18)
and to de

as de

e
one must remove the function \( \tilde{E}_1(\omega) \) as it allows for the removal of both major noise sources.

The function

\[ \tilde{E}_1(\omega) = \frac{\int M(\omega, \Omega') \, d\omega}{\int |\tilde{E}_2(\Omega')|^2 \, d\Omega'} \]

and

\[ \tilde{E}_2(\Omega') = \frac{\int M_2(\omega, \Omega') \tilde{E}_1(\omega)^* \, d\omega}{\int |\tilde{E}_1(\omega)|^2 \, d\omega}. \]

Next, equation (21) can be rearranged to give

\[ X(\Omega) = \frac{M_0(\omega, \Omega)}{\tilde{E}_1(\omega)\tilde{E}_2(\Omega - \omega)}, \]

which is the ideal form of \( X(\Omega) \). However, for actual data, measured or simulated, the weight function \( |\tilde{A}(\omega, \Omega)| = |\tilde{E}_1(\omega)\tilde{E}_2(\Omega - \omega)| \) should be used to significantly reduce the effects arising from noise. Thus, \( X(\Omega) \) can be written in terms of signum functions as in equation (30)

\[ X(\Omega) \propto \int \frac{M_0(\omega, \Omega)}{\tilde{E}_1(\omega)\tilde{E}_2(\Omega - \omega)} \, d\omega = \int M_0(\omega, \Omega) \text{sgn}[\tilde{E}_1(\omega)] \text{sgn}[\tilde{E}_2(\Omega' - \omega)] \, d\omega \]

where the asterisk denotes the conjugate complex. Using the signum function \( \text{sgn}(z) = z/|z| = \exp(i\varphi) \) generalized to complex numbers the solutions take the form

\[ \text{sgn}[\tilde{E}_1(\omega)] = \exp[i\varphi_1(\omega)] \]

\[ = \text{sgn}\left[ \int M_2(\omega, \Omega')\tilde{E}_1^*(\Omega') \, d\Omega' \right]. \]

\[ \text{sgn}[\tilde{E}_2(\Omega')] = \exp[i\varphi_2(\Omega')], \]

\[ = \text{sgn}\left[ \int M_2(\omega, \Omega')\tilde{E}_1^*(\omega) \, d\omega \right]. \]

which pertain to the spectral phase functions of the pulses. As \( \varphi_1 \) and \( \varphi_2 \) are unknown, the iterative factorization process, this is summarized in figure 5. A series of numerical tests has shown the convergence rate and results from this factorization method appear to be constant and without strong dependence on initial guesses.

Figure 4. Simulations: using the spectrograms from figure 2; left: a simulation of \(|\tilde{A}_1(\omega)|\) showing the effects carried forward from the spectrogram noise and the finite temporal sampling rate. Right: shows \(|\tilde{A}_1(\omega)|\) with the background noise removed. The noise is filtered from the function \( \tilde{A}_1(\omega) \) as it allows for the removal of both major noise sources.
Finally, one can compute the complex spectral pulse fields $\tilde{E}_1$ and $\tilde{E}_2$ as shown at the bottom of figure 5.

### 2.4. Simulations

Figure 6 shows a comparison between simulated and reconstructed spectrograms using the pulse retrieval method described in this work, these spectrograms are the same characteristically as those shown in figure 2, with the addition of standard uniformly distributed pseudo-random noise of 2% of the maximum spectrogram intensity.

Using the method presented in section 2.3. We were able to successfully reconstruct both pulses from the spectrogram data, the corresponding reconstructed pulses are shown in figure 7.

As a figure of merit, to quantify the accuracy of the reconstructed spectrograms, FROG error was calculated for each using

$$G = \frac{1}{N} \left[ \sum_{j=1}^{N} \left| I_{\text{meas}}(\Omega_j, \tau_j) - I_{\text{rec}}(\Omega_j, \tau_j) \right|^2 \right]^{1/2},$$  

where $I_{\text{meas}}$ and $I_{\text{rec}}$ are the measured/simulated and reconstructed spectrograms respectively with peaks normalized to one, $\mu$ is a minimizing scalar value for $G$, and $N^2$ is the grid size, in our case $512 \times 512$.

The linear term of $\phi_{\text{DE}}(\omega)$ can be ignored as it represents a time shift of the pulse and not a change in pulse shape. There are several optical media known to satisfy condition (33) over a broad spectral range. The dispersive properties of SF10 over the angular frequency range $-1.88 \times 10^4$ Hz, corresponding to a wavelength range of $600 - 1000$ nm, are shown in figure 8. The spectral phase change generated by the dispersive element can be expressed in terms of the refractive index as

$$\phi_{\text{DE}}(\omega) \approx \phi_{\text{DE,ideal}}(\omega).$$  

(33)

An important physical parameter for both simulations and experiments is the spectral phase modulation generated by the dispersive element. Primarily, equation (11), the point at which data analysis is initiated, was derived under the assumption that the spectral phase function of the dispersive element is a quadratic function with respect to $\omega$, as defined in equation (8), such that

$$\phi_{\text{DE}}(\omega) = \phi_{\text{DE,ideal}}(\omega).$$  

The material used throughout this work was SF10 with a TOD to GDD ratio of 0.66 fs at a wavelength of 800 nm. Hence, this ratio presents a source of systematic error, although its impact can be limited by restricting its value to below 1.3 fs. Our simulations using the levels of discrepancy shown in figure 8 displayed little impact on the reconstruction results, in comparison to noise sensitivity for example. All materials listed in table 1 have
the appropriate TOD to GDD ratios at 800 nm required to obtain reasonable FROG errors. Slightly improved FROG errors could be obtained by using one of the fluorides. Further into the infrared, BaF$_2$ would clearly provide the best performance of those listed.

The parameter $\Psi$ presents two important considerations or preconditions to our method. Firstly, $\Psi$ should be minimized as equation (12) is only valid for small changes in the pulse electric field resulting from the given dispersive

<table>
<thead>
<tr>
<th>Material</th>
<th>TOD/GDD$_{\lambda=800\text{nm}}$(fs)</th>
<th>TOD/GDD$_{\lambda=1100\text{nm}}$(fs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BaF$_2$</td>
<td>0.52</td>
<td>0.89</td>
</tr>
<tr>
<td>CaF$_2$</td>
<td>0.59</td>
<td>1.40</td>
</tr>
<tr>
<td>SF10</td>
<td>0.66</td>
<td>1.13</td>
</tr>
<tr>
<td>BK7</td>
<td>0.72</td>
<td>2.71</td>
</tr>
<tr>
<td>Fused silica</td>
<td>0.76</td>
<td>3.50</td>
</tr>
</tbody>
</table>

Figure 7. Simulations: (a) and (c) show the pulse $E_1$ and (b) and (d) show the pulse $E_2$. Black lines represent the ideal simulated pulses. Red lines represent reconstructed pulses subject to a noise level of 2% in the corresponding spectrograms. The pulses are defined as $E_1(f) = E_{01} \exp[(a_1 + ib_1)(f + \Delta f)^2] \{ \exp(-i\Delta t_{2\pi f}) + 0.7 \exp(i\Delta t_{2\pi f}) \}$ and $E_2(f) = E_{02} \{ \exp[a_2(f + \Delta f_1)^2] + \exp[a_2(f - \Delta f_2)^2] \} \exp(ib_2 f^2)$ with $a_1 = -0.394$ ps$^2$, $b_1 = 0.0563$ ps$^2$, $\Delta f = 0.133$ THz, $\Delta t = 250$ fs, $a_2 = -0.081$ ps$^2$, $b_2 = 0.162$ ps$^2$, $\Delta f_1 = 5.11$ THz, $\Delta f_2 = 2.67$ THz, and $f_{1,0} = 376.0$ THz, $f_{2,0} = 375.0$ THz are the center frequencies of the pulses.

Figure 8. The black line shows the SF10 spectral phase change for 1 mm thickness. The red line represents the ideal quadratic dispersion function for a center wavelength of 750 nm which corresponds to $\Psi = 173$ fs$^2$. The cyan dashed line shows the TOD to GDD ratio over the angular frequency range corresponding to the wavelength range 600 to 1000 nm.
element. This precondition can be quantified, in the form of a general guideline, using the complex transfer function of the dispersive element given by

$$\exp[i\phi_{DE}(\omega)] \approx \exp[i\phi_{DE, ideal}(\omega)]$$

$$= \exp \left( \frac{\Psi}{2} - \frac{\Psi}{2\omega^2} + \ldots \right)$$ (35)

For a small spectral phase change where

$$\frac{\Psi}{2\omega^2} \ll 1,$$ (36)

and in terms of the spectral bandwidth \(\Delta\omega\) of the dispersed pulse \(E_1\)

$$\frac{\Psi}{2} \Delta\omega_E^2 \ll 1,$$ (37)

one obtains the condition

$$\Psi(\Delta\omega_E)^2 \ll 8$$ (38)

or

$$\Psi(\Delta f_E)^2 \ll 0.2.$$ (39)

Secondly, with the consideration of noise, \(\Psi\) must be sufficiently large as to generate a difference between both spectrograms which is large in comparison to the given level of noise. That is to say, \(\Psi\) must be large enough to produce an acceptable signal-to-noise ratio (SNR) for the function \([I_0(\Omega, \tau) - I(\Omega, \tau)]\) as shown in figure 3.

For example, within our simulations shown in figure 7 the center wavelength of the pulses is approximately 800 nm and the spectral bandwidths are \(\Delta f_E = 5\) THz and \(\Delta f_{E_2} = 20\) THz for \(E_1\) and \(E_2\), respectively. The degree of GDD applied to one of the portions of \(E_1\), within the upper arm of the MZI, is approximately \(\Psi = 530\) fs\(^2\), which corresponds to 3.4 mm of SF10. Thus, in this case one gets \(\Psi(\Delta\omega_E)^2 = 0.5\) or \(\Psi(\Delta f_E)^2 = 0.013\) and the condition (38) is satisfied.

Table 2 provides the summary of the response of FROG error \(G\), as defined in equation (32), to the variation of \(\Psi\). In the noiseless situation the FROG error is reduced in response to less GDD, where 2% noise is present small values of GDD cause the FROG error to increase. From our simulations we find that \(\Psi = 440\) fs\(^2\) is a reasonable optimization in terms of resultant FROG error.

For very short pulses, accurate characterization is constrained due to the very thin SF10 layers which would be required in order to obtain acceptable FROG errors (less than 0.01). Table 3 summarizes the required adjustment of SF10 thickness for several categories of pulse duration, showing the maximum SF10 thickness allowed to obtain acceptable FROG errors. Generally, as pulse duration decreases the maximum allowed layer thickness also decreases, and this can be problematic for pulses approaching 10 fs as dispersive elements of micrometer scale thickness become necessary.

For the simulations shown, a center wavelength of approximately 800 nm (\(f = 375\) THz) was maintained. Of course, the center wavelength of the pulses can be changed. However, considerable FROG errors \((G > 0.01)\) may be generated in situations where parts of the pulse spectra, passing through the SF10, are outside of the range 380–1150 nm, where the TOD to GDD ratio is larger than 1.3 fs.

For the case of sub-10-fs pulses the reconstruction method presented here is not suitable in its current form. The bandwidth of such pulses is likely to be too large such that the spectral dispersion of SF10 is no longer approximately parabolic over the entire spectral range of the pulse. Hence, in general, sub-10-fs pulses may not be accurately characterized within tolerable FROG errors.

2.5. Uniqueness

The term uniqueness is defined as the unambiguous solution for the temporal pulse envelope function \([E(t)]\) and the instantaneous pulse frequency change as a function of time \(\omega(t) = d\omega/dt\) (chirp) without absolute time reference. Potential sources of ambiguities for arbitrary fields \(E_1\) and \(E_2\) that can be easily derived from equations (2) and (3) are listed below:

1. The spectral fields \(\tilde{E}(\omega)\) are only determined up to a complex factor \(\exp[i(\omega_1 + \omega_2)]\). Such a linear spectral phase function is equivalent to a time-shift. Since there is no absolute time reference in the experimental setup the appearance of this time-shift ambiguity is expected.

2. The pulse pair \((\tilde{E}_1(\omega), \tilde{E}_2(\omega))\) gives the same spectrogram, described by formula (2), as the pulse pair \((\tilde{E}_1(\omega + \Delta\omega), \tilde{E}_2(\omega - \Delta\omega))\). The second spectrogram, described by formula (3), neglecting a time-shift, remains also unchanged because \(\phi_{DE, ideal}(\omega)\) is a quadratic function with respect to \(\omega\). Hence, the method is not sensitive to the absolute pulse frequencies.
These initial points do not affect the determination of the temporal pulse envelope function and the chirp and, hence, are not considered as ambiguities, however;

3. The possible occurrence of frequency gaps in the spectral pulse envelopes can lead to relative spectral phase ambiguities in \( \hat{E}_1(\omega) \) and/or \( \hat{E}_2(\omega) \), which correspond to many different temporal pulse envelope functions for the same spectrograms. This topic is discussed in detail in [11, 19, 20]. VAMPIRE in general can account for this source of ambiguities under the condition that one of the pulses features sufficient spectral bandwidth so as to cover any spectral gaps in the other. So, as long as this condition is met then the issue of frequency gaps may not be considered as a source of ambiguity.

With reference to figure 5 and with respect to the two spectrograms obtained, there are two main considerations required in order to maintain uniqueness;

1. In regions of the spectrogram where \( I(\Omega, \tau) \) takes on very small values, the consequent SNR of \( I(\Omega, \tau) \) will be close to or less than one. In order to obtain \( \partial \phi_{\text{hk}}(\tau)/\partial \tau \) one must compute expression (13), therefore such regions where the SNR of \( I(\Omega, \tau) \) is low may produce severely deteriorated phase information. As a consequence the integration over \( \partial \phi_{\text{hk}}(\tau)/\partial \tau \) produces regions of undetermined phase and an overall relative phase ambiguity. Such a situation can be circumvented by modulating the spectral phase of one of the pulses i.e. one can apply chirp (dispersing the pulse by a piece of glass) to remove situations where \( I(\Omega, \tau) \) is close to zero. For VAMPIRE pulse \( E_2 \) is considered as the test pulse for a given external experiment (e.g. pump–probe experiment) thus it is \( E_{\text{in}} \) which should be given this minor spectral phase modulation prior to input into the VAMPIRE apparatus. In situations where characterization of both pulses is desired then \( E_{\text{in}} \) can be reconstructed simply by considering the dimensions and dispersive properties of the optical media in use.

2. With reference to figure 5 function

\[
M_0(\omega, \Omega) = X(\Omega)\hat{E}_1(\omega)\hat{E}_2(\Omega - \omega)
\]

is factorized in order to obtain the fields \( \hat{E}_1 \) and \( \hat{E}_2 \). This factorization process is nontrivial, however, the emerging solutions can be proven to be unique as shown in [10]. This can be achieved by considering the argument of \( M_0(\omega, \Omega) \)

\[
\arg[M_0(\omega, \Omega)] = \arg[X(\Omega)] \arg[\hat{E}_1(\omega)] \arg[\hat{E}_2(\Omega - \omega)],
\]

\[
\arg[M_0(\omega, \Omega)] = F(\Omega) + \varphi_1(\omega) + \varphi_2(\Omega - \omega),
\]

which can be uniquely decomposed in \( \varphi_1 \) and \( \varphi_2 \) despite the arbitrary function \( F \).[10]

In all other mathematical steps in the reconstruction sequence we find no further potential sources of ambiguities.

3. Experiment

A homemade Kerr lens mode-locked titanium:sapphire laser oscillator was used to produce ultrashort pulses with a duration of approximately 100 fs, the center wavelength was 800 nm, with a pulse energy of 5 nJ, and a repetition rate of 78.8 MHz. The pulses from the single laser source were divided into two pulses, \( E_{\text{in}} \) and \( E_2 \), using a beam splitter and then synchronously launched into the apparatus shown in figure 1. \( E_{\text{in}} \) was directed into the MZI while \( E_2 \) was simply directed upon the cylindrical lens. Using the cylindrical lens the pulses were focused into a 300 μm thick type I BBO crystal. The generated sum-frequency light, and only the sum-frequency light, was frequency resolved by a spectrometer. \( E_{\text{in}} \) is split by the MZI into two sub pulses, the one passing through the upper arm is subjected to dispersion, the other although not subject to dispersion is delayed in order to optimize the temporal separation between the two sub pulses \( (E_{\text{disp}}, E_1) \) exiting the MZI. Both spectrograms, shown in figures 9(a) and (b) were recorded by the same 8-bit CMOS camera.

Near perfect phase matching was achieved for 10 THz width pulses, \( E_1 \) and \( E_2 \), using a 300 μm thick type I BBO crystal with a spectral acceptance bandwidth of approximately \( \Delta f = 20 \) THz, such that equations (2) and (3), within the reconstruction sequence, could be used without spectral corrective functions. The degree of GDD applied to one of the portions of \( E_{\text{in}} \), generated by a 2 mm long piece of SF10 glass in the upper arm of the MZI, was approximately \( \Psi = 310 \) fs². Thus, with the spectral bandwidth of \( \Delta f_\Omega = 10 \) THz one gets \( \Psi(\Delta f_\Omega)^2 = 0.03 \) and the condition (38) was satisfied.

The reconstruction method described in this article is sensitive to potential phase mismatch. The function \( A_{\Omega}(\tau) \), which is simply the square-root of the spectrogram intensity, is directly affected by phase mismatch due to the incomplete spectrogram information which may result. Therefore, if there is a considerable phase mismatch, the function \( A_{\Omega}(\omega) \) in equation (14) is incorrectly obtained. This leads to a loss of accuracy in the factorization given by equations (15) to (19), and thus results in large FROG errors.

Figures 9(e) and (f) show the retrieved pulses in the time and frequency domains, respectively. The retrieval of two identical but independent pulses was analyzed. This was done partly as a simple test for the efficacy of our method, where successful reconstruction was expected to yield the two identical pulses. The results shown in figure 9 verified that this was the case, where modulus and phase showed agreement between \( E_1 \) and \( E_2 \) which corresponded to FROG errors less than 0.01 obtained for both spectrograms. Moreover, the use of two identical pulses allowed the comparison of our results to those obtained using the well-established SHG FROG, which requires two identical pulses by definition. The measured spectrogram shown in figure 9(a) was used to perform a SHG FROG pulse retrieval and again for this case a FROG error less than 0.01 was obtained. For the SHG FROG measurements, care was taken to balance \( E_1 \) and \( E_2 \) in terms of dispersion through the optical components within their distinct optical paths in the whole setup. A comparison of the pulses in the time and frequency domains shows good agreement between the reconstructions. With regards to the FROG errors obtained, our simplified VAMPIRE gave...
slightly higher FROG errors when compared to SHG FROG as shown in figure 9, although this was an expected outcome typical for dual spectrogram methods.

In summary, the reconstruction method described here has been experimentally validated by comparing the reconstruction results with the pulses obtained from a standard SHG FROG. Thus, the spectrographic reconstruction method, based on direct phase retrieval using Wigner–Ville function projections, has been shown to be feasible and reliable for the characterization of two independent ultrashort pulses. Within the experiment it was found to be resistant to experimental noise sources such as stray light, camera shot noise, spectral calibration errors, spatial inhomogeneities in the beams and all the possible sources of errors in our measurements.

4. Conclusion

Transparent dispersive media which exhibit an almost quadratic dispersion over a wide spectral range allow one to use a particular partial differential equation, frequently used in tomographic pulse reconstruction methods, for spectrographic methods. Hence, the non-interferometric spectrographic technique developed here works with a reconstruction method which is not based on an iterative Fourier transform algorithm or an ill-posed inversion problem in general. Therefore, it was possible to discuss the uniqueness conditions and to justify them mathematically within the constraints presented, and moreover, to reconstruct the pulse spectra analytically. Only for the final step of the reconstruction method was an iterative...
factorization algorithm used operating without Fourier transforms. In simulations, stagnation did not occur in the factorization algorithm and the total number of iterations required was mainly related to the grid size. Self-referenced operation, with both simulated and experimental data, was shown to be robust in the reconstruction of noisy ultrashort pulses.

The ability to reconstruct the pulse spectra analytically was also shown to be an advantageous feature with respect to the simulation of VAMPIRE, which provided a resolution of previous issues concerning experimental complexity [21]. In the previous version a full spectral sensitivity calibration of requiring reference pulses and making this method an advantageous alternative to techniques of previous issues concerning experimental complexity. ultrashort pulses can be reconstructed from double spectrograms alone without additional pulse spectra measurement, making this method an advantageous alternative to techniques requiring reference pulses and/or third-order nonlinear processes.

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References
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