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Damped 2-DOF subsystems of acoustic metamaterials

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Abstract. This paper introduces the theory of the mass-in-mass subsystems of metamaterials by a 2-Degree-Of-Freedom vibration model with linear characteristic springs and dampers, periodic excitation. The theory is described by a traditional method, by rotating vectors, using D'Alambert's principle. Formerly this method was used only for 1-Degree-Of-Freedom and now it has been developed for 2-Degree-Of-Freedom. Relationship between the phase shift and the amplitudes of the two masses' motion has been determined. Both depend on only the Lehr's damping ratio and the frequency rate. A $G_{21} = A_2/A_1$ factor has been introduced and called amplitude rate. The mass-in-mass subsystem has been described by the conducted results, so the motion and the forces are more apparent.

1. Introduction

The research of acoustic metamaterials is in progress. The mass-in-mass subsystem had been introduced during the theoretic foundation, that consists of two m_1, m_2 masses, one linear characteristic spring and the periodic exciting F_0 force loading the m_1 mass [1-4]. This model provides the base of the effective mass, without damping. We can approach the real material's model with adding a k_2 viscous damper ($F_{damp} = -k\dot{x}$) to the system that influence significantly the operation of the subsystem. With the aim of describing and understanding this, the most apparent way to use the rotating vector method [5]. First we start from the 1-Degree-Of-Freedom as it is well presented in the specified literature, than we extend it for 2-Degree-Of-Freedom. Through this it is easier to describe and understand the damped mass-in-mass subsystems of acoustic materials.

2. Model of 1-Degree-Of-Freedom (1-DOF) vibration with damping and excitation

The model of the 1-DOF, viscously damped, periodically excited vibration (Figure 1.) is consists of one mass, one viscous damping (damping force is proportional to velocity) and one spring with linear characteristic ($F_{spring} = -sx$) [6, 7]. The excitation is described by sin(t) function.

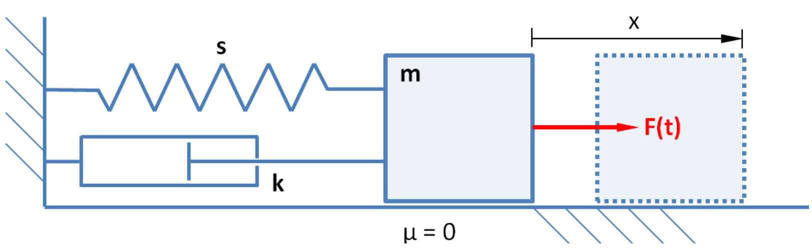

Figure 1. Model of 1-Degree-Of-Freedom, periodically excited, viscously damped vibration.

The motion equation of the model is based on the principle of linear momentum:

$$m\ddot{x} = F_0 \sin(\omega_g t) - k\dot{x} - sx \tag{1}$$

with some modification:

$$m\ddot{x} + k\dot{x} + sx = F_0 \sin(\omega_g t) \tag{2}$$

The equation of motion is a constant coefficient, linear inhomogeneous differential equation which solution consists of the general (x_h, free vibration) and the particular (x_p, excited vibration) solution ($x = x_h + x_p$). The solution of the homogenous equation $x_h = Ce^{-\beta t}\sin(\delta t + \psi)$ terminates fully by time because of damping, so we can focus only on the particular part [5-6].

As each part of the equation of motion rotates by the same ω_g angular velocity, change harmonically, the equation can be represented by rotating vectors. φ is the phase shift between exciting force and excited motion, and it is caused by damping. The displacement, velocity, acceleration vectors of a harmonic vibration is shown on Figure 2. The vectors of forces appearing in equation (1) are visible on Figure 3. The resultant force is represented by dashed line, showing the vectors in equilibrium position according to the principle of D'Alambert [5]:

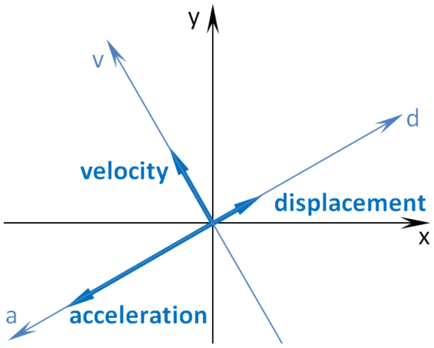

Figure 2. Displacement, velocity, acceleration.

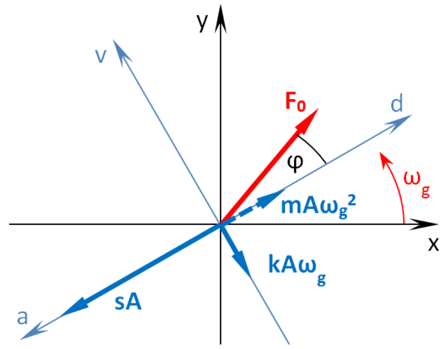

Figure 3. Equilibrium of force vectors.

The d and v directional equilibrium force vector equations by the principle of linear momentum [5]:

$$d: \quad 0 = F_0 cos\varphi + mA\omega_g^2 - sA \tag{3}$$

$$v: \quad 0 = F_0 sin\varphi - kA\omega_g \tag{4}$$

With some rearrangement:

$$d: \quad F_0 cos\varphi = A(s - m\omega_g^2) \tag{5}$$

$$v: \quad F_0 sin\varphi = kA\omega_g \tag{6}$$

Equation (6) dividing by (5) the phase shift value is:

$$g\varphi = \frac{F_0 sin\varphi}{F_0 cos\varphi} = \frac{kA\omega_g}{A(s - m\omega_g^2)} = \frac{k\omega_g}{s - m\omega_g^2} \tag{7}$$

Bringing out m mass, introducing $\beta = \frac{k}{2m}$ specific damping factor and $\omega^2 = \frac{s}{m}$:

$$tg\varphi = \frac{m\frac{k}{m}\omega_g}{m(\frac{s}{m} - \omega_g^2)} = \frac{\frac{k}{m}\omega_g}{\frac{s}{m} - \omega_g^2} = \frac{2\beta\omega_g}{\omega^2 - \omega_g^2} \tag{8}$$

Dividing both the denominator and the numerator by ω^2, introducing $D = \frac{k}{k_{krit}} = \frac{\beta}{\omega}$ damping ratio where $k_{kr} = 2m\omega$ is the critical damping factor:

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$$tg\varphi = \frac{2\frac{\beta\omega_g}{\omega\,\omega}}{\frac{\omega^2}{\omega^2}-\frac{\omega_g^2}{\omega^2}} = \frac{2D\frac{\omega_g}{\omega}}{1-\frac{\omega_g^2}{\omega^2}} \tag{9}$$

So the phase shift between the exciting force and the motion is [5]:

$$\varphi = arc\,tg\frac{2D\frac{\omega_g}{\omega}}{1-\left(\frac{\omega_g}{\omega}\right)^2} \tag{10}$$

Squaring and adding together equation (5) and (6):

$$F_0^2(cos^2\varphi + sin^2\varphi) = A^2\left[\left(s-m\omega_g^2\right)^2 + \left(k\omega_g\right)^2\right] \tag{11}$$

$$A = \frac{F_0}{\sqrt{\left(s-m\omega_g^2\right)^2+\left(k\omega_g\right)^2}} \tag{12}$$

Expanding the denominator and the numerator by $1/m$, entering the $\beta = k/2m$, $\omega^2 = s/m$ relationships and the $x_{st} = F_0/s$ static extension:

$$A = \frac{\frac{F_0}{s}\frac{s}{m}}{\sqrt{\left(\frac{s}{m}-\omega_g^2\right)^2+\left(\frac{k}{m}\omega_g\right)^2}} = \frac{x_{st}\omega^2}{\sqrt{\left(\omega^2-\omega_g^2\right)^2+\left(2\beta\omega_g\right)^2}} \tag{13}$$

Expanding by ω^2/ω^2, entering the $D = k/k_{kr} = k/2m\omega = \beta/\omega$ damping ratio:

$$A = \frac{x_{st}}{\sqrt{\left(1-\frac{\omega_g^2}{\omega^2}\right)^2+\left(2\frac{\beta\omega_g}{\omega\,\omega}\right)^2}} = \frac{x_{st}}{\sqrt{\left(1-\frac{\omega_g^2}{\omega^2}\right)^2+\left(2D\frac{\omega_g}{\omega}\right)^2}} \tag{14}$$

Based on that the dimensionless A/x_{st} scaling factor [5]:

$$\frac{A}{x_{st}} = \frac{1}{\sqrt{\left(1-\frac{\omega_g^2}{\omega^2}\right)^2+\left(2D\frac{\omega_g}{\omega}\right)^2}} \tag{15}$$

3. Model of 2-Degree-Of-Freedom (2-DOF), damped and excited vibration

The model of the 2-DOF, viscously damped, periodically excited vibration consists of two masses (m_1, m_2), two viscous dampers $(k_1, k_2$, where $F_{damping} = -k\dot{x})$ and two linear characteristic springs $(F_{spring} = -sx)$ as the Figure 4. shows [6].

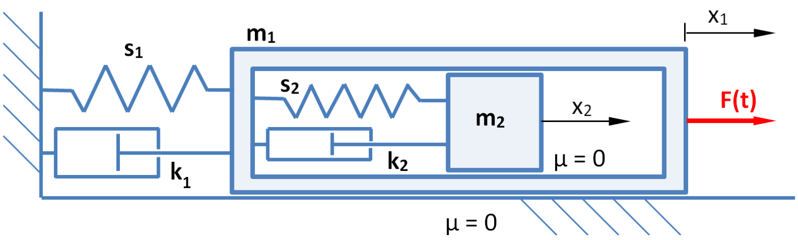

Figure 4. 2-DOF damped and excited vibration model.

The equation of motions written for the two masses:

$$\ddot{x}_1 + k_1\dot{x}_1 + k_2(\dot{x}_1 - \dot{x}_2) + s_1x_1 + s_2(x_1 - x_2) = F_0\sin(\omega_g t) \tag{16}$$

$$\ddot{x}_2 + k_2(\dot{x}_2 - \dot{x}_1) + s_2(x_2 - x_1) = 0 \tag{17}$$

Dissolving the brackets:

$$m_1\ddot{x}_1 + k_1\dot{x}_1 + k_2\dot{x}_1 - k_2\dot{x}_2 + s_1x_1 + s_2x_1 - s_2x_2 = F_0\sin(\omega_g t) \tag{18}$$

$$m_2\ddot{x}_2 + k_2\dot{x}_2 - k_2\dot{x}_1 + s_2x_2 - s_2x_1 = 0 \tag{19}$$

Equation (19) can be seen as equilibrium of five forces according to D'Alambert, after changing the senses. The equation (19) with opposite senses is represented on Figure 5. by rotating vectors. The exciting force does not move the m_2 mass directly, it is visible only for explanation.

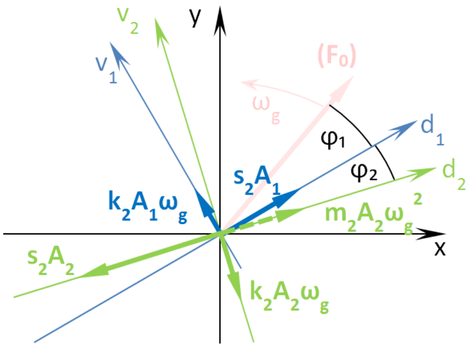

Figure 5. Forces on the m_2 mass.

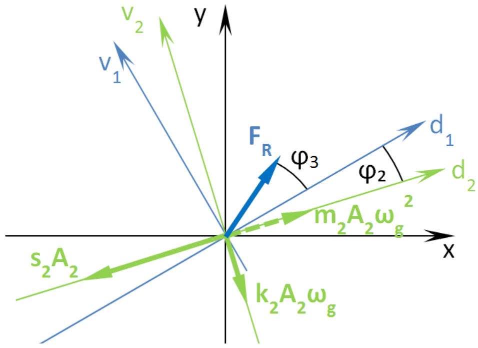

Figure 6. m_2 mass with resultant moving force.

The $k_2A_1\omega_g$, s_2A_1 force vectors are substituted by one F_R resultant force (Figure 6.), and now the problem is originate in the 1-DOF vibration. Based on that the m_2, k_2, s_2 vibrating system has a $\varphi_2 + \varphi_3$ phase shift to the F_R resultant force. The following relationships are used in the further reductions: $s_2 = \omega_2^2 m_2$, $D_2 = k_2/2m_2\omega_2 = \beta_2/\omega_2$. The φ_3 angle between the F_R resultant force and the d_1 axis, and the amplitude of F_R:

$$\varphi_3 = arc\ tg\frac{k_2A_1\omega_g}{s_2A_1} = arc\ tg2\frac{k_2}{2m_2\omega_2}\frac{\omega_g}{\omega_2} = arc\ tg2D_2\frac{\omega_g}{\omega_2} \tag{20}$$

$$|F_R| = \sqrt{\left(k_2A_1\omega_g\right)^2 + (s_2A_1)^2} = A_1\sqrt{\left(k_2\omega_g\right)^2 + s_2^2} \tag{21}$$

From the equation (21) bringing out s_2 and substituting D_2:

$$|F_R| = A_1s_2\sqrt{4\left(\frac{k_2}{2m_2\omega_2}\right)^2\left(\frac{\omega_g}{\omega_2}\right)^2 + 1} = A_1s_2\sqrt{4D_2^2\left(\frac{\omega_g}{\omega_2}\right)^2 + 1} \tag{22}$$

So the static extension of s_2:

$$x_{st2} = \frac{F_R}{s_2} = A_1\sqrt{4D_2^2\left(\frac{\omega_g}{\omega_2}\right)^2 + 1} \tag{23}$$

It is visible that only the amplitude of F_R resultant force depends on the motion of m_1, its angle with d_1 axis is independent of m_1's displacement. The φ_2 phase shift between the motions of the two masses:

$$\varphi_2 = arc\ tg\frac{2D_2\frac{\omega_g}{\omega_2}}{1-\left(\frac{\omega_g}{\omega_2}\right)^2} - \varphi_3 = arc\ tg\frac{2D_2\frac{\omega_g}{\omega_2}}{1-\left(\frac{\omega_g}{\omega_2}\right)^2} - arc\ tg2D_2\frac{\omega_g}{\omega_2} \tag{24}$$

Using the $arc\ tg(x) - arc\ tg(y) = arc\ tg\frac{x-y}{1+xy}$ trigonometrical relationship, the phase shift of m_2 to m_1 is (passing over the simplification):

$$\varphi_2 = arc\ tg\frac{\frac{2D_2\frac{\omega_g}{\omega_2}}{1-\left(\frac{\omega_g}{\omega_2}\right)^2} - 2D_2\frac{\omega_g}{\omega_2}}{1+\frac{2D_2\frac{\omega_g}{\omega_2}}{1-\left(\frac{\omega_g}{\omega_2}\right)^2}2D_2\frac{\omega_g}{\omega_2}} = arc\ tg\frac{\frac{\omega_g}{\omega_2}}{\frac{1}{2D_2\left(\frac{\omega_g}{\omega_2}\right)^2} - \frac{1}{2D_2} + 2D_2} \tag{25}$$

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Using equation (14) and (23) the amplitude of the attached mass' displacement:

$$A_2 = \frac{x_{st2}}{\sqrt{\left(1-\frac{\omega_g^2}{\omega_2^2}\right)^2+\left(2D_2\frac{\omega_g}{\omega_2}\right)^2}} = \frac{A_1\sqrt{4D_2^2\left(\frac{\omega_g}{\omega_2}\right)^2+1}}{\sqrt{\left(1-\frac{\omega_g^2}{\omega_2^2}\right)^2+\left(2D_2\frac{\omega_g}{\omega_2}\right)^2}} = A_1\sqrt{\frac{\left(2D_2\frac{\omega_g}{\omega_2}\right)^2+1}{\left(1-\frac{\omega_g^2}{\omega_2^2}\right)^2+\left(2D_2\frac{\omega_g}{\omega_2}\right)^2}} \quad (26)$$

If the m_1 mass stands still ($A_1 = 0\,m$), the m_2 mass cannot move either, in contrast with the undamped vibration where using $\omega_g = \omega_2$ excitation, setting s_1, m_1, s_2, m_2 values properly, m_1 mass could be stopped by moving m_2. From equation (26) the rate of amplitudes of the two masses (named *amplitude rate*, G_{21}) can be calculated:

$$G_{21} = \frac{A_2}{A_1} = \sqrt{\frac{\left(2D_2\frac{\omega_g}{\omega_2}\right)^2+1}{\left(1-\frac{\omega_g^2}{\omega_2^2}\right)^2+\left(2D_2\frac{\omega_g}{\omega_2}\right)^2}} \quad (27)$$

The A_2/x_{st2} scaling factor based on (15):

$$\frac{A_2}{x_{st2}} = \frac{1}{\sqrt{\left(1-\frac{\omega_g^2}{\omega_2^2}\right)^2+\left(2D_2\frac{\omega_g}{\omega_2}\right)^2}} \quad (28)$$

Before the representation of rotating vectors of m_1's motion, the equation (19) has to be rearranged and substituted to (18):

$$m_2\ddot{x}_2 = k_2\dot{x}_1 - k_2\dot{x}_2 + s_2x_1 - s_2x_2 \quad (29)$$

$$m_1\ddot{x}_1 + k_1\dot{x}_1 + s_1x_1 + m_2\ddot{x}_2 = F_0\sin(\omega_g t) \quad (30)$$

Rearranging equation (29) according to D'Alambert:

$$0 = F_0\sin(\omega_g t) - m_1\ddot{x}_1 - k_1\dot{x}_1 - s_1x_1 - m_2\ddot{x}_2 \quad (31)$$

The equation (30) is represented again by rotating vectors (Figure 7.). The exciting force is responsible not only for the excitation of m_1, k_1, s_1 system, but for the acceleration of m_2 as well. φ_1 is the phase shift between the F_0 and the displacement of m_1, φ_2 is still the phase shift between the displacements of m_1 and m_2. F_0, and $m_2\ddot{x}_{2max}$ vectors have to be factored to d_1, v_1 directions.

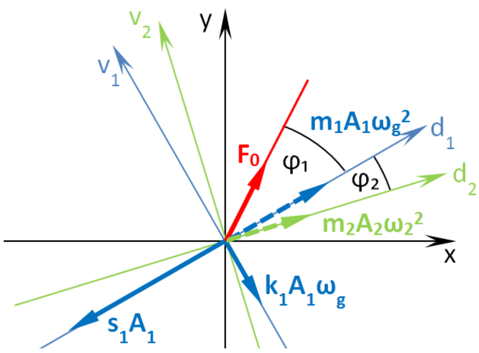

Figure 7. Forces of m_1 mass.

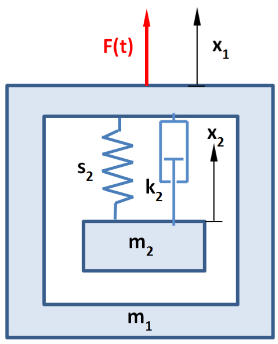

Figure 8. Mass-in-mass damped subsystem.

The d_1 and v_1 directional equilibrium of forces can be described according to D'Alambert, than F_0 has to be brought out:

$$d_1{:}\;\; s_1A_1 = F_0cos\varphi_1 + m_1A_1\omega_g^2 + m_2A_2\omega_g^2cos\varphi_2 \quad (32)$$

$$F_0cos\varphi_1 = s_1A_1 - m_1A_1\omega_g^2 - m_2A_2\omega_g^2cos\varphi_2 \quad (33)$$

$$v_1: \; F_0 sin\varphi_1 = k_1 A_1 \omega_g + m_2 A_2 \omega_g^2 sin\varphi_2 \tag{34}$$

Substituting $A_2 = G_{21} A_1$ based on (27), than bringing out A_1:

$$F_0 cos\varphi_1 = \; s_1 A_1 - m_1 A_1 \omega_g^2 - m_2 G_{21} A_1 \omega_g^2 cos\varphi_2 \tag{35}$$

$$F_0 sin\varphi_1 = k_1 A_1 \omega_g + m_2 G_{21} A_1 \omega_g^2 sin\varphi_2 \tag{36}$$

$$F_0 cos\varphi_1 = A_1 (\, s_1 - m_1 \omega_g^2 - m_2 G_{21} \omega_g^2 cos\varphi_2) \tag{37}$$

$$F_0 sin\varphi_1 = A_1 (k_1 \omega_g + m_2 G_{21} \omega_g^2 sin\varphi_2) \tag{38}$$

Using (36) and (37) the phase shift between the exciting force and the displacement of m_1, and the maximum displacement of m_1:

$$tg\varphi_1 = \frac{F_0 sin\varphi_1}{F_0 cos\varphi_1} = \frac{k_1 \omega_g + m_2 G_{21} \omega_g^2 sin\varphi_2}{s_1 - m_1 \omega_g^2 - m_2 G_{21} \omega_g^2 cos\varphi_2} \tag{39}$$

$$F_0^2 (cos^2\varphi_1 + sin^2\varphi_1) = A_1^2 \left[(\, s_1 - m_1 \omega_g^2 - m_2 G_{21} \omega_g^2 cos\varphi_2)^2 + (k_1 \omega_g + m_2 G_{21} \omega_g^2 sin\varphi_2)^2 \right] \tag{40}$$

$$A_1 = \frac{F_0}{\sqrt{(\, s_1 - m_1 \omega_g^2 - m_2 G_{21} \omega_g^2 cos\varphi_2)^2 + (k_1 \omega_g + m_2 G_{21} \omega_g^2 sin\varphi_2)^2}} \tag{41}$$

4. Damped mass-in-mass subsystem

The model of metamaterial's mass-in-mass subsystem with viscous damping is represented by Figure 8. The exciting force loads directly on the m_1 mass, while the connection between the two masses' displacements is ensured by the s_2 spring and k_2 damper. The relation between the displacements, the phase shift and the amplitude rate can be described by the equation (25) and (27) as well. Both variables depend on only the D_2 damping ratio and the ω_g/ω_2 frequency rate. The diagram of the φ_2 phase shift is shown on Figure 9.

It is visible that the highest phase shift can be achieved without damping. In that case, and when the exciting frequency is higher than the ω_2 natural frequency, the two masses vibrate in opposite phase. As the damping increases, the maximum value of the phase shift decreases and slightly moves right. If damping is higher than the critical damping, the maximum phase shift does not exceed 90^0. As the exciting frequency increases, the phase shift converges also to 90^0.

The $G_{21} = A_2/A_1$ amplitude rate is represented on Figure 10. The undamped vibration has the maximum value either, excited at the ω_2 natural angular frequency. As the damping increases, the difference between the two amplitudes decreases, and the maximum values moves left.

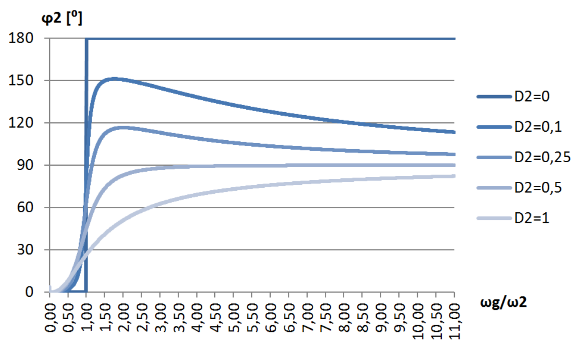

Figure 9. $\varphi_2(D_2, \frac{\omega_g}{\omega_2})$ phase shift.

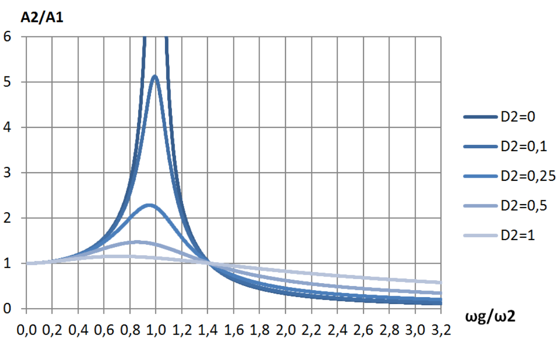

Figure 10. $G_{21}(D_2, \frac{\omega_g}{\omega_2}) = \frac{A_2}{A_1}$ amplitude rate.

5. Conclusion

For the analysis of the model of metamaterial's mass-in-mass subsystem the rotating vector method has been used. It is visible that stopping m_1 mass can be achieved only at $k_2 = 0 \; Ns/m$ damping. As

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the damping appears between the two masses, the $A_1 = 0\ m$ goal cannot be achieved even exciting at the ω_2 natural angular frequency. It is obvious, the damping is smaller, the bigger is the difference between the two masses, viewing both the phase shift and the amplitude rate. However the highest vibration absorption will be earned at closed to the ω_2 natural angular frequency as well.

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