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The Scaled Boundary Finite Element Method Applied to Electromagnetic Field Problems

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Abstract. Computation electromagnetic is an important research field of electromagnetic fields and microwave technology subjects. In this paper, the scaled boundary finite element method (SBFEM) is extended to solve one type of electromagnetic field problems-electrostatic field problems. Based on Laplace equation of electrostatic field, the derivations and solutions of SBFEM equations for both bounded and unbounded domain problems are expressed in details, and the solution for the inclusion of prescribed potential along the side-faces of bounded domain is also presented in details, then the total charges on the side-faces can be semi-analytically solved. The accuracy and efficiency of the method are illustrated by numerical examples of electromagnetic field problems with complicated field domains, potential singularities, inhomogeneous media and open boundaries. In comparison with analytic solution method and other numerical methods, the results show that the present method has strong ability to resolve potential field singularities analytically by choosing the scaling centre at the singular point, has the inherent advantage of solving the open boundary problems without truncation boundary condition, has efficient application to the problems with inhomogeneous media by placing the scaling centre in the bi-material interfaces, and produces more accurate solution than conventional numerical methods with far less number of degrees of freedom. The method in electromagnetic field calculation can have broad application prospects.

1. Introduction

With the rapid development of computer performance over half a century, computational electromagnetics has become a robust tool for many analyses of electromagnetic field problems. It can allow for a faster and cheaper design process, where the use of expensive and time-consuming prototypes is minimized and can also provide crucial information and understanding of a device’s electromagnetic operation, which may be difficult or even impossible to achieve by means of experiments or analytical calculations. Over the past decades, different numerical techniques have been used to analyze electromagnetic field problems, and remarkable achievements have been gained. Such techniques include finite difference method (FDM) [1], method of moments (MOM) [2], Green's function method [3], fast variation method [4], finite element method (FEM) [5], meshless method [6], boundary element method (BEM) [7] and so on. Among them, FEM is one of the undoubtedly

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dominant methods for modelling the electromagnetic problems, because of its powerful capability of simulating a large variety of problems with complex geometries, complicated material properties and various boundaries. However, FEM is also cumbersome in modelling some problems, such as problems involving infinite domain and singularity. For the first problems, the domain must be truncated in the FEM, which may violate the radiation condition and lead to inaccurate results, unless a sufficiently large domain is modelled. In the latter case, the singularity can only be accurately represented with many degrees of freedom or modelled with singular element. The computational cost may be high for both cases. One alternative to FEM is boundary element method (BEM). BEM discretises the boundaries only and thus reduces the modelled dimensions by one. BEM is an attracting technique especially for solving unbounded domain problems, because the radiation condition at infinity is satisfied automatically by a fundamental solution. However, difficulties are encountered in applying the BEM to many practical engineering problems owing to its reliance on the fundamental solution, which does not exist in some cases. Scaled boundary finite element method, originally established by Wolf and Song [8] for soil–structure interaction problems, is a semi-analytical method combining the advantages of FEM and BEM with unique properties of its own. The increasing popularity of SBFEM is also demonstrated by its range of applications, and recently expanded from computing the dynamic stiffness of an unbounded domain to simulation of other problems, such as fluid flow problems [9], acoustical problems [10], fracture mechanics problems [11], etc. SBFEM is quite general and can be applied to differential physical equations. So, this study extends the SBFEM to solve problems in electromagnetics and fully takes advantage of the following features of its own. (1) It reduces the modelled spatial dimensions by one and discretises the boundaries only as the BEM, but does not need fundamental solutions. (2) Its solution converges in the finite element sense in the circumferential direction, and more significantly, is analytical in the radial direction. Consequently, certain situations, such as the potential singularity point, the radiation condition at infinity in unbounded media, can be represented exactly. (3) No discretization of certain side-face boundaries and bi-material interfaces that are connected to scaling centre is needed, having efficient application to problems with inhomogeneous materials. This paper is organized as follows. In section 2, the fundamental equations of SBFEM for the electromagnetic field problems-electrostatic field problems in both bounded and unbounded domains are summarised. Section 3 addresses the analytical solution procedure. In section 4, the examples of electromagnetic field problems are solved and compared with other methods. Section 5 contains conclusions.

2. Formulation of scaled boundary finite element method

Electrostatics is one of the important parts in electromagnetic field. When device dimensions are much less than the wavelength of electromagnetic radiation at a particular frequency, the response of the system at that frequency can be considered quasi-static in that the emission, transmission, or absorption of electromagnetic radiation can be ignored. Therefore, electrostatics generally plays a very important role to improve the performance and reliability of microelectro-mechanical systems (MEMS) and electron devices in the design stage. Electrostatic field problems are governed by the Laplace equation, which may be expressed as (within $\Omega$)

$$\nabla^2 \phi = 0$$  \hspace{1cm} (1)

On the boundary of the domain, either the value of the potential or the electric intensity normal to the boundary must be specified. The boundary conditions may be specified as

$$\phi = \phi_0 \quad \text{on} \quad \Gamma_\phi$$  \hspace{1cm} (2)

$$\phi_n = E_n \quad \text{on} \quad \Gamma_n$$  \hspace{1cm} (3)

Equations (1) and (3) can be expressed in weighted residual form as

$$\int_{\Omega} \varepsilon \nabla^2 \nabla \phi d\Omega - \int_{r} \varepsilon w E d\Gamma = 0$$  \hspace{1cm} (4)

where $\varepsilon$ is permittivity of media and $w$ is any weighting function.
To apply the SBFEM to equation (4) for the two-dimensional problems, the so-called scaled boundary coordinate system is introduced. A typical scaled boundary coordinate system is shown in figure 1. A domain is represented by scaling a defining curve $S$ relative to a scaling centre $O(x_0, y_0)$. The scaling centre is chosen such that the whole boundary is visible from it. This can always be realized by dividing the domain into sub-domains with their own scaling centres. The circumferential coordinate $s$ is anticlockwise along the defining curve $S$ and the normalized radial coordinate $\xi$ is a scaling factor, defined as 1 at curve $S$ and 0 at the scaling centre. The whole solution domain $\Omega$ is in the range of $0 \leq \xi \leq 1$ and $0 \leq s \leq s_i$. The two straight sections $s = s_0$ and $s = s_i$ are called side-faces. They coincide, if the curve $S$ is closed. For bounded domain, $\xi_0 = 0$ and $\xi_1 = 1$, whereas, for unbounded domain, $\xi_0 = 1$ and $\xi_1 = \infty$. Any point in the two-dimensional plane can be specified by the scaled boundary coordinates $\xi$ and $s$. The mapping between this coordinate system and the Cartesian coordinate system can be expressed by the scaling equations (5) (see figure 1, $(x(s), y(s))$ is an arbitrary point on the curve $S$, $(x, y)$ is an interior point of the domain.)

$$x = x_0 + \xi x(s) : \quad y = y_0 + \xi y(s)$$

(5)

As the same time, a new modified SBFEM coordinate system with parallel side-faces $s = s_0$ and $s = s_i$ should be established in an unbounded domain. In this local coordinate system, the coordinate $s$ similar to the circumferential co-ordinate in the mentioned above the SBFEM coordinate system, measures the distance along the defining curve $S$. It should be noted that the defining curve will never be a closed curve. The origin of the horizontal co-ordinate $\xi$ is defined on the defining curve. Like the aforementioned SBFEM coordinate, only the defining curve needs to be discretized. The mapping between the new coordinate system and the Cartesian coordinate system can be expressed by the scaling equations (6) (see figure 2.)

$$x = x(s) + \xi \; : \; y = y(s)$$

(6)

Figure 1. The coordinate definition of SBFEM

Figure 2. The coordinate definition of SBFEM with parallel side-faces

As to the coordinate system in figure 1, the formulation of SBFEM is as follows:

The defining curve $S$ can be discretized by shape functions $N(s)$ as in the classical finite element method manner. A typical finite element with three nodes on the surface $S'$ (superscript $e$ for element) is shown in figure 1. Thus, for the element, an approximate solution $\phi_e(\xi, s)$ to equation (4) is sought in the form

$$\phi_e(\xi, s) = N(s) \varphi_e(\xi)$$

(7)

where vector $\varphi_e(\xi)$ represents radial nodal functions analogous to nodal values.

Using the scaled boundary transformation detailed by Wolf and Song (e.g. [8]), the Laplace operator $\nabla$ can be expressed as
\[ \nabla = b_1(s) \frac{\partial}{\partial \xi} + b_2(s) \frac{\partial}{\partial \xi} \quad (8) \]

where \( b_1(s) \) and \( b_2(s) \) are dependent only on the definition of \( S \).

The weighting function \( w \) can be formulated by employing the same shape functions as

\[ w(\xi) = N(s)w(\xi) = w(\xi)^T N(s)^T \quad (9) \]

Substituting equations (7), (8), (9) into equation (4) yields

\[ \int \varepsilon \left[ B_1(s)w(\xi)_{,\xi} + \frac{\partial}{\partial \xi} B_2(s)w(\xi) \right]^T \left[ B_1(s)\varphi_a(\xi)_{,\xi} + \frac{\partial}{\partial \xi} B_2(s)\varphi_a(\xi) \right] d\Omega \]

\[ - \int \varepsilon \left[ w(\xi)^T N(s)^T E \right] dT = 0 \quad (10) \]

with

\[ B_1(s) = b_1(s)N(s) \quad (11) \]
\[ B_2(s) = b_2(s)N(s) \quad (12) \]

Expanding and integrating the domain integrals containing \( w(\xi)_{,\xi} \) with respect to \( \xi \), using Green’s theorem, noting that \( d\Omega = \xi |J| d\xi ds \) (where \( J \) is Jacobian matrix between the two coordinate systems), and thinking that equation (11) is satisfied for any set of weighting function \( w(\xi) \) (e.g. [8]), the following relations are yielded

\[ E_{\theta} \xi \varphi_h(\xi)_{,\xi} + E_{\theta} \xi \varphi_h(\xi) \bigg|_{\xi=0} = - \int \varepsilon N(s)^T E d\Gamma \quad (13) \]
\[ E_{\theta} \xi \varphi_h(\xi)_{,\xi} + E_{\theta} \xi \varphi_h(\xi) \bigg|_{\xi=1} = \int \varepsilon N(s)^T E d\Gamma \quad (14) \]

\[ E_{\theta} \xi^2 \varphi_h(\xi)_{,\xi} + \xi (E_{\theta} + E_{\theta}^T - E_{\theta}^T - E_{\theta}) \varphi_h(\xi)_{,\xi} - \xi E_{\theta} \varphi_h(\xi)_{,\xi} + \xi F_s(\xi) = 0 \quad (15) \]

where

\[ E_{\theta} = \int \varepsilon B_1(s)^T B_1(s) |J| ds \quad (16) \]
\[ E_{\theta} = \int \varepsilon B_2(s)^T B_2(s) |J| ds \quad (17) \]
\[ E_{\theta} = \int \varepsilon B_2(s)^T B_2(s) |J| ds \quad (18) \]

\[ F_s(\xi) = \varepsilon N(s_0)^T E \left[ J(s_0) \right] + \varepsilon N(s_1)^T E \left[ J(s_1) \right] \quad (19) \]

Equations (13) and (14) indicate respectively the relationships between the nodal potential and the charge on the interior and exterior boundary for both bounded and unbounded domains, \( 0 = \xi_0 \leq \xi \leq \xi_1 = 1 \) for bounded domain and \( 0 = \xi_0 \leq \xi \leq \xi_1 = \infty \) for unbounded domain.

Equation (15) is the scaled boundary finite element equation and the non-homogeneous term \( F_s(\xi) \) at the right hand side of equation (15) is due to the prescribed charge of the side-faces. Equations (13)-(15) can be applied to the domain of the triangular corresponding to one finite element on the boundary. To model the total domain, an assemblage is as in the conventional finite element method.

As to coordinate system in figure 2, the formulation of SBFEM is obtained by using the same way as that in the coordinate system shown in figure 1. As a result, only the scaled boundary finite element equation is difference (the coefficients \( E_{\theta}, E_{\theta}, E_{\theta} \) are also difference, which are dependent only on the difference definition of curve \( S \)) and it is

\[ E_{\theta} \xi^2 \varphi_h(\xi)_{,\xi} + \xi (E_{\theta}^T - E_{\theta}) \varphi_h(\xi)_{,\xi} - E_{\theta} \varphi_h(\xi)_{,\xi} + \xi F_s(\xi) = 0 \quad (20) \]
3. Solution of scaled boundary finite element method

3.1. Analytical solution for vanishing non-homogeneous term

As to coordinate system shown in figure 1, for \( F_s(\xi) = \theta \), equation (15) becomes a homogeneous second-order ordinary differential equation, which can be transformed to first-order ordinary differential equations with twice the number of unknowns by introducing the variable

\[
X(\xi) = \begin{bmatrix} \varphi(\xi) \\ Q(\xi) \end{bmatrix}
\]

where

\[
Q(\xi) = E_0 \varphi(\xi) + E_1^T \varphi(\xi)
\]

This results in

\[
\xi X(\xi)_{\xi} = -ZX(\xi)
\]

with a Hamiltonian matrix

\[
Z = \begin{bmatrix} E_0^{-1} E_1^T & -E_0^{-1} \\ -E_2 + E_1 E_0^{-1} E_1^T & -E_1 E_0^{-1} \end{bmatrix}
\]

Equation (24) can be solved by eigenvalue method (e.g. [8]) and the solution is

\[
\varphi(\xi) = \Phi_{11} [\xi^{-\lambda_1}] c_1 + \Phi_{12} [\xi^{\lambda_1}] c_2
\]

\[
Q(\xi) = \Phi_{21} [\xi^{-\lambda_2}] c_1 + \Phi_{22} [\xi^{\lambda_2}] c_2
\]

where the real parts of all elements \( \lambda_i \) are negative, \( \Phi_{11}, \Phi_{12}, \Phi_{21}, \Phi_{22} \) are eigenvectors of equation (24) and \( c_1, c_2 \) are integration constants determined from the boundary conditions (for a bounded domain, \( c_2 = \theta \), and for an unbounded domain \( c_1 = \theta \) [8].)

As to coordinate system of figure 2, for \( F_s(\xi) = \theta \), equation (20) can be also solved by introducing the equation (22) and this results in

\[
X(\xi)_{\xi} = -ZX(\xi)
\]

Equation (27) can be also solved by eigenvalue method and the solution is

\[
\varphi(\xi) = \Phi_{21} [\xi^{-\lambda_2}] c_2
\]

\[
Q(\xi) = \Phi_{22} [\xi^{\lambda_2}] c_2
\]

where integration constants \( c_2 \) is determined from the boundary conditions.

3.2. Analytical solution for Prescribed side-face potential in bounded domain

In this paper, it is desirable to prescribe the potential along the side-faces. To simplify the nomenclature, in the discussion which follows the subscript \( h \) (denoting the approximate solution) will be dropped, so that \( \varphi(\xi) = \varphi_h(\xi) \). Supposing that the boundary discretized \( S \) have \( n \) nodal potential, the \( n \) nodal potential component function \( \varphi(\xi) \) can be partitioned into \( n_u \) which are unconstrained \( \varphi_u(\xi) \) and \( n_c \) \( (n_c \) has a maximum value 2) which are constrained \( \varphi_c(\xi) \). The scaled boundary finite element equation in potential (equation (15)) and the boundary equilibrium equation( equation (14)) can be partitioned in a similar manner (since the interior boundary becomes a scaling, equation (13) vanishes.)
Supposing $F_u(\xi) = \theta$, moving the known terms to the right-hand side, the upper portion of equation (30) becomes

$$E_{0uu} \varepsilon^2 \varphi_u(\xi)_{zz} + (E_{0uu} + E_{1uu}^T) \xi \varphi_u(\xi)_{zz} - E_{2uu} \varphi_u(\xi) =$$

$$-E_{0uc} \varepsilon^2 \varphi_c(\xi)_{zz} + (E_{0uc} + E_{1uc}^T) \xi \varphi_c(\xi)_{zz} - E_{2uc} \varphi_c(\xi)$$

In this work, the prescribed potential on the side-face takes the form

$$\varphi_u(\xi) = \varepsilon \varphi_{upi}$$

Equation (32) is non-homogenous differential equation and the quadratic eigenproblem for the constrained system becomes (reference to section 3.1)

$$\varphi_u(\xi) = \sum_{i=1}^{n} c_i \xi^{-\lambda} \varphi_u + \varphi_{upi} = \varphi_u + \varphi_{upi}$$

where the exponents $-\lambda$ and corresponding vectors $\varphi_u$ may be interpreted as independent modes. The integration constants $c_i$ represent the contribution of each mode to the solution, and are dependent on the boundary conditions. The $\sum_{i=1}^{n} c_i \xi^{-\lambda} \varphi_u$ and the vectors $\varphi_{upi}$ respectively represent the fundamental solution and the particular solution for equation (32). $\Phi_u$ contains the $n_u$ homogenous mode vectors $\varphi_u$ as columns. The particular solution $\varphi_{upi}$ can be obtained by substitution of equation (33) into equation (32)

$$\varphi_{upi} = -E_{2uu}^{-1}E_{2uc}\varphi_{upi}$$

The equivalent nodal total charges at the unrestrained degrees of freedom in equilibrium with this potential field is (substitution of equations (33) and (34) into equation (31).)

$$Q_u = \sum_{i=1}^{n_u} c_i r_u + r_{upi} = R_u c + r_{upi}$$

where $R_u$ contains the $n_u$ equivalent total charges vectors $r_u$ as columns. And $r_{upi}$ is the particular term

$$r_{upi} = E_{1uu}^T \varphi_{upi} + E_{1uc}^T \varphi_{upi}$$

Rearranging equation (34), the integration constants can be found in the terms of the nodal potential

$$c = \varphi_u^{-1}(\varphi_u - \varphi_{upi})$$

Substituting this equation into equation (36) and rearranging, the equilibrium requirement is reduced to

$$R_u \varphi_u^{-1} \varphi_u = Q_u - r_{upi} + R_u \varphi_u^{-1} \varphi_{upi}$$

Boundary conditions on the discretised boundaries place constraint on subsets of $\varphi_u$ and $Q_u$, and solution proceeds in the usual finite element manner. Once the complete set of boundary potential is found, equation (38) is used to obtain the integration constants. The unconstrained potential field is recovered by substituting equation (38) into equation (34). The prescription of potential along the two
side-faces requires the application of external charge along the side-faces. The variation of the charge can be recovered analytically by rearranging the bottom portion of equation (30) as

\[
F_{\text{sc}} \left( \xi \right) = \xi \left( E_{\text{exc}}^T \phi_{\text{exc}} \left( \xi \right) + E_{\text{exc}} \phi_{\text{exc}} \left( \xi \right) \right) + \left( E_{\text{exc}}^T + E_{\text{exc}}^T - E_{\text{exc}} \right) \phi_{\text{exc}} \left( \xi \right) + \left( E_{\text{exc}}^T - E_{\text{exc}} \right) \phi_{\text{exc}} \left( \xi \right)
\]

(40)

As the same time, the total charges \( Q \) on the side-faces can be obtained by integrating the equation (40) from \( \xi = 0 \) to \( \xi = 1 \).

4. Numerical examples

4.1. Constant electric field calculation

A simple electrostatic model-infinite straight grounding slot, which is shown in figure 3, is used for the comparison analysis of the accuracy of the present method. In this simulation, \( a = 3.0 \text{ m}, b = 1.0 \text{ m}, U_0 = 1.0 \text{ V} \) are assumed. 1281 measured nodes are uniformly distributed in the whole domain. The following relative error is defined as

\[
\text{error} = \left( \frac{1}{N} \sum_{i=1}^{N} \left( \phi_{\text{exact}} - \phi_{\text{calc}} \right)^2 / \sum_{i=1}^{N} \left( \phi_{\text{exact}} \right)^2 \right)^{1/2} \times 100\%
\]

(41)

where \( \phi_{\text{calc}} \) and \( \phi_{\text{exact}} \) are the numerical solution by the numerical method and the analytical method separately, and \( N \) is the number of nodes in the studied domain.

The scaling centre is selected at the centre of the domain, and four meshes of increasing accuracy are used. (1) The domain is discretized with 16 three-node quadratic line elements and 32 nodes with node interval of 0.250 cm on the boundary, as illustrated in figure 4. (2) The domain is discretized with 32 elements and 64 nodes with node interval of 0.125 cm. (3) The domain is discretized with 40 elements and 80 nodes with node interval of 0.100 cm. (4) The domain is discretized with 64 elements and 128 nodes with node interval of 0.0625 cm. This example has also been analyzed by Zhang [12] using the finite difference method (FDM) and radial basis function meshless method (RBF) by idealizing the domain with 21, 65, 96 and 341 degrees of freedom respectively, with node interval of 0.5, 0.25, 0.2, 0.1 cm in the entire domain respectively. Comparison on maximum and mean square error for the potential obtained by different methods are shown in table 1 (max represents maximum error and mean represents mean square error.). The results indicate that the SBFEM is more accurate and generally achieves uniform convergence.

Figure 3. Model of infinite grounding slot

Figure 4. Scaled boundary Finite element mesh
Table 1. Potential calculation of maximum and mean square error’s comparison between different methods

<table>
<thead>
<tr>
<th></th>
<th>FDM</th>
<th></th>
<th>RBF</th>
<th></th>
<th>SBFEM</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DOF</td>
<td>Max(V)</td>
<td>Mean</td>
<td>DOF</td>
<td>Max(V)</td>
<td>Mean</td>
</tr>
<tr>
<td>21</td>
<td>0.0329</td>
<td>1.6942%</td>
<td>21</td>
<td>0.0091</td>
<td>0.9157%</td>
<td>32</td>
</tr>
<tr>
<td>65</td>
<td>0.0082</td>
<td>0.4541%</td>
<td>65</td>
<td>3.7319×10⁻⁴</td>
<td>0.0401%</td>
<td>64</td>
</tr>
<tr>
<td>96</td>
<td>0.0055</td>
<td>0.3064%</td>
<td>96</td>
<td>3.9242×10⁻⁵</td>
<td>0.0043%</td>
<td>80</td>
</tr>
<tr>
<td>341</td>
<td>0.0014</td>
<td>0.0978%</td>
<td>341</td>
<td>8.2988×10⁻⁶</td>
<td>0.0005%</td>
<td>128</td>
</tr>
</tbody>
</table>

4.2. Computation of characteristic impedance of elliptical coaxial stripline

This example illustrates the benefits of SBFEM that can be achieved by taking advantage of the properties of the scaling centre and the side-face with prescribed potential boundary condition. Coaxial stripline have wide and practical applications to RF equipment. Characteristic impedance is one of its main design parameters. Many methods such as conformal transformations method, finite difference method, finite element method, boundary element method, integral equations method etc., have been applied in obtaining the characteristic impedance. In this example, the characteristic impedance of an elliptical coaxial stripline with outer elliptical and inner strip conductors is calculated by SBFEM, as illustrated in figure 5.

![Figure 5. Model of elliptical coaxial stripline](image)

![Figure 6. Scaled boundary finite element mesh](image)

The calculated domain is also governed by Laplace equation (1). Owing to the symmetry, only a quarter of elliptical coaxial stripline is modelled using a single bounded domain. The quarter model and boundary conditions are illustrated in figure 6. The scaling centre is positioned at the point at which the potential is discontinuous, which is at the edge of the strip conductor, and the strip conductor is modelled as a side-face with prescribed potential boundary condition, so that the total charges \( Q \) on the strip can be semi-analytically solved by integrating the equation (40) from \( \xi = 0 \) to \( \xi = 1 \). The boundary is discretized with three-node quadratic elements, and the mesh consists of six elements. The following formula can be used to calculate the total characteristic impedance \( Z_0 \) of a lossless transmission line

\[
Z_0 = 4Z_{q0} = 4 \left( \frac{v_0 C_0}{v_0} \right) = 4\Delta V \left/ \left( v_0 Q \right) \right. 
\]

where \( C_0 \) is the unit-length capacitance of transmission line of quarter mode, \( v_0 \) is the speed of light in the vacuum, \( \Delta V \) is the difference between outer conductor and inner strip conductor.

This example has also been analyzed by XIE etc., [13] using a boundary element method (BEM) and modelling the half of elliptical coaxial stripline with meshes consisting of 10 elements. Xu [14] provides exact impedance values using analytical conformal transformation. Table 2 lists the computed results by SBFEM, BEM and the analytical solution. The results show that the SBFEM results are more accurate, and the maximum relative error is less than 0.1%.
Table 2. Comparison of characteristic impedance values of elliptical coaxial stripline $Z_0(\Omega)$

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytic method.</td>
<td>51.20</td>
<td>62.49</td>
<td>71.68</td>
<td>79.10</td>
<td>85.03</td>
<td>89.72</td>
</tr>
<tr>
<td>$W/a=0.4$</td>
<td>BEM</td>
<td>51.44</td>
<td>62.76</td>
<td>71.97</td>
<td>79.40</td>
<td>85.33</td>
<td>90.01</td>
</tr>
<tr>
<td></td>
<td>SBFEM</td>
<td>51.20</td>
<td>62.49</td>
<td>71.68</td>
<td>79.10</td>
<td>85.01</td>
<td>89.70</td>
</tr>
<tr>
<td></td>
<td>Analytic method.</td>
<td>42.35</td>
<td>52.23</td>
<td>60.41</td>
<td>67.06</td>
<td>72.41</td>
<td>76.66</td>
</tr>
<tr>
<td>$W/a=0.5$</td>
<td>BEM</td>
<td>42.56</td>
<td>52.48</td>
<td>60.68</td>
<td>67.35</td>
<td>72.70</td>
<td>76.94</td>
</tr>
<tr>
<td></td>
<td>SBFEM</td>
<td>42.34</td>
<td>52.23</td>
<td>60.41</td>
<td>67.06</td>
<td>72.40</td>
<td>76.64</td>
</tr>
<tr>
<td></td>
<td>Analytic method.</td>
<td>35.69</td>
<td>44.32</td>
<td>51.51</td>
<td>57.38</td>
<td>62.11</td>
<td>65.86</td>
</tr>
<tr>
<td>$W/a=0.6$</td>
<td>BEM</td>
<td>35.83</td>
<td>44.49</td>
<td>51.70</td>
<td>57.58</td>
<td>62.31</td>
<td>66.06</td>
</tr>
<tr>
<td></td>
<td>SBFEM</td>
<td>35.69</td>
<td>44.32</td>
<td>51.51</td>
<td>57.38</td>
<td>62.11</td>
<td>65.86</td>
</tr>
</tbody>
</table>

4.3. Computation of characteristic impedance of inhomogeneous asymmetric stripline

This example illustrates the advantages of SBFEM that have efficient application to synthesis problems involving inhomogeneous material, singularity, and open boundary. Stripline is one of the most common and important interconnect structures used in a multi-layered package or board. It has the advantage of having less radiation compared with microstrip. In order to carry out rigorous analysis and design for stripline circuit, exact characteristic impedance for any strip is strongly needed. Many methods such as conformal mapping method, finite element method, boundary element method, method of moment, mixed spectral-space domain quasi-static etc., have been applied in calculating the characteristic impedance. In this example, the characteristic impedance of asymmetric stripline filled with inhomogeneous dielectric media on the two sides of the strip conductor is obtained by SBFEM, as illustrated in figure 7.

![Figure 7. Model of stripline](image1)

![Figure 8. Scaled boundary Finite element mesh](image2)

The calculated domain is also governed by Laplace equation. Owing to the symmetry, half of the stripline is modelled using a bounded domain and an unbounded domain with parallel side-faces. The half model and boundary condition are illustrated in figure 8. For the bounded domain, the scaling centre is positioned at the edge of the strip conductor and there is no discretization of bi-media interfaces. In this example, $b/b=1/6$, $\varepsilon_1=9.9$ and $\varepsilon_2=1.0$. And the boundary is also discretized with three-node quadratic elements, the mesh consists of 36 elements in bounded domain, and the mesh consists of 12 elements in unbounded domain. The following formula is used to calculate $Z$

$$Z = 2Z_0 = 2\left(\frac{C_0}{C}\right)^{1/2}Z_0 = 2\left(Q_0/Q\right)^{1/2}\Delta V/(v_0Q_0)$$  \hspace{1cm} (43)$$

where $C_0$ and $C$ are respectively the capacitance per unit length of the structure with or without air instead of dielectric $\varepsilon_1$, $v_0$ is the speed of light in the vacuum, $Q_0$ and $Q$ are respectively total charges on the strip conductor of quarter model with or without air instead of dielectric $\varepsilon_1$.  


The variation of the characteristic impedance of the stripline with \(d/b\) is obtained by the SBFEM and the conformal mapping technique [15]. The results are presented in figure 9 and the potential isolines of the SBFEM solution is illustrated in figure 10. The results show that the maximum relative error is less than 0.3% between the two methods.

Figure 9. Variation of the impedance with \(d/b\)

Figure 10. Potential isolines of SBFEM solution

5. Conclusion

This paper extends the application of scaled boundary finite element method to electromagnetic field problems. As applying to the electrostatic field problems governed by Laplace’s equation, the derivations and solutions of SBFEM are presented for both bounded and unbounded domains. Especially, this paper has developed the solution for the inclusion of prescribed potential along the side-faces of bounded domain then the total charges on those side-faces can be semi-analytically solved. Three examples are simulated by SBFEM. The first example shows that the new method obtains better accuracy and more uniform convergence with increasing mesh density compared to FDM and RBF techniques. The second example illustrates that the problem with singularity in the potential field is handled accurately and efficiently by placing the scaling centre at the singular point, and the new method obtains high accuracy for the complex boundary with few degrees of freedom compared to boundary element method. The third example indicates that the method has also strong ability to simulate the complex problems involving inhomogeneous materials, singularity and open boundary. Overall the features of the scaled boundary finite element method are shown to make it ideally suited to the analysis of other electromagnetic field problems.

References