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Polar duality and the reconstruction of quantum covariance matrices from partial data

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Polar Duality and the Reconstruction of Quantum Covariance Matrices from Partial Data

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Abstract

We address the problem of the reconstruction of quantum covariance matrices using the notion of Lagrangian and symplectic polar duality introduced in previous work. We apply our constructions to

Gaussian quantum states which leads to a non-trivial generalization of Pauli's reconstruction problem and we state a simple tomographic characterization of such states.

Keywords: Covariance matrix, Gaussian state, polar duality; Lagrangian plane; uncertainty principle

MSC classification 2020: 52A20, 52A05, 81S10, 42B35

1 Introduction

The covariance matrix is a fundamental concept in both classical and quantum mechanics, serving distinct purposes in each domain. In classical mechanics, the covariance matrix is employed to characterize statistical relationships and correlations between different variables within a system [5]. In quantum mechanics, the covariance matrix holds particular significance in the context of quantum correlations [6]. According to Born's rule, the quantum covariance matrix encapsulates all available statistical information about a quantum state. Moreover, covariance matrices serve as a powerful tool for detecting entanglement, playing a key role in identifying and analyzing quantum entangled states [7, 33].

Notably, the quantum covariance matrix fully characterizes Gaussian states, their Wigner distributions are parametrized by their covariance matrices and their center. This correspondence is central to the discussion in the present paper to address the problem of the determination of covariance matrices from partial data. This problem is not new and has been studied by many authors, see for instance Řeháček *et al.* [30]. We have initiated such a study from a wider point of view in [15, 20, 18] using the notion of polar duality from convex geometry; the present work considerably extends and synthesizes these preliminary works. More precisely, let Σ be an arbitrary real positive definite $2n \times 2n$ matrix; such a matrix can always be viewed as the covariance of the multivariate Gaussian distribution

$$\rho(z) = \frac{1}{(2\pi)^n \sqrt{\det \Sigma}} e^{-\frac{1}{2}\Sigma^{-1}(z-z_0)\cdot(z-z_0)} \tag{1}$$

(we are using the notation $z = (x, p) \in \mathbb{R}^n_x \times \mathbb{R}^n_p$); this distribution qualifies as a the Wigner distribution of a quantum state provided that we impose a constraint on Σ; the latter is usually chosen to be [3, 10, 22, 28]:

$$\textit{The eigenvalues of } \Sigma + \frac{i\hbar}{2} J \textit{ are all} \geq 0 \tag{2}$$

where J is the standard symplectic matrix. We have shown [11, 12, 22] that this condition is equivalent to

$$\textit{The covariance ellipsoid } \Omega_\Sigma \textit{ contains a quantum blob} \tag{3}$$

(a quantum blob is a symplectic ball with radius $\sqrt{\hbar}$); in [11, 22] we also formulated these conditions using the topological notion of symplectic capacity [9] which is closed related to Gromov's symplectic non-squeezing theorem. On the operator level the conditions (2) and (3) guarantees that the trace class operator with Weyl symbol $(2\pi\hbar)^n\rho$ is positive semidefinite and has trace one, and thus qualifies as a density operator [19].

The main results of this paper are

- Theorem 12 which states that a generalized Gaussian (and hence a covariance matrix) can be reconstructed from the knowledge of only two marginal distributions of along a pair of transverse Lagrangian planes; it provides a generalization of the solution of Pauli's reconstruction problem;

- Theorem 13 provides a geometric interpretation of the previous results; identifies Gaussian states with quantum blobs viewed as John ellipsoids of a convex set constructed using the notion of Lagrangian polar duality, which can be viewed as a geometric variant of the uncertainty principle:

- Theorem 16 uses the notion of symplectic polar duality $\Omega \longrightarrow \Omega^{\hbar,\omega}$ we introduces in [21, 17]. We prove that in order to prove that a phase space ellipsoid Ω is the covariance ellipsoid of a Gaussian quantum state it suffices that $\Omega^{\hbar,\omega} \cap \ell \subset \Omega \cap \ell$ for one Lagrangian subspace ℓ (and hence for all).

Notation and terminology We will denote by ω the standard symplectic form on $\mathbb{R}^{2n} \equiv \mathbb{R}^n_x \times \mathbb{R}^n_p$ that is, in matrix form, $\omega(z, z') = Jz \cdot z'$ where $z = (x,p)^T$, $z' = (x',p')^T$ and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The scalar product of two vectors $u, v \in \mathbb{R}^m$ is written $u \cdot v$. We denote by $\mathrm{Sym}_{++}(m, \mathbb{R})$ the convex set of all symmetric positive definite real $m \times m$ matrices.

The group of all automorphisms of the symplectic space $(\mathbb{R}^{2n}, \omega)$ leaving the standard symplectic form invariant is called the (standard) symplectic group and is denoted by $\mathrm{Sp}(n)$. The properties of $\mathrm{Sp}(n)$ and of its double covering, the metaplectic group $\mathrm{Mp}(n)$ are summarized in the Appendix A.

A linear subspace of $(\mathbb{R}^{2n},\omega)$ with dimension n is on which ω vanishes identically is called a Lagrangian subspace (or plane); the set of all Lagrangian subspaces of $(\mathbb{R}^{2n},\omega)$ will be denoted by $\mathrm{Lag}(n)$, and is called the Lagrangian Grassmannian of $(\mathbb{R}^{2n},\omega)$.

Let $B^{2n}(\sqrt{\hbar})$ be the ball with center 0 and radius $\sqrt{\hbar}$ in $\mathbb{R}^{2n}$ (equipped with the usual Euclidean norm). The image $S(B^{2n}(\sqrt{\hbar}))$ of that ball by $S\in\mathrm{Sp}(n)$ is called a *quantum blob.*

2 The RSUP for Gaussians

The Robertson–Schrödinger uncertainty principle (RSUP) is, as opposed to the elementary Heisenberg inequalities, a fundamental concept in quantum mechanics that describe the trade-off between the uncertainties in the measurements of two non-commuting observables, such as position and momentum. These inequalities are typically expressed in terms of standard deviations or (co)variances.

2.1 Multivariate Correlated Gaussians

We denote by ϕ_0 the standard Gaussian (called "fiducial state" by Littlejohn [24]) and by $W\phi_0$ its Wigner function:

$$\phi_0(x)=(\pi\hbar)^{-n/4}e^{-|x|^2/2\hbar}\ ,\quad W\phi_0(x)=(\pi\hbar)^{-n}e^{-|z|^2/\hbar}; \tag{4}$$

here $|x|^2=x_1^2\div\cdots\div+x_n^2$ and $|z|^2=|x|^2+|p|^2$. Consider now the set $\mathrm{Gauss}_0(n)$ of all centered generalized Gaussians: we have $\psi_{XY}\in\mathrm{Gauss}_0(n)$ if and only if

$$\psi_{X,Y}^{\gamma}(x)=i^{\gamma}\left(\tfrac{1}{\pi\hbar}\right)^{n/4}(\det X)^{1/4}e^{-\frac{1}{2\hbar}(X+iY)x\cdot x} \tag{5}$$

where X and Y are symmetric real $n\times n$ matrices such that $X>0$ and $\gamma\in\mathbb{R}$ is a real constant. This function is normalized to unity: $||\psi_{X,Y}^{\gamma}||_{L^2}=1$ and its Wigner transform is given by [10, 14]

$$W\psi_{X,YX,Y}^{\gamma}(z)=\left(\tfrac{1}{\pi\hbar}\right)^{n}e^{-\frac{1}{\hbar}Gz^2} \tag{6}$$

where G is the symmetric and symplectic matrix

$$G=\begin{pmatrix}X+YX^{-1}Y & YX^{-1}\\ X^{-1}Y & X^{-1}\end{pmatrix}\ . \tag{7}$$

(The result seems to back to Bastiaans, see [24].) Observe that $G = S^T S$ where

$$S = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{pmatrix} \in \mathrm{Sp}(n). \tag{8}$$

An almost trivial, but fundamental, observation is that every $\psi^{\gamma}_{X,Y} \in \mathrm{Gauss}_0(n)$ can be obtained from the standard Gaussian using elementary metaplectic transformations, namely

$$\psi^{\gamma}_{X,Y} = i^{\gamma}\widehat{V}_Y \widehat{M}_{X^{1/2},0}\phi_0. \tag{9}$$

In fact:

Proposition 1 *The metaplectic group* $\mathrm{Mp}(n)$ *acts transitively on Gaussians* $\psi_{X,Y} = \psi^0_{X,Y}$ *up to a unimodular factor:*

$$\mathrm{Mp}(n) \times \mathrm{Gauss}_0(n) \longrightarrow \mathrm{Gauss}_0(n).$$

Proof. To see this it is sufficient, in view of formula (9) to show that if $\widehat{S} \in \mathrm{Mp}(n)$ then $\widehat{S}\phi_0 = i^{\mu}\psi_{X,Y}$ for some $\mu \in \mathbb{R}$ and $\psi_{X,Y} \in \mathrm{Gauss}_0(n)$. In view of the pre-Iwasawa factorization (Appendix A, formula (76)) we have $S = \pi^{\mathrm{Mp}}(\widehat{S}) = V_P M_L R$ and hence

$$\widehat{S} = \pm\widehat{V}_P \widehat{M}_{L,0}\widehat{R}. \tag{10}$$

We claim that It follows that $\widehat{R}\phi_0 = i^{\mu}\phi_0$ for some $\mu \in \mathbb{R}$. In fact we have, by the symplectic covariance of the Wigner transform,

$$W(\widehat{R}\phi_0)(z) = W\phi_0(R^{-1}z) = W\phi_0(z)$$

the second equality in view of the rotational invariance of ϕ_0; it follows that

$$\widehat{S}\phi_0 = \pm i^{\mu}\widehat{V}_P \widehat{M}_{L,0}\widehat{R}\phi_0 = \pm i^{\mu}\psi_{L^2,P}$$

hence $\widehat{S}\phi_0 \in \mathrm{Gauss}_0(n)$ as claimed (this result can also be obtained using Fourier integral operators, at the cost of much more complicated calculations involving Cayley transforms (see *e.g.* [24]). There remains to show that $\widehat{S}\psi^{\gamma}_{X,Y} \in \mathrm{Gauss}_0(n)$ for all $\psi^{\gamma}_{X,Y} \in \mathrm{Gauss}_0(n)$. Every $\widehat{S} \in \mathrm{Mp}(n)$ can be written $\widehat{S} = \widehat{V}_P \widehat{M}_{L,m}\widehat{R}$ ($m \in \{0, 2\}$) corresponding to the pre-Iwasaw factorization of $S = \pi^{\mathrm{Mp}}(\widehat{S})$. For $\widehat{S'} \in \mathrm{Mp}(n)$; we have

$$\widehat{S'}\psi_{X,Y} = \widehat{S'}\psi\widehat{V}_Y \widehat{M}_{X^{1/2},0}\phi_0 = \widehat{S}\phi_0 = \widehat{V}_P \widehat{M}_{L,m}\widehat{R}\phi_0.$$

We claim that $\widehat{R}\phi_0 = i^\mu \phi_0$ for some $\mu \in \mathbb{R}$. We have

$$W(\widehat{R}\phi_0)(z) = W\phi_0(R^{-1}z) = W\phi_0(z)$$

the second equality in view of the rotational invariance of ϕ_0; hence

$$\widehat{S'}\psi_{X,Y} = \widehat{S}\phi_0 = i^\mu \widehat{V}_P \widehat{M}_{L,m} \widehat{R}\phi_0 = i^\mu \psi_{L^2,P} \in \mathrm{Gauss}_0(n).$$

■

2.2 The multivariate RSUP

A mixed Gaussian quantum state centered at z_0 is a density operator $\widehat{\rho}$ on $L^2(\mathbb{R}^n)$ whose Wigner distribution ρ is of the type

$$\rho(z) = \frac{1}{(2\pi)^n \sqrt{\det \Sigma}} e^{-\frac{1}{2}\Sigma^{-1}(z-z_0)\cdot(z-z_0)} \tag{11}$$

where Σ is the covariance (or noise) matrix; the condition $\widehat{\rho} \geq 0$ is equivalent to saying that the eigenvalues of the Hermitian matrix $\Sigma + \frac{i\hbar}{2}J$ are ≥ 0 [3, 10], which we write for short as the Löwner ordering:

$$\Sigma + \frac{i\hbar}{2}J \geq 0. \tag{12}$$

The state $\widehat{\rho}$ is pure if and only if $\det \Sigma = \left(\frac{\hbar}{2}\right)^{2n}$ (see below). The condition (12) is equivalent to the following three statements:

Claim 2 *The symplectic eigenvalues* $\lambda_1^\omega, ..., \lambda_n^\omega$ *of* Σ *are all* $\geq \frac{1}{2}\hbar$;

Claim 3 *The covariance ellipsoid* $\Omega_\Sigma = \{z : \frac{1}{2}\Sigma^{-1}z \cdot z < 1\}$ *contains a quantum blob* $Q = S(B^{2n}(\sqrt{\hbar}))$ $(S \in \mathrm{Sp}(n))$;

Claim 4 *The symplectic capacity of the covariance ellipsoid is at least* $\pi\hbar$.

Recall [10] that the symplectic eigenvalues of a positive definite matrix Σ are the numbers $\lambda_j^\omega > 0$, $(0 \leq j \leq n)$ such that the $\pm i\lambda_j^\omega$ are the eigenvalues of $J\Sigma$ (which are the same as those of the antisymmetric matrix $\Sigma^{1/2}J\Sigma^{1/2}$). The terminology "quantum blob" to denote a symplectic ball with radius $\sqrt{\hbar}$ was introduced in [12, 11]). For the notion of symplectic capacity and its applications to the uncertainty principle see [11]. The proof of the first claim is well-known and widespread in the literature, see for instance [3, 12, 22]. It is easily proven using Williamson's symplectic diagonalization theorem [10] which says that for every $\Sigma > 0$ there exists $S \in \mathrm{Sp}(n)$ such that

$$\Sigma = S^T D S \quad , \quad D = \begin{pmatrix} \Lambda^\omega & 0_{n\times n} \\ 0_{n\times n} & \Lambda^\omega \end{pmatrix} \tag{13}$$

where $\Lambda^\omega = \mathrm{diag}(\lambda_1^\omega, ..., \lambda_n^\omega)$. The proof of the second claim easily follows from a geometric argument; see [11, 22]. The case where $\lambda_j^\omega = \frac{1}{2}\hbar$ for all j is of particular interest; in this case we have $\Sigma = \frac{\hbar}{2} S^T S$ (and hence $\det \Sigma = \left(\frac{\hbar}{2}\right)^{2n}$) and so that

$$\rho(z) = \frac{1}{(\pi\hbar)^n} e^{-\frac{1}{2}S^{-1}(z-z_0)\cdot S^{-1}(z-z_0)} = W\phi_0(S^{-1}(z-z_0))$$

that is

$$\rho(z) = W(\widehat{S}\phi_0)(z-z_0) = W(\widehat{T}(z_0)\widehat{S}\phi_0)(z)$$

where $\widehat{S} \in \mathrm{Mp}(n)$ covers S and $\widehat{T}(z_0)$ is the Heisenberg–Weyl displacement operator [10]:

$$\widehat{T}(z_0)\widehat{S}\psi(x) = e^{\frac{i}{\hbar}(p_0\cdot x - \frac{1}{2}p_0\cdot x_0)}\psi(x-x_0).$$

From these considerations follows that the Wigner distribution of the Gaussian state $\widehat{\rho}$ is the Wigner function of a multivariate Gaussian $\widehat{T}(z_0)\psi_{X,Y}$ centered at z_0. We will write the covariance matrix in block-form

$$\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XP} \\ \Sigma_{PX} & \Sigma_{PP} \end{pmatrix} \quad , \quad \Sigma_{PX} = \Sigma_{XP}^T \tag{14}$$

with $\Sigma_{XX} = (\sigma_{x_j x_k})_{1\leq j,k\leq n}$, $\Sigma_{PP} = (\sigma_{p_j p_k})_{1\leq j,k\leq n}$, $\Sigma_{PP} = (\sigma_{x_j p_k})_{1\leq j,k\leq n}$.

Proposition 5 *Let Σ be the covariance matrix of the Gaussian state $\widehat{\rho}$; (i) This state is pure, that is there exists $\psi_{X,Y}$ such that $\rho = \psi_{X,Y}$, if and only if*

$$\Sigma_{XX}\Sigma_{PP} - (\Sigma_{XP})^2 = \frac{1}{4}\hbar^2 I_{n\times n} \tag{15}$$

$$\Sigma_{XX}\Sigma_{XP} = \Sigma_{PX}\Sigma_{XX} \ , \ \Sigma_{PX}\Sigma_{PP} = \Sigma_{PP}\Sigma_{XP}. \tag{16}$$

(ii) We have $(\Sigma_{XP})^2 \geq 0$, i.e. the eigenvalues of $(\Sigma_{XP})^2$ are ≥ 0 hence the generalizes Heisenberg inequality $\Sigma_{XX}\Sigma_{PP} \geq \frac{\hbar^2}{4} I_{n\times n}$ holds.

Proof. Suppose that $\widehat{\rho}$ is a pure state; then $\Sigma = \frac{\hbar}{2}S^T S$ and the conditions (15)–(16) are just a restatement of the relations (72) in APPENDIX A, taking into account the fact that $S^T S > 0$. To prove that $(\Sigma_{XP})^2 \geq 0$ we note that since $\Sigma_{XX}\Sigma_{XP} = \Sigma_{PX}\Sigma_{XX}$ we have $\Sigma_{XP} = \Sigma_{XX}^{-1}\Sigma_{PX}\Sigma_{XX}$ hence Σ_{XP} and Σ_{PX} have the same eigenvalues; since $\Sigma_{PX} = \Sigma_{XP}^T$ these eigenvalues must be real, hence those of $(\Sigma_{XP})^2$ are ≥ 0. ■

In particular, when $n = 1$ one recovers the usual saturated Robertson–Schrödinger uncertainty principle

$$\sigma_{xx}\sigma_{pp} - \sigma_{xp}^2 = \frac{\hbar^2}{4}$$

satisfied by all pure one-dimensional Gaussian states.

Note that when Σ is diagonal, *i.e.* the state is a tensor product of one-dimensional functions one recovers the well-known fact that the Heisenberg uncertainty inequalities are saturated (reduce to equalities): $\sigma_{x_j x j}\sigma_{p_j p_j} = \hbar^2/4$ for all $j = 1, ..., n$.

2.3 Orthogonal projections of the covariance ellipsoid

We begin with a general result. For $M \in \mathrm{Sym}_{++}(2n, \mathbb{R})$ we define the phase space ellipsoid

$$\Omega = \{z \in \mathbb{R}^{2n} : Mz \cdot z \leq \hbar\}. \tag{17}$$

Writing M in block-matrix form $M = \begin{pmatrix} M_{XX} & M_{XP} \\ M_{PX} & M_{PP} \end{pmatrix}$ where the blocks are $n \times n$ matrices the condition $M \in \mathrm{Sym}_{++}(2n, \mathbb{R})$ ensures us [34] that $M_{XX} > 0$, $M_{PP} > 0$, and $M_{PX} = M_{XP}^T$. Using classical formulas for the inversion of block matrices [31] the inverse of M is

$$M^{-1} = \begin{pmatrix} (M/M_{PP})^{-1} & -(M/M_{PP})^{-1}M_{XP}M_{PP}^{-1} \\ -M_{PP}^{-1}M_{PX}(M/M_{PP})^{-1} & (M/M_{XX})^{-1} \end{pmatrix} \tag{18}$$

where M/M_{PP} and M/M_{XX} are the Schur complements:

$$M/M_{PP} = M_{XX} - M_{XP}M_{PP}^{-1}M_{PX} \tag{19}$$

$$M/M_{XX} = M_{PP} - M_{PX}M_{XX}^{-1}M_{XP}. \tag{20}$$

The following results is well-known (see for instance [13] and the references therein):

Lemma 6 *The orthogonal projections $\Pi_{\ell_X}\Omega$ and $P = \Pi_{\ell_P}\Omega$ on the coordinate subspaces $\ell_X = \mathbb{R}_x^n \times 0$ and $\ell_P = 0 \times \mathbb{R}_p^n$ of the ellipsoid Ω are the ellipsoids*

$$\Pi_{\ell_X}\Omega = \{x \in \mathbb{R}_x^n : (M/M_{PP})x \cdot x \leq \hbar\} \tag{21}$$

$$\Pi_{\ell_P}\Omega = \{p \in \mathbb{R}_p^n : (M/M_{XX})p \cdot p \leq \hbar\}. \tag{22}$$

Consider now a Gaussian state (11) with covariance matrix $\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XP} \\ \Sigma_{PX} & \Sigma_{PP} \end{pmatrix}$; by definition the covariance (or Wigner) ellipsoid of this state is

$$\Omega_\Sigma = \{z : \tfrac{1}{2}\Sigma^{-1}z \cdot z \leq 1\}. \tag{23}$$

Setting $M = \frac{1}{2}\hbar\Sigma^{-1}$ the results above yield the inverse of the covariance matrix Σ:

$$\Sigma^{-1} = \begin{pmatrix} (\Sigma/\Sigma_{PP})^{-1} & -(\Sigma/\Sigma_{PP})^{-1}\Sigma_{XP}\Sigma_{PP}^{-1} \\ -\Sigma_{PP}^{-1}\Sigma_{PX}(\Sigma/\Sigma_{PP})^{-1} & (\Sigma/\Sigma_{XX})^{-1} \end{pmatrix}. \tag{24}$$

Formula (24) immediately follows from (18) using the relations

$$\Sigma_{XX} = \frac{\hbar}{2}(M/M_{PP})^{-1} \ , \ \Sigma_{PP} = \frac{\hbar}{2}(M/M_{XX})^{-1} \tag{25}$$

$$\Sigma_{XP} = -\frac{\hbar}{2}(M/M_{PP})^{-1}M_{XP}M_{PP}^{-1}. \tag{26}$$

Proposition 7 *The orthogonal projections on the canonical coordinate subspaces ℓ_X and ℓ_P of Ω_Σ are*

$$\Pi_{\ell_X}\Omega_\Sigma = \{x \in \mathbb{R}_x^n : \tfrac{1}{2}\Sigma_{XX}^{-1}x \cdot x \leq 1\} \tag{27}$$

$$\Pi_{\ell_P}\Omega_\Sigma = \{p \in \mathbb{R}_p^n : \tfrac{1}{2}\Sigma_{PP}^{-1}p \cdot p \leq 1\}. \tag{28}$$

Proof. The projection formulas (27) and (28) are a consequence of the corresponding formulas (21) and (22). ■

We next note the following remarkable fact showing that the notion of polar duality is related to the uncertainty principle (see APPENDIX C for a short review of polar duality):

Proposition 8 *(i) The ellipsoids $X = \Pi_{\ell_X}\Omega_\Sigma$ and $P = \Pi_{\ell_P}\Omega_\Sigma$ are such that $X^\hbar \subset P$ where*

$$X^\hbar = \{p \in \mathbb{R}_p^n : \sup_{x\in X}(p \cdot x) \leq \hbar\} \tag{29}$$

is the $\hbar$-polar dual of X. (ii) We have the equality $X^\hbar = P$ if and only if $\Omega_\Sigma = B^{2n}(\sqrt{\hbar})$ in which case $X = B_X^n(\sqrt{\hbar})$ and $P = B_P^n(\sqrt{\hbar})$ (the centered balls with radius $\sqrt{\hbar}$ in ℓ_X and ℓ_P, respectively)

Proof. (i) Recall [17, 21] that if.

$$X = \{x \in \mathbb{R}_x^n : Ax \cdot x \leq \hbar\} \tag{30}$$

with $A \in \mathrm{Sym}_{++}(n,\mathbb{R})$, then its $\hbar$-polar$X^{\hbar}$ is the ellipsoid

$$X^{\hbar} = \{p \in \mathbb{R}^n_p : A^{-1}p \cdot p \leq \hbar\}. \tag{31}$$

It follows that if

$$P = \{p \in \mathbb{R}^n_p : Bp \cdot p \leq \hbar\} \tag{32}$$

(with $B \in \mathrm{Sym}_{++}(n,\mathbb{R})$) the inclusion $X^{\hbar} \subset P$ holds if and only if $AB \geq I_{n\times n}$ (*i.e.* the eigenvalues of AB are all ≥ 1); this is equivalent to $BA \geq I_{n\times n}$. Since we have here $A = \frac{1}{2}\hbar\Sigma_{XX}^{-1}$ and $B = \frac{1}{2}\hbar\Sigma_{PP}^{-1}$ (formulas (27) and (21)) the inclusion $X^{\hbar} \subset P$ will hold if and only if $\frac{1}{4}\hbar^2\Sigma_{XX}^{-1}\Sigma_{PP}^{-1} \geq I_{n\times n}$ or, equivalently $\Sigma_{XX}\Sigma_{PP} \geq \frac{1}{4}\hbar^2$. But this is the generalized Heisenberg inequality of Proposition 5. (ii) We have $X^{\hbar} = P$ if and only if $AB = I_{n\times n}$. ■

Example 9 *Let us illustrate the result above in the case $n = 1$. Here $\Sigma_{XX} = \sigma_{xx} > 0$, $\Sigma_{PP} = \sigma_{pp} > 0$, and $\Sigma_{XP} = \Sigma_{PX} = \sigma_{xp}$ and the covariance ellipse is*

$$\Omega_\Sigma : \frac{\sigma_{pp}}{2D}x^2 - \frac{\sigma_{xp}}{D}px + \frac{\sigma_{xx}}{2D}p^2 \leq 1 \tag{33}$$

where $D = \sigma_{xx}\sigma_{pp} - \sigma_{xp}^2 \geq \frac{1}{4}\hbar^2$. The orthogonal projections Ω_X and Ω_P of Ω on the x and p coordinate axes are the intervals

$$\Omega_X = [-\sqrt{2\sigma_{xx}}, \sqrt{2\sigma_{xx}}] \ , \quad \Omega_P = [-\sqrt{2\sigma_{pp}}, \sqrt{2\sigma_{pp}}] \ . \tag{34}$$

Let $\Omega_X^{\hbar}$ be the polar dual of Ω_X: it is the set of all numbers p such that $px \leq \hbar$ for $-\sqrt{2\sigma_{xx}} \leq x \leq \sqrt{2\sigma_{xx}}$ and is thus the interval

$$\Omega_X^{\hbar} = [-\hbar/\sqrt{2\sigma_{xx}}, \hbar/\sqrt{2\sigma_{xx}}] \ .$$

Since $\sigma_{xx}\sigma_{pp} \geq \frac{1}{2}\hbar$ we have the inclusion

$$\Omega_X^{\hbar} \subset \Omega_P \tag{35}$$

which reduces to the equality $\Omega_X^{\hbar} = \Omega_P$ if and only if the Heisenberg inequality is saturated, i.e. $\sigma_{xx}\sigma_{pp} = \frac{1}{4}\hbar^2$ which is equivalent to $\sigma_{xp} = 0$.

3 Pauli's Problem and its Generalizations

3.1 Paulis' reconstruction problem

The Pauli reconstruction problem is a particular case of a larger class of phase retrieval problems; see Grohs and Liehr [27] for recent advances on this difficult topic. Pauli asked in [29] whether the probability densities $|\psi(x)|^2$ and $|\widehat{\psi}(p)|^2$ of a normed function $\psi \in L^2(\mathbb{R})$ uniquely determine ψ. The answer is in general negative: consider the correlated Gaussian

$$\psi(x) = \left(\tfrac{1}{2\pi\sigma_{xx}}\right)^{1/4} e^{-\frac{x^2}{4\sigma_{xx}}} e^{\frac{i\sigma_{xp}}{2\hbar\sigma_{xx}}x^2} \tag{36}$$

which has Fourier transform

$$\widehat{\psi}(p) = \left(\tfrac{1}{2\pi\sigma_{pp}}\right)^{1/4} e^{-\frac{p^2}{4\sigma_{pp}}} e^{-\frac{i\sigma_{xp}}{2\hbar\sigma_{pp}}p^2} \tag{37}$$

and thus

$$|\psi(x)|^2 = \left(\tfrac{1}{2\pi\sigma_{xx}}\right)^{1/2} e^{-\frac{x^2}{2\sigma_{xx}}} \quad , \quad |\widehat{\psi}(p)|^2 = \left(\tfrac{1}{2\pi\sigma_{xx}}\right)^{1/2} e^{-\frac{p^2}{2\sigma_{pp}}} \tag{38}$$

Since we have

$$\sigma_{xx}\sigma_{pp} - \sigma_{xp}^2 = \tfrac{1}{4}\hbar^2 \tag{39}$$

the covariance σ_{xp} can take any of the two values $\pm(\sigma_{xx}\sigma_{pp} - \frac{1}{4}\hbar^2)^{1/2}$ so the Pauli problem has two possible solutions $\psi^{\pm}$ ("Pauli partners": see Corbett's [4] review of Pauli's problem). The same argument works for multivariate Gaussians (5): setting$X = \frac{\hbar}{2}\Sigma_{XX}^{-1}$ and $Y = -\frac{2}{\hbar}\Sigma_{XP}\Sigma_{XX}^{-1}$ we have

$$\psi_{X,Y}(x) = \left(\tfrac{1}{2\pi}\right)^{n/4} (\det\Sigma_{XX})^{-1/4} \exp\left[-\left(\frac{1}{4}\Sigma_{XX}^{-1} + \frac{i}{2\hbar}\Sigma_{XP}\Sigma_{XX}^{-1}\right) x\cdot x\right] \tag{40}$$

from which one infers that

$$|\psi_{X,Y}(x)|^2 = \left(\tfrac{1}{2\pi}\right)^{n/2} (\det\Sigma_{XX})^{-1/2} \exp\left(-\frac{1}{2}\Sigma_{XX}^{-1}x\cdot x\right) \tag{41}$$

$$|\widehat{\psi}_{X,Y}(p)|^2 = \left(\tfrac{1}{2\pi}\right)^{n/2} (\det\Sigma_{PP})^{-1/2} \exp\left(-\frac{1}{2}\Sigma_{PP}^{-1}p\cdot p\right). \tag{42}$$

To find the covariance matrix Σ_{XP} one then uses the Robertson–Schrödinger formula (15)

$$\Sigma_{XX}\Sigma_{PP} - (\Sigma_{XP})^2 = \frac{\hbar^2}{4}I_{n\times n}$$

which has multiple solutions in Σ_{XP}.

Another way of seeing Pauli's problem is to use the Wigner formalism; recall that the Wigner transform of $\psi \in L^2(\mathbb{R}^n)$ is defined by

$$W\psi(x,p) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}p\cdot y}\psi(x+\tfrac{1}{2}y)\psi^*(x-\tfrac{1}{2}y)dy \tag{43}$$

and if $\psi, \widehat{\psi} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then the marginal properties

$$\int_{\mathbb{R}^n} W\psi(x,p)dp = |\psi(x)|^2 \ , \ \int_{\mathbb{R}^n} W\psi(x,p)dx = |\widehat{\psi}(p)|^2 \tag{44}$$

hold [14]. Since $W\psi^*(x,p) = W\psi(x,-p)$ the functions ψ and ψ^* have the same marginals, which is reflected by the relations (38) which are satisfied by both ψ and its complex conjugate. This idea extends to more general phase retrieval problems, see [27].

3.2 Lagrangian frames

We will henceforth call *Lagrangian frame* a pair (ℓ, ℓ') of transverse Lagrangian subspaces (see APPENDIX B), that is $(\ell, \ell') \in \mathrm{Lag}(n) \times \mathrm{Lag}(n)$ and $\ell \cap \ell' = 0$ (equivalently $\ell \oplus \ell' = \mathbb{R}^{2n}$ since $\dim \ell = \dim \ell' = n$). When ℓ and ℓ' are coordinate Lagrangian planes ℓ_X and ℓ_P we will call it the *canonical Lagrangian frame.* We denote by $\mathcal{F}_{\mathrm{Lag}}(n)$ The following property will be essential for our constructions to come (See [10]):

Lemma 10 *The symplectic group* $\mathrm{Sp}(n)$ *acts transitively on* $\mathcal{F}_{\mathrm{Lag}}(n)$*. In particular, every Lagrangian frame* $(\ell, \ell') \in \mathcal{F}_{\mathrm{Lag}}(n)$ *can be obtained from the canonical frame* (ℓ_X, ℓ_P) *by a symplectic transformation.*

Proof. Let (ℓ_1, ℓ_1') and (ℓ_2, ℓ_2') be two Lagrangian frames. Choose a basis $(e_{1i})_{1\leq 1\leq n}$ of ℓ_1 and a basis $(f_{1j})_{1\leq j\leq n}$ of ℓ_1' whose union $(e_{1i})_{1\leq 1\leq n} \cup (f_{1j})_{1\leq j\leq n}$ is a symplectic basis, that is $\omega(e_{1i}, e_{1j}) = \omega(f_{1i}, f_{1j}) = 0$ and $\omega(f_{1i}, e_{1j}) = \delta_{ij}$ for $1 \leq i, j \leq n$. Similarly, choose bases $(e_{2i})_{1\leq 1\leq n}$ and $(f_{2j})_{1\leq j\leq n}$ of ℓ_2 and ℓ_2' whose union is also a symplectic basis. The linear automorphism of $\mathbb{R}^{2n}$ defined by $S(e_{1i}) = e_{2i}$ and $S(f_{1i}) = f_{2i}$ for $1 \leq i \leq n$ is in $\mathrm{Sp}(n)$ and we have $(\ell_2, \ell_2') = (S\ell_1, S\ell_1')$. ■

3.3 Gaussian reconstruction by partial tomography

We are going to generalize the reconstruction procedure for Gaussians using the notion of Lagrangian frame introduced above. For this we need to define the integral of a real integrable function ρ on phase space along an affine subspace $\ell(z) = \ell + z$ where ℓ is a Lagrangian subspace of $(\mathbb{R}^{2n}, \omega)$. Assuming [8, 10] that ℓ is represented by the system of equations $Ax + Bp = 0$ with $A^T B$ (and $B^T A$) symmetric, and $A^T A + B^T B = I_{n\times n}$ (and $A^T A + B^T B = I_{n\times n}$) we parametrize $\ell(z)$ by $z(u) = (x(u), p(u))$, $u \in \mathbb{R}^n$, with

$$x(u) = -B^T u + x \quad , \quad p(u) = A^T u + p \tag{45}$$

where $z=(x,p)\in\mathbb{R}^{2n}$ is arbitrary. Taking into account the fact that $A^TA+B^TB=I_{n\times n}$ we then define the generalized line integral

$$\int_{\ell(z)}\rho(s)ds=\int_{\mathbb{R}^n}\rho(-B^Tu+x,A^Tu+p)du. \tag{46}$$

Applying this definition to the case where $\rho=W\psi$ with $\psi\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and choosing $\ell=\ell_P$ we have $B=0$ and $A=I_{n\times n}$ so that, in view of the marginal properties (44),

$$\int_{\ell_P(x,p)}W\psi(s)ds=\int_{\mathbb{R}^n}W\psi(x,u+p)du=|\psi(x)|^2$$

and, similarly, choosing $\ell=\ell_P$, and $A=0$, $B=I_{n\times n}$,

$$\int_{\ell_X}W\psi(s)ds=\int_{\mathbb{R}^n}W\psi(-u+x,p)du=|\widehat{\psi}(p)|^2.$$

We'll generalize these relations below, but we first prove that for fixed $z=(x,p)$ the integral (46) is independent of the choice of parametrization (45). To see this, we note that the Lagrangian subspace ℓ is the image of $\ell_X=\{(u,0):u\in\mathbb{R}^n\}$ by the symplectic rotation ([8, 10]; Appendix A)

$$R=\begin{pmatrix}-B^T & -A^T\\ A^T & -B^T\end{pmatrix}\in\operatorname{Sp}(n)\cap O(2n,\mathbb{R})$$

since $A^TA+B^TB=I_{n\times n}$ and B^TA symmetric. Let a new parametrization of $\ell(z)$ be

$$x'(u)=-B'^Tu+x\ ,\ \ p'(u)=A'^Tu+p$$

the matrices A' and B' satisfying relations similar to those of A and B, and let R' be the corresponding symplectic rotation. The product $R^{-1}R'$ leaves ℓ_X invariant; since it is a symplectic rotation we have $R^{-1}R'=\begin{pmatrix}H & 0\\ 0 & H\end{pmatrix}$ with $H\in O(n,\mathbb{R})$ hence the re-parametrization is

$$x'(u)=-B^THu+x\ ,\ p'(u)=A^THu+p$$

leading to the same value of the integral 46) since $d(Hu)=du$.

The following result relates the integral (46) to the notion of marginal value of the Wigner transform:

Proposition 11 *Let $\ell \in \mathrm{Lag}(n)$ and $z = (x, p)$ be as above. For $\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n))$ we have*

$$\int_{\ell(z)} W\psi(s)ds = |\widehat{U}\psi(Ax + Bp)|^2 \tag{47}$$

where $\widehat{U} \in \mathrm{Mp}(n)$ covers the symplectic rotation $U = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.

Proof. The Lagrangian subspace ℓ is the image of the momentum space ℓ_P by the symplectic rotation

$$U^T = U^{-1} = \begin{pmatrix} A^T & -B^T \\ B^T & A^T \end{pmatrix}$$

hence, using the covariance relation $W\psi \circ U^{-1} = W(\widehat{U}\psi)$ [10, 14],

$$\begin{aligned} \int_{\ell(z)} W\psi(s)ds &= \int_{\mathbb{R}^n} W\psi(U^{-1}((0, u) + U(x, p)))du \\ &= \int_{\mathbb{R}^n} W(\widehat{U}\psi)(0, u) + U(x, p))du \\ &= \int_{\mathbb{R}^n} W(\widehat{U}\psi)(Ax + Bp, Bx - Ap + u)du \\ &= \int_{\mathbb{R}^n} W(\widehat{U}\psi)(Ax + Bp, u)du \end{aligned}$$

hence (47) in view of the first marginal property (44). ■

Formula (47) is closely related to the definition of the symplectic Radon transform as given in our paper [16]. The result below shows that, however, we do not need the full power of the theory of inverse Radon transform to reconstruct Gaussians:

Theorem 12 *Let $(\ell, \ell') \in \mathcal{F}_{\mathrm{Lag}}(n)$ be a Lagrangian frame. The Wigner transform $W\psi_{X,Y}$ (and hence the Gaussian $\psi_{X,Y}$ itself, up to a unimodular factor) is uniquely determined by the knowledge of the integrals*

$$\int_{\ell(z)} W\psi_{X,Y}(s)ds \quad \text{and} \quad \int_{\ell'(z)} W\psi_{X,Y}(s)ds$$

for all $z \in \mathbb{R}^{2n}$.

Proof. It is similar to that of Proposition 11 above. In view of Lemma 10 the symplectic group acts transitively on $\mathcal{F}_{\mathrm{Lag}}(n)$ so we can find $S \in \mathrm{Sp}(n)$ such tat $(\ell, \ell') = S(\ell_X, \ell_P)$. Setting $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ let $\widehat{S} \in \mathrm{Mp}(n)$ be one of the two metaplectic operators covering S. Let $\ell(Sz) = S\ell_X + Sz$ and $\ell'(Sz) = S\ell_P + Sz$; we have, using the covariance relation $W\psi_{X,Y} \circ S = W(\widehat{S}^{-1}\psi_{X,Y})$,

$$\begin{aligned}\int_{\ell'(Sz)} W\psi_{X,Y}(s)ds &= \int_{\mathbb{R}^n} W\psi_{X,Y}\left[S((0,u)+(x,p))\right]du \\ &= \int_{\mathbb{R}^n} W(\widehat{S}^{-1}\psi_{X,Y})(x,u+p)du.\end{aligned}$$

that is, in view of the first marginal formula (44),

$$\int_{\ell'(Sz)} W\psi_{X,Y}(s)ds = |\widehat{S}^{-1}\psi_{X,Y}(x)|^2. \tag{48}$$

Similarly, using the second formula (44),

$$\int_{\ell(Sz)} W\psi_{X,Y}(s)ds = |F\widehat{S}^{-1}\psi_{X,Y}(p)|^2. \tag{49}$$

These values allow the determination of $\widehat{S}^{-1}\psi_{X,Y}$ and, hence, of $\psi_{X,Y}$. ■

3.4 Geometric Interpretation

We now consider the following situation: performing a large number of measurements on the coordinates $x_1, ..., x_k, p_{k+1}, ..., p_n$ (with $1 \leq k < n$) we identify this cloud of measurements with an ellipsoid X_ℓ carried by the Lagrangian subspace ℓ with coordinates $x_1, ..., x_k, p_{k+1}, ..., p_n$ and centered at the origin. We now ask whether to this ellipsoid we can in some way associate the covariance ellipsoid Ω_Σ of some pure Gaussian state with covariance matrix Σ. In view of the discussion above Ω_Σ has to be a quantum blob, *i.e.* the image of the phase space ball $B^{2n}(\sqrt{\hbar})$ by some $S \in \mathrm{Sp}(n)$. The answer is given by the following result, which actually holds for any Lagrangian subspace (not necessarily a coordinate subspace). It introduces a notion of polar duality between Lagrangian subspaces. (The basics of the notion of polar duality for convex sets are shortly reviewed in Appendix C.)

Theorem 13 *Let $(\ell,\ell') \in \mathcal{F}_{\mathrm{Lag}}(n)$ be a Lagrangian frame in $(\mathbb{R}^{2n},\omega)$. Let X_ℓ be a centered ellipsoid carried by ℓ and define the dual ellipsoid $(X_\ell)^\hbar_{\ell'} \subset \ell'$ by*

$$(X_\ell)^\hbar_{\ell'} = \{z' \in \ell' : \sup\nolimits_{z\in\ell} \omega(z,z') \leq \hbar\}. \tag{50}$$

(i) The John ellipsoid Ω of the product $X_\ell \times (X_\ell)^\hbar_{\ell'}$ is a quantum blob $S(B^{2n}(\sqrt{\hbar}))$; (ii) To that quantum blob corresponds the Gaussian pure state $\psi^{\pm}_{X,Y} = \pm i^\mu \widehat{S}\phi_0$ where $\widehat{S} \in \mathrm{Mp}(n)$ covers S.

Proof. (i) We recall [2] that the John ellipsoid of a convex set is the (unique) ellipsoid of maximal volume contained in that set. Let us first prove (i) for the particular case $(\ell,\ell') = (\ell_X,\ell_P)$ and $X_{\ell_X} = B^n_X(\sqrt{\hbar})$. In this case $(X_{\ell_X})^\hbar_{\ell_P} = B^n_P(\sqrt{\hbar})$. We claim that the John ellipsoid of $B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar})$ is the trivial quantum blob $B^{2n}(\sqrt{\hbar})$. To see this we first note that the inclusion

$$B^{2n}(\sqrt{\hbar}) \subset B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar}) \tag{51}$$

is obvious, and that we cannot have

$$B^{2n}(R) \subset B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar})$$

if $R > \sqrt{\hbar}$. Assume now that the John ellipsoid of $B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar})$ is defined by the inequality

$$Ax\cdot x + Bx\cdot p + Cp\cdot p \leq \hbar$$

where $A, C > 0$ and B are real $n\times n$ matrices. Since $B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar})$ is invariant by the transformation $(x,p) \longmapsto (p,x)$ so is its John ellipsoid and we must thus have $A = C$ and $B = B^T$. Similarly, $B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar})$ being invariant by the partial reflection $(x,p) \longmapsto (-x,p)$ we get $B = 0$ so the John ellipsoid of $B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar})$ is defined by $Ax\cdot x + Ap\cdot p \leq \hbar$. We next observe that $B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar})$ an its John ellipsoid are invariant under the transformations $(x,p) \longmapsto (Hx, HP)$ where $H \in O(n,\mathbb{R})$ so we must have $AH = HA$ for all $H \in O(n,\mathbb{R})$, but this is only possible if $A = \lambda I_{n\times n}$ for some $\lambda \in \mathbb{R}$. The John ellipsoid of $B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar})$ is thus of the type $B^{2n}(\sqrt{\hbar}/\sqrt{\lambda})$ for some $\lambda \geq 1$ and this concludes the proof in view of the inclusion (51) since the case $\lambda > R^2$ is excluded. Consider now, again when $(\ell,\ell') = (\ell_X,\ell_P)$, the case where $X_{\ell_X} = A(B^n_X(\sqrt{\hbar}))$ is an ellipsoid $(A \in \mathrm{Sym}_{++}(n,\mathbb{R}))$. We then have $(X_{\ell_X})^\hbar_{\ell_P} = A^{-1}(B^n_P(\sqrt{\hbar}))$ so that

$$X_{\ell_X} \times (X_{\ell_X})^\hbar_{\ell_P} = S_A(B^n_X(\sqrt{\hbar}) \times B^n_P(\sqrt{\hbar}))$$

where $S_A = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in \mathrm{Sp}(n)$ so that the John ellipsoid of $X_{\ell_X} \times (X_{\ell_X})^{\hbar}_{\ell_P}$ is here the quantum blob $S_A(B^{2n}(\sqrt{\hbar}))$. Let us finally consider the case of an arbitrary Lagrangian frame (ℓ, ℓ'). In view of the transitivity of the action of $\mathrm{Sp}(n)$ on $\mathcal{F}_{\mathrm{Lag}}(n)$ we can find $S \in \mathrm{Sp}(n)$ such that $(\ell, \ell') = S(\ell_X, \ell_P)$; set now $X_{\ell_X} = S^{-1}(X_\ell)$, this is an ellipsoid carried by $(\ell_X$. We claim that we have $(X_{\ell_X})^{\hbar}_{\ell_P} = S^{-1}((X_\ell)^{\hbar}_{\ell'})$. Let $z \in S^{-1}(X_\ell)^{\hbar}_{\ell'}$, that is $Sz \in (X_\ell)^{\hbar}_{\ell'}$. This is equivalent to the conditions $z \in S^{-1}\ell' = \ell_P$ and $\omega(Sz, z') \leq \hbar$ for all $z' \in X_\ell$. Since $\omega(Sz, z') = \omega(z, S^{-1}z')$ this is in turn equivalent to $z \in S^{-1}\ell' = \ell_P$ and $\omega(z, S^{-1}z') \leq \hbar$ for all $S^{-1}z' \in S^{-1}(X_\ell)$, that is to $z \in \omega(z, z'') \leq \hbar$ for all $z'' \in X_{\ell_X}$ which is the same thing as $z \in (X_\ell)^{\hbar}_{\ell'}$ which was to be proven. Summarizing, we have shown that

$$X_\ell \times (X_\ell)^{\hbar}_{\ell'} = S(X_{\ell_X} \times (X_{\ell_X})^{\hbar}_{\ell_P}).$$

Since the John ellipsoid of $X_{\ell_X} \times (X_{\ell_X})^{\hbar}_{\ell_P}$ is $S_A(B^{2n}(\sqrt{\hbar}))$, that of $X_\ell \times (X_\ell)^{\hbar}_{\ell'}$ is the quantum blob $Q = SS_A(B^{2n}(\sqrt{\hbar}))$. (ii) (*cf.* [12]). The result follows from the discussion above. Set $S' = SS_A$. To $Q = S'(B^{2n}(\sqrt{\hbar}))$ we associate the positive definite symplectic matrix $G = S'S'^T$. To the latter corresponds the phase space Gaussian $z \longmapsto (\pi\hbar)^{-n} e^{-\frac{1}{\hbar}Gz\cdot z}$, which is the Wigner transform of $i^{\mu}\widehat{S'}\phi_0$ for any $\mu \in \mathbb{R}$. ■

4 Polar Duality and Covariance Ellipsoid

4.1 Symplectic polar duality

Here is another variant of polar duality; it was introduced in our paper [17] and used in [20] to characterize covariance ellipsoids (we are following quite closely the presentation in the latter paper). Let Ω be a symmetric convex body in the phase space $(\mathbb{R}^{2n}, \omega)$ (for instance an ellipsoid). We call *symplectic* ($\hbar$-)*polar dual* $\Omega^{\hbar,\omega}$ of Ω the set

$$\Omega^{\hbar,\omega} = \{z' \in \mathbb{R}^{2n} : \sup_{z \in \Omega} \omega(z, z') \leq \hbar\}. \tag{52}$$

It is related to the usual polar dual

$$\Omega^{\hbar} = \{z' \in \mathbb{R}^{2n} : \sup_{z \in \Omega}(z \cdot z') \leq \hbar\}$$

by a symplectic rotation:

$$\Omega^{\hbar,\omega} = (J\Omega)^{\hbar} = J(\Omega^{\hbar}). \tag{53}$$

The symplectic polar dual is symplectically covariant in the sense that

$$(S\Omega)^{\hbar,\omega} = S(\Omega^{\hbar,\omega}) \textit{ for } S \in \mathrm{Sp}(n). \tag{54}$$

In fact, the condition $S \in \mathrm{Sp}(n)$ is equivalent to $S^T JS = J$ hence $JS = (S^T)^{-1}J$. It follows that

$$\begin{aligned}(S\Omega)^{\hbar,\omega} &= J(S(\Omega))^{\hbar} = J(S^T)^{-1}(\Omega^{\hbar}) \\ &= SJ(\Omega^{\hbar}) = S(\Omega^{\hbar,\omega})\end{aligned}$$

which is (54). In particular, since $B^{2n}(\sqrt{\hbar})^{\hbar} = B^{2n}(\sqrt{\hbar})$, we have

$$(S(B^{2n}(\sqrt{\hbar})))^{\hbar,\omega} = S(B^{2n}(\sqrt{\hbar})) \tag{55}$$

so the quantum blobs $S(B^{2n}(\sqrt{\hbar}))$, $S \in \mathrm{Sp}(n)$, are fixed points of the transformation $\Omega \longmapsto \Omega^{\hbar,\omega}$; it is easy to show that they are the only fixed points of this transformation.

Proposition 14 *Let Ω_Σ be the covariance ellipsoid associated with the covariance matrix Σ:*

$$\Omega_\Sigma = \{z \in \mathbb{R}^{2n} : Mz \cdot z \leq \hbar\} \quad , \quad M = \tfrac{1}{2}\hbar\Sigma^{-1}. \tag{56}$$

(i) Ω_Σ is quantized (that is, contains a quantum blob) if and only if we have the inclusion $\Omega_\Sigma^{\hbar,\omega} \subset \Omega_\Sigma$. (ii) The equality $\Omega_\Sigma^{\hbar,\omega} = \Omega_\Sigma$ holds if and only if there exists $S \in \mathrm{Sp}(n)$ such that $\Omega_\Sigma = S(B^{2n}(\sqrt{\hbar}))$ (i.e. if and only if Ω_Σ is a quantum blob).

Proof. (i) Suppose that there exists $S \in \mathrm{Sp}(n)$ such that $Q = S(B^{2n}(\sqrt{\hbar})) \subset \Omega$. By the anti-monotonicity of (symplectic) polar duality this implies that we have $\Omega^{\hbar,\omega} \subset Q^{\hbar,\omega} = Q \subset \Omega$, which proves the necessity of the condition. Suppose conversely that we have $\Omega^{\hbar,\omega} \subset \Omega$. Since

$$\Omega_\Sigma^{\hbar} = \{z \in \mathbb{R}^{2n} : M^{-1}z \cdot z \leq \hbar\} \tag{57}$$

we have, using (53),

$$\Omega_\Sigma^{\hbar,\omega} = \{z \in \mathbb{R}^{2n} : (-JM^{-1}J)z \cdot z \leq \hbar\} \tag{58}$$

hence the inclusion $\Omega_\Sigma^{\hbar,\omega} \subset \Omega_\Sigma$ is equivalent to $M \leq (-JM^{-1}J)$ ($\leq$ stands here for the Löwner ordering). Performing a symplectic diagonalization (13) of M and using the relations $JS^{-1} = S^T J$, $(S^T)^{-1}J = JS$ this is equivalent to

$$M = S^T DS \leq S^T(-JD^{-1}J)S$$

that is to $D \leq -JD^{-1}J$. In the notation in (13) this implies that we have $\Lambda^\omega \leq (\Lambda^\omega)^{-1}$ and hence $\lambda_j^\omega \leq 1$ for $1 \leq j \leq n$; thus $D \leq I$ and $M = S^T DS \leq S^T S$. The inclusion $S(B^{2n}(\sqrt{\hbar})) \subset \Omega$ follows. Let us next prove the statement (ii) The condition is sufficient since $S(B^{2n}(\sqrt{\hbar}))^{\hbar,\omega} = S(B^{2n}(\sqrt{\hbar}))$. Assume conversely that $\Omega_\Sigma^{\hbar,\omega} = \Omega_\Sigma$. Then there exists $S \in \mathrm{Sp}(n)$ such that$Q = S(B^{2n}(\sqrt{\hbar})) \subset \Omega_\Sigma$. It follows that $\Omega_\Sigma^{\hbar,\omega} \subset Q^{\hbar,\omega} = Q$ hence $\Omega_\Sigma^{\hbar,\omega} = \Omega_\Sigma \subset Q$ so that $\Omega = Q$. ■

Here is an example:

Example 15 *Consider again the covariance ellipse*

$$\Omega_\Sigma : \frac{\sigma_{pp}}{2D}x^2 - \frac{\sigma_{xp}}{D}px + \frac{\sigma_{xx}}{2D}p^2 \leq 1 \tag{59}$$

with $D = \sigma_{xx}\sigma_{pp} - \sigma_{xp}^2 = \frac{1}{4}\hbar^2$. *Here* $M = \frac{\hbar}{2D}\begin{pmatrix} \sigma_{pp} & -\sigma_{xp} \\ -\sigma_{xp} & \sigma_{xx} \end{pmatrix}$ *hence the symplectic polar dual of* Ω_Σ *is*

$$\Omega_\Sigma^{\hbar,\omega} : \frac{2\sigma_{pp}}{\hbar^2}x^2 - \frac{4\sigma_{xp}}{\hbar^2}px + \frac{2\sigma_{xx}}{\hbar^2}p^2 \leq 1.$$

The condition $\Omega_\Sigma^{\hbar,\omega} = \Omega_\Sigma$ *is equivalent to* $D = \frac{1}{4}\hbar^2$ *so that* Ω_Σ *is indeed a quantum blob.*

4.2 A tomographic result

We are going to prove a stronger statement, which can be seen as a "tomographic" result since it involves the intersection of the covariance ellipsoid with a (Lagrangian) subspace. Let us begin with a simple example in the case $n = 1$.

Theorem 16 *(i) The ellipsoid* Ω *contains a quantum blob* $Q = S(B^{2n}(\sqrt{\hbar}))$ $(S \in \mathrm{Sp}(n))$ *if and only if there exists* $\ell \in \mathrm{Lag}(n)$ *such that*

$$\Omega^{\hbar,\omega} \cap \ell \subset \Omega \cap \ell \tag{60}$$

in which case we have $\Omega^{\hbar,\omega} \cap \ell \subset \Omega \cap \ell$ *for all* $\ell \in \mathrm{Lag}(n)$. *(ii) The equality* $\Omega^{\hbar,\omega} \cap \ell = \Omega \cap \ell$ *holds (for some, and hence for all,* $\ell \in \mathrm{Lag}(n)$*) if and only if* Ω *is a quantum blob.*

Proof. (i) The necessity of the condition (60) is trivial (Proposition 14). Let us prove that this condition is also sufficient. Setting as usual $M = \frac{1}{2}\hbar\Sigma^{-1}$ and

$$\Omega_\Sigma = \{z : Mz \cdot z \leq \hbar\} = \{z : \tfrac{1}{2}\Sigma^{-1}z \cdot z \leq \hbar\} \tag{61}$$

we have

$$\Omega_\Sigma^{\hbar,\omega} = \{z \in \mathbb{R}^{2n} : (-JM^{-1}J)z \cdot z \leq \hbar\}. \tag{62}$$

Performing a symplectic diagonalization (13) of M we get

$$\Omega_\Sigma = S^{-1}\Omega_{\hbar D^{-1}/2}\ ,\quad \Omega_\Sigma^{\hbar,\omega} = S^{-1}(\Omega_{\hbar D^{-1}/2})^{\hbar,\omega} \tag{63}$$

where $\Omega_{\hbar D^{-1}/2}$ and its dual are explicitly given by

$$\begin{aligned}\Omega_{\hbar D^{-1}/2} &= \{z \in \mathbb{R}^{2n} : Dz \cdot z \leq \hbar\}\\ (\Omega_{\hbar D^{-1}/2})^{\hbar,\omega} &= \{z \in \mathbb{R}^{2n} : D^{-1}z \cdot z \leq \hbar\}.\end{aligned}$$

where we have used the identity $-JD^{-1}J = D^{-1}$. Let us first assume that $\ell = \ell_X = \mathbb{R}^n \times 0$. Then

$$\Omega_{\hbar D^{-1}/2} \cap \ell_X = \{x \in \mathbb{R}^n : \Lambda^\omega x \cdot x \leq \hbar\}$$

and

$$(\Omega_{\hbar D^{-1}/2})^{\hbar,\omega} \cap \ell_X = \{x \in \mathbb{R}^n : (\Lambda^\omega)^{-1}x \cdot x \leq \hbar\}.$$

Now, the condition

$$(\Omega_{\hbar D^{-1}/2})^{\hbar,\omega} \cap \ell_X \subset \Omega_{\hbar D^{-1}/2} \cap \ell_X$$

is equivalent to $(\Lambda^\omega)^{-1} \geq \Lambda^\omega$ that is to $D^{-1} \geq D$, which implies $(\Omega_{\hbar D^{-1}/2})^{\hbar,\omega} \subset \Omega_{\hbar D^{-1}/2}$, and $\Omega_{\hbar D^{-1}/2}$ contains a quantum blob in view of Proposition 14. We have thus proven our result in the case where $\Sigma = \hbar D^{-1}/2$ and $\ell = \ell_X$. For the general case we take $\ell = S^{-1}\ell_X$ where S is a Williamson diagonalizing matrix for Σ; in view of (63) we have

$$\begin{aligned}\Omega_\Sigma \cap \ell &= S^{-1}\Omega_{\hbar D^{-1}/2} \cap S^{-1}\ell_X = S^{-1}(\Omega_{\hbar D^{-1}/2} \cap \ell_X)\\ \Omega_\Sigma^{\hbar,\omega} \cap \ell &= S^{-1}(\Omega_{\hbar D^{-1}/2})^{\hbar,\omega} \cap S^{-1}\ell_X = S^{-1}((\Omega_{\hbar D^{-1}/2})^{\hbar,\omega} \cap \ell_X)\end{aligned}$$

and hence $\Omega_\Sigma^{\hbar,\omega} \cap \ell \subset \Omega_\Sigma \cap \ell$ if and only if $(\Omega_{\hbar D^{-1}/2})^{\hbar,\omega} \subset \Omega_{\hbar D^{-1}/2}$. It now suffices to apply Proposition 14. To prove (ii) it is sufficient to note that the equality

$$(\Omega_{\hbar D^{-1}/2})^{\hbar,\omega} \cap \ell_X = \Omega_{\hbar D^{-1}/2} \cap \ell_X$$

is equivalent to $(\Lambda^\omega)^{-1} = \Lambda^\omega$ that is to $\Lambda^\omega = I_{n\times n}$. Since we have in this case $M = S_0^T S_0$ in view of (13), the proof in the general case can be completed as above. ■

Example 17 *With the notation of the previous examples, we have*

$$\Omega_\Sigma : \frac{\sigma_{pp}}{2D}x^2 - \frac{\sigma_{xp}}{D}px + \frac{\sigma_{xx}}{2D}p^2 \leq 1 \tag{64}$$

$$\Omega_\Sigma^{\hbar,\omega} : \frac{2\sigma_{pp}}{\hbar^2}x^2 - \frac{4\sigma_{xp}}{\hbar^2}px + \frac{2\sigma_{xx}}{\hbar^2}p^2 \leq 1 \tag{65}$$

with $D = \sigma_{xx}\sigma_{pp} - \sigma_{xp}^2 = \frac{1}{4}\hbar^2$. *We have*

$$\Omega_\Sigma \cap \ell_X = [-\sqrt{2D/\sigma_{pp}}, \sqrt{2D/\sigma_{pp}}]$$

$$\Omega_\Sigma^{\hbar,\omega} \cap \ell_X = [-\hbar/\sqrt{2\sigma_{pp}}, \hbar/\sqrt{2\sigma_{pp}}]$$

and $\Omega_\Sigma^{\hbar,\omega} \cap \ell_X \subset \Omega_\Sigma \cap \ell_X$ *if and only if* $\hbar/\sqrt{2\sigma_{pp}} \leq \sqrt{2D/\sigma_{pp}}$ *which is equivalent to* $D \geq \frac{1}{4}\hbar^2$. *More generally, if* $\ell_a : p = ax$ *for any* $a \in \mathbb{R}$ *then* $\Omega_\Sigma \cap \ell_a$ *and* $\Omega_\Sigma^{\hbar,\omega} \cap \ell_a$ *are determined by the inequalities*

$$\Omega_\Sigma \cap \ell_a : \left(\frac{\sigma_{pp}}{2D} - \frac{\sigma_{xp}}{D}a + \frac{\sigma_{xx}}{2D}a^2\right)x^2 \leq 1$$

$$\Omega_\Sigma^{\hbar,\omega} \cap \ell_a : \left(\frac{2\sigma_{pp}}{\hbar^2} - \frac{4\sigma_{xp}}{\hbar^2}a + \frac{2\sigma_{xx}}{\hbar^2}a^2\right)x^2 \leq 1$$

and the inclusion

$$\Omega_\Sigma^{\hbar,\omega} \cap \ell_a \subset \Omega_\Sigma \cap \ell_a \tag{66}$$

is equivalent to the inequality

$$k\left(\sigma_{xx}a^2 - 2\sigma_{xp}a + \sigma_{pp}\right) \leq 0 \tag{67}$$

where $k = 2(\hbar^2/4 - D)/\hbar^2 D$. *Now* $\sigma_{xx}a^2 - 2\sigma_{xp}a + \sigma_{pp} > 0$ *for every* $a \in \mathbb{R}$ *(because* $\sigma_{xp}^2 - \sigma_{xx}\sigma_{pp} = -D < 0$ *since* $\Sigma > 0$*) and hence the inclusion (66) holds if and only if* $k \leq 0$, *that is, if and only if* $D \geq \hbar^2/4$ *which is the Robertson–Schrödinger inequality ensuring us that* Ω_Σ *contains a quantum blob (and is itself a quantum blob when* $D = \hbar^2/4$*).*

4.3 The case of mixed states

Sofar we have been considering pure states. Let us generalize our discussion to more general mixed states. We assume that $\widehat{\rho}$ is what we have called in [19] a "Feichtinger state", *i.e.* a density operator whose Wigner distribution ρ is regular enough to allow the existence of the covariance matrix

$$\Sigma = \int_{\mathbb{R}^{2n}} (z - \langle z \rangle)(z - \langle z \rangle)^T \rho(z) dz \tag{68}$$

($\langle z\rangle = \int_{\mathbb{R}^{2n}} z\rho(z)dz$ is the mean value vector). In order to represent a quantum state a necessary condition is that [3, 28]

$$\Sigma + \frac{i\hbar}{2}J \textit{ is semidefinite positive} \tag{69}$$

which we write for short as $\Sigma + \frac{i\hbar}{2}J \geq 0$ (this condition equivalent to the uncertainty principle in its strong Robertson–Schrödinger form, *ibid.*). One shows [28] that (69) implies that the covariance matrix Σ of a quantum state is always definite positive, and, conversely, that (69) is sufficient for Gaussian mixed states: a Gaussian of the type (11) introduced above, that is

$$\rho(z) = \frac{1}{(2\pi)^n\sqrt{\det\Sigma}}e^{-\frac{1}{2}\Sigma^{-1}(z-z_0)\cdot(z-z_0)} \tag{70}$$

is the Wigner distribution of a mixed quantum state if and only if the condition (69) holds. Recall that this contition is equivalent to saying that the covariance ellipsoid $\Omega_\Sigma : \frac{1}{2}\Sigma^{-1}z\cdot z \leq 1$ contains a quantum blob, from which follows that the orthogonal projetions of Ω_Σ : on any conjugate plane x_j, p_j has area at least $\pi\hbar$.

In view of Theoren 16 Ω_Σ contains a quantum blob $Q = S(B^{2n}(\sqrt{\hbar}))$if and only if there exists $\ell \in \mathrm{Lag}(n)$ such that $\Omega^{\hbar,\omega} \cap \ell \subset \Omega \cap \ell$ in which case we have $\Omega^{\hbar,\omega} \cap \ell \subset \Omega \cap \ell$ for all $\ell \in \mathrm{Lag}(n)$. It follows that:

Corollary 18 *The Gaussian 70 is the Wigner distribution of a mixed quantum state if and only its covariance ellipsoid satifies the condition* $\Omega_\Sigma^{\hbar,\omega} \cap \ell \subset \Omega_\Sigma \cap \ell$ *for some (and hence all)* $\ell \in \mathrm{Lag}(n)$.

Proof. It is just a restatemnt of Theoren 16. ■

5 Discussion and Conclusions

Theorem 13 shows that we can identify any pure Gaussian state with a geometric object, the Cartesian product $X_\ell \times (X_\ell)^\hbar_{\ell'}$. The physical interpretation of this correspondence is the following: once a cloud of position-momentum measurements is made on a given Lagrangian plane ℓ, the latter is approximated by an ellipsoid X_ℓ. One then chooses a transversal Lagrangian plane ℓ' and one calculates the polar dual $(X_\ell)^\hbar_{\ell'}$ of X_ℓ on ℓ'; the covariance ellipsoid of the Gaussian state we are looking for is then simply the maximal volume ellipsoid of the convex product $X_\ell \times (X_\ell)^\hbar_{\ell'}$, and the latter uniquely determines the state (which is here supposed to be centered at the origin;

the general case is trivially obtained using phase space translation or the Heisenberg displacement operator). Theorem 16, on the other hand, shows that one can test whether a covariance ellipsoid is that of a quantum state by intersecting it with a single arbitrary Lagrangian plane. This is typically a tomographic result which might have both theoretical and practical applications.

It would be interesting (and important!) to extend the approach and results of this paper to non-Gaussian states; non-Gaussian features are indispensable in many quantum protocols, especially to reach a quantum computational advantage (see the interesting discussions in Ra *et al.* [25] and Walschaers [32]). A possible approach could be to generalize the "geometric states" described by Theorem 13 to the case where X_ℓ no longer is an ellipsoid, but an arbitrary convex body. The Lagrangian plane ℓ would then be replaced with a Lagrangian submanifold of phase space (i.e. a n-dimensional submanifold where the tangent spaces are all Lagrangian). We will come back to these intriguing and potentially fruitful possibilities in future work.

An interesting point raised by one of the Reviewers is the question of what happens for reduced covariance matrices where obtaining purity or von entropy Neumann is possible? These questions will be answered in a forthcoming article (There are some delicate points to elucidate, and we have found they deserve to be discussed in a sequel of this work).

APPENDIX A. The Groups $\mathrm{Sp}(n)$, $U(n)$, and $\mathrm{Mp}(n)$

For details and proofs see [10]. The symplectic group $\mathrm{Sp}(n)$ consists of all linear automorphisms $S : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ such that $\omega(Sz, Sz') = \omega(z, z')$ for all $(z, z') \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$. A symplectic basis of $(\mathbb{R}_z^{2n}, \sigma)$ being chosen once for all, we can write this condition in matrix form $S^T JS = SJS^T = J$ and, writing $S \in \mathrm{Sp}(n)$ in block-matrix form

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{71}$$

where the entries A, B, C, D are $n \times n$ matrices, these conditions are then easily seen to be equivalent to the two following sets of equivalent conditions:

$$A^T C,\ B^T D \text{ symmetric, and } A^T D - C^T B = I \tag{72}$$

$$AB^T,\ CD^T \text{ symmetric, and } AD^T - BC^T = I. \tag{73}$$

It follows from the second of these sets of conditions that the inverse of S is

$$S^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}. \tag{74}$$

There are several ways to describe the generators of the group $\mathrm{Sp}(n)$. We will use here the following:

$$V_{-P} = \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \ , \ M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix} \quad , \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{75}$$

where $P = P^T$ and $\det L \neq 0$.

The subgroup $\mathrm{Sp}(n) \cap O(2n, \mathbb{R})$ of symplectic rotations is denoted by $U(n)$; this notation comes from the fact that $U(n)$ is identified with the unitary group $U(n, \mathbb{C})$ via the monomorphism $A + iB \hookrightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.

The symplectic group is a connected Lie group contractible to $U(n) \equiv U(n, cC)$ and therefore has covering groups $\mathrm{Sp}_q(n)$ of all orders; the double covering $\mathrm{Sp}_2(n)$ has a unitary representation in $L^2(\mathbb{R}^n)$, the metaplectic group $\mathrm{Mp}(n)$. The latter is generated by the operators $\widehat{V}_{-P}$, $\widehat{M}_{L,m}$, and $\widehat{J}$ defined by

$$\widehat{V}_{-P}\psi(x) = e^{\frac{i}{2\hbar}Px\cdot x}\psi(x) \quad , \widehat{M}_{L,m}\psi(x) = i^m\sqrt{|\det L|}\psi(Lx)$$

(the integer m corresponding to a choice of $\arg \det L$), and $\widehat{J}$ being essentially the Fourier transform:

$$\widehat{J}\psi(x) = \left(\frac{1}{2\pi i\hbar}\right)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}x\cdot x'}\psi(x')dx'.$$

Denoting by π^{Mp} the covering projection $\mathrm{Mp}(n) \longrightarrow \mathrm{Sp}(n)$ we have

$$\pi^{\mathrm{Mp}}(\widehat{V}_{-P}) = V_{-P} \ , \ \pi^{\mathrm{Mp}}(\widehat{M}_{L,m}) = M_L \quad , \ \pi^{\mathrm{Mp}}(\widehat{J}) = J.$$

We will also use the following factorization results: given $S \in \mathrm{Sp}(n)$ written in block form (71) we have the pre-Iwasawa factorization [3, 26]: There exist unique matrices $P = P^T$ and $L = L^T > 0$ and $R \in \mathrm{Sp}(n) \cap O(2n)$ such that

$$S = V_P M_L R. \tag{76}$$

These matrices are given by

$$P = -(CA^T + DB^T)(AA^T + BB^T)^{-1} \tag{77}$$

$$L = (AA^T + BB^T)^{-1/2}; \tag{78}$$

writing $R = \begin{pmatrix} E & F \\ -F & E \end{pmatrix}$ the $n \times n$ blocks E and F are given by

$$E = (AA^T + BB^T)^{-1/2}A \quad , \quad F = (AA^T + BB^T)^{-1/2}B. \tag{79}$$

The matrix R is a symplectic rotation: $R \in \mathrm{Sp}(n) \cap O(2n, \mathbb{R})$.

APPENDIX B. Lagrangian Subspaces

By definition a *Lagrangian coordinate subspace* is a n-dimensional subspace $\ell_{(\alpha,\beta)}$ of the $(\mathbb{R}^{2n}, \omega)$ given by the relations $x_{(\alpha)} = 0$ and $p_{(\beta)}$ where α and β form a partition of the set of integers $\{1, ..., n\}$. Thus, for instance, $x_1 = 0$ and $p_1 = 0$ defines coordinate Lagrangian subspaces in $\mathbb{R}^4$. The choices $\alpha = \emptyset$ and $\beta = \emptyset$ correspond to the canonical coordinate planes ℓ_X and ℓ_P, respectively. Let $\ell_{(\alpha,\beta)}$ be a Lagrangian coordinate subspace; we assume for notational simplicity that $\alpha = \{1, ..., k\}$, $\beta = \{k + 1, ..., n\}$ $(1 \leq k < n)$. It is represented by the equation

$$Ax + Bp = 0 \tag{80}$$

where A and B are the diagonal matrices $A = I_{k\times k} \otimes 0_{(n-k)\times(n-k)}$ and $B = 0_{k\times k} \otimes I_{(n-k)\times(n-k)}$. A remarkable feature of coordinate Lagrangian subspaces is that the symplectic form ω vanishes identically on them if $z, z' \in \ell_{(\alpha,\beta)}$ then $\omega(z, z) = 0$. This motivates the following definition [8, 10]: a n-dimensional subspace ℓ of $\mathbb{R}^{2n}$ is called a Lagrangian subspace (or plane) if $\omega(z, z) = 0$ for all $z, z' \in \ell$. (Lagrangian subspaces intervene in many areas of mathematical physics; for instance they are the tangent spaces to the invariant tori of classical mechanics [1, 8]). In the case $n = 1$ Lagrangian planes are just the straight lines through the origin in the phase plane. In the general case they are represented by equations $Ax + Bp = 0$ where $\mathrm{rank}(A, B) = n$ and $A^T B = B^T A$ [8]. It turns out that every Lagrangian subspace can be obtained from any Lagrangian coordinate subspace using a symplectic transformation. This follows from the fact the symplectic group $\mathrm{Sp}(n)$ acts transitively on the set $\mathrm{Lag}(n)$ of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega)$ (see [10] for a proof using symplectic bases). In particular:

The action $U(n) \times \mathrm{Lag}(n) \longrightarrow \mathrm{Lag}(n)$ *is transitive*

where $U(n) \subset \mathrm{Sp}(n)$ is the group of symplectic rotations (see Appendix A).

Let $\ell_{(\alpha,\beta)} \in \mathrm{Lag}(n)$ be a Lagrangian coordinate subspace. It follows that There exists non-unique) $S_{(\alpha,\beta)}, S'_{(\alpha,\beta)} \in \mathrm{Sp}(n)$ such that

$$\ell_{(\alpha,\beta)} = S_{(\alpha,\beta)}\ell_X = S'_{(\alpha,\beta)}\ell_P$$

(notice that we can take $S'_{(\alpha,\beta)} = S_{(\alpha,\beta)}J$).

APPENDIX C. $\hbar$-Polar Duality

Let $X \subset \mathbb{R}^n_x$ be a convex body: X is compact and convex, and has non-empty interior $\operatorname{int}(X)$. If $0 \in \operatorname{int}(X)$ we define the $\hbar$-polar dual $X^\hbar \subset \mathbb{R}^n_p$ of X by

$$X^\hbar = \{p \in \mathbb{R}^m : \sup_{x\in X}(p\cdot x) \leq \hbar\} \tag{81}$$

where $\hbar$ is a positive constant (we have $X^\hbar = \hbar X^o$ where X^o is the traditional polar dual from convex geometry). The following properties of polar duality are obvious:

- $(X^\hbar)^\hbar = X$ (reflexivity) and $X \subset Y \Longrightarrow Y^\hbar \subset X^\hbar$ (anti-monotonicity),
- For all $L \in GL(n,\mathbb{R})$:
$$(LX)^\hbar = (L^T)^{-1}X^\hbar \tag{82}$$
(scaling property). In particular $(\lambda X)^\hbar = \lambda^{-1}X^\hbar$ for all $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

We can view X and $X^\hbar$ as subsets of phase space by the identifications $\mathbb{R}^n_x \equiv \mathbb{R}^n_x \times 0$ and $\mathbb{R}^n_p \equiv 0 \times \mathbb{R}^n_p$. Writing $\ell_X = \mathbb{R}^n_x \times 0$ and $\ell_P = 0 \times \mathbb{R}^n_p$ the transformation $X \longrightarrow X^\hbar$ is a mapping $\ell_X \longrightarrow \ell_P$. With this interpretation formula (82) can be rewritten in symplectic form as

$$(M_{L^{-1}}X)^\hbar = M_{L^T}X^\hbar \tag{83}$$

where $M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}$ is in $\operatorname{Sp}(n)$. Notice that $M_L : \ell_X \longrightarrow \ell_X$ and $M_L : \ell_P \longrightarrow \ell_P$.

Suppose now that X is an ellipsoid centered at the origin:

$$X = \{x \in \mathbb{R}^n_x : Ax\cdot x \leq \hbar\} \tag{84}$$

where $A \in \operatorname{Sym}_{++}(n,\mathbb{R})$. The polar dual $X^\hbar$ of X is the ellipsoid

$$X^\hbar = \{p \in \mathbb{R}^n_p : A^{-1}p\cdot p \leq \hbar\}. \tag{85}$$

In particular the polar dual of the ball $B^n_X(\sqrt{\hbar}) = \{x : |x| \leq \sqrt{\hbar}\}$ is

$$(B^n_X(\sqrt{\hbar}))^\hbar = B^n_P(\sqrt{\hbar}).$$

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References

[1] V. I. Arnold. *Mathematical Methods of Classical Mechanics.* Graduate Texts in Mathematics, second edition, Springer-Verlag, 1989

[2] K. M. Ball. Ellipsoids of maximal volume in convex bodies. *Geom. Dedicata.* 41(2), 241–250 (1992)

[3] B. Dutta, N. Mukunda, and R. Simon. The real symplectic groups in quantum mechanics and optics. *Pramana J. of Phys.* 45(6), 471–497 (1995)

[4] J. V. Corbett. The Pauli problem, state reconstruction and quantum-real numbers. *Rep. Math. Phys.* 57(1), 53–68 (2006)

[5] D. Bogar, D. González, D. Gutiérrez-Ruiz, and J. D. Vergara. Classical analogs of the covariance matrix, purity, linear entropy, and von Neumann entropy. *Phys. Rev. A* 105, 062412, (2022)

[6] A. Carmi and E. Cohen. On the Significance of the Quantum Mechanical Covariance Matrix. *Entropy* 20, 500 (2018)

[7] O. Gittsovich and O. Gühne. Quantifying entanglement with covariance matrices. *Phys. Rev.* A 81, 032333 (2010)

[8] M. de Gosson. *The Principles of Newtonian and Quantum Mechanics: the Need for Planck's Constant h; with a foreword by B. Hiley.* Imperial College Press, 2001; Second edition (revised and augmented), 2016

[9] M. de Gosson, The "symplectic camel principle" and semiclassical mechanics, *J. Phys. A: Math. Gen* 35(3) (2002)

[10] M. de Gosson. *Symplectic Geometry and Quantum Mechanics.* Birkhäuser, 2006

[11] M. de Gosson. The Symplectic Camel and the Uncertainty Principle: The Tip of an Iceberg? *Found. Phys.* 39(2), 194–214 (2009)

[12] M. de Gosson. Quantum blobs. *Found. Phys.* 43(4), 440–457 (2013)

[13] M. de Gosson. Quantum Polar Duality and the Symplectic Camel: a New Geometric Approach to Quantization. *Found. Phys.* 51, Article number: 60 (2021)

[14] M. de Gosson. *The Wigner Transform*, World Scientific, series Advanced Texts in Mathematics, 2017

[15] M. de Gosson. The Pauli Problem for Gaussian Quantum States: Geometric Interpretation. *Mathematics* 9(20), 2578 (2021)

[16] M. de Gosson. Symplectic Radon Transform and the Metaplectic Representation. *Entropy*, 24(6), 761 (2022)

[17] M. de Gosson. Polar Duality Between Pairs of Transversal Lagrangian Planes; Applications to Uncertainty Principles. *Bull. sci. math* 179, 103171 (2022)

[18] M. de Gosson. Geometric Quantum States and Lagrangian Polar Duality: Quantum Mechanics Without Wavefunctions. In: Nielsen, F., Barbaresco, F. (eds.) Geometric Science of Information. GSI Lecture Notes in Computer Science, vol 14072. Springer, Cham 2023

[19] M. de Gosson. *Quantum Harmonic Analysis*, An Introduction. De Gruyter 2022

[20] M. de Gosson and C. de Gosson. Symplectic Polar Duality, Quantum Blobs, and Generalized Gaussians. *Symmetry* 14(9), 1890 (2022)

[21] M. de Gosson and C. de Gosson. Pointillisme à la Signac and Construction of a Quantum Fiber Bundle Over Convex Bodies. *Found. Physics* 3, 43 (2023)

[22] M. de Gosson and F. Luef. Symplectic Capacities and the Geometry of Uncertainty: the Irruption of Symplectic Topology in Classical and Quantum Mechanics. *Phys. Reps.* 484, 131–179 (2009)

[23] J. Hilgevoord. The standard deviation is not an adequate measure of quantum uncertainty. *Am. J. Phys.* 70(10), 983 (2002)

[24] R. G. Littlejohn. The semiclassical evolution of wave packets, *Phys. Reps*. 138(4–5) 193–291 (1986)

[25] Y. S. Ra, A. Dufour, M. Walschaers, C. Jacquard, T. Michel, C. Fabre, and N. Treps. Non-Gaussian quantum states of a multimode light field. *Nat. Phys.* 16, 144–147 (2020)

[26] M. Benzi and N. Razouk. On the Iwasawa decomposition of a symplectic matrix. *Applied Mathematics Letters* 20, 260–265 (2007)

[27] P. Grohs and L. Liehr. Non-uniqueness theory in sampled STFT phase retrieval. *SIAM J. on Math. Anal.* 55(5), 4695–4726 (2023)

[28] F. J. Narcowich. Geometry and uncertainty. *J. J. Math. Phys.* 31, 354–364 (1990)

[29] W. Pauli. *General principles of quantum mechanics*, Springer Science & Business Media, 2012 [original title: *Prinzipien der Quantentheorie*, publ. in : Handbuch der Physik, v.5.1, 1958]

[30] J. Řeháček, S. Olivares, D. Mogilevtsev, Z. Hradil, M. G. A. Paris, S. Fornaro, V. D'Auria, A. Porzio, and S. Solimen. Effective method to estimate multidimensional Gaussian states. *Phys. Rev.* A 79, 032111 (2009)

[31] Tzon-Tzer Lu and Sheng-Hua Shiou. Inverses of 2×2 Block Matrices, *Comput. Math. Appl.* 43, 119–129 (2002)

[32] M. Walschaers. Non-Gaussian Quantum States and Where to Find Them. *PRX Quantum* 2, 030204 (2021)

[33] M. M. Wolf, G. Giedke, O. Krüger, R. F Werner, and J. I. Cirac. Gaussian entanglement of formation. *Phys. Rev. A*, 69(5), 052320 (2004).

[34] F. Zhang. *The Schur Complement and its Applications*, Springer, Berlin, 2005