PAPER

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Spherical type integrable classical systems in a magnetic field

A Marchesiello¹, L Šnobl² and P Winternitz³

1 Department of Applied Mathematics, Czech Technical University in Prague, Faculty of Information Technology, Thákurova 9, 160 00 Prague 6, Czechia
2 Department of Physics, Czech Technical University in Prague, Faculty of Nuclear Sciences and Physical Engineering, Břehová 7, 115 19 Prague 1, Czechia
3 Centre de recherches mathématiques and Département de mathématiques et de statistique, Université de Montréal, CP 6128, Succ Centre-Ville, Montréal (Québec) H3C 3J7, Canada

E-mail: marchant@fit.cvut.cz, Libor.Snobl@fjfi.cvut.cz
and wintern@crm.umontreal.ca

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Abstract
We show that four classes of second order spherical type integrable classical systems in a magnetic field exist in the Euclidean space $E_3$, and construct the Hamiltonian and two second order integrals of motion in involution for each of them. For one of the classes the Hamiltonian depends on four arbitrary functions of one variable. This class contains the magnetic monopole as a special case. Two further classes have Hamiltonians depending on one arbitrary function of one variable and four or six constants, respectively. The magnetic field in these cases is radial. The remaining system corresponds to a constant magnetic field and the Hamiltonian depends on two constants. Questions of superintegrability—i.e. the existence of further integrals—are discussed.

Keywords: integrability, superintegrability, classical mechanics, magnetic field

1. Introduction

A finite dimensional classical or quantum system is superintegrable if it is Liouville integrable and allows more integrals of motion than degrees of freedom. The best-known superintegrable systems are the Kepler–Coulomb system with the potential $V = Q/r$, and the harmonic oscillator $V(r) = \frac{1}{2}\omega^2 r^2$. These are also the only spherically symmetric superintegrable systems (in any dimension $N$), and the only ones that were known before about 1940 [9, 10, 20]. For a recent review of the field with many references see [19]. Let us just recall that in classical
mechanics Liouville integrability means that there exist \( N \) integrals of motion \( X_j \) (including the Hamiltonian) that Poisson commute pairwise, are well defined functions on phase space, and are functionally independent. The system is superintegrable if it allows \( k \) further independent integrals \( Y_a \) that Poisson commute with the Hamiltonian, but not necessarily with each other, nor with the integrals \( X_j \). The integer \( k \) satisfies \( 1 \leq k \leq N - 1 \), where \( k = 1 \) and \( k = N - 1 \) correspond to minimal and maximal superintegrability, respectively. In quantum mechanics, the definitions are similar, but of course the Hamiltonian and other integrals are operators, usually assumed to be Hermitian. Poisson brackets are replaced by commutators. The operators are usually assumed to be polynomials or convergent series in the elements of the Heisenberg algebra, i.e. they are elements of the enveloping algebra of the Heisenberg algebra, or its generalisation.

Maximally superintegrable systems are of special interest in classical physics, because all finite trajectories in these systems are closed in configuration space and the motion is periodic. Quite recently, an application of superintegrability to absolute optical instruments has been proposed [30]. In quantum mechanics, the energy levels are degenerate, and it has been conjectured that maximally superintegrable systems are exactly solvable [29].

Let us mention that the concepts of integrability and superintegrability are not restricted to scalar particles. For the case of spin 1/2 or arbitrary spin, see e.g. [4, 5], [21–23, 25, 26, 31].

This article is part of a research program the purpose of which is to study integrable and superintegrable systems in three-dimensional Euclidean space \( \mathbb{E}_3 \) in the presence of magnetic fields. We refer to previous articles [15, 16] for a review of earlier work and for further motivation. It was shown in [14] that without magnetic field, second order integrability in \( \mathbb{E}_3 \)—i.e. the existence of two commuting integrals that are second order polynomials in the momenta—is related to the separation of variables in the Hamilton–Jacobi or Schrödinger equation. Thus, 11 classes of pairs of second order operators exist, each leading to separation in one of the 11 separable coordinate systems in \( \mathbb{E}_3 \) [7, 8]. It has also been shown that the connection between second order superintegrability and separability no longer holds in \( \mathbb{E}_2 \) [2, 3, 6, 18, 27, 28], nor in \( \mathbb{E}_4 \) [1, 15, 16]. However, the leading (second order) part of the integrals of motion still lies in the enveloping algebra of the Euclidean Lie algebra. The number and structure of the relevant pairs of leading order terms is not yet known—one hand, the structure of some of the 11 classes present without magnetic field may not allow any generalization to nonvanishing magnetic fields; on the other hand, we found in [15] that it is not always possible to bring the leading order terms to the simple structure of [14].

[15] was devoted to integrability and superintegrability where the second order integrals responsible for integrability were of the Cartesian type—i.e. they started out as \( X_1 = p_1^2 + \ldots, X_2 = p_2^2 + \ldots \). The purpose of this article is to study the spherical case, starting out as \( X_1 = l_1^2 + \ldots, X_2 = l_2^2 + l_3^2 + \ldots \). In this article, we restrict attention to the case of classical mechanics, and assume that the vector potential corresponds to the case of minimal electromagnetic coupling.

The structure of the paper is as follows. In section 2, we describe our problem and the equations that govern it in Cartesian coordinates. In section 3, we express the determining equations for the integrals of motion in spherical coordinates. In section 4, we introduce the structure of spherical-type integrals, and solve the determining equations for integrals of this form. Some details of the calculation are relegated to the appendix. In the last section, we conclude with a summary of the results and outline of future research.
2. Formulation of the problem

Let us consider a particle moving under the influence of a static electromagnetic field. In Cartesian coordinates, it is described by the Hamiltonian

\[ H = \frac{1}{2} (p + \vec{A}(\vec{x}))^2 + W(\vec{x}), \]

(1)

where \( \vec{p} = (p_1, p_2, p_3) \equiv (p_x, p_y, p_z) \) are the linear momenta, \( \vec{x} = (x_1, x_2, x_3) \) Cartesian spatial coordinates; the vector potential \( \vec{A}(\vec{x}) = (A_1(\vec{x}), A_2(\vec{x}), A_3(\vec{x})) \equiv \vec{A}(\vec{x}) \) and the electrostatic potential \( W(\vec{x}) \) are functions only of the coordinates \( \vec{x} \). The mass and electric charge of the particle are set to numerical values \( 1 \) and \( -1 \) respectively, by a suitable choice of the physical units.

The Newtonian equations of motion of the Hamiltonian (1) are gauge invariant—i.e. they are the same for the potentials

\[ \vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \nabla \chi, \quad W'(\vec{x}) = W(\vec{x}) \]

(2)

for any choice of the function \( \chi(\vec{x}) \). It is worth mentioning that the Hamiltonian equations of motion depend on the choice of gauge, since the relation between the velocity and the momentum is gauge dependent: \( \vec{v} = \vec{p} + \vec{A}(\vec{x}) \). Thus, the physically relevant quantities are the electrostatic potential \( W(\vec{x}) \) and the magnetic field

\[ \vec{B}(\vec{x}) = \nabla \times \vec{A}(\vec{x}), \quad \text{i.e.} \quad B_j(\vec{x}) = \epsilon_{jkl} \partial_k A_l(\vec{x}), \]

(3)

where \( \epsilon_{jkl} \) is the completely antisymmetric tensor with \( \epsilon_{123} = 1 \).

Let us recall the structure of the integrals of motion of at most second order in momenta—see e.g [16]. Any such integral of motion has the form

\[ X = \sum_{j=1}^{3} h'(\vec{x}) p_j^A p_j^A + \sum_{j,k,l=1}^{3} \frac{1}{2} \epsilon_{jkl} n'(\vec{x}) p_k^A p_l^A + \sum_{j=1}^{3} s'(\vec{x}) p_j^A + m(\vec{x}), \]

(4)

where we have denoted

\[ p_j^A = p_j + A_j(\vec{x}), \]

(5)

and \( h'(\vec{x}), n'(\vec{x}), s'(\vec{x}), j = 1, 2, 3 \), \( m(\vec{x}) \) are real functions on the configuration space \( \mathbb{R}^3 \). They must satisfy determining equations which arise from the vanishing of the Poisson bracket between the integral and the Hamiltonian,

\[ \{H, X\}_{PB} = 0, \]

(6)

as coefficients of various powers of the linear momenta \( p_1, p_2, p_3 \), and read, order by order:

- third order in momenta

\[ \begin{align*}
\partial_i h^1 &= 0, \quad \partial_i h^1 = -\partial_i n^2, \quad \partial_i h^1 = -\partial_i n^2, \\
\partial_j h^2 &= -\partial_j n^3, \quad \partial_j h^2 = 0, \quad \partial_j h^2 = -\partial_j n^1, \\
\partial_i h^3 &= -\partial_i n^2, \quad \partial_i h^3 = -\partial_i n^1, \quad \partial_i h^3 = 0, \\
\nabla \cdot \vec{n} &= 0;
\end{align*} \]

(7)
second order in momenta
\[ \partial_\mu s^1 = n^2 B_2 - n^3 B_3, \]
\[ \partial_\mu s^2 = n^3 B_3 - n^1 B_1, \]
\[ \partial_\mu s^3 = n^1 B_1 - n^2 B_2, \]
\[ \partial_\mu s^1 + \partial_\nu s^2 = n^1 B_2 - n^2 B_1 + 2(h^1 - h^2) B_3, \]
\[ \partial_\mu s^3 + \partial_\nu s^3 = n^1 B_1 - n^3 B_2 + 2(h^3 - h^1) B_3; \]
\[ \partial_\mu s^3 + \partial_\nu s^2 = n^2 B_3 - n^3 B_2 + 2(h^2 - h^3) B_1, \]
(the conditions (8) imply \( \nabla \cdot \vec{s} = 0 \)),

first order in momenta
\[ \partial_\mu m = 2h^1 \partial_\mu W + n^1 \partial_\nu W + n^2 \partial_\zeta W + s^1 B_2 - s^2 B_3, \]
\[ \partial_\mu m = n^3 \partial_\mu W + 2h^2 \partial_\mu W + n^1 \partial_\nu W + s^1 B_1 - s^3 B_2, \]
\[ \partial_\mu m = n^2 \partial_\mu W + n^1 \partial_\nu W + 2h^3 \partial_\mu W + s^2 B_1 - s^1 B_2; \]

zero order in momenta
\[ \vec{s} \cdot \nabla W = 0. \]

As in the case without magnetic field, the highest order conditions (7) imply that the functions \( h^i(\vec{x}) \) and \( n^i(\vec{x}) \) are the following polynomials depending on 20 real constants \( \alpha_{abc} \) (see [16]):

\[
\begin{align*}
    h^1(\vec{x}) &= y^2 \alpha_{66} - y \alpha_{66} + z^2 \alpha_{35} - y \alpha_{16} + z \alpha_{95} + \alpha_{11}, \\
    h^2(\vec{x}) &= x^2 \alpha_{66} - x \alpha_{66} + z^2 \alpha_{44} + x \alpha_{26} - z \alpha_{42} + \alpha_{22}, \\
    h^3(\vec{x}) &= x^2 \alpha_{35} - x \alpha_{35} - y^2 \alpha_{44} - x \alpha_{35} + y \alpha_{34} + \alpha_{33}, \\
    n^1(\vec{x}) &= -x^2 \alpha_{36} + x \alpha_{46} + x \alpha_{45} - 2yz \alpha_{44} - x \alpha_{25} + x \alpha_{36} + y \alpha_{24} - z \alpha_{34} + \alpha_{23}, \\
    n^2(\vec{x}) &= x \alpha_{36} - 2x \alpha_{55} - y^2 \alpha_{46} + y \alpha_{45} - x \alpha_{15} + y \alpha_{14} - y \alpha_{36} + z \alpha_{35} + \alpha_{13}, \\
    n^3(\vec{x}) &= -2x \alpha_{56} + x \alpha_{46} + y \alpha_{46} - z^2 \alpha_{45} + x \alpha_{16} - y \alpha_{26} - z \alpha_{14} + z \alpha_{25} + \alpha_{12}.
\end{align*}
\]

In other words, the leading order terms in the integral (4) must be elements of the universal enveloping algebra of the Euclidean group in three spatial dimensions, i.e.

\[ X = \sum_{1 \leq a \leq b \leq 6} \alpha_{abc} Y^A_a p^A_b + \sum_{j=1}^3 s^j(\vec{x}) p^A_j + m(\vec{x}), \]

where

\[ Y^A = (p^A_1, p^A_2, p^A_3, \ell^A_1, \ell^A_2, \ell^A_3), \quad \ell^A_j = \sum_{1 \leq k \leq 3} \epsilon_{ijk} p^A_i. \]

We note that although the explicit expression for the integral (12) involves the vector potential and thus depends on the choice of gauge, the conditions (8)–(10), as well as their solutions, i.e. the functions \( s^j(\vec{x}) \) and \( m(\vec{x}) \), are gauge invariant.

We also observe that, contrary to the case without magnetic field, the determining equations governing odd and even order terms in the integral (4) do not separate. This is a consequence of the presence of the vector potential, which implies terms of first order in momenta in the Hamiltonian (1).
The classical Hamiltonian system (1) is integrable (in the Liouville sense) if in addition to the Hamiltonian it possesses two functionally independent well-defined integrals of motion \( X_1, X_2 \) in involution. In this paper, we shall consider systems possessing two commuting integrals of the following leading order form in the momenta:

\[
X_1 = \left( l^A_i \right)^2 + \text{lower order terms}, \\
X_2 = \left( L^A \right)^2 + \text{lower order terms},
\]

where \( (L^A)^2 = \sum_{i=1}^3 (l^A_i)^2 \). In the absence of magnetic field, such integrals imply separability of the Hamilton–Jacobi equation (and correspondingly, also of quantum mechanical Schrödinger equation) in spherical coordinates [7, 8, 14]; hence, we speak of spherical-type integrals.

In contrast to the case of Cartesian-type integrals [15, 32], no classification of integrable systems with spherical-type integrals and nonvanishing magnetic field is known in the literature. Thus, we investigate the classification problem in this paper, as a necessary prerequisite for the construction of superintegrable systems with integrals of the spherical type.

3. The determining equations in spherical coordinates

We start by rewriting our system, the integrals of motion (4) and their determining equations (8)–(10) in spherical coordinates. We assume the relation between Cartesian and spherical coordinates in the following form:

\[
x = R \sin(\theta) \cos(\phi), \quad y = R \sin(\theta) \sin(\phi), \quad z = R \cos(\theta).
\]

The transformation of the momenta follows from the coordinate-invariant structure of the canonical 1-form on the phase space:

\[
\lambda = p_x dx + p_y dy + p_z dz = p_R dR + p_\theta d\theta + p_\phi d\phi.
\]

It implies that the momenta transform as the components of differential 1-forms, i.e.

\[
p_x = \sin(\theta) \cos(\phi) p_R + \frac{\cos(\theta) \cos(\phi)}{R} p_\theta - \frac{\sin(\theta)}{R \sin(\theta)} p_\phi, \\
p_y = \sin(\theta) \sin(\phi) p_R + \frac{\cos(\theta) \sin(\phi)}{R} p_\theta + \frac{\cos(\phi)}{R \sin(\theta)} p_\phi, \\
p_z = \cos(\theta) p_R - \frac{\sin(\theta)}{R} p_\phi.
\]

In order to write the Hamiltonian, we also have to introduce the vector potential and the magnetic field. The components of the vector potential appear in the Hamiltonian in combination with the linear momenta; thus, we introduce the vector potential 1-form:
\begin{align}
A = A_1(\vec{x})d\!x + A_2(\vec{x})d\!y + A_3(\vec{x})d\!z &= A_R(R, \theta, \phi)d\!R + A_\theta(R, \theta, \phi)d\!\theta + A_\phi(R, \theta, \phi)d\!\phi. 
\end{align}

The transformation of the components of the vector potential then follows:
\begin{align}
A_1(\vec{x}) &= \sin(\theta) \cos(\phi)A_R(R, \theta, \phi) + \frac{\cos(\theta) \cos(\phi)}{R}A_\theta(R, \theta, \phi) - \frac{\sin(\phi)}{R \sin(\theta)}A_\phi(R, \theta, \phi), \\
A_2(\vec{x}) &= \sin(\theta) \sin(\phi)A_R(R, \theta, \phi) + \frac{\cos(\theta) \sin(\phi)}{R}A_\theta(R, \theta, \phi) + \frac{\cos(\phi)}{R \sin(\theta)}A_\phi(R, \theta, \phi), \\
A_3(\vec{x}) &= \cos(\theta)A_R(R, \theta, \phi) - \frac{\sin(\theta)}{R}A_\theta(R, \theta, \phi). 
\end{align}

The components of the magnetic field in any coordinate system are the components of the magnetic field 2-form \( \mathbf{B} = d\mathbf{A} \):
\begin{align}
\mathbf{B} &= B_x(\vec{x})d\!y \wedge d\!z + B_y(\vec{x})d\!z \wedge d\!x + B_z(\vec{x})d\!x \wedge d\!y \\
&= B_R(R, \theta, \phi)d\!\theta \wedge d\!\phi + B_\theta(R, \theta, \phi)d\!\phi \wedge d\!R + B_\phi(R, \theta, \phi)d\!R \wedge d\!\theta. 
\end{align}

(Let us recall that the relation \( \mathbf{B} = d\mathbf{A} \) implies
\begin{align}
B_R(R, \theta, \phi) &= \partial_\theta A_\phi(R, \theta, \phi) - \partial_\phi A_\theta(R, \theta, \phi),
\end{align}
and similarly for the remaining components. Consequently, the relation between the components of the magnetic field in Cartesian and spherical coordinates is
\begin{align}
B_x(\vec{x}) &= \frac{\cos(\phi)}{R^2}B_R(R, \theta, \phi) + \frac{\cos(\theta) \cos(\phi)}{R \sin(\theta)}B_\theta(R, \theta, \phi) - \frac{\sin(\phi)}{R}B_\phi(R, \theta, \phi), \\
B_y(\vec{x}) &= \frac{\sin(\phi)}{R^2}B_R(R, \theta, \phi) + \frac{\cos(\theta) \sin(\phi)}{R \sin(\theta)}B_\theta(R, \theta, \phi) + \frac{\cos(\phi)}{R}B_\phi(R, \theta, \phi), \\
B_z(\vec{x}) &= \frac{\cos(\theta)}{R^2 \sin(\theta)}B_R(R, \theta, \phi) - \frac{1}{R}B_\phi(R, \theta, \phi). 
\end{align}

The Hamiltonian (1) expressed in spherical coordinates reads
\begin{align}
H = \frac{1}{2} \left( \left( p_R + A_R(R, \theta, \phi) \right)^2 + \left( \frac{p_\theta + A_\theta(R, \theta, \phi)}{R} \right)^2 + \left( \frac{p_\phi + A_\phi(R, \theta, \phi)}{R \sin(\theta)} \right)^2 \right) + W(R, \theta, \phi). 
\end{align}

The equations of motion, as well as the conditions on the integrals, are derived using the Poisson brackets,
\begin{align}
\{ f, g \}_{PB} = \sum_{j=1}^{3} \left( \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial x_j} \frac{\partial f}{\partial p_j} \right) = \sum_{\alpha \in \{ R, \theta, \phi \}} \left( \frac{\partial f}{\partial q_\alpha} \frac{\partial g}{\partial p_\alpha} - \frac{\partial g}{\partial q_\alpha} \frac{\partial f}{\partial p_\alpha} \right). 
\end{align}

The Newtonian equations of motion, which in Cartesian coordinates read
\begin{align}
\ddot{x} &= -\dot{x} \times \ddot{B} - \nabla W,
\end{align}
become, in spherical coordinates,
\begin{align}
\dot{R} &= R \sin(\theta)^2 \dot{\phi}^2 + R \dot{\theta}^2 + \phi \ddot{B}_\theta - \partial_\theta B_\phi - \partial_\phi W, \\
\dot{\theta} &= \sin(\theta) \cos(\theta) \dot{\phi}^2 - 2R \frac{\dot{R}}{R} \dot{\theta} - \dot{\phi} \frac{\dot{R}}{R} B_\phi + \frac{R}{R^2} B_\theta - \frac{1}{R} \partial_\theta W, \\
\dot{\phi} &= -2 \frac{\dot{R}}{R} \frac{\cos(\theta) \dot{\phi}}{\sin(\theta)} - \frac{\dot{\theta}}{R^2 \sin^2(\theta)} B_\phi + \frac{R}{R^2 \sin^2(\theta)} B_\theta - \frac{1}{R^2 \sin^2(\theta)} \partial_\phi W. 
\end{align}
In order to express the conditions for the integrals of motion (4) in spherical coordinates (15), we again introduce covariantized momenta,

\[ p_R^A = p_R + A_R(R, \theta, \phi), \quad p_\theta^A = p_\theta + A_\theta(R, \theta, \phi), \quad p_\phi^A = p_\phi + A_\phi(R, \theta, \phi), \]

and express the integral (4) in spherical coordinates as

\[ X = h^R(R, \theta, \phi)(p_R^A)^2 + h^\theta(R, \theta, \phi)(p_\theta^A)^2 + h^\phi(R, \theta, \phi)(p_\phi^A)^2 
+ n^R(R, \theta, \phi)p_R^A p_R^A + n^\theta(R, \theta, \phi)p_\theta^A p_\theta^A + n^\phi(R, \theta, \phi)p_\phi^A p_\phi^A 
+ s^R(R, \theta, \phi)p_R^A + s^\theta(R, \theta, \phi)p_\theta^A + s^\phi(R, \theta, \phi)p_\phi^A + m(R, \theta, \phi). \]

The integral (27) is assumed to be well-defined everywhere except possible isolated singular points or lines. Thus we require all functions \( h^R, h^\theta, h^\phi, n^R, n^\theta, n^\phi, s^R, s^\theta, s^\phi \) and \( m \) to be periodic in \( \phi \) with period \( 2\pi \).

The functions \( h^R, h^\theta, h^\phi, n^R, n^\theta, n^\phi, s^R, s^\theta, s^\phi \) can be obtained from their Cartesian counterparts \( h^j, n^j, \)
\( j = 1, 2, 3 \). They are expressed in terms of the same 20 constants \( \alpha_{ab} \) as follows:

\[ h^R(R, \theta, \phi) = \alpha_{33} + \alpha_{12} \sin^2(\theta) \cos(\phi) \sin(\phi) + \alpha_{13} \sin(\theta) \cos(\theta) \cos(\phi) 
+ (\alpha_{22} - \alpha_{11}) \sin^2(\theta) \sin(\phi) + (\alpha_{11} - \alpha_{33}) \sin^2(\theta) + \alpha_{23} \cos(\theta) \sin(\theta) \sin(\phi), \]

\[ h^\theta(R, \theta, \phi) = \alpha_{55} + \frac{\alpha_{11}}{R^2} + (\alpha_{11} - \alpha_{22}) \frac{\sin^2(\theta) \sin^2(\phi)}{R^2} - \left( \alpha_{15} + \alpha_{24} \right) \frac{\cos(\theta) \sin^2(\phi)}{R} 
+ \frac{\alpha_{12} \cos(\phi) \sin(\phi)}{R^2} - \frac{\sin(\theta) \cos(\phi)}{R} + \frac{\sin(\theta) \sin(\phi)}{R} + \left( \alpha_{44} - \alpha_{55} \right) \sin^2(\phi) 
+ \frac{\alpha_{15} \cos(\theta) + (\alpha_{33} - \alpha_{11}) \sin^2(\theta) + (\alpha_{22} - \alpha_{11}) \frac{\sin^2(\phi)}{R^2}}{R^2} 
+ \left( \alpha_{25} - \alpha_{14} \right) \frac{\cos(\theta) \cos(\phi) \sin(\phi)}{R} - \frac{\sin(\theta) \cos(\theta) \cos(\phi)}{R^2} 
- \frac{\alpha_{21} \cos(\theta) \sin(\theta) \sin(\phi)}{R^2} - \frac{\sin^2(\theta) \cos(\phi) \sin(\phi)}{R^2} - \alpha_{45} \sin(\phi) \cos(\phi), \]

\[ h^\phi(R, \theta, \phi) = \alpha_{66} + \frac{\alpha_{44}}{\sin^2(\theta)} - \alpha_{55} \sin^2(\phi) - \alpha_{44} \cos^2(\phi) + (\alpha_{11} - \alpha_{22}) \frac{\sin^2(\phi)}{(R \sin(\theta))^2} 
+ \frac{\alpha_{45} \cos(\phi) \sin(\phi)}{\sin^3(\theta)} - \frac{\alpha_{24} \cos(\theta) \cos(\phi)}{R \sin^2(\theta)} - \frac{\alpha_{46} \cos(\theta) \cos(\phi)}{\sin(\theta)} - \frac{\alpha_{16} \sin(\phi)}{R \sin(\theta)} 
- \frac{\alpha_{56} \cos(\theta) \sin(\phi)}{\sin(\theta)} + \frac{\alpha_{26} \cos(\phi)}{R \sin(\theta)} + \frac{\alpha_{22}}{(R \sin(\theta))^2} + \frac{\alpha_{35} - \alpha_{44}}{\sin^2(\phi)} 
+ (\alpha_{14} - \alpha_{25}) \frac{\cos(\theta) \cos(\phi) \sin(\phi)}{R \sin^2(\theta)} + (\alpha_{15} + \alpha_{24}) \frac{\cos(\theta) \sin^2(\phi)}{R \sin^2(\theta)} 
- \frac{\alpha_{12} \sin(\phi) \cos(\phi)}{(R \sin(\theta))^2} - \alpha_{45} \sin(\phi) \cos(\phi), \]

\[ n^4(R, \theta, \phi) = \alpha_{56} \cos(\phi) - \alpha_{46} \sin(\phi) + 2(\alpha_{44} - \alpha_{55}) \frac{\cos(\theta) \sin(\phi) \cos(\phi)}{\sin(\theta)} \]
\[ - 2(\alpha_{24} + \alpha_{15}) \frac{\cos(\phi) \sin(\phi)}{R \sin(\theta)} + (\alpha_{15} + \alpha_{24}) \frac{\sin(\theta) \cos(\phi) \sin(\phi)}{R} \]
\[ - 2\alpha_{12} \frac{\cos(\theta) \sin^2(\phi)}{R^2 \sin(\theta)} + \alpha_{12} \frac{\cos(\theta)}{R^2 \sin(\theta)} + (\alpha_{25} - \alpha_{14}) \frac{\sin(\theta) \sin^2(\phi)}{R} \]
\[ + (\alpha_{16} + \alpha_{34}) \frac{\cos(\theta) \cos(\phi)}{R} + (\alpha_{26} + \alpha_{35}) \frac{\cos(\theta) \sin(\phi)}{R} + 2(\alpha_{14} - \alpha_{25}) \frac{\sin^2(\phi)}{R \sin(\theta)} \]
\[ + 2\alpha_{45} \frac{\cos(\theta) \sin^2(\phi)}{\sin(\theta)} - \alpha_{45} \frac{\cos(\theta)}{\sin(\theta)} + \frac{(\alpha_{25} - \alpha_{14})}{R \sin(\theta)} + \frac{\alpha_{13}}{R^2} + (\alpha_{14} - \alpha_{36}) \frac{\sin(\theta)}{R} \]
\[ - \alpha_{23} \frac{\cos(\phi)}{R^2} + 2(\alpha_{22} - \alpha_{11}) \frac{\cos(\theta) \sin(\phi) \cos(\phi)}{R^2 \sin(\theta)}. \]

The functions (28) satisfy the highest order determining equations of the form
\[ \partial \kappa = 0, \quad \partial h^\phi = -R^2 \partial \kappa \kappa^\phi, \quad \partial h^\theta = -R^2 \sin^2(\theta) \partial \kappa \kappa^\theta. \]
\[ \partial \kappa^\phi = \frac{1}{R^2} \partial h^\phi - \frac{2}{R} h^\phi, \quad \partial h^\phi = -\frac{1}{R} h^\phi, \quad \partial h^\theta = -\sin^2(\theta) \left( \partial \kappa n^\theta + \frac{n^\theta}{R} \right), \]
\[ \partial h^\phi = -\frac{\cos(\theta) \sin^2(\theta)}{\sin(\theta)^2} \left( \partial \kappa n^\phi + \frac{\cos(\theta)n^\phi}{R} + \frac{n^\phi}{R} \right), \quad \partial h^\theta = -\frac{1}{\sin^2(\theta)} \left( \partial \kappa n^\theta + \frac{n^\theta}{R} + 2 \cos(\theta) n^\theta \right). \]

The determining equations for the remaining functions \( s^R, s^\theta, s^\phi \) and \( m \) become


From now on, we shall restrict our attention to the integrals chosen in (14).

4. Spherical-type integrals

Let us first consider the integral $X_1$ of (14). In spherical coordinates, it takes the form

$$X_1 = (p_0^a)^2 + s_R^0(R, \theta, \phi) p_1^a + s_R^0(R, \theta, \phi) p_1^0 + s_R^0(R, \theta, \phi) p_1^0 + m_1(R, \theta, \phi),$$

(i.e. we have $\alpha_{00} = 1$, all the remaining $\alpha_{ij}$ vanish). For the integral of the form (33) we can completely solve (30), finding

$$s_R^0(R, \theta, \phi) = \sin(\theta) \partial_\theta \rho_1(\phi) + \cos(\theta) \rho_2(\phi),$$

$$s_R^0(R, \theta, \phi) = \partial_\phi \sigma(\phi) + \frac{\cos(\theta)}{R} \partial_\theta \rho_1(\phi) - \frac{\sin(\theta)}{R} \rho_2(\phi),$$

$$s_R^0(R, \theta, \phi) = -\frac{1}{R \sin(\theta)} (R \cos(\theta) \sigma(\phi) + \rho_1(\phi)) + \tau(R, \theta)$$

(34)


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\[96x404\]
where the functions $\rho_{1}(\phi), \rho_{2}(\phi), \sigma(\phi), \tau(\mathbf{R}, \theta)$ are arbitrary functions of the indicated variables. Equations (30) for the integral $X_1$ do not constrain the component $B_\phi$ of the magnetic field. The equations (31) and (32) expressed in terms of the functions $\rho_{1}(\phi), \rho_{2}(\phi), \sigma(\phi), \tau(\mathbf{R}, \theta), B_\phi(\mathbf{R}, \theta, \phi)$ are rather complicated, and we postpone their discussion till the second integral $X_2$ in involution is considered below.

The integral $X_2$ of (14) in spherical coordinates takes the form

$$X_2 = (\rho_0^1)^2 + \left(\frac{\rho_\phi^1}{\sin(\theta)}\right)^2 + s_R^2(B_\mathbf{R}, \theta, \phi) p_R + s_\phi^2(B_\mathbf{R}, \theta, \phi) p_\phi + s_\phi^2(B_\mathbf{R}, \theta, \phi) p_\phi + m_2(B_\mathbf{R}, \theta, \phi),$$

(i.e. we have $\alpha_{44} = \alpha_{55} = \alpha_{66} = 1$; all the remaining $\alpha_\phi$ vanish). We impose that both $X_1, X_2$ are integrals of motion, and also that their Poisson bracket $\{X_1, X_2\}_{PB}$ should vanish. Thus, the three Poisson brackets lead to three sets of conditions: two as in (30)–(32) for the two choices of $\alpha_\phi$ described above, the remaining one analogous but coming from $\{X_1, X_2\}_{PB} = 0$. Order by order, it reads:

- **second order in momenta**

$$B_R = \frac{1}{2} \left( \partial_\phi s_2^\phi - \partial_\theta s_1^\phi - \frac{1}{\sin^2(\theta)} \partial_\theta s_1^\phi \right),$$

$$\partial_\phi s_2^\phi = \frac{1}{\sin^2(\theta)} \partial_\theta s_1^\phi + \frac{\cos(\theta)}{\sin^2(\theta)} s_1^\phi, \quad \partial_\theta s_2^\phi = 0,$$

$$\partial_\phi s_1^\phi = 0;$$

- **first order in momenta**

$$s_1^\phi \partial_\phi s_2^\phi - s_1^\phi \partial_\theta s_1^\phi = 2 s_1^\phi \partial_\theta s_2^\phi + s_2^\phi \partial_\phi s_1^\phi - s_2^\phi \partial_\theta s_2^\phi - s_2^\phi \partial_\theta s_1^\phi = 0,$$

$$2 \partial_\theta m_2 = \frac{-2}{\sin^2(\theta)} \partial_\phi m_1 = \frac{-2}{\sin^2(\theta)} s_1^\phi B_\theta + 2 s_1^\phi B_\theta + s_1^\phi \partial_\phi s_2^\phi + s_2^\phi \partial_\phi s_1^\phi = 0,$$

$$2 \partial_\theta m_2 = \frac{2}{\sin^2(\theta)} s_1^\phi B_\theta - 2 s_1^\phi B_\theta - s_2^\phi \partial_\theta s_1^\phi = 0,$$

- **zero order in momenta**

$$\partial_\phi s_2^\phi = \frac{1}{\sin^2(\theta)} \partial_\theta s_1^\phi + \frac{\cos(\theta)}{\sin^2(\theta)} s_1^\phi, \quad \partial_\theta s_2^\phi = 0,$$

$$\partial_\phi s_1^\phi = 0;$$

$$\partial_\theta s_1^\phi = 0;$$

$$\partial_\theta s_2^\phi = 0.$$
\[(s_1^R s_2^φ - s_1^φ s_2^R)B_R + (s_1^R s_2^θ - s_1^θ s_2^R)B_θ + (s_1^R s_2^φ - s_1^φ s_2^R)B_φ \]
\[-s_2^R \partial R m_1 - s_2^θ \partial θ m_1 - s_2^φ \partial φ m_1 + s_1^R \partial R m_2 + s_1^θ \partial θ m_2 + s_1^φ \partial φ m_2 \]
\[= 0. \quad (39)\]

The highest order conditions (29) are satisfied for both \(X_1\) and \(X_2\), due to the chosen structure of the Hamiltonian (22) and the integrals (33), (36). Similarly to the case above where only the integral \(X_1\) was considered, all second order conditions (30) and (37) can be solved, leading to the following structure of the functions \(s_1^R, s_1^θ, s_1^φ, s_2^R, s_2^θ, s_2^φ\) and the magnetic field \(B_R, B_θ, B_φ\):

\[s_1^R(R, θ, φ) = 0, \quad s_1^θ(R, θ, φ) = \partial_φ σ(φ), \quad \sin^2(θ) = \frac{1}{2} \frac{\cos(θ)}{\sin(θ)} σ(φ), \]
\[s_1^φ(R, θ, φ) = \cos(θ) S, \quad s_2^θ(R, θ, φ) = -\sin(θ) \frac{S}{R}, \quad s_2^φ(R, θ, φ) = ω(R), \]
\[B_R(R, θ, φ) = -\sin(θ) \cos(θ)ω(R) - \frac{1}{2} \partial_θ τ(θ) - \frac{\partial_φ σ(φ) + σ(φ)}{2 \sin^2(θ)}, \]
\[B_θ(R, θ, φ) = \sin^2(θ) \frac{2}{2} \partial_θ ω(R), \quad B_φ(R, θ, φ) = 0. \quad (40)\]

In the solution (40) the functions \(σ(φ), τ(θ), ω(R)\) are arbitrary functions of the specified variables; \(S\) is an arbitrary constant. As one should expect, the solution (40) possesses significantly less freedom than the solution of only one set of conditions (30) presented in equations (34) and (35) (and the solution (40) is contained in (34) and (35) as a particular subcase).

Once we know the general solution (40) of the second order conditions, i.e. (30) and (37), we need to consider the lower order conditions, i.e. (31) and (32) for our two choices of \(α_0\) together with (38) and (39). We observe that the functions \(m_1(R, θ, φ)\) and \(m_2(R, θ, φ)\) show up in (31), (38) and (39) only through their first derivatives. Their compatibility conditions,
\[\partial_μ (\partial_ν m_1(R, θ, φ)) = \partial_ν (\partial_μ m_1(R, θ, φ)), \quad (41)\]
\[\partial_μ (\partial_ν m_2(R, θ, φ)) = \partial_ν (\partial_μ m_2(R, θ, φ)), \quad μ, ν = R, θ, φ, \quad (42)\]
after substituting the relations (31) for the derivatives of \(m_1(R, θ, φ)\) and \(m_2(R, θ, φ)\), give conditions independent of \(m_1\) and \(m_2\). We also substitute (31) into (38) and (39), obtaining equations independent of the functions \(m_1\) and \(m_2\) as well. Once these equations, together with the compatibility conditions (41) and (42), are solved, the (local) existence of the functions \(m_1(R, θ, φ)\) and \(m_2(R, θ, φ)\)—i.e. the solutions of (31)—follows. Thus, from this point on, the functions \(m_1(R, θ, φ)\) and \(m_2(R, θ, φ)\) do not appear in the rest of our paper, other than in the presentation of our results.

This is the stage where our calculation starts to split into various subcases. The major splitting is based on the equation
\[\sin(θ) S \partial_φ σ(φ) = 0. \quad (43)\]
It arises as one of the first order terms (38) in \(\{X_1, X_2\}_R\) after using equations (40) and (31). Explicitly, equations (38) now read
\[
\begin{align*}
\sin(\theta) S \partial_\phi \sigma &= 0, \\
\frac{\cos(\theta)}{R} S \partial_\phi \sigma + \omega \partial_\phi^2 \sigma &= 0, \\
\frac{\sin(\theta)}{R} \left( 2 \sin(\theta) \cos(\theta) \omega + \partial_\theta \tau \right) - \cos(\theta) \sin(\theta) S \partial_k \omega \\
+ \frac{S}{R \sin(\theta)} \sigma + \frac{\cos(\theta)}{\sin(\theta)} \omega \partial_\phi \sigma &= 0.
\end{align*}
\] (44)

Thus, we have to consider separately

- **Case A** \( S = 0 \) and \( \sigma(\phi) \) is assumed to be a nonconstant function.
- **Case B** \( \sigma(\phi) = s_0 = \text{const.} \)

### 4.1. Case A: \( S = 0 \)

In this case, we immediately find that the conditions (38), the compatibility conditions (42) and the equations (31) for the integral \( X_2 \) imply

\[
\omega(R) = 0, \quad W(R, \theta, \phi) = U(R) + \frac{V(\theta, \phi)}{R^2}, \quad m_2(R, \theta, \phi) = 2V(\theta, \phi),
\] (45)

thus, we have \( s_2^S = s_2^R = s_2^\phi = 0 \), and the condition (32) for \( X_2 \) is satisfied, trivially.

In order to solve the compatibility conditions (41) in terms of the function \( V(\theta, \phi) \) we redefine

\[
\tau(\theta) = \sin^2(\theta) \partial_\theta \tilde{\tau}(\theta),
\] (46)

and introduce two new arbitrary functions \( F(\theta) \) and \( G(\phi) \). The angular part \( V(\theta, \phi) \) of the potential then acquires the form

\[
V(\theta, \phi) = \frac{\sigma(\phi)}{4 \sin(\theta)} \partial_\theta \left( \sin(\theta) \partial_\theta \tilde{\tau}(\theta) \right) + \frac{1}{\sin^4(\theta)} \left( \frac{1}{4} \tilde{\tau}(\theta) \partial^2_{\phi\phi} \sigma(\phi) - G(\phi) \right) \\
+ \frac{\sigma(\phi)^2}{4 \sin^4(\theta)} + \frac{1}{8 \sin^4(\theta)} \partial_\phi \left( \sigma(\phi) \partial_\phi \sigma(\phi) \right) + \frac{1}{4} F(\theta).
\] (47)

It must meet the only remaining condition (39) (that became equivalent to (32) for the integral \( X_1 \)), which takes the form

\[
\frac{\sin^3(\theta) \partial_\theta \tilde{\tau}(\theta) - \cos(\theta) \sigma(\phi)}{\sin(\theta)} \partial_\phi V(\theta, \phi) + \partial_\phi \sigma(\phi) \partial_\theta V(\theta, \phi) = 0.
\] (48)

The magnetic field (40) after substituting (46) and (45) becomes

\[
B_R(R, \theta, \phi) = -\frac{1}{2} \left( \partial_\theta \left( \sin^2(\theta) \partial_\theta \tilde{\tau}(\theta) \right) + \frac{1}{\sin^4(\theta)} \left( \partial^2_{\phi\phi} \sigma(\phi) + \sigma(\phi) \right) \right),
\]

\[B_\theta(R, \theta, \phi) = 0, \quad B_\phi(R, \theta, \phi) = 0.
\] (49)

Thus, we need to solve the sole remaining equation (48), into which the function \( V(\theta, \phi) \) expressed in terms of four single-variable functions \( \tilde{\tau}(\theta), F(\theta), G(\phi), \sigma(\phi) \), i.e. (47), is substituted.
This computation is somewhat cumbersome, and we present its outline in the appendix. We find two solutions. In order to make the presentation of our results more comprehensible, we label the integration constants so that

- \( \kappa_j \) are present in the magnetic field,
- \( \lambda_j \) are present in the electrostatic potential \( W \) but not in the magnetic field,
- \( \mu_j \) are present in the integrals but not in the Hamiltonian (if any).

(i) The solution corresponding to functions \( \sigma, \tilde{\tau} \) as in equations (A.7) and (A.8) is

\[
B_\phi(R, \theta, \phi) = 0, \quad B_\theta(R, \theta, \phi) = -\frac{1}{2} \sin(\theta) (\kappa_1 - 3 \kappa_2 \cos^2(\theta)), \quad B_\rho(R, \theta, \phi) = 0
\]

with the integrals determined by

\[
\begin{align*}
V(\theta, \phi) &= \frac{\kappa_1 \kappa_2}{4} \sin(\theta)^2 + \frac{\kappa_2^2}{8} \cos(\theta)^4 - \frac{\kappa_2}{2} (\lambda_1 \cos(\theta) - \lambda_2 \sin(\phi) \sin(\theta) - \lambda_3 \cos(\phi) \sin(\theta)) + \frac{1}{4} \lambda_4 \\
\end{align*}
\]

(ii) The second solution corresponds to \( \sigma(\phi) \) as in (A.10) and \( \tilde{\tau}(\theta) = \frac{\kappa_2}{4} \frac{1}{\sin(\theta)} \). In spherical coordinates, its magnetic field and electrostatic potential read

\[
B_\phi(R, \theta, \phi) = \frac{4 \kappa_1 \kappa_2 - \kappa_3^2}{16 \sin^2(\theta) \left( \sqrt{\kappa_1 \cos^2(\phi) + \kappa_2 \sin^2(\phi) + \kappa_3 \sin(\phi) \cos(\phi)} \right)},
\]

\[
B_\theta(R, \theta, \phi) = 0,
\]

\[
B_\rho(R, \theta, \phi) = 0,
\]

\[
W(R, \theta, \phi) = U(R) - \frac{\lambda_1}{R^2 \sin^2(\theta) \left( \kappa_1 \cos^2(\phi) + \kappa_2 \sin^2(\phi) + \kappa_3 \sin(\phi) \cos(\phi) \right)}.
\]
where $U(R)$ is an arbitrary function of $R$, and $\kappa_1, \kappa_2, \kappa_3$ and $\lambda_1$ are arbitrary constants such that the square root in the denominator of $B_R$ is nonzero and real-valued, and the magnetic field is nonvanishing. The integrals are defined by the following formulae for the functions $s_1^R, s_1^\theta, s_1^\phi, s_2^R, s_2^\theta, s_2^\phi, m_1$ and $m_2$

\[
\begin{align*}
  s_1^R(R, \theta, \phi) &= 0, \\
  s_1^\theta(R, \theta, \phi) &= \frac{\kappa_1 \sin(2\phi) - \kappa_3 \cos(2\phi) - \kappa_2 \sin(2\phi)}{4 \sqrt{\kappa_1 \cos^2(\phi) + \kappa_2 \sin^2(\phi) + \kappa_3 \sin(\phi) \cos(\phi)}}, \\
  s_1^\phi(R, \theta, \phi) &= \frac{\cos(\theta) \sqrt{\kappa_1 \cos^2(\phi) + \kappa_2 \sin^2(\phi) + \kappa_3 \sin(\phi) \cos(\phi)}}{2 \sin(\theta)}, \\
  s_2^R(R, \theta, \phi) &= 0, \\
  s_2^\theta(R, \theta, \phi) &= 0, \\
  s_2^\phi(R, \theta, \phi) &= 0, \\
  m_1(R, \theta, \phi) &= \frac{128 \sin^2(\theta) \cos(\phi) \left((\kappa_1 - \kappa_2) \cos(\phi) + \kappa_3 \sin(\phi)\right) \lambda_1 + 4\kappa_1 \kappa_2^2 - \kappa_1^2 \kappa_2}{64 \kappa_2 \sin^2(\phi) \left(\kappa_1 \cos^2(\phi) + \kappa_2 \sin^2(\phi) + \kappa_3 \sin(\phi) \cos(\phi)\right)}, \\
  m_2(R, \theta, \phi) &= -\frac{2\lambda_1}{\sin^2(\theta) \left(\kappa_1 \cos^2(\phi) + \kappa_2 \sin^2(\phi) + \kappa_3 \sin(\phi) \cos(\phi)\right)}. \quad (54)
\end{align*}
\]

We note that the integration constant $K_S$, which is present in the functions $\tilde{r}(\theta)$ and $\sigma(\phi)$, cancels out in all physically relevant quantities, and is thus irrelevant.

In Cartesian coordinates, the magnetic field and the electrostatic potential read

\[
\begin{align*}
  \vec{B}(\vec{x}) &= \frac{(4\kappa_1 \kappa_2 - \kappa_1^2)}{16 \sqrt{\kappa_1 x^2 + \kappa_2 y^2 + \kappa_3 xy}} \vec{x}, \\
  W(\vec{x}) &= U(\sqrt{x^2 + y^2 + z^2}) - \frac{\lambda_1}{\kappa_1 x^2 + \kappa_2 y^2 + \kappa_3 xy}. \quad (55)
\end{align*}
\]

The direction of the magnetic field $\vec{B}$ is radial; however, its magnitude is generically not a function of $R$ alone. We shall call such fields ‘radial’.

The potential $W$, the magnetic field $\vec{B}$ and the integrals $X_1, X_2$ are singular whenever

\[
\kappa_1 x^2 + \kappa_2 y^2 + \kappa_3 xy = 0 \quad (56)
\]

holds. The magnetic field and the integrals $X_1, X_2$ become imaginary when

\[
\kappa_1 x^2 + \kappa_2 y^2 + \kappa_3 xy < 0.
\]

Thus, the quadratic form defined by the left-hand side of (56) needs to be positive—i.e. we arrive at the conditions

\[
\kappa_1 + \kappa_2 \geq 0, \quad 4\kappa_1 \kappa_2 \geq \kappa_3^2.
\]

If sharp inequalities hold, the potential $W$ and the magnetic field $\vec{B}$ are singular only along the $z$-axis. If $4\kappa_1 \kappa_2 = \kappa_3^2$ the magnetic field vanishes. The other possibility $\kappa_1 + \kappa_2 = 0$ implies $\kappa_1 = \kappa_2 = \kappa_3 = 0$, the magnetic field is ill-defined everywhere, and we rule out this case. To sum up, only the constants $\kappa_1, \kappa_2, \kappa_3$ satisfying
\[ \kappa_1 + \kappa_2 > 0, \quad 4\kappa_1\kappa_2 > \kappa_3^2, \]  
\hfill (57)

define physically relevant systems.

### 4.2. Case B: \( \sigma(\phi) = s_0 = \text{const.} \)

In this case, we have two major subcases: \( S \neq 0 \) and \( S = 0 \).

- **First subcase: \( S \neq 0 \).** Considering equations (38), out of which \( m_1 \) and \( m_2 \) were eliminated using (31), we find

\[
\begin{align*}
    s_1^R(R, \theta, \phi) &= 0, \quad s_1^\theta(R, \theta, \phi) = 0, \\
    s_1^\phi(R, \theta, \phi) &= -\kappa_1 (R \sin(\theta))^2 - \frac{1}{2} \mu_1 + \mu_2, \\
    s_2^R(R, \theta, \phi) &= \cos(\theta)S, \quad s_2^\theta(R, \theta, \phi) = -\sin(\theta)\frac{S}{R}, \\
    s_2^\phi(R, \theta, \phi) &= -\kappa_1 R^2 - \frac{1}{2} \mu_1, \\
    B_\phi(R, \theta, \phi) &= 0, \quad B_R(R, \theta, \phi) = \kappa_1 R^2 \cos(\theta) \sin(\theta), \quad B_\theta(R, \theta, \phi) = -\kappa_1 R \sin^2(\theta),
\end{align*}
\]  
\hfill (58)

where \( \kappa_1, \mu_1 \) and \( \mu_2 \) are integration constants. We are assuming that the magnetic field is nonvanishing; thus, we have \( \kappa_1 \neq 0 \).

The remaining conditions are (31), (32) and (39). The conditions (32) and (39) restrict the functional dependence of \( W \) as follows:

\[ W(R, \theta, \phi) = W(R \sin(\theta)). \]  
\hfill (59)

Finally, the equations (31), together with their compatibility conditions (41) and (42), imply the following structure of the potential \( W(R, \theta, \phi) \) and of the functions \( m_1(R, \theta, \phi), m_2(R, \theta, \phi) \):

\[
\begin{align*}
    W(R, \theta, \phi) &= -\frac{\kappa_1^2}{8} (R \sin(\theta))^2 - \frac{\lambda_1}{(R \sin(\theta))^2}, \\
    m_1(R, \theta, \phi) &= \frac{\kappa_1^2}{4} (R \sin(\theta))^4 + \frac{\kappa_1 (\mu_1 - 2\mu_2)}{4} (R \sin(\theta))^2, \\
    m_2(R, \theta, \phi) &= \frac{\kappa_1 (\kappa_1 R^2 + \mu_1)}{4} R^2 \sin^2(\theta) - 2 \frac{\lambda_1}{\sin^2(\theta)},
\end{align*}
\]  
\hfill (60)

where \( \lambda_1 \) is another integration constant.

In Cartesian coordinates, the system has constant magnetic field

\[ \vec{B}(x) = (0, 0, \kappa_1), \]  
\hfill (61)

and the potential reads

\[ W(\vec{x}) = -\frac{\kappa_1^2}{8} (x^2 + y^2) - \frac{\lambda_1}{x^2 + y^2}. \]  
\hfill (62)
We observe that in the functions $s^R, s^\theta, s^\phi, s^R, s^\theta, s^\phi, m_1$ and $m_2$ there are two parameters $\mu_1$ and $\mu_2$ which are present neither in the magnetic field nor in the electrostatic potential. This indicates existence of two first order integrals, which turn out to be

$$\tilde{X}_1 = p^\lambda_o - \frac{\kappa_1}{2} \left( R \sin(\theta) \right)^2 = x^t - \frac{\kappa_1}{2} \left( x^2 + y^2 \right),$$

$$X_3 = \cos(\theta) \ p^\lambda_R - \sin(\theta) \ R \ p^\lambda_o = p^\lambda_x.$$  \hfill (63)

The integral $X_3$ can be expressed in terms of $\tilde{X}_1$ as

$$X_1 = \left( \tilde{X}_1 \right)^2 - \frac{\mu_1 - 2 \mu_2}{2} \tilde{X}_1;$$  \hfill (64)

thus, we can replace $X_1$ by the first order integral $\tilde{X}_1$.

On the other hand, the integral $X_3$ is independent of $H, \tilde{X}_1$ and $X_2$, i.e. the system defined by (58) and (60) is (at least) minimally superintegrable.

The integral $X_3$ allows us also to express the integral $X_2$ in a more convenient way, independently of $S$:

$$\tilde{X}_2 = X_2 - SX_3 = (p^\lambda_o)^2 + \frac{\left( p^\lambda_R \right)^2}{\sin^2(\theta)} - \kappa_1 R^2 p^\lambda_o + \frac{\kappa_1^2}{4} \left( R^2 \sin(\theta) \right)^2 - \frac{2 \lambda_1}{\sin^2(\theta)}$$

$$= \left( \tilde{X}_1 \right)^2 + \left( \tilde{X}_1 \right)^2 - \left( x^2 + y^2 + z^2 \right) \left( \frac{\kappa_1 \tilde{X}_1}{4} (x^2 + y^2) + 2 \frac{\lambda_1}{x^2 + y^2} \right).$$  \hfill (65)

Looking for additional integrals at most quadratic in momenta, we find one more integral:

$$X_4 = \frac{1}{2} \left\{ \tilde{X}_2, X_3 \right\}_{P,B}$$

$$= \cos(\theta) \ R \ \left( \frac{p^\lambda_o}{\sin^2(\theta)} \right) - \frac{\kappa_1 R^2 \sin^2(\theta)}{2} \right)^2 + \sin(\theta) \ p^\lambda_R p^\lambda_o - \frac{2 \lambda_1 \cos(\theta)}{R \sin^2(\theta)}$$

$$= \frac{\kappa_1^2}{4} - \kappa_1 \tilde{X}_1 + \frac{\kappa_1^2}{4} (x^2 + y^2) z - 2 \frac{\lambda_1 z}{x^2 + y^2}.$$

However the integral (66), similarly to the $z$-component of the Laplace–Runge–Lenz vector, is not independent of $H, \tilde{X}_1, \tilde{X}_2, X_3$. It satisfies the relation

$$X_4^2 = 2H \left( X_2 - \tilde{X}_2^2 + 2 \lambda_1 \right) - \tilde{X}_2 X_3^2 + \kappa_1 \left( \tilde{X}_3^2 - \tilde{X}_1 \tilde{X}_2 \right) - 2 \kappa_1 \lambda_1 \tilde{X}_1.$$  \hfill (67)

Thus, we conclude that the system with the magnetic field (61) and the electrostatic potential (62) possesses four independent integrals at most quadratic in momenta—i.e. it is minimally but not maximally quadratically superintegrable.

- Second subcase: $S = 0$. The magnetic field (40) takes the form

$$B^R(R, \theta, \phi) = - \sin(\theta) \cos(\theta) \omega(R) - \frac{1}{2} \partial_\theta \tau(\theta) - \frac{s_0}{2 \sin^2(\theta)},$$

$$B^\theta(R, \theta, \phi) = \frac{1}{2} \sin^2(\theta) \partial_\phi \omega(R), \quad B^\phi(R, \theta, \phi) = 0.$$  \hfill (68)

The equation (32) for the two integrals considered, $X_1, X_2$, together with the assumption of nonvanishing magnetic field, imply
\[ W(R, \theta, \phi) = W(R, \theta). \]  
(Solving the remaining equations—first, the compatibility conditions (41) and (42), finding \( W(R, \theta, \phi); \) next, integrating (31) to find \( m_1(R, \theta, \phi) \) and \( m_2(R, \theta, \phi) \)—we find the complete solution. The constant \( s_0 \) can be absorbed into \( \tau(\theta) \) without loss of generality, and we redefine \( \omega(R) = R^2 \chi(R) \) for notational convenience. The magnetic field and the electrostatic potential read

\[ B_R(R, \theta, \phi) = -\sin(\theta)\chi(R)R^2\cos(\theta) - \frac{1}{2} \partial_\theta \tau(\theta), \]
\[ B_\theta(R, \theta, \phi) = R \sin^2(\theta) \left( \frac{1}{2} R \partial R \chi(R) + \chi(R) \right), \]
\[ B_\phi(R, \theta, \phi) = 0. \]

\[ W(R, \theta, \phi) = -\frac{1}{8} R^2 \sin^2(\theta) \chi(R)^2 - \frac{1}{4} \tau(\theta) \chi(R) + F_1(R) + \frac{1}{R^2} F_2(\theta). \]

The corresponding integrals (33) and (36) are defined by

\[ s_1^R(R, \theta, \phi) = 0, \quad s_1^\theta(R, \theta, \phi) = 0, \quad s_1^\phi(R, \theta, \phi) = R^2 \sin^2(\theta) \chi(R) + \tau(\theta), \]
\[ s_2^R(R, \theta, \phi) = 0, \quad s_2^\theta(R, \theta, \phi) = 0, \quad s_2^\phi(R, \theta, \phi) = R^2 \chi(R), \]
\[ m_1(R, \theta, \phi) = \frac{(R \sin(\theta))^4}{4} \chi(R)^2 + \frac{(R \sin(\theta))^2}{2} \tau(\theta) \chi(R) + \frac{1}{4} \tau(\theta)^2, \]
\[ m_2(R, \theta, \phi) = \frac{1}{4} R^2 \sin^2(\theta) \chi(R)^2 + 2 F_2(\theta). \]

In Cartesian coordinates, the magnetic field (70) is given by

\[ B_x(\vec{\chi}) = \frac{1}{2} \left( \frac{z}{R} \partial R \chi(R) - \frac{1}{R} \partial x \tau(\theta) \right) x, \]
\[ B_y(\vec{\chi}) = \frac{1}{2} \left( \frac{z}{R} \partial R \chi(R) - \frac{1}{R} \partial y \tau(\theta) \right) y, \]
\[ B_z(\vec{\chi}) = \frac{1}{2} \left( \frac{z}{R} \partial R \chi(R) - \frac{1}{R} \partial z \tau(\theta) \right) z - \frac{1}{2} R \partial R \chi(R) - \chi(R). \]

We notice that for \( \chi(R) = \frac{\kappa_2}{R} \), the last two terms in \( B_z \) cancel each other and consequently the magnetic field is directed along the line connecting the point \( \vec{\chi} \) with the coordinate origin:

\[ \vec{B}(\vec{\chi}) = -\frac{1}{R^3} \left( \kappa_1 \frac{z}{R} + \frac{1}{2 \sin(\theta)} \partial x \tau(\theta) \right) \cdot \vec{\chi}. \]

If in addition \( \tau(\theta) = \kappa_1 \cos^2(\theta) + 2 \kappa_2 \cos(\theta) + \kappa_3 \), the magnetic field becomes central,

\[ \vec{B}(\vec{\chi}) = \frac{\kappa_2}{R^3} \vec{\chi}, \]

and the electrostatic potential reads

\[ W(R, \theta, \phi) = F_1(R) + \frac{F_2(\theta)}{R^2} + \frac{(\sin^2(\theta) - 2)\kappa_1^2}{8R^2} - \frac{(2 \cos(\theta) \kappa_2 + \kappa_3) \kappa_1}{4R^2}. \]
By a redefinition of the arbitrary function $F_2(\theta)$, the potential takes the separated form

$$W(R, \theta, \phi) = F_1(R) + \frac{F_2(\theta)}{R^2}. \quad (76)$$

As above, spurious integration constants $\kappa_1, \kappa_3$, present neither in the magnetic field nor in the electrostatic potential, help us to identify a first order integral,

$$\tilde{X}_1 = p_\theta^1 + \kappa_2 \cos(\theta), \quad (77)$$

related to $X_1$ as follows:

$$X_1 = (\tilde{X}_1)^2 + (\kappa_1 + \kappa_3)\tilde{X}_1 + \frac{1}{4}(\kappa_1 + \kappa_3)^2. \quad (78)$$

In (74), we recognize the magnetic field of a magnetic monopole, see [16]. The existence and order of eventual additional integrals of motion depends on the functions $F_1(R), F_2(\theta)$ in the potential $W(R, \theta, \phi)$—see [12, 13, 17, 24].

5. Conclusions

Let us sum up the main results of this article as theorems. The article is part of a series in which we study the integrability properties of the Hamiltonian (1). The first result concerns the general form of the integrals of motion, and is proven above in section 2 (it has already been used in earlier articles [15, 16]).

**Theorem 1.** The general form of a second order integral of motion for the Hamiltonian system (1) is given by equation (12). The coefficients $\alpha_{ab}$ are arbitrary real constants, and $s^j(\vec{x}), j = 1, 2, 3$ and $m(\vec{x})$ are real functions satisfying (8)–(10).

It is interesting to compare this result with the results for a purely scalar potential (i.e. $\vec{A}(\vec{x}) = 0$) considered in [14].

(i) In both cases, the leading part of the integral $X$ lies in the enveloping algebra of the Euclidean Lie algebra $e_3$.

(ii) If the vector potential satisfies $\vec{A}(\vec{x}) = 0$, then so does the magnetic field $\vec{B}(\vec{x}) = 0$. Hence, in the absence of a magnetic field even and odd terms in the integral (12) Poisson commute with $H$ separately. It follows that without loss of generality we can put $s^j(\vec{x}) = 0$ (or $\alpha_{ab} = 0$ and $m(\vec{x}) = 0$) in $X$ for $\vec{B}(\vec{x}) = 0$, but not in the presence of a magnetic field—i.e. $\vec{B}(\vec{x}) \neq 0$.

(iii) In both cases, the determining equations for the functions in the integrals are linear if the potentials are known, but nonlinear if the potentials are to be determined (usually from compatibility conditions).

(iv) For a system to be second order integrable in both cases, we need two integrals $X_1, X_2$ of the form (12), and they must be in involution. The leading order parts in both cases can be classified into conjugacy classes under Euclidean transformations.

(v) For integrals of arbitrary order $N$, the Poisson commutation relation (6) yields a polynomial of order $N + 1$, and hence we obtain $N + 2$ sets of determining equations generalizing the four sets (7)–(10) for $N = 2$. The set multiplying terms of the order $N + 1$ will not depend on the potentials $\vec{A}$ and $W$, i.e. is the same as in the absence of the magnetic field. Hence, for arbitrary value of $N$, we can write the leading term in an integral of motion as...
\[ X = \sum_{1 \leq a_1 \leq a_2 \leq \ldots \leq a_9 \leq 6} \alpha_{a_1 a_2 \ldots a_9} (Y^A)_{a_1} (Y^A)_{a_2} \ldots (Y^A)_{a_9} + \text{lower order terms}. \]  

(79)

All other principal results of this article are contained in section 4 and the Appendix, and concern integrable and superintegrable systems of the spherical type. Again, we sum them up as a theorem.

**Theorem 2.** Four classes of second order spherical type integrable systems exist. For all of these, the Hamiltonian \( H \) has the form (22) and the two further integrals in involution have the form \( X_1, X_2 \) as in (33) and (36). To specify them completely, we must specify the functions \( B_\alpha(R, \theta, \phi) \) and \( W(R, \theta, \phi) \) describing the system, and the functions \( s_\alpha^0 \) and \( m_j \) for each integral (where \( \alpha = R, \theta, \phi \) and \( j = 1, 2 \)).

The four classes are given, respectively, by

**Class I.** The system: \( B_\alpha \) and \( W \) as in (50), or equivalently (52)

\[
\vec{B}(\vec{x}) = -\frac{1}{2(x^2 + y^2 + z^2)} \left( \kappa_1 - \frac{3\kappa_2 z^2}{x^2 + y^2 + z^2} \right) \vec{x}, \quad (\kappa_1, \kappa_2) \neq (0, 0),
\]

\[
W(\vec{x}) = U \left( \sqrt{x^2 + y^2 + z^2} + \frac{\kappa_2^2 z^4}{8(x^2 + y^2 + z^2)^3} + \frac{\kappa_2 \kappa_1 (x^2 + y^2)}{4(x^2 + y^2 + z^2)^3} \right) + \frac{\kappa_2 (\lambda_2 y + \lambda_3 x - \lambda_4 z)}{2(x^2 + y^2 + z^2)^2} + \frac{\lambda_4}{4(x^2 + y^2 + z^2)}.
\]

The integrals: \( s_\alpha^0 \) and \( m_j \) as in (51).

**Class II.** The system: \( B_\alpha \) and \( W \) as in (53), or equivalently (55)

\[
\vec{B}(\vec{x}) = \frac{(4\kappa_1 \kappa_2 - \kappa_3^2)}{16\sqrt{\kappa_1 x^2 + \kappa_2 y^2 + \kappa_3 z^2}}, \quad \kappa_1 + \kappa_2 > 0, \quad 4\kappa_1 \kappa_2 > \kappa_3^2,
\]

\[
W(\vec{x}) = U \left( \sqrt{x^2 + y^2 + z^2} + \frac{\lambda_1}{\kappa_1 x^2 + \kappa_2 y^2 + \kappa_3 z^2}. \right.
\]

The integrals: \( s_\alpha^0 \) and \( m_j \) as in (54).

**Class III.** The system: \( B_\alpha \) as in (58), or equivalently (61), \( W \) as in (60), or equivalently (62)

\[
\vec{B}(\vec{x}) = (0, 0, \kappa_1), \quad W(\vec{x}) = -\frac{\kappa_1}{8} (x^2 + y^2) - \frac{\lambda_1}{x^2 + y^2}, \quad \kappa_1 \neq 0.
\]

The integrals: \( s_\alpha^0 \) as in (58), \( m_j \) as in (60).

**Class IV.** The system: \( B_\alpha \) as in (70) or (72), \( W \) as in (70):

\[
B_1(\vec{x}) = \frac{1}{2} \left( \frac{z}{R} \frac{\partial \kappa(R)}{\partial R} - \frac{1}{R^3 \sin(\theta)} \frac{\partial \theta}{\partial \tau(\theta)} \right) x,
\]

\[
B_2(\vec{x}) = \frac{1}{2} \left( \frac{z}{R} \frac{\partial \kappa(R)}{\partial R} - \frac{1}{R^3 \sin(\theta)} \frac{\partial \theta}{\partial \tau(\theta)} \right) y,
\]

\[
B_3(\vec{x}) = \frac{1}{2} \left( \frac{z}{R} \frac{\partial \kappa(R)}{\partial R} - \frac{1}{R^3 \sin(\theta)} \frac{\partial \theta}{\partial \tau(\theta)} \right) z - \frac{1}{2} \frac{\partial \kappa(R)}{\partial \tau(\theta)} - \chi(R),
\]

\[
W(\vec{x}) = -\frac{1}{8} R^3 \sin^2(\theta) \chi(R)^2 - \frac{1}{4} \frac{\tau(\theta)}{\chi(R)} + F_1(R) + \frac{1}{R^2} F_2(\theta).
\]

The integrals: \( s_\alpha^0 \) and \( m_j \) as in (71).
Comments on properties of the above introduced classes of systems:

- **Class I.**
  The magnetic field $B_\alpha$ is radial, and depends on two constants $\kappa_1, \kappa_2$. The potential $W$ depends on one arbitrary function $U(R)$ and four constants in addition to $\kappa_1$ and $\kappa_2$.

- **Class II.**
  - The system depends on one arbitrary function $U(R)$ and four constants $\kappa_1, \kappa_2, \kappa_3$ and $\lambda_1$. For specific choices of the function $U(R)$ and the constants, further integrals may exist—making the system superintegrable.
  - The conditions on the integration constants $\kappa_1, \kappa_2, \kappa_3$ imply that the magnetic field, the electrostatic potential and the integrals are well defined and real-valued, and that the magnetic field is nonvanishing.

- **Class III.**
  - The magnetic field $B_\alpha$ depends on one constant, $\kappa_1 \neq 0$. The electrostatic potential $W$ depends on two constants, $\kappa_1$ and $\lambda_1$.
  - Equations (61) and (62) show that the considered system corresponds to a constant magnetic field $\vec{B}$ oriented along the $z$-axis with an electrostatic potential $W$ that is a generalized inverted harmonic oscillator in the $(x, y)$ plane.
  - This system was overlooked in [16], though it has two first order integrals of the cylindrical type given in (63). This system allows one further independent second order integral, namely $X_2$ (or $\tilde{X}_2$, equivalently), and hence is at least minimally superintegrable. We are investigating the possibility that it is actually maximally superintegrable, i.e. it allows one more integral, either a higher order polynomial, or not a polynomial at all (either in general, or for particular choices of the constants).

- **Class IV.**
  - The magnetic field $B_\alpha$ depends on two arbitrary functions of one variable, $\chi(R)$ and $\tau(\theta)$. The electrostatic potential $W$ also depends on $\chi$ and $\tau$ and on two further functions $F_1(R)$ and $F_2(\theta)$.
  - For a special choice of the functions
    \[ \chi(R) = \frac{\kappa_1}{R^2}, \quad \tau(\theta) = \kappa_1 \cos^2(\theta) + 2\kappa_2 \cos(\theta) + \kappa_3, \]
    the functions $B_\alpha$ and $W$ can be simplified to (74) and (76), and we obtain the magnetic field of a monopole. The field $B_\alpha$ is completely specified by the constant $\kappa_2$, but $W$ still involves two arbitrary functions.
  - Thus, the integrable magnetic monopole is imbedded into a family of integrable systems, depending on two arbitrary functions. For special choices of these functions, the monopole is known to be superintegrable.

A related problem was investigated in [11], namely the breaking of the $O(4)$ symmetry of the hydrogen atom by scalar and vector potentials (in quantum mechanics). All potentials were found that leave the separation of variables—related to the $O(4)$ symmetry and thus to superintegrability—intact. The vector and electrostatic potentials defining the Hamiltonian separable in spherical coordinates obtained in [11] depend on a total of six arbitrary functions of one variable each. They seem to be more general than those obtained in the present article (in principle, those potentials should be special cases of the present ones). However, it turns out that the magnetic field $\vec{B}$ associated with the vector potential of [11] vanishes, $\vec{B} = 0$. In
other words, the system of [11] may be of interest in quantum mechanics—in particular, in the context of the Bohm–Aharonov effect—but in classical mechanics is equivalent to a system with only a scalar potential, and thus by definition excluded from our analysis here.

Further work on spherical-type integrable systems in a magnetic field is in progress in two directions:

(i) To determine which subclasses of the four classes found above can be extended to superintegrable systems, to find the additional integrals and particle trajectories.

(ii) To find all quantum integrable and superintegrable systems of the spherical type, and analyze them. In particular, to verify whether the conjecture that all maximally superintegrable systems are exactly solvable [29] also holds in the presence of magnetic fields.

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Appendix. Solution of equation (48)

Let us sketch, in this rather technical appendix, the method used to solve the equation (48) of Case A above. The complexity of the expressions arising forces us to present just the conceptual outline—all the formulae mentioned below are too cumbersome (anything between five lines and 1–2 pages) to be presented explicitly. We mention that assistance of computer algebra systems Maple and Mathematica was essential for effective manipulation of these expressions.

For the reader’s convenience, we recall that (48) reads

\[
\sin^3(\theta) \partial_\theta \tilde{\tau}(\theta) - \frac{\cos(\theta) \sigma(\phi)}{\sin(\theta)} \partial_\phi V(\theta, \phi) + \partial_\phi \sigma(\phi) \partial_\theta V(\theta, \phi) = 0,
\]

where the substitution (47)

\[
V(\theta, \phi) = \frac{\sigma(\phi)}{4 \sin(\theta)} \partial_\theta \left( \sin(\theta) \partial_\theta \tilde{\tau}(\theta) \right) + \frac{1}{\sin^2(\theta)} \left( \frac{1}{4} \tilde{\tau}(\theta) \frac{1}{\sin^2(\theta)} \sigma(\phi) - G(\phi) \right) + \frac{\sigma(\phi)^2}{4 \sin^4(\theta)} + \frac{1}{8 \sin^4(\theta)} \partial_\phi (\sigma(\phi) \partial_\phi \sigma(\phi)) + \frac{1}{4} F(\theta)
\]

is assumed.

Thus, the equation (48) involves two functions \( \tilde{\tau}, F \) of the variable \( \theta \), and two functions \( \sigma, G \) of the variable \( \phi \), together with their derivatives. Thus, it is neither ODE nor PDE.

We first observe that after the substitution the resulting equation depends only on the first derivative of \( F(\theta) \), not on \( F(\theta) \) itself. We solve for \( \partial_\theta F(\theta) \)—this is possible because \( \sigma(\phi) \) is assumed to be nonconstant and smooth; thus, we can divide by \( \partial_\phi \sigma(\phi) \), at least locally.

Next, we differentiate the obtained equation of the form

\[
\partial_\theta F(\theta) = \text{function of } (\tilde{\tau}(\theta), \sigma(\phi), G(\phi), \theta, \phi, \ldots)
\]

(A.1)
with respect to $\phi$, obtaining an equation that must be satisfied by the remaining functions $\tilde{\tau}, \sigma, G$. It contains the first and second derivative of $G(\phi)$. Provided $\sin^3(\theta) \partial_\theta \tilde{\tau}(\theta) - \cos(\theta) \sigma(\phi) \neq 0$—which again holds at least locally, due to nonconstancy of $\sigma(\phi)$—we solve it with respect to $\partial^2_{\phi\phi} G(\phi)$, thus obtaining an equation of the form

$$\partial^2_{\phi\phi} G(\phi) = \text{function of } (\partial_\theta G(\phi), \tilde{\tau}(\theta), \sigma(\phi), \theta, \phi, \ldots).$$  \hspace{1cm} (A.2)

We differentiate (A.2) with respect to $\theta$ and attempt to solve the result with respect to $\partial_\theta G(\phi)$. Here, we are forced to consider a first special subcase,

$$\tilde{\tau}(\theta) = C \left( \frac{\cos(\theta)}{\sin(\theta)} \right)^2,$$  \hspace{1cm} (A.3)

for which the derivative of (A.2) is independent of $\partial_\theta G(\phi)$. Going through the case in a similar fashion as for the generic case described below, we find that it leads either to a contradiction with nonconstant $\sigma(\phi)$, or to the case (A.10) considered below.

Once the expression for $\partial_\theta G(\phi)$ is found, we substitute it back into (A.2), and find an equation involving only $\tilde{\tau}(\theta), \sigma(\phi)$ and their derivatives—up to $\partial^4_{\phi\phi\theta\theta} \tilde{\tau}(\theta), \partial^4_{\phi\phi\phi\theta} \sigma(\phi)$. We solve it with respect to $\partial^4_{\phi\phi\theta\theta} \sigma(\phi)$; this is possible provided that

$$(\cos(\theta) \partial_\theta + \sin^2(\theta) \partial^3_{\phi\phi\theta})^3 \neq 0,$$  \hspace{1cm} (A.4)

which again holds due to nonvanishing $\partial_\theta \sigma(\phi)$.

Similarly as above, we differentiate the resulting equation

$$\frac{d^5}{d\phi^5} \sigma(\phi) = \text{function of } \left( \tilde{\tau}(\theta), \ldots, \frac{d^4}{d\phi^4} \tilde{\tau}(\theta), \sigma(\phi), \ldots, \frac{d^4}{d\phi^4} \sigma(\phi), \theta, \phi \right) \hspace{1cm} (A.5)$$

with respect to $\theta$, and eliminate $\frac{d^4}{d\phi^4} \sigma(\phi)$ from it (after checking that the coefficient of $\frac{d^4}{d\phi^4} \sigma(\phi)$ cannot vanish). Substituting the expression obtained for $\frac{d^4}{d\phi^4} \sigma(\phi)$ into (A.5) we obtain an equation which, after dividing by the nonvanishing factors $(\partial_\theta \sigma(\phi))^2$ and $\sin^3(\theta) \partial_\theta \tilde{\tau}(\theta) - \cos(\theta) \sigma(\phi)$, turns out to be the following fifth order linear ODE for $\tilde{\tau}(\theta)$:

$$\cos^2(\theta) \sin^3(\theta) \frac{d^3}{d\theta^3} \tilde{\tau}(\theta) + 3 \sin^2(\theta) \cos(\theta)(2 \cos^2(\theta) + 1) \partial_\theta^4 \tilde{\tau}(\theta) + (7 \cos(\theta)^4 + 2 \cos^2(\theta) + 3) \sin(\theta) \partial_\theta^3 \tilde{\tau}(\theta) - 6(\cos(\theta)^2 + 1) \cos(\theta) \partial_\theta^2 \tilde{\tau}(\theta) + 2 \sin^3(\theta)(4 \sin^2(\theta) - 7) \partial_\theta \tilde{\tau}(\theta) = 0.$$  \hspace{1cm} (A.6)

Its general solution reads

$$\tilde{\tau}(\theta) = K_1 + K_2 \frac{\cos(\theta)}{\sin(\theta)} + K_3 \left( \sin(\theta) + \frac{1}{\sin(\theta)} \right) + K_4 \left( \frac{1}{\sin^2(\theta)} \right)$$  \hspace{1cm} (A.7)

(where $K_1$ is irrelevant, since only derivatives of $\tilde{\tau}(\theta)$ appear in the integrals of motion and potentials. It can thus be set to any convenient value, e.g. $K_1 = 0$). Next, we insert the solution (A.7) into (A.5) and we find an expression in which all dependence on $\theta$ is explicit, in terms of powers of $\sin(\theta)$ and $\cos(\theta)$, but their coefficients still depend on $\phi$. Thus, each of these must vanish, and we have a set of ODEs for the function $\sigma(\phi)$; these involve $\frac{d^4}{d\phi^4} \sigma(\phi), \ldots, \frac{d^4}{d\phi^4} \sigma(\phi)$.

Solving for the highest order derivatives, we gradually reduce the order until we arrive at an
equation which we can solve explicitly. We find that if \( K_2 \neq 0 \) or \( K_3 \neq 0 \) or \( K_4 \neq 0 \), we must have

\[
\sigma(\phi)''' + \sigma(\phi)' = 0,
\]

which implies

\[
\sigma(\phi) = \sigma_1 + \sigma_2 \sin(\phi) + \sigma_3 \cos(\phi), \quad \sigma_i \in \mathbb{R} \tag{A.8}
\]

with \( \sigma_2 \) or \( \sigma_3 \) nonvanishing, since \( \sigma(\phi) \) is assumed to be nonconstant. Integrating the remaining two quadratures, i.e. (A.1) and

\[
\partial_\phi G(\phi) = \text{function of } (\tilde{\tau}(\theta), \sigma(\phi)), \tag{A.9}
\]

we find the solution given in (50)–(52).

The other possibility is \( K_2 = K_3 = K_4 = 0 \) in (A.7). We find

\[
\sigma(\phi) = -\frac{K_5}{2} - \frac{\sqrt{\kappa_1 \cos^2(\phi) + \kappa_2 \sin^2(\phi) + \kappa_3 \sin(\phi) \cos(\phi)}}{2}. \tag{A.10}
\]

Finding the functions \( G(\phi) \) and \( F(\phi) \) via a straightforward integration, we arrive at the solution presented in formulae (53) and (54).

**ORCID iDs**

L. Šnobl  
https://orcid.org/0000-0002-7270-6251

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