LETTER

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Convergence from divergence

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Abstract
We show how to convert divergent series, which typically occur in many applications in physics, into rapidly convergent inverse factorial series. This can be interpreted physically as a novel resummation of perturbative series. Being convergent, these new series allow rigorous extrapolation from an asymptotic region with a large parameter, to the opposite region where the parameter is small. We illustrate the method with various physical examples, and discuss how these convergent series relate to standard methods such as Borel summation, and also how they incorporate the physical Stokes phenomenon. We comment on the relation of these results to Dyson’s physical argument for the divergence of perturbation theory. This approach also leads naturally to a wide class of relations between bosonic and fermionic partition functions, and Klein–Gordon and Dirac determinants.

Keywords: non-perturbative physics, Borel summation, Painlevé

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Physics often requires approximation in the form of an expansion in which a physical parameter becomes large or small compared to other scales. This is a generic feature of perturbation theory, and much physical information is encoded in the relation between the characteristic divergence of perturbation theory and non-perturbative physics [1, 2]. It is less well-known that divergent series can readily be converted to convergent expressions, such as continued fractions or inverse factorial series [3–5]. For example, a series in inverse powers of a large parameter $x$ may be re-written as a series in inverse Pochhammer symbols, $(x)_{m+1} \equiv \Gamma(x + m + 1)/\Gamma(x) = x(x + 1) \ldots (x + m)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{c_n}{x^{n+1}} = \sum_{m=0}^{\infty} \sum_{l=0}^{m} (-1)^{l+m} S(l,m) \frac{c_l}{(x)_{m+1}}$$

(1)

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where \( S^{(1)}(m, l) \) are Stirling numbers of the first kind [6]. This can be viewed as a resummation of the series expansion, and often this resummed series converges, even if the original series diverges [4]. Unfortunately, convergence is typically slow, and such series are not accurate near a Stokes line, so (1) is typically not competitive with other resummation methods [7]. The new results reported in this paper concern physical applications of a radically new form of inverse factorial series, introduced recently in [8], which is both rapidly convergent and valid near a Stokes line.

To illustrate the basic idea, consider the digamma function,

\[
\psi(1+x) = \ln x + \int_0^\infty dp \, e^{-xp} \left( \frac{1}{p} - \frac{1}{e^p - 1} \right).
\]

The remarkable ‘binary rational identity’ (valid for all \( p \in \mathbb{C}/\{0\} \) [8])

\[
\frac{1}{p} = \frac{1}{(e^p - 1)} + \sum_{k=1}^\infty \frac{1}{2^k (e^{p/2^k} + 1)}
\]

and successive integrations leads to a ‘binary rational’ inverse factorial series expansion

\[
\psi(1+x) = \ln x + \sum_{k=1}^\infty \sum_{m=0}^\infty \frac{m!}{2^{m+1}(2^k x + 1)^{m+1}}
\]

in which the large parameter \( x \) is rescaled by factors of \( 2^k \). An efficient method for the integrations by parts is the Horn expansion [5, 8]: given a Borel integral \( f(x) = \int_0^\infty dp \, e^{-xp} F(p) \), change variables to \( s = e^{-p} \), so that \( f(x) = \int_0^1 ds \, s^{x-1} \varphi(s) \), where \( \varphi(s) = F(-\ln s) \). Then integrating by parts successively one finds the inverse factorial series expansion: \( f(x) = \sum_{m=0}^\infty (-1)^m \varphi^{(m)}(1)/(x)_m \), with coefficients expressed in terms of derivatives of \( \varphi(s) \) evaluated at \( s = 1 \). Even when these coefficients cannot be found in closed form it is a simple matter to construct a table of coefficients.

The expansion (4) has the dual advantage of being rapidly convergent, and also having the correct Stokes jump as the phase of \( x \) is rotated. In fact, it gives a global representation valid everywhere in the cut plane. Figure 1 shows the convergence even at extremely small values of \( x \), contrasted with the truncated asymptotic expansions.

Expression (4) involves a double infinite sum, but because of the rapid convergence of these binary rational series, we find in practice that only a small number of terms is needed in order to obtain excellent numerical agreement. For example, consider the generalization of (4) to the physical example of the one-loop Euler–Heisenberg (EH) effective Lagrangian [9–11], which encodes vacuum polarization effects in quantum electrodynamics (QED). (In fact the EH effective Lagrangian can be expressed in terms of the log of the Barnes gamma function, which involves an integral of the log gamma function [11], so these examples are closely related.) In the weak (magnetic) field limit the exact (Borel) integral representation of the EH effective Lagrangian is expanded as a divergent series:

\[
\frac{\mathcal{L}(x)}{m^2} = -\frac{1}{8\pi^2 x^2} \int_0^\infty \frac{dp}{p^2} e^{-xp} \left( \frac{1}{p} - \frac{1}{e^{p/3}} \right)
\]

\footnote{For a comprehensive discussion, with explicit comparisons, see [7].}
Here \( x \equiv \frac{m^2}{(eB)} \), with \( m \) and \( e \) the electron mass and charge, and \( B \) the external magnetic field strength. But we can rewrite the identity (3) as

\[
\left( \coth p - \frac{1}{p} - \frac{p}{3} \right) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left( \tanh \left( \frac{p}{2^k} \right) - \frac{p}{2^k} \right)
\]

which implies that

\[
\frac{L(x)}{m^4} = -\frac{1}{8\pi^2 x^2} \sum_{k=1}^{\infty} \frac{1}{4^k} \int_0^{\infty} \frac{dp}{p^2} e^{-2^k p} (\tanh p - p).
\]

This generates a new inverse factorial expansion, with the asymptotic parameter \( x \) effectively rescaled by \( 2^k \) for each term in the \( k \) sum. A convergent binary rational expansion of (8) is obtained by systematically removing the ‘instanton pole’ terms from the integrand, and matching near \( p = 0 \). We subtract pole terms with the same residues, and then add back regular corrections to match the small \( p \) behavior. For example, the leading pole in (8) is at \( p = \pm i\pi/2 \), so we obtain a leading approximation of the form:

\[
\frac{\tanh p - p}{p^2} \approx -\frac{8}{\pi^2} + \frac{4}{\pi^2} + \left( \frac{4}{\pi^2} - \frac{1}{3} \right) p + \ldots.
\]

The advantage of such an expansion is that for the pole term of the form \( \frac{1}{1+e^{-2p}} \) the \( p \) integrals can now be done in closed form, yielding binary rational expansions that are given in equation (45) of [8]. It is straightforward to extend such an expansion, but the excellent agreement of the blue-dashed curve in figure 2 is obtained already using only this leading approximation. The resulting binary rational weak-field expansion behaves very much like the digamma example plotted in figure 1. The imaginary part is more interesting, and is shown...
in figure 2. Figure 2 shows (with just three terms of the binary rational \(k\) sum, and six of the other sum) that when the phase of the magnetic field \(B\) is rotated, to become an electric field \((B \rightarrow e^{\pm i\pi/2}E)\), the binary rational factorial series encodes the correct imaginary non-perturbative contribution which is associated with the genuine physical process of particle production from the QED vacuum:

\[
\text{Im} \left( \frac{\mathcal{L} \left( \frac{\alpha}{m^4} \right)}{m^4} \right) = \frac{1}{8\pi^3} \left( \frac{eE}{m^2} \right)^2 \text{Li}_2 \left( e^{-m^2\pi/(eE)} \right)
\]  

(10)

which follows directly from (5) [10, 11]. The comparison in figure 2 may be made because the Euler–Heisenberg (EH) effective Lagrangian (5) can be expressed in closed-form in terms of the logarithm of the Barnes Gamma function [11], whose analytic continuation properties are known [6].

At first sight, this result appears to be in conflict with Dyson’s physical argument [12] that QED perturbation theory (as an expansion in the fine structure constant \(\alpha = e^2/\hbar c\)) should be divergent, because if it were a convergent series in \(\alpha\) it would not be able to describe the expected physical instability as the phase of \(\alpha\) is changed from 0 to \(\pm \pi\). In mathematical terms, Dyson’s argument is that a convergent series in \(\alpha\) cannot describe a non-perturbative Stokes jump. The resolution of this apparent contradiction is that this argument refers to a series expansion in powers of \(\alpha = 1/x\), whereas the above examples show clearly that a binary rational factorial series correctly describes this non-perturbative effect, even though the binary rational factorial series is convergent. The key fact is that the inverse Pochhammer symbols \(1/(x)_n\) of the binary rational factorial expansion encode the proper analytic continuation properties, while the powers \(1/x^n\) of the divergent expansion do not, even when truncated at optimal order. Indeed, since the binary rational expansion is a convergent representation of the function, it must encode the correct Stokes phenomenon.
In these two examples, the integral representations (2) and (5) are in Borel form, with Borel transforms having an infinite sequence of equally spaced poles on the imaginary axis. Physically, the sum over poles in (5) is precisely a QFT instanton sum when the phase of $x$ rotates by $\pm \pi/2$ [10, 13–16]. But the poles in (5) are at integer multiples of $i\pi$, while those of (8) are at half-odd-integer multiples of $i\pi$. This is because before rescaling $p$ by $2^k$ we have used the fact that every integer can be expressed as a half-odd-integer times a power $2^k$, exactly once. (This interpretation also leads to generalizations from ‘binary rational’ to ‘n-ary rational’ inverse factorial series expansions, with even faster convergence, if desired.) Thus the binary rational expansion, which is a resummation of standard perturbation theory, also produces a nontrivial rearrangement of the non-perturbative instanton sum.

More general physical problems (e.g. for special functions such as Bessel functions or Painlevé functions (see below), or in quantum mechanics, quantum field theory and matrix models [1, 2, 17, 18]) involve Borel transforms with cuts in the Borel plane. For these we use a generalized binary rational identity in terms of polylog functions (written here for $s < 0$ and $p > 0$)

$$
ps = \frac{1}{\Gamma(-s)} \left[ \text{Li}_{s+1}(e^{-p}) - \sum_{k=1}^{\infty} 2^{2k} \text{Li}_{s+1} \left( -e^{-p/2^k} \right) \right].
$$

(11)

Note that (3) is obtained for $s = -1$. The analytic continuation properties are given by properties of the polylog functions. An alternative representation is $p^s = \zeta(s, p) - \sum_{k=1}^{\infty} \frac{1}{2k} \zeta \left( s, \frac{1}{2} + \frac{k}{2} \right)$. Mathematically, the binary rational identities (3), (11) have a natural interpretation in terms of Umbral Calculus [19], as geometric Riemann sum approximations to the ‘proper-time’ integral:

$$
p^s = \frac{1}{\Gamma(-s)} \int_0^{\infty} \frac{dt}{t^{1+s}} e^{-pt}.
$$

(12)

The identity (3) can also be interpreted as expressing the Bose–Einstein distribution as a classical term minus a series of Fermi–Dirac distributions with rescaled temperature. This has numerous thermodynamic consequences. E.g. thermodynamic quantities for non-relativistic Bose ($+$) or Fermi ($-$) ideal gases are expressed in terms of polylog functions [20, 21] (here $\beta$ is the inverse temperature, $\mu$ the chemical potential, and $n$ is related to the spatial dimension and the particular physical quantity):

$$
f_{n}^{\pm} (\beta, \mu) = \frac{\beta^{n+1}}{n!} \int_0^{\infty} \frac{e^n \, d\varepsilon}{\varepsilon^{\beta} (\varepsilon - \mu) + 1} = \pm \text{Li}_{n+1} \left( \pm e^{\beta \mu} \right).
$$

(13)

The number densities in 3 dimensions are: $\lambda_3 (n)_{\text{Bose}} = \text{Li}_{3/2} (e^{\beta \mu})$, and $\lambda_3 (n)_{\text{Fermi}} = -\text{Li}_{3/2} (-e^{\beta \mu})$. Iterating the polylog duplication relation, $\text{Li}_{n} (q) + \text{Li}_{n} (-q) = 2^{1-n} \text{Li}_{n} (q^2)$, we learn that the bosonic and fermionic densities are related as (see figure 3):

$$
\langle n \rangle_{\text{Bose}} (\beta \mu) = \sum_{k=0}^{\infty} \frac{1}{2k+1} \langle n \rangle_{\text{Fermi}} (2^k \beta \mu).
$$

(14)

Similarly, for relativistic ideal gases, thermodynamic quantities are expressed in terms of the functions
The binary rational identity (3) implies, for example for $l = 1$:

$$
g^+_l (\beta, m, \mu) \equiv \frac{\beta^l}{\Gamma(l)} \int_0^\infty \left[ \frac{dq_q^{-1}}{e^{\beta(\sqrt{q^2 + m^2} - \mu)} + 1} - (\mu \to -\mu) \right]$$

$$
h^+_l (\beta, m, \mu) \equiv \frac{\beta^{l-1}}{\Gamma(l)} \int_0^\infty \left[ \frac{dq_q^{-1}/\sqrt{q^2 + m^2}}{e^{\beta(\sqrt{q^2 + m^2} - \mu)} + 1} + (\mu \to -\mu) \right].$$

(15)

The binary rational identity (3) implies, for example for $l = 1$:

$$
g^+_1 (\beta, m, \mu) \equiv \frac{\pi \mu}{\sqrt{m^2 - \mu^2}} - \sum_{k=1}^\infty g^-_1 \left( \frac{\beta}{2k}, m, \mu \right)$$

$$
h^+_1 (\beta, m, \mu) \equiv \frac{\pi / \beta}{\sqrt{m^2 - \mu^2}} - \sum_{k=1}^\infty \frac{1}{2k} h^-_1 \left( \frac{\beta}{2k}, m, \mu \right).$$

(16)

(Higher values of the index $l$ can be reached by taking derivatives with respect to $m$ and $\mu$.) These expressions naturally isolate the non-analytic term $\sqrt{m^2 - \mu^2}$ for bosons, whose physical importance for Bose–Einstein condensation has been emphasized in [22]. These Bose–Fermi relations can also be understood at the level of the partition function using the novel logarithmic identity (obtained from (11) by differentiating wrt $s$ at $s = 0$):

$$\ln p = \ln (1 - e^{-p}) = \sum_{k=1}^\infty \ln \left[ \frac{1}{2} \left( 1 + e^{-p/2^k} \right) \right].$$

(17)

The binary rational identities (3), (11) and (17) have interesting implications for trans-series expansions [17, 23], wherein a function $F(p)$ is expanded in terms of powers (and possibly iterations) of three basic ‘trans-monomial’ objects: $1/p$, $e^{-p}$, and $\ln p$. These are asymptotically independent functions. However, expressions (3), (11) and (17) show that powers of $p$ and logarithms of $p$ can indeed be expressed in terms of exponentials, but they require an infinite number of different exponentials. The logarithmic expression (17) has potential...
applications for the computation of entropy, partition functions and effective actions, as an alternative representation of the logarithm compared to conventional proper-time or replica methods.

For example, the effective actions for bosonic or fermionic fields on the hyperbolic manifold $H^2$ are basic building blocks for studying strong-coupling expansions of one-loop corrections for Wilson loop minimal surfaces in $\text{AdS}_5 \times S^5$ (see, e.g. [24]). They can be expressed as:

$$\Gamma_{\text{Bose}}(m) = \frac{V}{2\pi} \left( \zeta'(-1) + \frac{\ln 2}{12} + \int_0^{\frac{m^2 + 1}{2}} dx \, \psi \left( \frac{1}{2} + \sqrt{x} \right) \right)$$

$$\Gamma_{\text{Dirac}}(m) = \frac{V}{2} \left( -2\zeta''(-1) - \sqrt{m^2} + \int_0^{m^2} dx \, \psi \left( 1 + \sqrt{x} \right) \right)$$

where $V$ is the volume of $H^2$ [24]. The binary rational identities, combined with an integral representation of $\psi(x)$ (see the DLMF entry 5.9.E15 at [6]), imply another identity

$$\psi(1+x) = 2 \ln 2 + \sum_{k=1}^{\infty} \frac{1}{2^k} \psi \left( \frac{1}{2} + \frac{x}{2^k} \right). \quad (18)$$

Therefore, the effective actions are related as:

$$\frac{\pi}{V} \Gamma_{\text{Dirac}}(m) = -2\zeta'(-1) - \sqrt{m^2} + m^2 \ln 2 + \sum_{k=1}^{\infty} 2^k \left( \frac{2\pi}{V} \Gamma_{\text{Bose}} \left( \sqrt{\frac{m^2}{4^k} - \frac{1}{4}} \right) - \zeta'(-1) - \frac{\ln 2}{12} \right) \quad (19)$$

implying relations for the strong-coupling expansions. Relations of this kind are generic for QFT on constant curvature surfaces because of the appearance of $1/(e^\theta \pm 1)$ type factors in the heat kernel expressions [25], which can be related using the binary rational identity (3).

So far, we have described examples where an explicit Borel representation is known analytically. But another significant application of these new convergent binary rational series occurs when no such explicit Borel transform is known. To illustrate this numerical approach, consider the asymptotic expansion of the solution of the first Painlevé equation ($\text{PI}$). The Painlevé equations have numerous applications in physics, including random matrix theory, fluid mechanics, statistical physics, matrix models, and 2d quantum gravity [1, 2, 18, 26–30].

The standard form of $\text{PI}$, $y''(z) = 6y^2(z) + z$, can be converted to Boutroux-like form,

$$h''(x) + \frac{1}{x} h'(x) - h(x) - \frac{1}{2} h^2(x) - \frac{392}{625\Lambda^4} = 0 \quad (20)$$

more suitable for asymptotic analysis, by defining $y(z) = i \sqrt{\frac{2}{\xi}} \left( 1 - \frac{4}{\xi^2} + h(x) \right)$, where $x = e^{x/2} (24z)^{5/4}/30$ is the natural variable in terms of which the trans-series representation is [17, 31, 34, 35]:

$$h(x) = \sum_{k=0}^{\infty} C^k \xi^k(x) h_k(x). \quad (21)$$

The sum over $k$ is an ‘instanton sum’ [18, 31–34], with exponentially small ‘instanton’ factors $\xi(x) = e^{-x}/\sqrt{x}$, where $C$ is a trans-series parameter, and $h_k(x)$ represents the fluctuations about the $k$-instanton term. It is straightforward to generate many terms in these fluctuation expansions, each of which is a divergent series. We can convert the trans-series expression (21) into a unique global convergent binary rational inverse factorial expansion by the
following procedure [37]: (i) from the large-order behavior of the expansion coefficients we learn that the associated Borel transform has two square root branch cuts along the real axis, from $(-\infty, -1]$ and $[+1, +\infty)$; (ii) for each instanton sector, use the conformal map $p = \frac{2z}{z^2 + 1}$ to map the Borel transform $H_\beta(p)$ of $h_\beta(x)$ to the unit disc [36], then re-expand about $z = 0$, and re-express in terms of $p$ using the inverse map $z = p/(1 + \sqrt{1 - p^2})$. This procedure yields an optimal and global representation of the Borel transform in the doubly-cut plane; (iii) use the Cauchy kernel, and the resurgent properties of the trans-series to express the necessary Laplace transforms as integrals wrapped around the cuts. This step is efficiently done numerically by rotating the cuts to vertical lines emanating from the branch points at integer values of $p$; (iv) convert the resulting asymptotic expansions to binary rational expansions using the binary rational identity (3). Note that (3) can clearly be generalized by shifting and rescaling, $p \to \alpha + \beta p$, and there is in fact a unique optimal choice of $\beta$ for each instanton series. The output of these algorithmic steps is a unique, global and convergent representation of the function $h(x)$ in the appropriate sector of the physical $x$ plane. As an illustration see figure 4, which shows the smoothed Stokes jump [38, 39] in the (convergent) binary rational expansion as the phase of the variable $x$ is varied across the Stokes line: $\sqrt{\frac{6}{5\pi}} = 0.618$ (horizontal line). This shows again that the convergent binary rational expansion correctly encodes the Stokes phenomenon.

Figure 4. The jump across the Stokes line in the convergent binary rational expansion of the solution to the first Painlevé equation, as described in the text. The plot shows the imaginary part of the difference between the convergent binary rational expansion and the least-term truncation, $S_\theta(x) = \sqrt{x} e^x (h_{\text{binary rational}}(x) - h_{\text{least term}}(x))$, for $x = 14 e^{i\theta}$. Note that as the phase $\theta$ of the variable $x$ is varied across the Stokes line at $\theta = 0$, we clearly see the jump corresponding to the magnitude of the Stokes constant: $\sqrt{\frac{6}{5\pi}} = 0.618$ (horizontal line). This shows again that the convergent binary rational expansion correctly encodes the Stokes phenomenon.

5 Further details and examples in [37].
6 See footnote 4.
A novelty of this approach is that it is constructive, producing a provably convergent and unique expression which still correctly encodes the important non-perturbative physics of the Stokes phenomenon. Furthermore, since these new series are geometrically convergent, we may rigorously extrapolate from an asymptotic region with a large parameter to the opposite region of small parameter. This method is therefore ideally adapted to the study of dualities, and general relations between weak-coupling and strong-coupling. The analysis has also revealed new numerical procedures for evaluating Stokes constants with extremely high precision, potentially also those associated with subleading Borel singularities [37].

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