Escaping superoscillations

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Escaping superoscillations

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Abstract

A detailed classical model supports and elaborates an old prediction that in some circumstances gamma radiation could emerge from a box containing only red light. The ‘red light’, travelling nondispersively in a thin tube, is represented by a superoscillatory function: a band-limited function which in some regions oscillates faster than its fastest Fourier component. A weakly transmitting window opens and closes as the superoscillations pass by, releasing light into a half-plane. Detailed analysis of the spectrum of emerging light shows that the superoscillations escape, and propagate into the far field as ‘gamma radiation’, described precisely by a causal scattering amplitude.

Keywords: diffraction, scattering, weak measurement, optics

(Some figures may appear in colour only in the online journal)

1. Introduction

In a celebrated preprint [1], still unpublished after nearly thirty years, the authors proposed an apparently paradoxical scenario, in which a box containing red light would, when a window is opened, release gamma radiation. The prediction relies on the existence of band-limited functions (representing waves of red light), which possess, counterintuitively, regions where they vary arbitrarily faster than their fastest Fourier component: where they ‘superoscillate’. In such a region, the red light locally resembles light of much higher frequency (‘gamma radiation’), which would emerge from a window small enough to release only the superoscillations.

The physical argument in [1] inspired the mathematical theory of superoscillations [2–8], which has now reached a high level of sophistication [9, 10]. A central feature of superoscillations is that they occur where functions are relatively exponentially weak. There are applications, for example to sub-wavelength microscopy [11–14], and connections to earlier studies, for example in radar [15–17] and phase singularities in waves [18–22].

The emphasis in [1], and in an updated version [23], is on the quantum-physics question of how high-energy photons can be released from light composed only of low-energy ones; this is leading to new insights into conservation laws in quantum mechanics. Our aim here is...
different, and complementary: to develop a precise classical wave model showing how superoscillations can emerge and whether they can propagate into the far-field. The result is that they can: gamma radiation will indeed be released.

In the model, the ‘red light’ (section 2) is a superoscillatory wave travelling along the $x$ axis in a 1D tube: the ‘box’, mimicking a waveguide. The ‘window’ (section 3) is a segment of the $x$ axis that is opened and closed as the superoscillation passes by, weakly releasing a pulse of light into the 2D space ($x, z > 0$) (figure 1).

The waves will be solutions of the wave equation with speed $c = 1$, representing light, and the window corresponds to a time-dependent boundary condition at $z = 0$. The spectrum of the escaping pulse is calculated in section 4, and includes superoscillations, which constitute a fraction, calculated in section 5, of the emitted power.

The detailed form of the escaping pulse is explored in section 6, using an exact propagator formalism, leading to a description in terms of a time- and direction-dependent scattering amplitude, somewhat analogous to that from an atomic nucleus emitting real gamma radiation. Approximations corresponding to the superoscillatory regime reveal that the pulse escaping into the far field indeed contains the anticipated high frequencies: the ‘gamma radiation’.

**Figure 1.** A low-frequency wave, travelling along $x$ in a waveguide, possesses superoscillations passing by a window that opens and closes, releasing them into $z > 0$.

**Figure 2.** The superoscillatory region of the red light (2.1), and its approximation (2.2); outside this region, the amplitude rises to a maximum value of $a^N = 10^{10}$.

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**Figure 2.** The superoscillatory region of the red light (2.1), and its approximation (2.2); outside this region, the amplitude rises to a maximum value of $a^N = 10^{10}$. 

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2. The red light

For this we take a bandlimited function of \( x - t \) which is superoscillatory near \( x = t \), representing a wave travelling in the positive \( x \) direction with speed \( c = 1 \). For definiteness, and without loss of essential generality, we choose the standard superoscillatory function [1, 13]

\[
\psi_{\text{red}}(x, t) = \left( \cos \left( \frac{x - t}{N} \right) + i \sin \left( \frac{x - t}{N} \right) \right)^N = \sum_{m=-N/2}^{N/2} c_m \exp (im(x - t))
\]

\[
(a > 1, N \text{ even}, k_m = \frac{2m}{N}, c_m = \frac{N(a^2-1)^{\frac{N}{2}}(-1)^m + \frac{1}{2}}{2^{\frac{N}{2}}((\frac{1}{2}N+m)!((\frac{1}{2}N-m)!)}^{m})}
\]

(2.1)

This is band-limited, because \( |k_m| \leq 1 \). It is a rigidly moving polychromatic packet, with associated frequencies \( \omega_m = k_m \) which are all bounded; that is why we call it ‘red’. The packet is periodic with period \( \Delta x = \Delta t = \pi N \), but this feature is irrelevant for our present purpose.

The packet is superoscillatory, because near the origin it oscillates with wavenumber \( a \) [13]. We will need one order of approximation beyond the lowest, to capture the rapid ‘anti-gaussian’ increase away from the superoscillatory region, namely

\[
\psi_{\text{red,app}}(x, t) = \exp (ia(x - t)) \exp \left( \frac{(x - t)^2}{2X^2} \right), \quad X = \sqrt{\frac{N}{a^2 - 1}}
\]

(2.2)

The parameter \( a \) gives the superoscillatory frequency magnification, and \( \sqrt{N} \) is proportional to the width of the interval where superoscillations occur. The approximation (2.2) accurately represents the superoscillations, as illustrated in figure 2 over the interval \( -2X < x < 2X \), for \( t = 0 \).

3. The window

To capture the superoscillations, we select the region near \( x = 0 \) with a gaussian window of width \( L \), and open and close it with a gaussian switching function, over an interval near \( t = 0 \), whose duration we choose, mainly for notational convenience, to also be \( L \) (i.e. \( L/c \) with \( c \) reinstated). Thus the windowed red light, whose escape into \( z > 0 \) we are interested in, is, neglecting a coupling constant representing the transparency of the window,
\[
\psi(x, z = 0, t) = \psi_{\text{red}}(x, t) \exp \left( -\frac{(x^2 + t^2)}{2L^2} \right),
\]
(3.1)

and for the approximation (2.2),
\[
\psi_{\text{app}}(x, z = 0, t) = \exp(i \alpha (x - t)) \exp \left( + \frac{(x - t)^2}{2X^2} - \frac{(x^2 + t^2)}{2L^2} \right).
\]
(3.2)

These are boundary conditions at \(z = 0\) for the light released into the halfplane \(z > 0\).

For the window to select the superoscillations, we must choose its width to dominate the antigaussian increase of \(\psi_{\text{red}}\), that is, the ratio
\[
\lambda \equiv \frac{L}{X} = L \sqrt{\frac{a^2 - 1}{N}}
\]
(3.3)

must be small enough. As figure 3 shows, with \(\lambda = 1/2\) the windowed approximation captures the exact windowed wave very accurately.

Choosing the window to open and close near the fixed origin means that the windowed wave is no longer a function of the single variable \(x-t\). Therefore the wavenumber and frequency are decoupled, and, as we will see, this is a central feature, enabling the light to escape, including its superoscillations.

The window must faithfully transmit the red light, including its superoscillations. A conventional window or curtain, briefly opened and closed to release light from an illuminated room out into the dark night, allows people outside to see, accurately transmitted, what is inside. This is incoherent light, and (largely) geometrical optics, and any distortion is not evident. But for coherent waves, especially those containing superoscillations, the requirement of accurate reproduction is more stringent. Escape represents leakage of the red light from the tube, corresponding to a nonhermitian boundary condition for propagation along the \(x\) axis, threatening to distort both the wave inside and the wave that escapes. Superoscillations are particularly vulnerable, because they arise from near-perfect destructive interference between the fourier components, and are easily destroyed; for phase noise, this has been studied in detail [24]. One effect of nonhermiticity is to convert the modes in the red light, that is, the terms labelled \(m\) in (2.1), into resonances, whose frequencies \(k_m\) will generally be changed, in particular by acquiring imaginary parts. This will cause the wave to be distorted, unless the transparency of the window is very small, resulting in low intensity of the escaping gamma radiation. The leakage from the tube can be compensated by increasing the power of the source of red light.

4. The spectrum of escaping light

To understand how the light propagates into the halfplane \(z > 0\), we need the spectrum of the light at \(z = 0\), in terms of transverse (i.e. along \(x\)) wavenumber \(q\) and frequency \(\omega\). This is the space-time Fourier transform of \(\psi(x, 0, t)\) in (3.1), namely \(\tilde{\psi}(q, \omega)\), defined by
\[
\psi(x, 0, t) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dq \tilde{\psi}(q, \omega) \exp(i (qx - \omega t)).
\]
(4.1)

Each pair \(q, \omega\) corresponds to a plane wave, so the escaping wave is
\[
\psi(x, z, t) = \int_{-\infty}^{\infty} d\omega \int_{[-\omega]}^{[\omega]} dq \tilde{\psi}(q, \omega) \exp \left( i \left( qx + z \sqrt{\omega^2 - q^2} - \omega t \right) \right).
\]
(4.2)
The square root is positive real if $|q| < |\omega|$, corresponding to a wave travelling towards positive $z$, that is, escaping. If $|q| > |\omega|$, the square root is positive imaginary, representing evanescent waves. (Waves with $|q| < |\omega|$ and a negative real square root would correspond to light travelling towards negative $z$, which is not the region we consider.)

In the escaping light, we are interested in the superoscillations. These are the plane waves with frequencies $|\omega| > 1$, that is, frequencies greater than those in $\psi_{\text{red}}$, given by (2.1). More generally, we will study frequencies $|\omega| > \Omega$, to understand the distribution of superoscillations. ‘Gamma radiation’, represented by the fast superoscillations in (2.1), corresponds to $\Omega = a$. These waves of interest are the shaded regions of the spectrum in figure 4.

To proceed further, we need the spectrum of the windowed light. From (4.1) and (3.1), this is, exactly,

$$\bar{\psi}(q, \omega) = \frac{L^2}{2\pi} \sum_{-\frac{1}{2}N}^{\frac{1}{2}N} c_m \exp \left[ -\frac{1}{2}L^2 \left( (q - k_m)^2 + (\omega - k_m)^2 \right) \right],$$

and, for the approximation in (3.2),

$$\bar{\psi}_{\text{app}}(q, \omega) = \frac{L^2}{2\pi \sqrt{1 - 2\lambda^2}} \times \exp \left[ -\frac{L^2}{4(1 - 2\lambda^2)} \left( (q - a)^2 + (\omega - a)^2 - \lambda^2(\omega - q)^2 \right) \right],$$

with $\lambda$ given by (3.3). The approximation shows that the spectrum is centred around $\omega = q = a$, indicating the window has indeed selected the superoscillatory gamma radiation in the red light (with gaussian broadening corresponding to the line width of gamma radiation emitted by an atomic nucleus). Figure 5 is an illustration, calculated from the exact formula but visually indistinguishable from its approximate counterpart.

(With a larger window width $L$, the separate peaks in (4.3) are resolved, and the approximation (4.4) fails. This phenomenon is familiar in the shifted pointer wavefunctions in weak measurement [25–28].)

An important feature of the spectrum (both exact and approximate), to be exploited in the following, is symmetry under interchange of $q$ and $\omega$:

$$\bar{\psi}(q, \omega) = \bar{\psi}(\omega, q).$$

### 5. How much of the escaping light superoscillates?

The escaping light, including all frequencies and transverse wavenumbers $|q| < |\omega|$ is represented by the shaded region in figure 4 with $\Omega = 0$. From the symmetry (4.5), this is half the totality of light including all frequencies. The other half—not studied here—consists of evanescent waves clinging close to the $x$ axis.

Here we seek the fraction $F(\Omega)$ of escaping light with frequencies $|\omega| > \Omega$. This is

$$F(\Omega) = \frac{\int_{|\omega| > \Omega} d\omega \int_{-|\omega|}^{+|\omega|} dq |\bar{\psi}(q, \omega)|^2}{\int_{-\infty}^{\infty} d\omega \int_{-|\omega|}^{+|\omega|} dq |\bar{\psi}(q, \omega)|^2}.$$  

The superoscillatory fraction corresponds to $\Omega = 1$. To calculate the integrals, we again use the symmetry (4.5), as represented by figure 6.
Thus
\[ F(\Omega) = 1 - \frac{\int_{-\Omega}^{\Omega} d\omega \int_{-\Omega}^{\Omega} dq |\bar{\psi}(q, \omega)|^2}{\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dq |\bar{\psi}(q, \omega)|^2}. \] (5.2)

For the exact spectrum (4.3), the double integrals separate and can be evaluated analytically, with the result
\[ F(\Omega) = 1 - \frac{A}{B}, \quad \text{where} \]
\[ A = \sum_{-N}^{N} \sum_{-N}^{N} c_m c_n \exp \left[ -\frac{1}{2} L^2 (k_m - k_n)^2 \right] \]
\[ B = \sum_{-N}^{N} \sum_{-N}^{N} c_m c_n \exp \left[ -\frac{1}{2} L^2 (k_m - k_n)^2 \right] \]
\[ \times \left[ \text{erf} \left( L \left( \Omega + \frac{1}{2} (k_m + k_n) \right) \right) + \text{erf} \left( L \left( \Omega - \frac{1}{2} (k_m + k_n) \right) \right) \right]^2. \] (5.3)

For the approximate spectrum (4.4), the fraction is
\[ F_{\text{app}}(\Omega) = 1 - \frac{1}{\pi \sqrt{1 - 2\lambda^2}} \times \int_{-\Omega L}^{\Omega L} d\omega \int_{-\Omega L}^{\Omega L} dq \exp \left[ -\frac{1}{1 - 2\lambda^2} \left( (q - aL)^2 + (\omega - aL)^2 - \lambda^2 (q - \omega)^2 \right) \right]. \] (5.4)

It is easy to calculate the double integral numerically. But it seems that in general it cannot be evaluated analytically in closed form, except for small \( \lambda \), where
\[ F_{\text{app}}(\Omega) \rightarrow 1 - \frac{1}{4} \left( \text{erf} \left( (a - \Omega) L \right) - \text{erf} \left( (a + \Omega) L \right) \right)^2. \] (5.5)

As figure 7 shows, the approximation captures the behaviour of the exact function \( F(\Omega) \). Superoscillations correspond to \( \Omega > 1 \). It is clear (also from figure 5), that most of the escaping light is superoscillatory: the gamma radiation really does escape.
6. The propagating field

It is possible to calculate the escaping light using (4.2), as a superposition of plane waves labelled by $\omega$ and $q$. But with the combination of positive and negative $\omega$, and branch cuts, it is easy to lose track of causality. Therefore we prefer the more physically transparent spacetime representation, propagating the wave directly in $r, t$.

We seek the solution of

\begin{equation}
\nabla^2 \psi (x, z, t) = \partial_t^2 \psi (x, z, t)
\end{equation}

satisfying the time-dependent boundary condition (3.1) or (3.2) for the exact or approximate waves at $z = 0$, and propagating causally towards $z > 0$. This can be accomplished using the causal 2D propagator

\begin{equation}
G (x, z, t) = \cos \frac{\theta}{\pi r} \partial_t \left( \frac{t}{\sqrt{t^2 - r^2}} \Theta (t - r) \right), \quad (x = r \cos \theta, z = r \sin \theta),
\end{equation}

in which $\Theta$ denotes the unit step, and which satisfies (6.1) and

\begin{equation}
G (x, 0, t) = \delta (x) \delta (t)
\end{equation}

(normalization is easily checked by integrating over $x$ first). The asymmetry of the propagator, with a power-law tail decaying for $r < t$, is a familiar feature of dispersionless wave propagation in even dimensions [29].

By superposition, the wave propagating for $z > 0$ is obtained exactly after integrating by parts and using $\cos \theta = z/r$:

\begin{align*}
\psi (x, z, t) &= \int_{-\infty}^{\infty} d x' \int_{\sqrt{x'^2 + z^2}}^{\infty} d r' \psi (x - x', 0, t - t') G (x', z, t') \\
&= \partial_t \int_{-\infty}^{\infty} d x' \int_{\sqrt{x'^2 + z^2}}^{\infty} d r' \psi (x - x', 0, t - t') \\
&\quad \times \frac{\pi r}{\sqrt{r^2 - x'^2}} \\
\end{align*}

This can be computed for any position and time. But we are particularly interested in the escape of gamma radiation into far field, where the formula simplifies because the initial wave $\psi (x, 0, t)$ is localized in $x$ and $t$. Use of the inequalities and definitions

![Figure 5. The power spectrum of escaping light, from (4.3).](image-url)
leads to

\[
\psi(x, z, t) \rightarrow \frac{S(\rho, \theta)}{\sqrt{r}}, \quad r \rightarrow \infty
\]

in which

\[
S(\rho, \theta) \equiv \partial_{\rho} s(\rho, \theta), \quad s(\rho, \theta) = -\frac{\cos \theta}{\pi \sqrt{r}} \int_{-\infty}^{\infty} d\xi \int_{0}^{\infty} \frac{dr}{\sqrt{r}} \psi(\xi, 0, -\rho - \tau + \xi \sin \theta).
\]

(6.7)

Figure 6. Application of the symmetry (4.5) to calculate the fraction of escaping light with \( \omega > \Omega \).

\[
|x - x'| \ll |x|, \quad |t - t'| \ll |t|, \quad |r - t| \ll |r|,
\]

\[
x - x' \equiv \xi, \quad t' \equiv \sqrt{x'^2 + z'^2 + \tau}, \quad r - t \equiv \rho
\]

\[\text{(6.5)}\]

is a scattering amplitude, and \(|S|^2\) is the scattering cross section, which in this 2D case has the dimensions of length. With slight abuse of terminology, we will denote by ‘scattering amplitude’ both the functions \(S(\rho, \theta)\) and \(s(\rho, \theta)\).

\(S\) and \(s\) can be evaluated exactly for both the exact and approximate boundary conditions (3.1) and (3.2). To see this, observe that the integrand in (6.7) involves an exponential quadratic in both \(\xi\) and \(\tau\). Therefore the \(\xi\) integral is gaussian and can be evaluated immediately. The remaining integral over \(\tau\) is of the form

\[
g(A) = \exp \left( -\frac{1}{2} A^2 \right) \int_{0}^{\infty} \frac{du}{\sqrt{\pi}} \exp \left( -\frac{1}{2} u^2 - Au \right) = \int_{-\infty}^{\infty} dv \exp \left( -\frac{1}{2} (v - A)^2 \right)
\]

\[
= \exp \left( -\frac{1}{2} A^2 \right) \left\{ \begin{array}{ll}
\sqrt{\frac{1}{2} A K_{1/2} \left( \frac{1}{2} A^2 \right)} & (\text{Re} A > 0) \\
\sqrt{-\frac{1}{2} A} \left( K_{1/4} \left( \frac{1}{2} A^2 \right) + \pi \sqrt{2} I_{1/2} \left( \frac{1}{2} A^2 \right) \right) & (\text{Re} A < 0)
\end{array} \right.
\]

\[\text{(6.8)}\]

(The analytical evaluation is straightforward in Mathematica™, and the two forms provide the analytic continuation necessary to accommodate the fact that the fractional-order Bessel functions \(K\) and \(I\) have branch cuts but the integral \(g\) is a single-valued function of complex \(A\). Numerical evaluation is also easy, because the Bessel functions are standard in Mathematica.)
Straightforward calculation now gives the exact scattering amplitude as

\[ s(\rho, \theta) = -\frac{L^{3/2} \cos \theta}{\sqrt{\pi(1 + s^2)^{3/2}}} \times \sum_{m=-N/2}^{N/2} c_m \exp \left( -k_m^2 r^2 \right) g \left( \frac{\rho/L - i k_m (1 + s)}{\sqrt{1 + s^2}} \right), \quad (s \equiv \sin \theta). \]  

(6.9)

The corresponding approximate amplitude is

\[ s_{\text{app}}(\rho, \theta) = -\frac{L^{3/2} \cos \theta \exp \left( -a^2 r^2 / (1 - 2\lambda^2) \right)}{\sqrt{\pi \left( 1 - 2\lambda^2 \right) \left( 1 + r^2 - \lambda (1 - s)^2 \right)^{3/2}}} g \left( A(\rho, \theta) \right) \]

where \( A(\rho, \theta) = \frac{\sqrt{1 - 2\lambda^2 \rho^2 / L^2 - \lambda (1 - s)^2}}{\sqrt{1 + r^2 - \lambda (1 - s)^2}} \) and \( s \equiv \sin \theta \).

(6.10)

Computation shows that this approximation accurately reproduces the exact amplitude (6.9) if the window width \( L \) is chosen according to (3.3) with \( \lambda \leq 1/3 \) (slightly more restrictive than \( \lambda = 1/2 \) which suffices for the window itself—see figure 3).

Figure 8 shows a sample scattering amplitude \( s \) at a fixed time. (Calculations with \( s_{\text{app}} \) give a visually indistinguishable picture.) The oscillations correspond to the superoscillations in \( \psi_{\text{red}} \), indicating graphically the escape of the gamma radiation.
To see these escaping superoscillations, \( N \) needs to be quite large. In the approximation (3.2) for \( t = 0 \), the superoscillatory wavelength \( 2\pi/a \) must be smaller than the width of the windowed red light. Taking the \( 1/e \) width, and using (2.2) and (3.3), leads to

\[
N > \frac{\pi^2 (1 - \lambda^2)}{2a^2 \lambda^2}.
\]

(6.11)

For figure 8 this gives \( N > 15\pi^2/4 \approx 39 \), which is comfortably satisfied.

To get more insight into the approximate amplitude (6.10), we employ a further approximation, based on the following asymptotics of \( g(A) \) for complex \( A \), derived in the appendix:

\[
g(A) \approx \sqrt{\frac{\pi}{A}} \exp\left(-\frac{1}{4}A^2\right) + \sqrt{\frac{2\pi}{A}} \Theta(-\text{Re } A)
\]

(6.12)

The first term dominates if \( \text{Re}(A^2) < 0 \), i.e. if

\[
\frac{|\rho|}{L} < \frac{aL(1 + s)}{1 - 2\lambda^2}.
\]

(6.13)

This is satisfied for the cases of interest here, where the window releases many superoscillations. Then the envelope of the scattered pulse is a symmetric function of \( \rho \); the resulting approximation of (6.10), denoted \( s_{\text{app},1} \), is

\[
s_{\text{app},1}(\rho, \theta) = -L^{1/2} \cos \theta \exp\left(-a^2 L^2 (1 - s)^2 / (2F)\right)
\]

\[
\times \frac{\exp\left(-(1-2\lambda^2)|\rho/\sqrt{L-aL(1+s)}|/F\right)}{\sqrt{(1-2\lambda^2)|\rho/\sqrt{L-aL(1+s)}|} F \equiv 1 + s^2 - \lambda^2 (1 - s)^2, s \equiv \sin \theta}.
\]

(6.14)

By contrast, when the second term in (6.12) is significant the pulse is not symmetric, reflecting the lack of symmetry of the causal propagator (6.2).

For a quantitative analysis, we examine the frequency \( f(s) \) (spatial and temporal), in different directions \( \theta \). From the phase in the exponential in \( s_{\text{app},1} \), this is

\[
f(s) = \frac{a(1 + s)}{1 + s^2 - \lambda^2(1 - s)^2}.
\]

(6.15)

The mean frequency, over directions \( -\pi/2 \leq \theta \leq \pi/2 \), weighted with the cross section \( \langle s_{\text{app},1}\rangle^2 \), and evaluated at the centre \( \rho = 0 \) of the pulse, is

\[
\bar{f} = \int_{-1}^{1} \frac{ds \sqrt{1 - s^2} f(s) \exp\left(-\frac{a^2 L^2 (1 - s)^2}{1 + s^2 - \lambda^2 (1 - s)^2}\right)}{1 + s} / (1 + s).
\]

(6.16)

As figure 9 illustrates, this is remarkably close to the superoscillation ‘gamma’ radiation frequency \( a \), for all \( \lambda \), and increasingly close as \( a \) increases.

In contrast to the gamma radiation from a real nucleus, our escaping superoscillations are strongly directional, and more so for increasing \( a \). The cross section \( \langle S \rangle^2 \) (or \( \langle |S|^2 \rangle \)), either near the pulse maximum \( \rho = 0 \) or when integrated over \( \rho \), gets sharper in angle as \( a \) increases, and approaches \( \theta = 90^\circ \). To a good approximation,

\[
\int_{-\infty}^{\infty} d\rho |S(\rho, \theta)|^2 \propto \cos^2 \theta \exp\left(-\frac{a^2 L^2 (1 - \sin \theta)^2}{F (\sin \theta)^2}\right),
\]

(6.17)
whose maximum for large $a$ is close to
\[ \theta_{\text{max}} = \frac{\pi}{2} - \sqrt{\frac{2}{aL}}, \]  
(6.18)
and almost independent of $\lambda$. This concentration near $\theta = 90^\circ$, i.e. towards positive $x$, is unsurprising, because the original ‘red light’ superoscillations are travelling in the positive $x$ direction before they escape. For the case illustrated in figure 8, $aL = 2.44$, which is not large, (6.18) gives $\theta_{\text{max}} \approx 38^\circ$, and a more accurate calculation gives $\theta_{\text{max}} \approx 44^\circ$.

7. Concluding remarks

Our detailed analysis of a consistent model for releasing superoscillatory light has demonstrated that the ‘gamma radiation’ escapes and propagates into the far field, in the form of a pulse described, exactly and at several levels of approximation, by a time- and direction-dependent scattering amplitude. The light is released into a half-plane but the analysis of the analogous 3D model should be qualitatively similar.

It is natural to ask about the origin of the energy required to convert red light into the escaping gamma radiation. At the quantum level, this is studied in detail elsewhere [1, 23]. For our classical model, it is clear from the analysis of the spectrum in section 4 where the higher frequencies come from: they are introduced into the band-limited ‘red light’ by the spacetime structure (opening and closing) of the window. This converts the superoscillations, which are low frequencies masquerading as high frequencies, into genuine high frequencies in the spectrum of escaping light.

Our model has been designed to represent non-dispersive waves such as light, because the low-to-high frequency conversion is more dramatic when there is no dispersion, and because the original ‘red to gamma’ claim [1] was expressed in terms of light. But similar phenomena will surely occur for dispersive waves, for example water waves, or waves representing quantum particles with non-zero mass.
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Appendix. Asymptotics of $g(A)$

This is the derivation of (6.12). For large $|A|$, the integral over $v$ in (6.8) is governed by its three saddle-points, at $v = 0$ and $v = \pm \sqrt{(-A)}$. The contributing saddles are determined by the deformation of the real-axis contour into paths of steepest descent. For Re$A > 0$, the steepest-descent contour, shown in figure A1(a), passes only through the saddle at $v = 0$. For Re$A < 0$, the contour, shown in figure A1(b), passes through all three saddles. Then standard saddle-point asymptotics [30, 31], applied to the contributing saddles, gives (6.12). The transition between the two cases is Re$A = 0$, which is a Stokes line [32–34] for the $v$ integral in (6.8). Complementing this is the antiStokes line Re$A^2 = 0$ (i.e. Re$A = \text{Im}A$ & Re$A < 0$), where the two contributions in (6.11) exchange dominance.

The accuracy of these approximations for rather modest values of $A$ is illustrated in figure A2. In figure A2(a), the second term in the asymptotics (6.12) is significant, and the profile envelope is asymmetric. In figure A2(b), the first term dominates and the profile envelope is symmetric.

Figure A1. Contours for the $v$ integral in $g(A)$ (equation (6.8)): (a) Re$A > 0$; (b) Re$A < 0$.

Figure A2. Comparison of $g(A)$ (equation (6.8)) with the approximation (6.10).
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