LETTER

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To cite this article: AElgart et al 2018 J. Phys. A: Math. Theor. 51 01LT02

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Droplet localization in the random XXZ model and its manifestations

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Received 12 August 2017, revised 25 October 2017
Accepted for publication 31 October 2017
Published 6 December 2017

Abstract
We examine many-body localization properties for the eigenstates that lie in the droplet sector of the random-field spin-$\frac{1}{2}$ XXZ chain. These states satisfy a basic single cluster localization property (SCLP), derived in Elgart et al (2018 J. Funct. Anal. (in press)). This leads to many consequences, including dynamical exponential clustering, non-spreading of information under the time evolution, and a zero velocity Lieb–Robinson bound. Since SCLP is only applicable to the droplet sector, our definitions and proofs do not rely on knowledge of the spectral and dynamical characteristics of the model outside this regime. Rather, to allow for a possible mobility transition, we adapt the notion of restricting the Hamiltonian to an energy window from the single particle setting to the many body context.

Keywords: many-body localization, XXZ spin chain in a random field, exponential clustering, Lieb–Robinson bound

Introduction
Noninteracting electrons in strongly disordered media are characterized by Anderson localization [1]; all electronic states are localized, dc transport is absent. Evidence from perturbative [2–7] and numerical [8–11] approaches suggests that some features of localization persist in one-dimensional spin systems and particle systems in the presence of weak interactions, even if the system is locally constrained [12]. Numerical studies [10, 11, 13–15] and the renormalization group approach [16] support the existence of transition from a many-body localized
(MBL) phase to delocalized phases as the strength of interactions increases. See [17] for a review of recent work on MBL.

Rigorous results on localization in truly many-body systems that are uniform in the particle number are scarce (e.g. [18–20]); the behavior of quasi-free systems is better understood (e.g. [21]). In this announcement we report on new findings for the one-dimensional disordered XXZ spin chain, in the ‘zero temperature localization regime’.

Results

The infinite XXZ chain in a random field is given by the Hamiltonian

\[ H = H_\omega = H_0 + \lambda B_\omega, \quad H_0 = \sum_i h_{i,i+1}, \quad B_\omega = \sum_i \omega_i N_i, \]

acting on quantum spin-\(1/2\) configurations on the one-dimensional lattice \(\mathbb{Z}\). The local next-neighbor Hamiltonian \(h_{i,i+1}\) is given by

\[ h_{i,i+1} = (I - S_i^z S_{i+1}^z) - \Delta^{-1} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y), \]

where \(S_i^x, S_i^y, S_i^z\) are the spin matrices (\(\sigma^x, \sigma^y, \sigma^z\) are the standard Pauli matrices) and \(\Delta > 0\) is a parameter.

The Heisenberg chain is given by \(\Delta = 1\) and we get the Ising chain in the limit \(\Delta \to \infty\). In this letter we consider \(\Delta > 1\), the Ising phase of the XXZ chain.

The local magnetic fields \(N_i = S_i^z\) are projections onto the down-spin state (or local number operators) at site \(i\). The positive parameter \(\lambda\) describes the strength of the disordered longitudinal magnetic field \(B_\omega\). We choose the parameters \(\{\omega_i\}\) as independent identically distributed random variables with a bounded density supported on \([0, \omega_{\text{max}}]\). In particular, the random field \(B_\omega\) is non-negative. We have normalized both \(H_0\) and \(B\) so that the ground state energy of \(H\) is \(E_0 = 0\), independent of the random parameters \(\omega\), with the ground state given by the all-spins-up configuration. The random field fills in all spectral gaps of \(H_0\) other than the ground state gap \(1 - \Delta^{-1}\), so that the spectrum of \(H\) consists with probability one of the vacuum energy \(0\) and the ray \([1 - \Delta^{-1}, \infty)\).

The XXZ chain preserves the total particle number \(N = \sum_i N_i\) and its restriction to the \(N\)-particle sector is unitarily equivalent to an \(N\)-body discrete Schrödinger operator \(H_N\) over the ordered lattice points

\[ \mathcal{X}_N = \{ x = (x_1, \ldots, x_N) \in \mathbb{Z}^N : x_1 < x_2 < \ldots < x_N \} \]

of \(\mathbb{Z}^N\). It is given by

\[ H_N = -\frac{1}{2} (-\Delta)^{-1} \mathcal{L}_N + (1 - \Delta^{-1}) W + \lambda V_\omega, \]

where \(\mathcal{L}_N\) is the graph Laplacian on \(\mathcal{X}_N\),

\[ (\mathcal{L}_N \psi)(x) = \sum_{y \sim x} (\psi(y) - \psi(x)), \]

\(W\) is the attractive (in the Ising phase \(\Delta > 1\)) next-neighbor interacting potential,

\[ W(x) = 1 + \# \{ j : x_{j+1} \neq x_j + 1 \}, \]

and \(V_\omega\) is the \(N\)-body random potential given by
\begin{equation}
(V_\omega \psi)(x) = \left( \sum_{j=1}^{N} \epsilon_j \right) \psi(x).
\end{equation}

It follows from the positivity of \(-L_N, W, V_\omega\) that for energies in the droplet band
\[
\hat{I} = [1 - \Delta^{-1}, 2 (1 - \Delta^{-1})]
\]
only only the simply connected ‘droplet’ configurations (e.g. \([22]\))
\[
X_{\hat{I}_{1}} = \{ x = (x_1, x_1 + 1, \ldots, x_1 + N - 1) \}
\]
(the ‘edge’ of \(X_N\)) are classically accessible for all values of \(N\). A (naive) removal of the classically forbidden region for each \(N\) maps the problem to the well studied one dimensional Anderson model (with a correlated random potential). The latter is characterized by complete Anderson localization, for all non-zero values of \(\lambda\). The density of states in the droplet band rapidly decreases with \(N\).

The rigorous passage from the whole \(X_N\) space to the edge \(X_{\hat{I}_{1}}\) is implemented by means of Schur complementation (the Feshbach map). It expresses the edge-restricted Green’s function \(G_E\) of the full operator \(H_N\) at energy \(E\) in terms of the Green’s function of an effective Hamiltonian \(K_E\) defined on \(X_{\hat{I}_{1}}\).

The operator \(K_E\) is comprised of two parts: The first one is simply the restriction of \(H_N\) to the edge as in the naive description, while the second part encodes the influence of the bulk. The technical difficulties associated with the addition of the second term are two-fold: On one hand, it is non-local and non-linear (in \(E\)), on the other, it is statistically dependent on the randomness associated with the first term. Both issues can have potentially fatal consequences as far as localization is concerned: Non-locality allows for hopping between distant sites while strongly correlated randomness can amplify the effect of resonances, suppressed in the non correlated case.

It turns out that \(K_E\) is in fact a quasi-local operator for energies in the droplet band, i.e. its kernel exhibits rapid spatial decay. This property can in turn be used to overcome the correlations issue as well. A careful analysis based on the fractional moment method (e.g. \([23]\)) yields the following basic result on droplet localization, formulated in terms of the finite volume Hamiltonians
\[
H^{(L)} = \sum_{i=-L}^{L-1} h_{i+1} + \lambda \sum_{i=-L}^{L} \omega_i \mathcal{N}_i + \beta (\mathcal{N}_{-L} + \mathcal{N}_L)
\]
on \(H = \otimes_{j=-L}^{L} \mathbb{C}^2\), where the presence of boundary terms ensures preservation of the droplet regime (for \(2\beta \geq 1 - \Delta^{-1}\)). We will generally write \(H\) for \(H^{(L)}\), with the understanding that all bounds are uniform in \(L\).

**Theorem 1 (\([19]\)).** For any \(1 < \gamma < 2\) consider the subinterval 1 of \(\hat{I}\) of the form
\[
I = [1 - \Delta^{-1}, \gamma (1 - \Delta^{-1})]
\]
Let \(\sigma_I\) denote the collection \(\{E\}\) of all eigenvalues of \(H\) in the interval 1, and let \(\{\psi_E\}\) be the corresponding normalized eigenvectors. Then, if \(\lambda \sqrt{\Delta - 1} \min (1, \Delta - 1)\) is sufficiently large, there exist constants \(C < \infty\) and \(m > 0\) such that
\[
\mathbb{E} \left( \sum_{E \in \sigma_I} \| \mathcal{N}_i \psi_E \| \| \mathcal{N}_j \psi_E \| \right) \leq C e^{-m|j-i|}.
\]
(1)

Here \(\mathbb{E}\) is the expectation with respect to the randomness.
This result confirms that the droplet eigenstates of $H$ are localized quasi-particles. The fact that one can perform the summation in equation (1) above, indicates that there are not too many of these states (per unit length of the system) in the interval $I$. Theorem 1 applies, e.g. to the regime of large disorder $\lambda$ (at fixed $\Delta > 1$) as well as to the semiclassical regime of large $\Delta$ (at fixed $\lambda > 0$).

Our goal in this letter is to draw some conclusions concerning the dynamics of $H$ based exclusively on equation (1) above. For completely localized many-body systems, the dynamical manifestation of localization is often expressed in terms of the non-spreading of information [24], i.e. for $X$ supported at site $i$, all times $t$, and each integer $\ell$, the existence of an observable $X_{\ell}(t)$ supported on $[i - \ell, i + \ell]$ such that

$$\|\tau_t(X) - X_{\ell}(t)\| \leq C \|X\| e^{-m\ell}.$$ 

An alternative (and, if formulated in terms of general local observables, equivalent) description is the zero-velocity Lieb–Robinson (LR) bound

$$\|\|\langle \tau_t(X), Y \rangle\| \leq C \|X\| \|Y\| e^{-m|\ell-j|},$$

where $X, Y$ are local observables supported at sites $i$ and $j$, respectively, and $\tau_t(A) = e^{itH}Ae^{-itH}$, e.g. [24–27].

The difficulty in even formulating our results is due to the fact that we only know the structure of the eigenstates in the droplet interval $I$ and cannot assume complete localization for all energies. In fact, numerical studies suggest the presence of a mobility edge for $\lambda$ sufficiently small, [10, 11, 13, 14]. The expectation is that for all energies below this mobility edge, the spreading of the initial wave packet is blocked, while it might be possible for higher energies. To express this phenomenon mathematically, we consider a set $J \subset \mathbb{R}$ and the subspace $V_J$ spanned by the eigenstates of $H$ with energies in $J$. For any operator $X$ in $\mathcal{H}$ we will denote by $X_{\ell}$ its restriction to $V_J$, i.e. $X_{\ell} = P_J XP_J = \sum_{E, F \in \sigma_{\ell}} \langle \psi_E | X | \psi_F \rangle | \psi_E \rangle \langle \psi_F |$, where $P_J$ is the spectral projection of $H$ onto $J$. All our results below hold uniformly in time.

Theorem 2 (Non spreading of information). Equation (1) implies that for any $\gamma \in (1, 2)$ and $I_0 = [0, \gamma (1 - \Delta^{-1})]$, and any observable $X$ supported at site $i$, there exists an observable $X_{\ell}(t)$ supported on $[i - \ell, i + \ell]$ such that

$$\mathbb{E} \left( \|\tau_t(X) - X_{\ell}(t)\|_{I_0} \right) \leq C \|X\| e^{-m\ell}. \tag{2}$$

Theorem 3 (Zero velocity LR bound). Let $X, Y$ be observables in $\mathcal{H}$ supported at $i$ and $j$, respectively. Equation (1) implies that for $I$ as in theorem 1,

$$\mathbb{E} \left( \|\tau_t(X), Y)\| \right) \leq C \|X\| \|Y\| e^{-m|\ell-j|}. \tag{3}$$

Note that the interval $I$ where equation (3) holds is smaller than the set $I_0$ in theorem 2, namely it does not include the ground state energy 0. In fact, the zero velocity LR bound does not hold as stated for $I_0$. Instead, for all $t$,

$$\mathbb{E} \left( \|\tau_t(X_{\ell}), Y)\| \right) \leq C \|X\| \|Y\| e^{-m|\ell-j|}. \tag{4}$$

with the correction term $T = \tau_t(X) P_0 Y - YP_0 \tau_t(X)$, where $P_0$ denotes the ground state (vacuum) projection for $H$. This correction is not zero for some observables $X, Y$, even for $t = 0$. The correction terms are due to the interaction between the vacuum and one
quasi-particle sector present in the energy window $I_0$. It is thus natural to ask when (and in what form) a zero-velocity LR bound will hold in a given energy window.

The crucial property of the Hamiltonian used in the usual LR bound is its locality (or, more generally, quasi-locality). It is tempting to replace the LR bound on $[\tau_t(X), Y]$ with the one for $[\tau_t^2(X), Y]$, where the truncated evolution $\tau_t^2$ is defined by $\tau_t^2(A) = e^{ith} Ae^{-ith}$, with $H_I = \sum_{E \in \sigma_I} E |\psi_E\rangle \langle \psi_E|$, the restriction of $H$ to the energy window $I$ of localization. This approach indeed works well in the one particle setting, since localization in $I$ implies that the operator $H_I$ is quasi-local (i.e. its matrix elements $H_I(x,y)$ decay rapidly in $|x - y|$). This feature is, however, lost in the many body picture (even $H^2$, say, is no longer a local operator).

Instead, we define quasi-locality on any subspace $V$ of $\mathcal{H}$ as follows: To each lattice site $i$ we assign a family $\mathcal{A}_i$ of observables (i.e. operators) on $V$. We say that the collection of these families is quasi-local if

$$\|[X_i, Y_j]\| \leq Ce^{-m|j-i|} \|X_i\| \|Y_j\|$$

for any $X_i$ in $\mathcal{A}_i$ and $Y_j$ in $\mathcal{A}_j$. An operator $K$ on $V$ is quasi-local if it can be written as $K = \sum_i X_i$, where $X_i \in \mathcal{A}_i$, for all $i$.

In this language, theorem 3 establishes (for $t = 0$) that restrictions of local observables to the subspace $V_I$ are a quasi-local collection of observables and thus the restricted Hamiltonian $H_I$ is quasi-local (as $H$ is a local operator in $\mathcal{H}$). On the other hand, restrictions of local observables to the subspace $V_{I_0}$ are not quasi-local (as equation (4) attests), a feature consistent with the failure of the usual zero-velocity LR bound in this energy interval.

Using theorem 3, equation (4) and the analysis presented below in the proof of theorem 2, one obtains an LR-type bound in $I_0$ but for higher order commutators. Namely, let $X, Y$, and $Z$ be observables in $\mathcal{H}$ supported on $i$, $j$ and $k$, respectively. Then

$$\mathbb{E} \|[\tau_t(X_k), \tau_s(Y_j), Z_i]\| \leq C \|X\| \|Y\| \|Z\| e^{-\tilde{m}R} \tag{5}$$

for all $t$ and $s$, where $R = \min(|i-j|, |i-k|, |j-k|)$.

The number of commutators taken is related to the number of quasi-particles in the corresponding energy interval: in $I_0$ we have eigenstates which correspond to either zero or one quasi-particles, hence the double commutator in equation (5).

We next turn our attention to another manifestation of dynamical localization, namely the exponential clustering property. For a pair of observables $X, Y$ we define the correlator $R_{XY}(\psi)$ of a (normalized) state $\psi$ as

$$R_{XY}(\psi) = |\langle \psi, XY \rangle \psi \rangle^2 - |\langle \psi, X \psi \rangle \langle \psi, Y \psi \rangle|^2.$$

We will denote by $S_X$ the support of the observable $X$. We have the following result:

**Theorem 4 (Dynamical exponential clustering [19]).** Equation (1) implies that for all local observables $X$ and $Y$ with $\ell = \min S_Y - \max S_Y > 0$,

$$\mathbb{E} \left( \sum_{R \in \sigma_I} R^{(\tau_t)(X), Y}(\psi_E) \right) \leq C \|X\| \|Y\| e^{-\tilde{m}R} \tag{6}.$$

The interesting feature here is that the correlator $R^{(\tau_t)(X), Y}(\psi_E)$ is not equal to the one defined in $V_I$, i.e. $R_{(\tau_t)(X), Y}(\psi_E)$ (for which the result above also holds). In particular, we can no longer consider $H_I$ as a quasi-local operator (since the correlator accounts also for the ‘spills’ outside $V_I$). The exponential decay that we see in equation (6) is due to the special structure of the correlator, as we shall see now by considering a more general form of the correlator above, where these spills become explicit. We first observe that one can write $R_{XY}(\psi) = |\text{tr} PXYPY|$ where
\( P = |\psi\rangle\langle\psi| \) and \( \bar{P} = 1 - P \). One can consider a more general operator of this form, by replacing \( P \) with \( P_I \), the spectral projection of \( H \) onto \( I \), i.e. \( P_I = \sum_{E \in \sigma_I} |\psi_E\rangle\langle\psi_E| \). Then the correlator \( R_{XY}(\psi_E) \) is closely related to the diagonal elements of the operator \( R_{X,Y} \). Analysis similar to the one performed in [19], augmented by an argument of Hastings [28, 29] (which combines LR bounds and the Fourier transform), yields the following bound. Let \( X, Y \) be observables in \( \mathcal{H} \) supported on \( i \) and \( j \), respectively. Then equation (1) implies that for all \( 0 < \alpha < 1 \),

\[
\mathbb{E} \left( \| R_{T_i(X,Y)}(I) - T \| \right) \leq C_\alpha \|X\| \|Y\| e^{-\alpha \beta \|}.
\]

where \( T = P_I \left( \tau_{\alpha}^I(X) P_0 Y + \tau_{\alpha}^I(Y) P_0 X \right) P_I \).

The diagonal contributions in \( T \) are exponentially small in \( |i - j| \) by equation (1), since a straightforward computation gives \( |\langle \psi_0 |X| \psi_E\rangle| = |\langle \psi_0 |X| \psi_E\rangle| \leq \|X\| \|N \psi_E\| \), and similarly for \( |\langle \psi_E |Y| \psi_0\rangle| \). This explains why we do not see these corrections in equation (6). The interesting feature in the structure of \( T \) is that it involves only \( P_0 \), but the dynamical evolution in the second term sits in the ‘wrong’ place - it is \( \tau_{\alpha}^I(Y) \) and not \( \tau_{\alpha}^I(X) \)! In fact this term encodes information about states above the energy window \( I \), and appearance of the evolution in the wrong spot is related to the reduction of this data to \( P_0 \) via Hastings’ argument mentioned earlier.

**Proofs**

We sketch the proofs of our main new results, theorems 2 and 3. Further details shall be given elsewhere.

**Theorem 2:** We will only consider the case when the interval \( I_0 \) is replaced by \( I \) (the general case is similar but longer). Let \( S \) be the complement of the set \( [i - \ell/2, i + \ell/2] \), and let \( T = [i - \ell, i + \ell] \cap S \). Given a subset \( \mathcal{R} \) of \( \mathcal{E} \), we define projections \( P_\mathcal{R} \) by \( P_\mathcal{R} = \prod_{E \in \mathcal{R}} (1 - N_E) \) and \( P_\mathcal{R} = 1 - P_\mathcal{R} \).

We first decompose \( X = X^{+} + X^{-} + X^{-} + X^{-} \), where \( X^{+} = P_+ (i) X P_+ (i) \), etc. Clearly, it suffices to approximate the dynamical evolution for each one of these four components separately. Since for any \( X \), \( e^{t \mathcal{L}} = e^{t \mathcal{L}} e(1 - P_-(i)) \) for some constant \( e \), and \( \tau_{\alpha}^I = e \), we can reduce the problem to the approximation of the three remaining components (i.e. we can assume that \( X^{+} = 0 \)). In this case we claim that the operator \( X_\ell(t) \) is given by

\[
X_\ell(t) = P_+ (T) Y(t),
\]

where \( Y(t) \) is supported on \([i - \ell/2, i + \ell/2] \), satisfies \( \| Y(t) \| \leq \| X \| \), and is uniquely defined by the relation

\[
P_+ (S) \tau_{\alpha} (X_\ell) P_+ (S) = P_+ (S) Y(t).
\]

To verify this choice of \( X_\ell(t) \), it essentially suffices to prove that

\[
\tau_{\alpha} (X_\ell) - (P_+ (S) \tau_{\alpha} (X_\ell) P_+ (S))_j
\]

is exponentially small in \( \ell \), in expectation. To this end, we bound

\[
\| \tau_{\alpha} (X_\ell) - (P_+ (S) \tau_{\alpha} (X_\ell) P_+ (S))_j \|_l \leq \| (\tau_{\alpha} (X_\ell) P_+ (S))_j \| + \| (P_+ (S) \tau_{\alpha} (X_\ell))_j \| .
\]

\[\text{Elgart et al (to be published).}\]
For the $X_{+,-}$ and $X_{-,+}$ components, the contributions to $\|\tau_i (X_j) P_{-} (S)\|$ can be bounded as
\[
\|X\| \mathbb{E} \|P_{-} (i e^{-itH} P_j P_{-} (S))\| \leq \|X\| \sum_{j \in S} \sum_{E \in \sigma_j} \|\mathcal{N}_j \psi_E\| \|\mathcal{N}_j \psi_E\| \\
 \leq C \|X\| e^{-\tilde{m} t},
\]  
(10)

where in the last step we have used equation (1). For $X_{+,+}$, the contribution to $\|\tau_i (X_j) P_{-} (S)\|$ can be bounded as
\[
\|P_j X_{+,+} P_{-} (R)\| + \|X\| \|P_{-} (R) e^{-itH} P_j P_{-} (S)\|
\]
where $R$ is the complement of $[i - \ell/4, i + \ell/4]$. The first term is small in expectation since
\[
\|P_j P_{-} (i e^{-itH} P_j P_{-} (S))\|^2 \leq \sum_{j \in R} \sum_{E \in \pi_j} \|\mathcal{N}_j \psi_E\| \|\mathcal{N}_j \psi_E\|
\]
while the second one is small in expectation as
\[
\|P_{-} (R) e^{-itH} P_j P_{-} (S)\| = \|P_{-} (R) P_{-} (R^c) e^{-itH} P_j P_{-} (S)\|
\]
\[
\leq \sum_{j \in R^c} \sum_{E \in \pi_j} \|\mathcal{N}_j \psi_E\| \|\mathcal{N}_j \psi_E\|
\]
where $R^c = [i - \ell/4, i + \ell/4]$, thus completing the proof.

**Theorem 3:** As in the argument in the proof of theorem 2, we can assume that $X_{+,+} = Y_{+,+} = 0$. It suffices to prove that $\mathbb{E} \| (X g (H) Y_j) \| \leq C \|X\| \|Y\| e^{-\tilde{m} |i-j|}$ for any function $g$ supported in $\mathcal{I}$ and bounded by 1.

Let $|j - j'| = 2\ell$, and let $S_i$ be the complement of the set $[i - \ell/2, i + \ell/2]$, and similarly for $S_j$. Inserting $1 = P_{-} (S_i) + P_{+} (S_i)$ and $1 = P_{-} (S_j) + P_{+} (S_j)$, we get
\[
X g (H) Y = \sum_{a = \pm} \sum_{b = \pm} X P_{a} (S_i) g (H) P_{b} (S_j) Y.
\]

We estimate the norms of the terms on the right hand side separately. If one of the indices $a$ or $b$, say $a$, is $-$, using $[P_{-} (S_i), X] = 0$ and $X_{+,+} = 0$, we get
\[
\|\left( X P_{-} (S_i) g (H) P_{b} (S_j) Y \right)\| \leq \| Y \| \| P_{-} X P_{-} (S_j) P_{b} \|
\]
\[
\leq \| X \| \| Y \| \left( \| P_{-} X P_{-} (S_i) N_j \| + \| N P_{-} (S_j) P_{b} \| \right).
\]

The expectation of the right hand side is bounded by $C \|X\| \|Y\| e^{-\tilde{m} t}$. On the other hand, if both indices are $+$, we bound the corresponding contribution as $\|X\| \|Y\| \| P_{+} (S_i) g (H) P_{-} (S_j) \|$. But
\[
P_{+} (S_i) g (H) P_{+} (S_j) = P_{+} (S_i) P_{-} (S_j^c) g (H) P_{-} (S_j^c) P_{+} (S_j),
\]
where $A^c$ stands for the complement of a set $A$. Hence
\[
\| P_{+} (S_i) g (H) P_{+} (S_j) \| \leq \| P_{-} (S_j^c) g (H) P_{-} (S_j^c) \|
\]
which is again exponentially small in expectation.
Conclusions

Our results, theorems 2 and 3 above, give a very detailed picture for the structure and dynamical behavior of eigenstates with energies in the droplet spectrum of the Ising phase of the random XXZ model. These states have the structure of a single cluster of down spins in a sea of up spins, up to relatively small corrections, and thus can be referred to as droplet states. To investigate their dynamical properties, we construct a suitable restriction of their Heisenberg evolution to the energy window of interest and identify the associated quasi-local propagator. The droplet states do not participate in transport, as is evident in the non spreading of the wave packets formed from these states. They are also characterized by the uniform exponential clustering property, i.e. by the rapid decay of the dynamical 2-point correlator associated with them. Our other finding is that the Lieb–Robinson bound is not equivalent to the non spreading of information when we restrict the initial state to the energy window $I_0$ (this equivalence was known to hold on the full Hilbert space). In fact, only higher order commutators enjoy an LR bound in this setting.

While the methods used to prove our result are perturbative in nature, requiring an asymptotic $(\lambda, \Delta)$-regime, it is plausible that droplet localization holds for any non-zero value of $\lambda$ and $\Delta > 1$.

We only studied the first (droplet) band of energies $\tilde{I} = I_1$ in the Ising phase of the random XXZ model, where eigenstates behave effectively as one quasi-particle states. The next band $I_2 = [2(1 - \Delta^{-1}), 3(1 - \Delta^{-1})]$ of energies comprises eigenstates that form one and two quasi-particles, and so forth. We expect appropriately modified forms of theorem 1 to hold in the $k$th band and that this will lead to an extension of the results on non spreading of information and exponential clustering presented in this announcement. It is natural to expect that the Lieb–Robinson bound in the form of a commutator of order $k + 1$ will hold in this setting as well.

Unfortunately, these results are not informative for an extensive energy regime (i.e. for energies of order $cL$ where $L$ is the system size). For the same reason, we cannot show the existence of an extensive set of local integrals of motion (LOIM), a property frequently used in the literature to characterize the fully MBL regime. (See [30], which provides numerical evidence for the emergence of local integrability in situations where, in the thermodynamic limit, only a non-vanishing fraction of eigenstates is localized). Thus it is probably unrealistic to expect that this approach, while effective in obtaining zero temperature localization, can yield insight about the full regime where MBL is expected to hold.

Nonetheless, we believe that the ideas presented here will be useful in understanding the transport properties of interacting systems that have a mobility edge, as in the quantum Hall effect. So far the best known results [31] were only established in this context for gapped local Hamiltonians, due in large part to the poor understanding of the dynamics associated with a mobility gap.

Acknowledgments

AK was supported in part by the NSF under grant DMS-1301641.

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