LETTER

On divergences in non-minimal $\mathcal{N} = 4$ conformal supergravity

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We review the question of quantum consistency of $\mathcal{N}=4$ conformal supergravity in 4 dimensions. The UV divergences and anomalies of the standard \textquote{\textquote{minimal\textquote{}} conformal supergravity where the complex scalar $\phi$ is not coupled to the Weyl graviton kinetic term can be cancelled by coupling this theory to $\mathcal{N}=4$ super Yang–Mills with gauge group of dimension 4. The same turns out to be true also for the \textquote{\textquote{non-minimal\textquote{}}\textquote{}} $\mathcal{N}=4$ conformal supergravity with the action (recently constructed (Butter et al 2017 Phys. Rev. Lett. 118 081602)) depending on an arbitrary holomorphic function $f(\phi)$. The special case of the \textquote{\textquote{non-minimal\textquote{}}\textquote{}} conformal supergravity with $f = e^{2\phi}$ appears in the twistor-string theory. We show that divergences and anomalies do not depend on the form of the function $f$ and thus can be cancelled just as in the \textquote{\textquote{minimal\textquote{}}\textquote{}} $f = 1$ case by coupling the theory to four $\mathcal{N}=4$ vector multiplets.

Abstract

We review the question of quantum consistency of $\mathcal{N}=4$ conformal supergravity in 4 dimensions. The UV divergences and anomalies of the standard \textquote{\textquote{minimal\textquote{}} conformal supergravity where the complex scalar $\phi$ is not coupled to the Weyl graviton kinetic term can be cancelled by coupling this theory to $\mathcal{N}=4$ super Yang–Mills with gauge group of dimension 4. The same turns out to be true also for the \textquote{\textquote{non-minimal\textquote{}}\textquote{}} $\mathcal{N}=4$ conformal supergravity with the action (recently constructed (Butter et al 2017 Phys. Rev. Lett. 118 081602)) depending on an arbitrary holomorphic function $f(\phi)$. The special case of the \textquote{\textquote{non-minimal\textquote{}}\textquote{}} conformal supergravity with $f = e^{2\phi}$ appears in the twistor-string theory. We show that divergences and anomalies do not depend on the form of the function $f$ and thus can be cancelled just as in the \textquote{\textquote{minimal\textquote{}}\textquote{}} $f = 1$ case by coupling the theory to four $\mathcal{N}=4$ vector multiplets.

Keywords: conformal supergravity, conformal anomalies, super Yang–Mills

Conformal supergravities (CSGs) are $\mathcal{N} \leq 4$ supersymmetric extensions of the $(C_{\text{med})}^2$ Weyl gravity in 4 dimensions [1, 2]. Like Weyl gravity they are non-unitary having higher derivative kinetic terms and thus ghosts but are formally power-counting renormalizable with one coupling constant. The corresponding one-loop beta-functions were found to be non-zero [3] implying non-vanishing conformal anomaly. As the Weyl symmetry here is gauged, this means quantum inconsistency [4]. The same conclusion was reached also from the analysis of chiral $SU(4)$ R-symmetry gauge anomalies [5], in agreement with the fact that all anomalies should belong to the same $\mathcal{N}=4$ superconformal multiplet.

Remarkably, it was observed that the standard $\mathcal{N}=4$ CSG theory of [2] can be made UV finite [4, 6, 7] and thus anomaly-free [5] by coupling it [8] to exactly four $\mathcal{N}=4$ super Maxwell multiplets (e.g. to $U(2)$ $\mathcal{N}=4$ super YM theory). This was shown directly at the one-loop order but should be true to all orders as the beta-function in $\mathcal{N} > 1$ conformal supergravity and conformal anomaly of SYM may receive contributions only from the first loop (as
follows from formal superspace arguments as in the SYM case, see [7]). In the present case of $\mathcal{N} = 4$ there is also another reason for one-loop exactness: the conformal anomaly is tied by supersymmetry with $SU(4)$ chiral anomaly which has one-loop origin.

In the $SU(1, 1)$ invariant $\mathcal{N} = 4$ CSG of [2] the 4-derivative complex scalar $\varphi = \phi + i\psi$ did not couple to Weyl graviton and $SU(4)$ gauge field kinetic terms. It was conjectured in [4, 6, 7] that there may exist a ‘non-standard’ (non-$SU(1, 1)$ invariant) version of $\mathcal{N} = 4$ CSG. If one assumes that $\varphi$ may ‘non-minimally’ couple to Weyl term,

$$f(\varphi)(C_{mnkl})^2 + ... = (1 + k_1 \phi + k_2 \phi^2 + ...)(C_{mnkl})^2 + ...,$$

then there will be additional contributions to the beta-function that may cancel against the ‘minimal’ $\mathcal{N} = 4$ CSG beta-function. This finiteness conjecture was, however, in an apparent contradiction with the chiral anomaly count [5] as ‘non-minimal’ couplings should not contribute to the chiral anomaly (see, e.g. [9]).

This would suggest that either (i) a ‘non-minimal’ theory does not exist as non-minimal scalar couplings are inconsistent with $\mathcal{N} = 4$ supersymmetry, or (ii) a ‘non-minimal’ $\mathcal{N} = 4$ CSG exists but its UV divergences and thus anomalies are the same (i.e. non-vanishing) as in the ‘minimal’ theory.

It is the ‘minimal’ $\mathcal{N} = 4$ CSG that appeared (as the coefficient of the log cutoff term) in the quantum effective action of $\mathcal{N} = 4$ SYM coupled (in the standard $SU(1, 1)$ covariant way [8]) to the conformal supergravity background [10–12] or in the classical action of the 5d $\mathcal{N} = 8$ gauged supergravity evaluated on the solution of the AdS$_5$ Dirichlet boundary problem [10]. However, one should expect that there should be another inequivalent version of $\mathcal{N} = 4$ CSG with a non-minimal coupling $f = e^{4\varphi}$ was provided [7] by dimensional reduction to 4d from 10d conformal supergravity [13] (see equation (4.23) in [7] and (1) and (3) below).

Another strong indication that a ‘non-minimal’ CSG should exist came from the twistor-string theory [14, 15] with closed-string or singlet gauge sector describing a theory with $\mathcal{N} = 4$ CSG spectrum. Twistor-string arguments suggested exponential dependence on the scalar and the 3-point scalar-graviton amplitudes were consistent with $e^{2\varphi}(C_{mnkl})^2 + c.c.$ terms in the action [15] (see also [16]). Furthermore, it was conjectured in [15] that in general the action of $\mathcal{N} = 4$ CSG may contain an arbitrary holomorphic function: if $W = \varphi + ... + \theta^a C_{a}^-- + ... + \theta^a \theta^b \varphi$ is a linearized chiral $\mathcal{N} = 4$ superfield strength, then the action may have the following structure

$$\int d^4 x d^4 \theta E f(W) + c.c. \rightarrow \int d^4 x \sqrt{g} f(\varphi)(C_{mnkl})^2 + c.c..$$

It is an extra assumption of manifest $SU(1, 1) \approx SL(2, R)$ invariance (that includes constant shifts of $\varphi$) that fixes the function $f$ to be constant, i.e. leads to the ‘minimal’ CSG. Such ‘non-minimal’ $\mathcal{N} = 4$ CSG with the action depending on an arbitrary holomorphic function was indeed constructed recently in [19]. As we shall explain below, the puzzle about divergences versus anomalies of such ‘non-minimal’ theory is resolved according to point (ii) above: the divergences do not actually depend on a particular form of the ‘non-minimal’ function $f$, i.e. are the same as in the ‘minimal’ theory, in agreement with the chiral anomaly count [5] as required by $\mathcal{N} = 4$ supersymmetry. Thus there is no ‘non-minimal’ $\mathcal{N} = 4$ CSG theory which is UV finite by itself but it as in the ‘minimal’ case it can be made finite and thus consistent by coupling it to four $\mathcal{N} = 4$ vector multiplets.

Let us first review some basic relations [4, 6, 7]. We shall concentrate only on terms involving the Weyl tensor $C_{mnkl}$, $SU(4)$ gauge field $F_{mn}$ and the scalar $\varphi$ of $\mathcal{N} = 4$ CSG. The Lagrangian of the $\mathcal{N} = 4$ CSG contains the following ‘minimal’ terms:

$$L = \frac{1}{2} F_{mn} F^{mn} + \frac{1}{2} C_{mnkl} C^{mnkl} + \frac{1}{2} \varphi^2 - \frac{1}{2} \varphi \Box \varphi - \frac{1}{8} \varphi^4.$$

Note that the kinetic terms of the Weyl gravity and the $SU(4)$ gauge field have opposite signs in the $\mathcal{N} = 4$ CSG action. This is consistent with the fact that integrating out the ‘matter’ $\mathcal{N} = 4$ vector multiplet coupled to conformal supergravity background induces the $\Box \varphi$ term with positive (‘asymptotically-free’) sign and the $\varphi^4$ term with the negative (usual ‘non-asymptotically-free’) sign.
\[
\mathcal{L} = \frac{2}{g^2} L_{\text{min}}, \quad L_{\text{min}} = \varphi^* D^4 \varphi + \frac{1}{2} (C_{\text{min}})^2 - \frac{1}{4} (F_{\text{min}})^2 + \ldots .
\] (1)

In what follows we will suppress the internal index \( r = 1, \ldots, 15 \) on the \( SU(4) \) field strength. The log UV divergent part of the effective action is then

\[
\Gamma_\infty = - \frac{1}{(4\pi)^2} \log \Lambda \int d^4 \sqrt{g} b_4, \quad b_4 = 2\beta L_{\text{min}},
\] (2)

where the beta-function coefficient is equal to \( \beta = -1 \) in the ‘minimal’ \( \mathcal{N} = 4 \) CSG. For completeness, let us recall that the conformal anomaly depends also on the \( a \)-coefficient of the Euler number density: \( < T^a_a > = - a R^\alpha R^a_\alpha + c (C^2_{\text{min}} - F^2_{\text{min}} + \ldots) = \beta_1 R^\alpha R^a_\alpha + \beta_2 (R^2_{\text{min}} - \frac{1}{2} R^2 - \frac{1}{2} F^2_{\text{min}} + \ldots) \) where \( \beta_1 = c - a, \beta_2 \equiv \beta = 2c \). One finds \([4, 7]\) that \( \beta_1 = c - a \) vanishes separately for \( \mathcal{N} = 4 \) SYM and \( \mathcal{N} = 4 \) CSG theories which should be a consequence of their maximal \( \mathcal{N} = 4 \) supersymmetry. The possibility of the cancellation of the \( \mathcal{N} = 4 \) CSG beta-function or \( c \)-anomaly by coupling to \( \mathcal{N} = 4 \) SYM (four \( \mathcal{N} = 4 \) vector multiplets) is a non-trivial consequence of the negative sign of \( c_{\mathcal{N}=4 \text{ SYM}} \); \( c = c_{\mathcal{N}=4 \text{ CSG}} + 4 c_{\mathcal{N}=4 \text{ SYM}} = 1 + 4 \times \frac{1}{4} = 0 \).

For a discussion of this cancellation from AdS\(_5\) perspective \([10]\) see section 5 in \([20]\). Similar statement is true in 6 dimensions: conformal \( a \)-anomaly of (2,0) conformal supergravity is cancelled by coupling it to 26 (2,0) tensor multiplets \([24]\).

Next, let us assume that the CSG action may contain some non-minimal scalar couplings. The ones that may contribute to one-loop divergences may be parametrized as \([6, 7]\)

\[
L = \varphi^* D^4 \varphi + \frac{1}{2} (C_{\text{min}})^2 - \frac{1}{4} (F_{\text{min}})^2 + \varphi \left[ a_1 (C_{\text{min}})^2 + b_1 C_{\text{min}}^{\mu
u} C_{\text{min}}^{\mu
u} - a_2 (F_{\text{min}})^2 - b_2 F_{\text{min}}^{\mu
u} F_{\text{min}}^{\mu\nu} \right] + \varphi^2 \left[ c_1 (C_{\text{min}})^2 - c_2 (F_{\text{min}})^2 \right],
\] (3)

where \( \varphi \) stands for any of the two real components of the complex scalar \( \varphi^* \). Then it is straightforward to find that the additional contributions of the non-minimal terms (3) to the one-loop divergence in (2) (coming from one-loop diagrams with two Weyl gravitons or two \( SU(4) \) gauge fields on external lines) are \([6, 7]\)

\[
\Delta b_4 = (4a_1^2 - 4b_1^2 + c_1)(C_{\text{min}})^2 - (4a_2^2 - 4b_2^2 - c_2)(F_{\text{min}})^2.
\] (4)

Let us now consider the particular form of the non-minimal couplings that actually appear in the \( \mathcal{N} = 4 \) CSG action constructed in \([19]\). As in the ‘minimal’ CSG \([2]\) the 4-derivative (Weyl weight 0) complex scalar that parametrizes the \( SU(1,1)/U(1) \) coset may be described by a doublet \( \Phi_\alpha \) of complex scalars subject to the \( SU(1,1) \) invariant constraint \( \Phi_0^a \Phi_0^a \equiv \Phi_1^a \Phi_1^a - \Phi_2^a \Phi_2^a = 1 \) which is also invariant under the ‘non-dynamical’ \( U(1) \) gauge symmetry (with composite gauge field coupled chirally to fermions). In the ‘minimal’ theory

\(^3\)An AdS\(_5\) explanation of why the combination of \( \mathcal{N} = 4 \) CSG and four \( \mathcal{N} = 4 \) vector multiplets is anomaly-free or why \( c_{\mathcal{N}=4 \text{ CSG}} = (2) \times 2 \times 2 \times 4 \times 4 \) involves their indirect relation to \( \mathcal{N} = 8 \) 5d supergravity: (i) the partition functions of 5d fields in AdS\(_5\) and of the corresponding 4d conformal fields at the boundary are closely related \([20-22]\) (by factor of \(-2\)) and thus the \( a = c \) ‘anomalies’ of \( \mathcal{N} = 8 \) 5d supergravity and \( \mathcal{N} = 4 \) 4d CSG are also related by factor of \(-2\); (ii) the \( \mathcal{N} = 8 \) 5d supergravity may be viewed \([23]\) as a product of two ‘doubletons’—\( \mathcal{N} = 4 \) vector multiplets—and thus their anomalies are related by factor of \(-2\).

\(^4\)Here we use Minkowski-metric notation and the dual tensors are defined as \( F_{\mu
u} = \frac{1}{2} \epsilon_{\mu
u\alpha\beta} F^\alpha\beta, \epsilon_{\mu
u\alpha\beta} \epsilon^{\mu
u\alpha\beta} = -4! \).

\(^5\)Note that due to 4-derivative scalar kinetic term here one gets logarithmic UV divergences from both the scalar tadpole graphs and the mixed scalar-graviton and scalar-vector loops. Compared to similar expressions in \([6, 7]\) here we have \( b_1 \to \epsilon b_1, \epsilon b_2 \to \epsilon b_2 \) as we use Minkowski notation where \( \epsilon = -1 \). It is sufficient to consider the linearized expansion, \( C_{\text{min}}^{\mu
u} \to 0, C_{\text{min}}^{\mu
u} \to 0, F_{\text{min}}^{\mu
u} \to \epsilon (\partial^2 V_{\text{min}}) + \ldots, F_{\min}^{\mu\nu} = 2 (\partial^2 V_{\text{min}}) + \ldots \). A short-cut is just to expand \( C_{\text{min}}^{\mu
u}, F_{\min}^{\mu\nu} \) near background values, integrate out their fluctuations and use that for \( L = \varphi^* D^4 \varphi + U \varphi^4 \) one gets in (2) \( \Delta b_4 = - U \).
the $SU(1, 1)$ is the off-shell global symmetry that acts only on the scalars:\footnote{The action of the $\mathcal{N} = 4$ vector multiplet coupled to conformal supergravity is invariant under this $SU(1, 1)$ combined with a duality rotation of the vector field. Once the vector multiplet fields are integrated out, this $SU(1, 1)$ becomes an off-shell symmetry of the resulting induced action, i.e. of the minimal CSG action \cite{11, 12}.} Choosing the $U(1)$ gauge as $\Phi^*_1 = \Phi_1$, one may introduce the physical complex scalar as $\varphi = \Phi_2/\Phi_1$ so that
\begin{equation}
\Phi_1 = (1 - |\varphi|^2)^{-1/2}, \quad \Phi_2 = \varphi (1 - |\varphi|^2)^{-1/2}.
\end{equation}

The $\mathcal{N} = 4$ CSG action of \cite{19} depends on an arbitrary holomorphic function $\mathcal{H}(\Phi_1, \Phi_2)$ of the scalars $\Phi_\alpha$ that is homogeneous of degree 0 in its arguments. Its presence breaks the global $SU(1, 1)$ symmetry so it is natural to use the explicit parametrization in terms of $\varphi$. In the gauge (5) we thus get $\mathcal{H}(\Phi_1, \Phi_2) \rightarrow f(\varphi)$, where $f$ is a holomorphic function of the complex scalar $\varphi$. The relevant terms in the Lagrangian of \cite{19} that generalize the ‘minimal’ one $L_0$ in (1) are then given by (we ignore modification of the kinetic term of $\varphi$ as its is not relevant for the computation of the one-loop $C^2 + F^2$ divergences)
\begin{equation}
L_{\text{non-min}} = \frac{1}{16} \left[ f(\varphi)(C_{mnkl})^2 - f(\varphi)(F_{mn}^-)^2 + c.c. \right] + \varphi^* D^4 \varphi + \ldots,
\end{equation}
where $C_{mnkl} = C_{mnkl} - i C^*_{mnkl}$, $F_{mn}^- = F_{mn} - i F^*_{mn}$. For $f = 1$ this reduces (up to a total derivative) to $L_{\text{min}}$ in (1).

To find the terms that may contribute to $C^2 + F^2$ one-loop divergences (2) it is sufficient to expand $f$ to first two orders in $\varphi$,
\begin{equation}
f(\varphi) = 1 + k_1 \varphi + k_2 \varphi^2 + O(\varphi^3), \quad \varphi = \phi + i \psi,
\end{equation}
\begin{equation}
-\frac{1}{16} f(\varphi)(F_{mn}^-)^2 + c.c. = -\frac{1}{4} \left[ 1 + k_1 \phi + k_2 (\phi^2 - \psi^2) \right] (F_{mn}^-)^2 - \frac{1}{2} (k_1 \psi + 2 k_2 \phi \psi) F^m n F^*_{mn} + \ldots.
\end{equation}

and similarly for the $f(\varphi) C_{mnkl} C_{mnkl}^*$ term in (6). The mixed $\phi \psi$ term can not contribute to one-loop divergences. Comparing (6) and (8) to the non-minimal coupling ansatz in (3) we conclude that for the real part $\phi$ of $\varphi$ the constants in (3) are $a_1 = a_2 = \frac{1}{2} k_1$, $b_1 = b_2 = 0$, $c_1 = c_2 = \frac{1}{2} k_2$, while for the imaginary part $\psi$ they are $a_1 = a_2 = 0$, $b_1 = b_2 = \frac{1}{2} k_1$, $c_1 = c_2 = -\frac{1}{2} k_2$. Summing up the contributions of $\phi$ and $\psi$ to the divergent term (4) we find that they cancel each other. The reason why that happens can be traced back to the holomorphicity of the scalar couplings in (6) dictated by the $\mathcal{N} = 4$ supersymmetry\footnote{Indeed, to explain this cancellation one may use explicitly the holomorphicity of the scalar coupling: given an (abelian) theory like $\varphi^* \partial^4 \varphi + f(\varphi)(F_{mn}^-)^2 + c.c.$ one does not generate $F_{mn}^2$ dependent quantum corrections as propagators for both the scalar field and the vector field strength involve both conjugate components while the vertices are chiral. We thank Roiban for suggesting this argument.}.

We conclude that the one-loop divergences of the ‘non-minimal’ CSG theory do not depend on the function $f$ in (6) and thus are the same (2) as in the ‘minimal’ theory. This means formally that only the constant part of the function $f$ in (6) may be deformed by renormalization. The quantum consistency of the theory, i.e. the preservation of the $\mathcal{N} = 4$ superconformal gauge symmetry requires the cancelation of the divergence, and that can be achieved again by coupling the CSG theory (in the usual $SU(1, 1)$ covariant way \cite{8} not depending on the choice of the function $f$) to four $\mathcal{N} = 4$ vector multiplets \cite{4, 5, 7}.

This suggests that a twistor-string theory that describes a coupled system of $\mathcal{N} = 4$ SYM and ‘non-minimal’ $\mathcal{N} = 4$ CSG can be quantum-consistent only for the SYM gauge group of dimension 4. A world-sheet explanation of this still remains an open problem (see \cite{15}).
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