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Logarithmic conformal field theory, log-modular tensor categories and modular forms

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Abstract
The two pillars of rational conformal field theory and rational vertex operator algebras are modularity of characters, and the interpretation of its category of modules as a modular tensor category. Overarching these pillars is the Verlinde formula. In this paper we consider the more general class of logarithmic conformal field theories and $C_2$-cofinite vertex operator algebras. We suggest logarithmic variants of those pillars and of Verlinde’s formula. We illustrate our ideas with the $W_p$-triplet algebras and the symplectic fermions.

Keywords: logarithmic VOA, category, modular forms, Verlinde’s formula

1. Introduction

Vertex operator algebras (VOAs) have profoundly influenced mathematics and mathematical physics since their inception in the mid 1980s. We mention their roles underlying Moonshine [FLM, Bo], clarifying conformal field theories (CFTs) [GabG], the chiral de Rham complex in geometry [MSV], geometrical Langlands [F], among many others.

In particular, Monstrous Moonshine concerns the representation theory of two very special VOAs: the Moonshine module $V^t$ and its orbifold $(V^t)^{36}$ by its automorphism group, the Monster. These are examples of VOAs with semi-simple representation theory—we call these strongly-rational. We want similar interpretations of the more modern moonshines, but we know that the class of strongly-rational (super-)VOAs is too narrow.

We review the theory of strongly-rational VOAs in section 2.3 below. Two pillars of this theory are on the one hand the modularity of characters and, more generally, the automorphicity of chiral blocks for all surfaces, and on the other hand the interpretation of the modules of the VOA as the objects in a modular tensor category. Overarching these pillars is the Verlinde
formula, which expresses the tensor (fusion) product coefficients and dimensions of those spaces of chiral blocks, in terms of a quantity (the $S$ matrix) shared by both pillars. This picture is completed with a rich collection of examples of strongly rational VOAs. There are fundamental questions which remain unanswered (e.g. the possible rationality of orbifolds and cosets, and the possible equivalence with completely rational conformal nets), but over all one must be satisfied with the rather complete picture which has arisen.

The natural generalization of strongly-rational VOAs are the strongly-finite ones. They correspond to $C_2$-cofinite VOAs, and logarithmic CFTs. In section 2.4 we review what is currently known about them. Strongly-rational VOAs are to finite groups, as strongly-finite VOAs are to finite-dimensional Hopf algebras. With this analogy in mind, the theory of strongly-finite VOAs should be a fairly gentle but far-reaching nonsemi-simple generalization of that of strongly-rational VOAs. Nevertheless, very few nonrational examples have been identified, and most of the work in the area has focused on studying those isolated examples. We are trying to see the forest via some individual trees.

The main purpose of this paper is to describe the general picture of strongly-finite VOAs that is evolving. We review the representation theory of Hopf algebras in section 2.1, where we also explain why representation theory is naturally categorical, and give an abstract meaning of character. Most of section 2 as well as section 3 describe the standard examples and general theory of strongly-finite VOAs.

For the strongly-rational VOAs, the fundamental general result on modularity is due to Zhu [Z], who showed that the characters are closed under modular transformations. Unfortunately this fails in general for strongly-finite VOAs. However, Miyamoto [Miy2] modified Zhu’s arguments and found that modularity is restored if the characters are supplemented by finitely many pseudo-characters. This fundamental result is not so well understood, and we review it in section 2.5. Though mathematically elegant and natural from the associative algebra perspective, it is not at all natural from the CFT perspective. Quantum field theory teaches that correlation functions should arise from field insertions and ordinary traces. Moreover, it is hard to do computations within Miyamoto’s picture—it is difficult even to determine in practice the dimension of the resulting modular group representation. Is there another way to recover Miyamoto’s functions, in a manner more compelling from a standard VOA or CFT point-of-view? We propose in section 3.1.1 a way to do exactly this.

For strongly-rational VOAs, the fundamental result on the categorical pillar was Huang’s proof that the representation category is a modular tensor category. We propose a formulation of strongly-finite representation theory, i.e. the definition of a nonsemi-simple modular tensor category, in section 3.1.2; it is probably equivalent to that of [GaiR2]. Thanks primarily to work of Huang and collaborators (see [HLZ]), the main thing left to prove here is rigidity. This categorical formulation has several interesting consequences, some of which we describe in section 3.1.2.

In the strongly-rational setting, Moore–Seiberg [MS], Turaev [T], and Bakalov–Kirillov [BK] explain how to obtain from the modular tensor category, a tower of mapping class group representations and (abstract) spaces of chiral blocks associated to the surfaces with marked points (called the modular functor), and the associated 3-dimensional topological field theory. This is just the chiral theory; Fuchs, Runkel, Schweigert and collaborators have explained how to obtain a full CFT from this chiral data [FRS, FFRS]. The analogue of Moore–Seiberg et al for the strongly-finite theory should be Lyubashenko [Ly1, Ly2, KL], which we review in section 2.7, and Fuchs et al are currently extending their work to the strongly-finite setting [FSS, FS2]. An important question here is to show that Lyubashenko’s $SL(2,\mathbb{Z})$-representation is equivalent to that on Miyamoto’s space of (pseudo-)characters.
In the strongly-rational setting, a key to understanding Verlinde (the arch connecting the modular pillar to the categorical one) is the (numerical) link invariant

\[ S^{\infty}_{ij} := \begin{array}{c} M_i \hspace{1cm} \bigcirc \hspace{1cm} M_j \end{array} \in \mathbb{C} \]  

(1)

associated to the Hopf link. In the strongly-finite setting, too many of these now vanish, and more significant, the fusion rules are no longer diagonalizable in general so Verlinde’s formula must be modified. In the strongly-finite setting, two rings must be distinguished: the Grothendieck ring spanned by the simple modules, which sees only the composition factors of all the modules, and the tensor ring which is spanned by the indecomposable modules and which sees the full splendor of the tensor product. We propose in section 3.1.3 to replace the Hopf link by the open Hopf link—the associated link invariants

\[ \Phi_{U,W} = \begin{array}{c} U \hspace{1cm} \bigcirc \hspace{1cm} W \end{array} \in \text{End}(W) \]  

(2)

are now matrix-valued and give representations of the tensor ring. These seem sufficient to recover the Jordan blocks in the Grothendieck ring (i.e. the indecomposables in the regular representation of the Grothendieck ring)—this is a purely categorical statement, and should be possible to prove as in [T]. To recover the Jordan blocks in the tensor ring, the open Hopf links are not sufficient.

In the strongly-rational world, the modular $S$-matrix, which describes how the characters $\chi_M(\tau)$ transform under $\tau \mapsto -1/\tau$, coincides up to a scalar factor with the matrix of Hopf link invariants. Much more delicate is to see what this relation becomes in the strongly-finite world. Our proposal, again involving the open Hopf links, is given in section 3.1.

Perhaps the development of the theory is limited more than anything else by the lack of independent examples. Again, there should be a far richer zoo of strongly-finite VOAs than strongly-rational ones, but at present the opposite is emphatically the case. We describe some possibilities in section 3.4.

We are aware of the reluctance, not only by physicists but by many mathematicians, towards categorical formulations. In fact we have little sympathy for abstract nonsense done for its own sake. But in representation theory especially, it seems an indispensable tool: it helps obtain interesting results that do not need category theory to formulate. The simplest illustration of the importance of categories to the theory of VOAs is the categorical interpretation of VOA extensions [HKL]. We hope this paper helps make the point that this is just the tip of the iceberg.

We would like to note, that there is a recent appearance of strongly-finite but not strongly-rational VOAS in physics. Namely the Schur-index of protected states of four-dimensional supersymmetric gauge theories is expected to be the character of a vertex operator algebra [BLLPRR]. In the instance of so-called Argyres-Douglas theories the best known strongly-finite VOAs, the triplet algebras, appear as quantum Hamiltonian reductions and coset VOAs of the VOAs of Argyres-Douglas theories [C2]. As we will mention later in conjecture 3.3, the semi-simplification of a log-modular tensor category is expected to be a modular tensor category. This category seems to describe physical invariants as well [FPYY].
2. Background

2.1. Representation theory of associative algebras

We will assume the reader is familiar with the finite-dimensional representations over \( \mathbb{C} \) of a finite group \( G \). The theory is semi-simple: \( G \) has finitely many irreps \( \rho_1, \ldots, \rho_r \) (up to equivalence), and every finite-dimensional representation \( \rho \) is (up to equivalence) the direct sum \( \rho \cong \bigoplus_{i=1}^r m_i \rho_i \) of irreps in a unique way. We have a tensor product \( \rho \otimes \rho' \) of representations, which (up to equivalence at least) we can completely capture by the tensor product multiplicities \( T^n_k \) defined by \( \rho_i \otimes \rho_j \cong \bigoplus_k T^n_{ij} \rho_k \). Letting \( [\rho] \) denote the equivalence class of \( \rho \), and writing \( [\rho \otimes \rho'] = [\rho] + [\rho'] \) and \( [\rho \otimes \rho'] = [\rho][\rho'] \), we get a ring structure on the \( \mathbb{Z} \)-span of the irreps \([\rho]_i\), with structure constants \( T^n_{ij} \). We call this the Grothendieck ring of \( G \). We also have a trivial 1-dimensional representation \( \rho_1 = 1 \), which is the tensor unit. And we have duals \( \rho^* \), which we can completely capture (up to equivalence) by an order 1 or order 2 permutation \( i \mapsto i^* \) on the index set \( \{1, \ldots, r\} \). All of this data is equivalent to the character table of \( G \).

The usefulness of character tables to finite groups is clear. But different groups (e.g. the dihedral group \( D_3 \) = \( \{ (a, b) \mid a^2 = b^3 = abab = 1 \} \) and the group \( Q_8 = \langle i, j \mid i^2 = j^2, i^4 = 1, ij = ji \rangle \) can have the same character table. How can we enhance the character theory, to get something representation theoretic which more closely characterizes \( G \)?

The key phrase, used over and over in the previous paragraphs, was ‘up to equivalence’. We can enhance this data by including explicitly the intertwiners \( T \in \text{Hom}_G(\rho, \rho') \), i.e. linear maps \( T : \mathbb{C}^d \rightarrow \mathbb{C}^{d'} \) (where \( d, d' \) are the dimensions of \( \rho, \rho' \) respectively) satisfying \( \rho \circ T = T \circ \rho' \). Put another way, what we really have is a category: the objects are the representations (or modules) and the arrows between the objects are the intertwiners. This category has additional structure: direct sums, tensor products, complete reducibility, duals, etc. The result is called a tensor category (in fact a fusion category, which is even better). We review basic categorical notions in section 2.6.

In fact, character tables can be surprisingly ineffective. The representation theory of the groups \( D_3 \) and \( Q_8 \) are quite different. For one thing, the determinant of the unique 2-dimensional \( D_3 \)-irrep is nontrivial, whereas the determinant of the unique 2-dimensional \( Q_8 \)-irrep is trivial. This means that character tables cannot see the determinant of irreps. Also, the 2-dimensional irrep of \( D_3 \) can be realised as

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\]

which is real and orthogonal. Such a representation equals, not merely is equivalent to, its own dual. This cannot be done for \( Q_8 \). So the character table by itself cannot determine if representations are self-dual, merely whether they are equivalent to their duals.

These differences can be seen by tensor categories. Another realisation of the 2-dimensional \( D_3 \)-irrep is

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
i & 0 \\
0 & -i
\end{pmatrix}.
\]

In the tensor category the equivalence \( \rho^* \cong \rho \) here is replaced with the equality \( \rho^* = \psi \otimes \rho \), where \( \psi \) is the 1-dimensional irrep defined by \( \psi(a) = 1, \psi(b) = -1 \). The matrix representation (3) is a different object in the category (even though it is an equivalent representation), and for it \( \rho^* = \rho \). In both cases the equivalence \( \rho \cong \rho^* \) is replaced by something stronger. Tensor categories also see the determinant \( \text{det} \rho \).
Although tensor categories distinguish $D_4$ and $Q_8$, they alone cannot distinguish all groups: e.g. there are groups of order $2^{15} 3^4 \cdot 5 \cdot 7 \approx 92$ million which are identical as tensor categories [BeG] (surely there are much smaller examples).

However, we can say more. We know $\rho \otimes \rho' \cong \rho' \otimes \rho$ (after all, they have identical characters $\chi_{\rho} \chi_{\rho'} = \chi_{\rho'} \chi_{\rho}$). Hence there is an invertible intertwiner $c_{\rho,\rho'} \in \text{Hom}_{G}(\rho \otimes \rho', \rho' \otimes \rho)$ called a braiding. For finite groups, this is just $c_{\rho,\rho'}(u \otimes v) = v \otimes u$. Of course this braiding obeys $c_{\rho,\rho'} \circ c_{\rho',\rho} = \text{Id}_{\rho' \otimes \rho}$. A tensor category together with choices of such braidings is called a symmetric tensor category. Tannaka-Krein duality says that the symmetric tensor category of $G$ uniquely determines $G$.

The moral: in representation theory, consider the category of representations, not just the representations up to equivalence. The category of representations of $G$ should be regarded as a gentle enhancement of the combinatorics of characters, which carries much more information. In short: representation theory is categorical.

The representations of finite groups can be recast as modules of the group algebra $\mathbb{C}G$. More generally [ASS], one can consider the finite-dimensional modules over an associative algebra $A$. Let $\text{Mod}^{\text{fin}}(A)$ denote the category of $A$-modules with intertwiners $\text{Hom}_{A}(U, V)$, i.e. linear maps $T : V \rightarrow U$ satisfying $a \cdot T(v) = T(a \cdot v)$ for all $a \in A$ and $v \in V$. When $A$ is finite-dimensional, it will have finitely many irreducibles, and direct sum of modules will exist, but semi-simplicity (complete reducibility) of modules will usually fail. If $A$ is e.g. a Hopf algebra (as $\mathbb{C}G$ is), then one can define tensor products of modules using comultiplication, a tensor unit (i.e. an analogue of trivial representation) using the counit, and duals (using the antipode). In fact in that case $\text{Mod}^{\text{fin}}(A)$ is an example of a finite tensor category (discussed in section 2.6). If e.g. the Hopf algebra $A$ is cocommutative (as $\mathbb{C}G$ is), we can define braidings $c_{U,V}$, but these may not satisfy $c_{U,V} \circ c_{V,U} = \text{Id}_{U \otimes V}$. In fact this latter possibility turns out to be by far the most interesting: the resulting braided tensor categories are discussed in section 2.6.

The characters of a finite group $G$ form a basis for the space of class functions $f(h)$, i.e. maps $f : G \rightarrow \mathbb{C}$ satisfying $f(k^{-1}hk) = f(h)$. The group algebra $\mathbb{C}G$ is a bimodule over itself, equivalently it carries a representation of $G \times G$: it acts on itself by both left- and right-multiplication. This bimodule is $\mathbb{C}G^{\text{reg}} \cong \bigoplus_{d} \mathbb{C}M \otimes M^{*}$, where the sum runs over the finitely many irreducible $G$-modules, and $M^{*}$ denotes the dual or contragredient. This bimodule $\mathbb{C}G$ is central to the whole representation theory of $G$: we can restrict this $G \times G$ representation to $G \times 1$ or $1 \times G$, in which case it is the (left- or right-)regular representation $\mathbb{C}G^{\text{reg}} \cong \bigoplus_{d} \dim_{\mathbb{C}} M_{M}$.

Or we can restrict it to the diagonal subgroup $\{(g,g)\} \cong G$ in $G \times G$, in which case we get the adjoint representation $\mathbb{C}G^{\text{adj}} \cong \bigoplus_{d} \mathbb{C}M \otimes M^{*}$.

Abstractly, a class function for the group $G$ is an intertwiner $f \in \text{Hom}(\mathbb{C}G^{\text{adj}}, \mathbb{C})$. This interpretation of character (or class function) underlies the space in Lyubashenko on which the modular group is to act, as we will see in section 2.7 below.

Usually $A$ will not have a semi-simple representation theory: each $A$-module will always decompose uniquely into a direct sum of indecomposables, but not all indecomposables will be simple.

For example, consider the polynomial algebra $A = \mathbb{C}[x]$. Any $n$-dimensional module of $A$ is completely determined by how $x$ acts, which we can think of as an $n \times n$ matrix $X$. An indecomposable module corresponds to $X$ being a Jordan block $B_{\lambda, \mu}$. It is simple iff $n = 1$.

We say $A$ has finite representation type if it has finitely many indecomposables. For example, $CG$ has finite type. We say $A$ has tame representation type if it has finitely many 1-parameter families of indecomposable modules of each dimension. For example, the polynomial algebra $\mathbb{C}[x]$ is tame: for each dimension $n$ there is a 1-parameter family of indecomposables
(parametrized by the eigenvalue $\lambda$). Though harder than finite type, tame $\mathcal{A}$ are under control and all indecomposables could be classifiable.

The remaining type is wild-type. This means that as the dimension $n$ grows, so does the number of parameters needed to parametrise the dimension $n$ indecomposables. The simplest and prototypical example is the group algebra $\mathbb{C}[F_2]$ of the free group $F_2 = \langle a, b \rangle$. A representation corresponds to a choice of 2 invertible matrices $A, B$ of the same size $n \times n$: $a \mapsto A$ and $b \mapsto B$. It is easy to verify that there are $n^2 + 1$-dimensional families of pairwise inequivalent $n$-dimensional indecomposable $F_2$-representations. Parametrizing all indecomposables of a wild-type algebra such as $\mathbb{C}[F_2]$ would be very complicated and probably useless.

Given an $\mathcal{A}$-module $M$, by its socle we mean the largest semi-simple submodule. The Loewy diagram of $M$ has bottom row the socle $soc(M)$ of $M$, on top of that is put the socle of $M/soc(M)$, etc. Writing the Loewy diagram horizontally essentially gives the composition series of $M$: $M: 0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \cdots \hookrightarrow M_n = M$ where $M_n$ are all submodules of $M$ and each $M_n/M_{n+1}$ is simple (called a composition factor of $M$). The composition factors are the union (keeping multiplicities) of all simple modules appearing in each row of the Loewy diagram.

For example the Loewy diagram of the $x \mapsto B_{3,\alpha}$ indecomposable of $\mathbb{C}[x]$ consists of a vertical line of length $d$ with the $x \mapsto \lambda$ simple in each spot. The composition factors are $\lambda$ (with multiplicity $d$).

When semi-simplicity is lost, much of the role played by the simple modules in the semi-simple case is shared by simple modules and their projective covers. Given any algebra $\mathcal{A}$ (e.g. a VOA), an $\mathcal{A}$-module $P$ is called projective if for every $\mathcal{A}$-modules $M, N$ and surjective intertwiner $f : M \to N$ and any intertwiner $h : P \to N$, there is an intertwiner $h' : P \to M$ such that $h = f \circ h'$. An intertwiner $h : P \to M$ is called a projective cover of an $\mathcal{A}$-module $M$ if $P$ is a projective $\mathcal{A}$-module and any intertwiner $g : N \to P$ for which $h \circ g$ is surjective, is itself surjective.

For example, the simple modules of $\mathbb{C}[x]$ have no projective cover. Choosing instead $\mathcal{A} = \mathbb{C}[x]/(x^3)$ say, the only simple module is $x \mapsto 0$, and its projective cover is $x \mapsto B_{0,3}$. In this example, up to isomorphism the only other indecomposable is $x \mapsto B_{0,2}$. This algebra is finite type though not semi-simple.

2.2. VOA basics

We assume the reader has at least a vague understanding of the definition of a VOA and its modules [FLM, FHL, Z]. This section collects some more technical aspects which may help the reader through some of the following material.

For reasons of simplicity, we restrict attention in this paper to VOAs $\mathcal{V}$ of CFT-type—this means $\mathcal{V}$ has $L_0$-grading $\mathcal{V} = \bigoplus_{n=0}^\infty \mathcal{V}(n)$ with $\mathcal{V}(0) = \mathbb{C}1$ and $\dim \mathcal{V}(n) < \infty$. We also require $\mathcal{V}$, when regarded as a $\mathcal{V}$-module in the usual way, to be simple and isomorphic to its contragradient $\mathcal{V}^*$.

Unless otherwise stated, we will assume throughout that $\mathcal{V}$ is a strongly-finite VOA, by which we mean it is $C_2$-cofinite (in addition to the aforementioned conditions). $C_2$-cofiniteness [Z] is a standard condition implying $\mathcal{V}$ has finitely many simple modules, a tensor product, convergent characters, etc. If $\mathcal{V}$ is in addition regular (complete reducibility of weak modules), then $\mathcal{V}$ is called strongly-rational.

Let $\text{Mod}^\geq(\mathcal{V})$ denote the category of grading-restricted weak $\mathcal{V}$-modules $M$, by which we mean $L_0$ decomposes $M$ into a direct sum of finite-dimensional generalized eigenspaces $M(h)$ with eigenvalues $h$ bounded from below. The hypothesis of $C_2$-cofiniteness implies each such $h$ is rational. The conformal weight $h_M$ of $M$ is the smallest eigenvalue, when it exists.
$L_0$ is in fact diagonalizable over $M$, $M$ is called ordinary. For convenience we will refer to any grading-restricted weak $V$-module, simply as a $V$-module.

Let $M, M'$ be $V$-modules. By a homomorphism $f : M \rightarrow M'$ we mean a linear function satisfying $f(v^M(v, z)w) = v^{M'}(v, z)f(w)$ for all $v \in V$ and $w \in M$, or equivalently $f \circ v^M_w = v^{M'}_w \circ f$ for all $v \in V$ and $n \in \mathbb{Z}$. For example, $L_0$ is always in the centre of $\text{End}(M)$.

Let $M, M', M''$ be any $V$-modules. The notion of logarithmic intertwiner $\mathcal{Y}$ of type $\left(\frac{M'}{M, M''}\right)$ was introduced in [Mil], building upon [FHL]. Among other things, for each $u \in M$, $\mathcal{Y}$ has an expansion

$$\mathcal{Y}(u, z) = \sum_{i=0}^{N} \sum_{n \in \mathbb{Q}} \alpha^{(i)}(u)z^{n-1}(\log z)^i$$

for some $N$ depending on $\mathcal{Y}$, where for $u \in M(k)$, each $u^{(i)}$ maps the generalized $L_0\subseteq$eigenspace $M'(\ell)$ into $M''(k + \ell - n - 1)$. When $M, M', M''$ are indecomposable, then the sum in (4) is over $n \in h_M + h_{M'} - h_{M''} + \mathbb{Z}$.

Let $M, N$ be any indecomposable $V$-modules with conformal weights $h_M, h_N$, and choose any intertwiner $\mathcal{Y}$ of type $\left(\frac{M}{N, M}\right)$. Define the zero-mode $o_{\mathcal{Y}}(v) = v^{(0)}_{k-1}$ for $v \in N(k)$. For any $v \in N$ and $\tau$ in the upper half-plane $\mathbb{H}$, write $q$ for $e^{2\pi i \tau}$ and define the 1-point function

$$Z^\mathcal{Y}(v, \tau) = \text{Tr}_M(o_{\mathcal{Y}}(v)q^{L_0-\tau/24}).$$

These are central to the modularity of our story. We restrict here to intertwiners of type $\left(\frac{M}{N, M}\right)$, as otherwise the traces would not mean anything.

An important case of (5) is $N = V$. In this case the intertwiner space is 1-dimensional, with basis given by the map $Y_M$ defining the $V$-action on $M$. Then

$$\text{ch}[M](v, \tau) := Z^\mathcal{Y}(v, \tau) = q^{-\tau/24} \sum_{n=0}^{\infty} \text{Tr}_{M(n+h_M)}o(v) q^{n+h_M}.$$  

for the zero-mode $o = o_{\mathcal{Y}}$, is called the character of $M$. Most important is the specialisation $v = 1$, which we call the graded-dimension of $M$:

$$\text{ch}[M](1, \tau) := Z^\mathcal{Y}(1, \tau) = q^{-\tau/24} \sum_{n=0}^{\infty} \dim(M(n + h_M)) q^{n+h_M}.$$  

The term ‘character’ is usually used for (7), but this is misleading considering the analogy with finite groups.

Write $\left[ M \right]$ for the equivalence class of $V$-modules isomorphic to $M$. Define $\left[ M \right] + \left[ N \right] = \left[ M \oplus N \right], \left[ M \right]\left[ N \right] = \left[ M \otimes N \right]$, and $\left[ M \right]^* = \left[ M^* \right]$. Extend all this in the usual way to formal combinations over the integers, and the result is the tensor product ring, which we will denote $\text{Fus}\{\mathcal{V}\}$. In the nonrational case, this is a very big ring, and for most or all purposes it suffices to consider the subring $\text{Fus}^{\text{simp}}\{\mathcal{V}\}$ generated by both simple and projective modules. In all strongly-rational examples we have seen, this ring is finite-dimensional (over $\mathbb{Z}$), and we expect this to hold in general.

A different smaller version of $\text{Fus}^{\text{full}}\{\mathcal{V}\}$ is the Grothendieck ring $\text{Fus}_0\{\mathcal{V}\}$. Recall that each $V$-module $M$ has a composition series, which describes how to build it up by extending by simple modules, called its composition factors. Write $\left[ \left[ M \right] \right]$ for the formal sum of all equivalence classes of composition factors of $M$ (with multiplicities). Then the definitions
[[\mathbf{M}]] + [[\mathbf{N}]] := [[\mathbf{M} \oplus \mathbf{N}]], [[\mathbf{M}]] [[\mathbf{N}]] := [[\mathbf{M} \otimes \mathbf{N}]] are well-defined and define a ring structure on the \( \mathbb{Z} \)-span of the simple classes \([[\mathbf{M}]]\).

We prefer to avoid calling any of these the fusion ring, as the term ‘fusion ring’ is used in different senses in the literature.

2.3. The rational modular story

A strongly-rational VOA \( \mathcal{V} \) possesses the following three fundamental properties:

\begin{enumerate}[(Cat)]
\item\hspace{1em} its category of modules \( \text{Mod}^\mathcal{V} (\mathcal{V}) \) is a modular tensor category [H2];
\item\hspace{1em} its torus 1-point functions (5) are modular [Z, Miy1]; and
\item\hspace{1em} Verlinde’s formula is true [V, MS, H1].
\end{enumerate}

These will now be explained in more detail.

\text{(Cat):} In this case all \( \mathbf{M} \in \text{Mod}^\mathcal{V} (\mathcal{V}) \) are ordinary and there are only finitely many isomorphism classes of simple modules. We choose representatives \( \mathbf{M}_1, \ldots, \mathbf{M}_n \) of these classes, with \( \mathbf{M}_1 = \mathcal{V} \) the VOA itself. Every \( \mathcal{V} \)-module will be isomorphic to a unique direct sum of these simple \( \mathbf{M}_i \). The fusion coefficients \( \mathcal{N}_{ij}^k \in \mathbb{Z}_{\geq 0} \) are the dimensions of the spaces of intertwining operators of type \( (\mathbf{M}_i, \mathbf{M}_j) \) \( \otimes \mathbf{M}_k \).

Huang and Lepowsky have defined tensor products of modules (see e.g. [HL] and references therein), realizing those fusion rules in the sense that

\[
\mathbf{M}_i \otimes \mathbf{M}_j \cong \bigoplus_{k=1}^n \mathcal{N}_{ij}^k \mathbf{M}_k.
\]

This makes \( \text{Mod}^\mathcal{V} (\mathcal{V}) \) into a modular tensor category, defined in section 2.6. These fusion coefficients are also given by \( \mathcal{N}_{ij}^k = \dim(\text{Hom}_\mathcal{V}(\mathbf{M}_i \otimes \mathbf{M}_j, \mathbf{M}_k)) \).

\text{(Mod):} For a \( \mathcal{V} \)-module \( \mathbf{M} \), recall the character \( \text{ch}[\mathbf{M}](v, \tau) \) defined in (6) for any \( v \in \mathcal{V} \) and \( \tau \in \mathbb{H} \). Zhu has shown [Z] that for fixed \( v \in \mathcal{V}[k] \) (using the \( \mathbb{L}_{01} \)-grading introduced in [Z]), these \( \text{ch}[\mathbf{M}](v, \tau) \) form the components of a vector-valued modular form of weight \( k \) for the modular group \( \text{SL}(2, \mathbb{Z}) \). Further they are holomorphic in \( \mathbb{H} \) and meromorphic at the cusps. Especially, the graded-dimensions form a vector-valued modular form of weight \( 0 \) for \( \text{SL}(2, \mathbb{Z}) \). We stress that in many examples graded-dimensions are not linearly independent whereas the characters always are. This is the reason for preferring the more general characters over the more familiar graded-dimensions. This modularity (and linear independence) means, for any \( v \in \mathcal{V}[k] \), the modular \( S \)-transformation \( \tau \mapsto -1/\tau \) defines a matrix \( S^\mathcal{V} \) via

\[
\text{ch}[\mathbf{M}](v, -1/\tau) = \tau^k \sum_{j=1}^n S^\mathcal{V}_{ij} \text{ch}[\mathbf{M}_j](v, \tau).
\]

More generally, choose any \( \mathcal{V} \)-module \( \mathcal{N} \) and vector \( v \in \mathcal{N}[k] \). Then Miyamoto showed that Zhu’s methods naturally generalize to show that the 1-point functions \( Z^\mathcal{V}(v, \tau) \), as \( \mathcal{V} \) runs over a basis of intertwiners of type \( (\mathcal{N}, \mathbf{M}) \) for all simple modules \( \mathbf{M} \), form a vector-valued modular form of weight \( k \in h_\mathcal{N} + \mathbb{Z} \) for \( \text{SL}(2, \mathbb{Z}) \).

\text{(Ver):} When \( \mathcal{V} \) is strongly-rational, the Grothendieck ring \( \text{Fus}^\mathcal{V} (\mathcal{V}) \), full tensor ring \( \text{Fus}^{\text{full}} (\mathcal{V}) \) and simple-projective subring \( \text{Fus}^{\text{simp}} (\mathcal{V}) \) all coincide. For each \( i \), define the fusion matrix \( \mathcal{N}_i \) by \( (\mathcal{N}_i)_{jk} = \mathcal{N}_{ij}^k \). They yield a representation of the tensor ring:

\[
\mathcal{N}_i \mathcal{N}_j = \sum_k \mathcal{N}_i^k \mathcal{N}_k.
\]
In 1988, Verlinde [V] conjectured that the fusion coefficients $N_{ij}^k$ are related to the matrix $S^x$ by what is now called *Verlinde’s formula*:

$$N_{ij}^k = \sum_{\ell=1}^n S_{i\ell}^x S_{j\ell}^x (S^{x^{-1}})_{\ell k}.$$  \hfill (8)

This is one of the most exciting outcomes of the mathematics of rational CFT. As $S^x$ is a unitary matrix, $(S^{x^{-1}})_{\ell k}$ here can be replaced with the complex conjugate $S_{\ell k}^*$. The Verlinde formula, thus has three aspects:

(V1) There is a matrix $S^\otimes$ simultaneously diagonalizing all fusion matrices $N_i$. Each diagonal entry $\rho_i(M_i) := (S^{\otimes^{-1}}N_i S^{\otimes})_i$ defines a one-dimensional representation of the tensor ring. All of these $\rho_i$ are distinct.

(V2) The Hopf link invariants $(1)$ for any $1 \leq i,j \leq n$ give one-dimensional representations of the fusion ring:

$$\frac{S_{ij}^\otimes}{S_{ii}^\otimes S_{jj}^\otimes} = \sum_{k=1}^n N_{ij}^k \frac{S_{ik}^\otimes}{S_{kk}^\otimes}.$$

Moreover, each representation $\rho_i$ appearing in (V1) equals one of these $S$-matrices: these three $S$-matrices are essentially the same. More precisely, $S_{ij}^\otimes / S_{ii}^\otimes = S_{ij}^\otimes$, and $S^\otimes = S^x$ works in (V1). For this choice of $S^\otimes$, $\rho_i(M_i) = S_{ii}^\otimes / S_{ii}^\otimes$.

(V3) The *deepest fact*: these three $S$-matrices are essentially the same. More precisely, $S_{ij}^\otimes / S_{ii}^\otimes = S_{ij}^\otimes$, and $S^\otimes = S^x$ works in (V1). For this choice of $S^\otimes$, $\rho_i(M_i) = S_{ii}^\otimes / S_{ii}^\otimes$.

Much of (V1)-(V3) is automatic in any modular tensor category. In particular, theorem 4.5.2 in [T] says that in any such category,

$$N_{ij}^k = D^{-2} \sum_{\ell} \frac{S_{i\ell}^w S_{j\ell}^w S_{\ell k}^{w*}}{S_{\ell\ell}^w},$$

where $D^2 = \sum_j S_{jj}^w$. So the remaining content of (8) is that $S^x = D^{-1}S^w$. Categorically, the space of 1-point functions $\text{ch}[M]$ is identified with the space $\text{Hom}(1, \mathcal{H})$ where

$$\mathcal{H} \cong \bigoplus_i M_i^* \otimes M_i$$

(throughout this paper, $M^*$ denotes the *contragredient* or dual of $M$). This should remind us of the definition of class function from section 2.1, and that is no accident. This object $\mathcal{H}$ is naturally a Hopf algebra and a Frobenius algebra. Using this structure, section 6 of [Ly1] shows $\text{End}(\mathcal{H})$ carries a projective action of $\text{SL}(2, \mathbb{Z})$; it is the coend of [Ly1], and [Sh] suggests to interpret it (or its dual) as the adjoint algebra of the category. The $\text{SL}(2, \mathbb{Z})$-action on the $\text{ch}[M]$ correspond to the subrepresentation on $\text{Hom}(1, \mathcal{H})$, which is generated by $S$-matrix $D^{-1}S^w$ and $T$-matrix coming from the ribbon twist. The other subrepresentations $\text{Hom}(M_j, \mathcal{H})$ correspond to the projective $\text{SL}(2, \mathbb{Z})$-representations on the space of 1-point functions with insertions $v \in M_j$, defined in (5). As in [FS1], the character $\text{ch}[M]$ of (6) can be interpreted categorically as the partial trace of the ‘adjoint’ action $\mathcal{H} \otimes M \to M$.

It is far easier to construct modular tensor categories, than to construct VOAs or CFTs. This can be used to probe just how complete the lists of known (strongly-rational) VOAs and CFTs are. More precisely, given a fusion category (essentially a modular tensor category without the
braiding), taking its Drinfeld double or centre construction yields a modular tensor category. Fusion categories can be constructed and classified relatively easily, using subfactor methods, and their doubles computed, at least when their fusion rules are relatively simple. Examples of this strategy are provided in e.g. [EG1]. This body of work suggests that the zoo of known modular tensor categories is quite incomplete, and hence that the zoo of known strongly-rational VOAs and CFTs is likewise incomplete.

2.4. The finite logarithmic story

An important challenge is to extend the aforementioned results beyond the semi-simplicity of the associated tensor categories. A VOA is called logarithmic if at least one of its indecomposable modules is not simple. See e.g. [CR4, FS1] for introductions to logarithmic VOAs. The name refers to logarithmic singularities appearing in their correlation functions and operator product algebras of intertwining operators. This paper focusses on strongly-finite VOAs, i.e. logarithmic VOAs with only finitely many simple modules (see section 2.2 for the formal definition). This section reviews their basic theory.

Though strongly-finite VOAs have finitely many simples, they usually have uncountably many indecomposables. Much of the following theory persists even when simplicity or CFT-type is lost, provided $C_2$-cofiniteness is retained.

Any strongly-rational VOA is strongly-finite. The best studied class of nonrational but strongly-finite VOAs are the $\mathcal{W}_p$-triplet algebras, parameterized by $p \in \mathbb{Z}_{>1}$. They have central charge
$$c = 1 - 6\frac{(p - 1)^2}{p},$$
and are generated by the conformal vector and 3 other states. The symplectic fermions form a logarithmic superVOA with $c = -2d$ for any $d \in \mathbb{Z}_{>0}$ (the number of pairs of fermions); their even part $SF_+^d$ is a strongly-finite VOA [Ab]. The $\mathcal{W}_{p^2}$-models are $C_2$-cofinite [TW2] but not simple, so are not strongly-finite.

A tensor product theory for strongly-finite VOAs has been developed by Huang, Lepowsky and Zhang (see e.g. [HLZ] and references therein), see also Miyamoto [Miy3]. The corresponding tensor category $\text{Mod}_{g \cdot r}(V)$ is braided. Rigidity of this category is proven so far only for the $\mathcal{W}_p$ models [TW1] and symplectic fermions $SF_+^d$ [DR].

There have been different proposals in the literature for what replaces modular tensor category here, e.g. [KL, FS1]. See section 3.1.2 for our preference. In this nonsemi-simple setting, we will in general only have the inequality $N_{U,V,W} \leqslant \dim \text{Hom}_V(U \otimes V, W)$ for the tensor product multiplicities, where now $U, V, W$ are indecomposables.

Fix representatives $M_1, \ldots, M_n$ of isomorphism classes of simple $V$-modules as before. Let $P_i$ be their projective covers. Likewise fix representatives of isomorphism classes $M_\lambda$ of indecomposable $\bar{V}$-modules. Any $\bar{V}$-module $M \in \text{Mod}_\bar{V}(V)$ is isomorphic to a direct sum of finitely many $M_\lambda$, in a unique way. Recall from section 2.2 the full tensor ring $\text{Fus}_{\text{full}}(V)$, the subring $\text{Fus}_{\text{simp}}(V)$, and the Grothendieck ring $\text{Fus}_\text{gr}(V)$. These will no longer be isomorphic in general—in fact $\text{Fus}_{\text{full}}(V)$ will usually have an uncountable basis. We avoid calling any of these the fusion ring, as the term ‘fusion ring’ is used in different senses in the literature.

A serious problem here is that the $\mathbb{C}$-span of graded-dimensions $\text{ch}[M_i](\tau) = F_{M_i}(\tau, 1)$ is no longer $\text{SL}(2, \mathbb{Z})$-invariant, although we still have that each $\text{ch}[M_i] \left( \frac{n + 1}{n + 2} \right)$ lies in the $\mathbb{C}[\tau]$-span of the characters. Many authors interpret this as saying that e.g. $S^V$ is now $\tau$-dependent. We prefer to say that the ordinary trace functions (or characters) (6) must be augmented by
pseudo-trace functions (or pseudo-characters) associated to indecomposable $V$-modules. The main result here is due to Miyamoto [Miy2]:

(\textbf{Mod})' The $\mathbb{C}$-span of (pseudo-)characters (suitably defined) is $SL(2,\mathbb{Z})$-invariant.

The dimension of the resulting $SL(2,\mathbb{Z})$-representation is given by $\dim \mathbb{A}_m/\mathbb{A}_m - \dim \mathbb{A}_{m-1}/\mathbb{A}_{m-1}$, where $A_k = A_k(V)$ is the $k$th Zhu algebra (a finite-dimensional associative algebra), and $m$ is sufficiently large. It is possible to state explicitly how large $m$ must be, but this is not terribly useful as the algebras $A_k(V)$ are very hard to identify in practise. We review Miyamoto’s theory in section 2.5.

The CFT philosophy is that traces with field insertion are the natural objects to look at. This point of view has first appeared in the context of boundary CFT of $bc$-ghost and then symplectic fermions [CQS2, CRo] and then later in the symplectic fermion super VOA [Ru]. Computations of them in the spirit of Miyamoto using symmetric linear functions has so far also only been done in the symplectic fermion case [AN].

There have been several proposals for a strongly-finite Verlinde formula beyond rationality, mainly in the physics literature, see [R, GabR, FHST, GaiT, FK]. All these proposals are guided by analogy to the rational setting and they do not connect to the tensor category point of view (recall that equation (8) is a theorem in any modular tensor category, when $S^k$ there is replaced with $D^{-1}S^m$). Define the tensor resp. Grothendieck matrices $N^\delta$ resp. $N^\delta$, with entries $(N^\delta_{M',M''}) = N^\delta_{M',M''}$ in $Fus^{\text{simp}}(V)$ and $(N^\delta_{M',M''})_{i,j} = N^\delta_{M',M''}M_i$ in $Fus^r(V)$. As before, they define representations of $Fus^{\text{simp}}(V)$ respectively $Fus^r(V)$.

(V1)' There are matrices $S^r \otimes \text{resp. } S^r \otimes$ which simultaneously put the $N^\delta$ resp. $N^\delta$ into block diagonal form: e.g.

$$S^r \otimes \text{diag}(B^r_1(M), B^r_2(M), \ldots, B^r_j(M))$$

where for each $h$, $M_i \mapsto B^h(M_i)$ defines an indecomposable representation of the Grothendieck ring $Fus^r(V)$ (and similarly for $Fus^{\text{simp}}(V)$).

(V1)' is not deep: the fusion matrices $N^\delta$ form a representation of $Fus^r(V)$, so $S^r \otimes$ is the change-of-basis which decomposes that representation into a direct sum of indecomposables. In [R, PRR], the matrices $N^\delta$ and $N^\delta$ for the $W_p$ models have been explicitly block-diagonized: we review this work in section 3.3.2. We also should block-diagonalise the full tensor ring $Fus^\text{full}(V)$, in the sense of [CR1, CR2]. To our knowledge this has never been explored.

To our knowledge, no analogue of (V2)' has been explored. The Hopf link invariants of (V2) are still defined, but vanish on projective modules (at least if the category is rigid). If they did not vanish they would no longer yield tensor or Grothendieck ring representations. The reason for this is that $\dim \text{End}(M_i)$ can be larger than one. We propose a general (V2)' in section 3.1.2.

Nothing general about (V3)' is known or has been explicitly conjectured. However, Lyubashenko [Ly1] explained that the modular group acts also on certain nonsemi-simple braided tensor categories, [FGST1] observed that the $SL(2,\mathbb{Z})$ action on the center of the restricted quantum group $U_q(sl_2)$ at $2p$th root of unity $q$, coincides with the one on the space of trace and pseudo-trace functions of the $W_p$-triplet algebra.

The categorical $SL(2,\mathbb{Z})$-action discussed last section extends here as follows. Write $\mathcal{C} = \text{Mod}^\text{Fr}(V)$. According to [Sh], the coend/Hopf algebra/Frobenius algebra/adjoint algebra $\mathcal{H}$ is simply $UR(V)$, where $U: Z(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor from the Drinfeld centre of $\mathcal{C}$, and $R$ is its right adjoint. The space of formal 1-point functions on the torus is $\text{Hom}_\mathcal{F}(V, \mathcal{H})$. The ordinary characters (6) span a subspace (of dimension equal to the number of simple $V$-modules). We turn to this next.
2.5. Miyamoto’s modularity theorem

Let \(\mathcal{V}\) be strongly-finite. Miyamoto [Miy2] copies from Zhu [Z] the definition of \textit{formal 1-point functions}. These are, among other things, complex-valued functions \(F(f, \tau)\) where \(\tau \in \mathbb{H}\) and \(f = f(\tau)\) is a finite combination of vectors in \(\mathcal{V}\) with coefficients which are modular forms of \(\text{SL}(2, \mathbb{Z})\) (so polynomials in \(E_{4}(\tau)\) and \(E_{6}(\tau)\)). Let \(\mathcal{F}(\mathcal{V})\) denote the space of all formal 1-point functions.

It is easy to show directly from the definition that \(\mathcal{F}(\mathcal{V})\) carries an action of \(\text{SL}(2, \mathbb{Z})\) through Möbius transformations on \(\tau \in \mathbb{H}\) as usual. The main result in [Z] is that, for strongly-rational \(\mathcal{V}\), this space \(\mathcal{F}(\mathcal{V})\) has a basis over \(\mathbb{C}[E_{4}, E_{6}]\) given by the characters \(\chi_{M}[\nu, \tau]\) in (6). The main result (theorem 5.5) of [Miy2] is that, when \(\mathcal{V}\) is strongly-finite, \(\mathcal{F}(\mathcal{V})\) is finite-dimensional, and spanned by the pseudo-trace functions \(S^{M, \phi}(u, \tau)\), where \(M\) is a ‘generalized Verma module interlocked with a symmetric linear functional \(\phi\) of the \(n\)th Zhu algebra \(A_{n}(\mathcal{V})\) for \(n\) sufficiently large (we will try to explain this shortly). These functions \(S^{M, \phi}(u, \tau)\) are quite difficult to compute in practice. Even the dimension of \(\mathcal{F}(\mathcal{V})\) is hard to determine in practice. Although Miyamoto’s approach is natural from the associative algebra point-of-view, it is not all so from the quantum field theory one. (We address this directly in section 3.1.1 below.) This makes it hard for many researchers to understand, despite its obvious importance. For this reason, in this section we will supply some of the background and motivation for Miyamoto’s work.

Let \(S^{(j)}\) denote the basis of \(\mathcal{F}(\mathcal{V})\) found by Miyamoto. Fix \(u \in \mathcal{V}\) with \(L[0]u = ku\), then the vector \(\mathcal{X}(\tau)\) with components \(\mathcal{X}(\tau)_{\nu} = S^{(j)}(u, \tau)\) is a weight-k vector-valued modular form. Taking \(u = 1\), the vacuum, recovers what could be called graded-pseudo-dimensions, e.g. the familiar graded-dimensions \(\chi_{M}(\tau) = q^{h_{M} - c/24} \sum_{n=0}^{\infty} \dim M_{n} q^{n}\).

Incidentally, the presence of \(\mathbb{C}[E_{4}, E_{6}]\) in the arguments of formal 1-point functions is not significant. But the presence there of \(u \in \mathcal{V}\) is vital. It is common in the literature to drop \(u\) from the character (6) and consider only the graded-dimensions \(\chi_{M}(\tau)\). The reason for considering 1-point functions is to avoid accidental linear dependencies between characters. For example, a module and its contragredient will always have the same graded-dimensions, but usually different characters (1-point functions).

Here is the complication in the strongly-finite case. Choose any indecomposable \(\mathcal{V}\)-module \(M\), and let \(M(n)\) \((n = 0, 1, \ldots)\) denote the generalized \(L_{0}\)-eigenspace of eigenvalue \(h_{M} + n\). On each subspace \(M(n), \ L_{0} - c/24\) acts as \((n + h_{M} - c/24)\text{Id} + L_{\text{nil}}\) where \(L_{\text{nil}}^{k} = 0\) for some \(k\) independent of \(n\). We obtain for the usual character

\[
\operatorname{ch}[M][\nu, \tau] = \sum_{n=0}^{\infty} \text{Tr}[M|\nu] (o(\nu) q^{n-h_{M} + c/24} \sum_{j=0}^{k-1} \left(\frac{2\pi i \tau}{j}\right)^{j} \text{Tr}[M|\nu] (o(\nu) L_{\text{nil}}^{j}).
\]

It is not hard to show that on \(M(n)\) the nilpotent operator \(L_{\text{nil}}\) commutes with all operators \(o(\nu)\). This means for any \(j > 0\) and any \(\nu \in \mathcal{V}\), \((o(\nu) L_{\text{nil}}^{j})^{k} = o(\nu)^{k} L_{\text{nil}}^{k} = 0\). But the trace of a nilpotent operator is always zero, so (9) collapses to

\[
\operatorname{ch}[M][\nu, \tau] = \sum_{n=0}^{\infty} \text{Tr}[M|\nu] (o(\nu) q^{n-h_{M} - c/24}.
\]

Thus the ordinary trace can never see the nilpotent part of \(L_{0}\), and \(\operatorname{ch}[M][\nu, \tau] = \operatorname{ch}[N][\nu, \tau]\) whenever \(\mathcal{V}\)-modules \(M, N\) have the same composition factors.

Nor can we obtain any more 1-point functions if we insert an endomorphism \(f\) of \(\mathcal{V}\)-module \(M\) by definition, \(f\) will commute with all zero-modes and hence with \(L_{0}\), so \(f \circ o(\nu) \circ L_{\text{nil}}\) is still nilpotent.
What we need is a way to generalize the trace of $M(n)$-endomorphisms so that the terms $o(v)L^j_{nil}$ with $j > 0$ can contribute. We need to see the off-diagonal parts of the endomorphisms.

There is a classical situation where this happens. Let $A$ be a finite-dimensional associative algebra with 1 over $\mathbb{C}$. A linear functional $\phi : A \to \mathbb{C}$ is called symmetric if $\phi(ab) = \phi(ba)$ for all $a, b \in A$. Let $SLF(A)$ denote the space of all symmetric linear functionals on $A$—it can be naturally identified with the dual space $(A/[A,A])^*$. Now suppose $W$ is a finitely-generated projective $A$-module. Then an $A$-coordinate system of $W$ consists of finitely many $u_i \in W$ and the same number of $f_i \in \text{Hom}_A(W,A)$ such that $w = \sum f_i(w)u_i$ for any $w \in W$. Given an $A$-coordinate system, we can associate any endomorphism $\alpha \in \text{End}_A(W)$ with a matrix $[\alpha]$ whose $ij$th entry is $[\alpha]_{ij} = f_i(\alpha(u_j)) \in A$. Fix any symmetric linear functional $\phi \in SLF(A)$ and $A$-coordinate system $\{u_i,f_i\}$, and define the pseudo-trace $\text{Tr}_W^\phi : \text{End}_A(W) \to \mathbb{C}$ by

$$\text{Tr}_W^\phi(\alpha) = \phi(\text{Tr}(\alpha)) = \sum_i \phi(f_i(\alpha(u_i))).$$

The pseudo-trace $\phi^\prime_W$ is independent of the choice of $A$-coordinate system, and lies in $SLF(\text{End}_A(W))$. It satisfies $\text{Tr}_W^\phi(\alpha \circ \beta) = \text{Tr}_W^\phi(\beta \circ \alpha)$ for any $\alpha \in \text{Hom}_A(V,W), \beta \in \text{Hom}_A(W,V)$.

To apply this generalized notion of trace to our VOA setting, we need to find a finite-dimensional associative algebra $A$ and $\mathcal{V}$-modules $M$ for which the generalized eigenspaces $M_\alpha$ are projective $A$-modules. In [Miy2], $A$ is related to the $n$th Zhu algebra $A_n(V)$ for $n$ sufficiently large, and $M$ are certain $\mathcal{V}$-modules. [AN] makes a different choice: it is elementary that a $\mathcal{V}$-module $M$ (and in fact all of its generalized eigenspaces $M_\alpha$) is a module over the (finite-dimensional associative) algebra $\text{End}_\mathcal{V}(M)$, so choose some subalgebra $A$ of $\text{End}_\mathcal{V}(M)$ so that $M$ is projective over $A$.

In both of these (related) cases, we have a finite-dimensional associative algebra $A$ and a symmetric linear functional $\phi \in SLF(A)$, and a (logarithmic) $\mathcal{V}$-module $M$ such that each generalised $L_\mathcal{V}$-eigenspace $M_\alpha$ is a finite-dimensional projective $A$-module. We can then define the pseudo-trace function

$$\chi_M^\phi(\mathcal{V},\tau) = \sum_{n=0}^{\infty} q^{h_{\mathcal{V}} + n - c/24} \sum_{j=0}^{k-1} \frac{(2\pi i \tau)^j}{j!} \text{Tr}_M^\phi(\phi^j L_{nil}^j).$$

This is a 1-point function in $\mathcal{F}(\mathcal{V})$. In the Miyamoto case, these span $\mathcal{F}(\mathcal{V})$; in the simpler Arike–Nagatomo case, there is no such theorem yet.

Let us make this more concrete with an example arising in the $W_p$ models. Suppose there are submodules $M, M'$ of a projective module $P$ such that $P/M' \cong M$, as $\mathcal{V}$-modules. Suppose in addition that $M$ is a submodule of $M'$. This happens for example with $M = 0, M' = P$, but it also happens in $W_p$ with $M = \text{soc}(P)$ and $M' = \text{rad}(P)$ when $P = P_p^s, s \neq p$. Then in each generalized eigenspace $P(n)$ of $L_0$, the operator $o(v)q^s$ has matrix form

$$\begin{pmatrix} A & B & C \\ 0 & D & B' \\ 0 & 0 & A \end{pmatrix} \begin{pmatrix} q^n & 0 & 2\pi iq^s \tau \\ 0 & q^s & 0 \\ 0 & 0 & q^s \end{pmatrix}$$

(10)

where the first row/column refers to the subspace $M$, the second row/column refers to any lift to $P$ of $M'/M$, and the third row/column refers to any lift to $P$ of $P/M'$. Note that $C$ can be regarded as an operator on $M$. In this context, Miyamoto has a pseudo-trace $\text{ptr}_{W(p)}$ of this operator, given by $2\pi i q^s \text{Tr} C$. In particular, we see an off-diagonal part of the endomorphism.
More general 1-point functions, by inserting intertwiners, has been studied recently in this picture [Fi].

2.6. Braided tensor categories

The books [T, EGNO] are good references on tensor categories. Some features for non-semisimple ones relevant for us are included in [EO, CGP, GKP1].

Let \( C \) be a monoidal category, so it has a tensor product, and a tensor unit \( 1 \). For simplicity we assume it to be strict. This comes with a warning: the monoidal category of a VOA is not strict, meaning that associativity isomorphisms are not trivial. One thus needs to verify always that theorems also hold in the non-strict case (this is called coherence). A very useful theorem is here that every braided monoidal category is braided equivalent to a free braided monoidal category and in the latter morphisms composed out of braidings and associativity only depend on its braid diagram, see [JS].

If \( V \) is strongly-finite, then \( \text{Mod}^{sq}(V) \) will always be \( \mathbb{C} \)-linear, additive and abelian. The former means that the \( \text{Hom}(U, V) \) are vector spaces over \( \mathbb{C} \); additive means there are direct sums; the exact meaning of the latter is a little technical but it holds whenever the category is equivalent as an additive category to the category of left modules over an associative algebra with 1, which is the case here. \( \text{Mod}^{sq}(V) \) will also have \( \text{Hom}(1, 1) = \mathbb{C} \). Assume \( C \) obeys those 4 properties.

The category is called rigid if for each object \( M \) in the category, there is a dual \( M^* \) and morphisms \( b_V \in \text{Hom}(1, V \otimes V^*) \) (the co-evaluation) and \( d_V \in \text{Hom}(V^* \otimes V, 1) \) (the evaluation) such that

\[
(Id_V \otimes d_V) \circ (b_V \otimes Id_V) = Id_V, \quad (d_V \otimes Id_{V^*}) \circ (Id_{V^*} \otimes b_V) = Id_{V^*}.
\]

Rigidity is required for the categorical trace and dimension, as we will see shortly. Given any objects \( V, W \), the braiding \( c_{V,W} \) is an intertwiner in \( \text{Hom}(V \otimes W, W \otimes V) \) satisfying

\[
c_{U \otimes V, W} = (Id_V \otimes c_{U,W}) \circ (c_{U,V} \otimes Id_W), \quad c_{V \otimes W, U} = (c_{U,W} \otimes Id_V) \circ (Id_U \otimes c_{V,W})
\]

and also for any intertwiners \( f \in \text{Hom}(V, V'), g \in \text{Hom}(W, W') \),

\[
(g \otimes f) \circ c_{V,W} = c_{V',W'} \circ (f \otimes g).
\]

This property is called naturality of braiding. It implies \( c_{V,1} = c_{1,V} = 1 \) as well as the Yang–Baxter equation. If the category \( C \) is in addition additive, then \( (U \oplus V) \otimes W \) resp. \( W \otimes (U \oplus V) \) are isomorphic with \( U \otimes W \oplus V \otimes W \) resp. \( W \otimes U \oplus W \otimes V \), and using these isomorphisms the naturality (12) of the braiding implies that we can make the identifications

\[
c_{U \otimes V, W} = c_{U,W} \otimes c_{V,W}, \quad c_{W,U \otimes V} = c_{W,U} \otimes c_{W,V}.
\]

Given any module \( V \) in our category, the twist \( \theta_V \in \text{Hom}(V, V) \) satisfies, for any module \( V \) and intertwiner \( f \in \text{Hom}(V, V) \),

\[
\theta_{V \otimes W} = c_{W,V} \circ c_{V,W} \circ (\theta_V \otimes \theta_W), \quad \theta_V \circ f = f \circ \theta_V
\]

This implies \( \theta_1 = 1 \). Define morphisms

\[
b'_V := (Id_{V^*} \otimes \theta_V) \circ c_{V,V^*} \circ b_V \in \text{Hom}(1, V^* \otimes V)
\]

\[
d'_V := d_V \circ c_{V,V^*} \circ (\theta_V \otimes Id_{V^*}) \in \text{Hom}(V \otimes V^*, 1).
\]

Computations are greatly simplified using a graphical calculus, e.g.
For any simple object \( C \), the topological representation of \( SL_2 \) is always a direct sum of simples and projectives (as it is for category \([T]\) we mean a strict braided tensor category equipped with a twist \( \theta \) and \( \epsilon \)) we mean a finite tensor category which is in addition semi-simple. By a ribbon category \([T]\) we mean a strict braided tensor category equipped with a twist \( \theta \in \text{End}(X) \).

In a ribbon finite tensor category, the categorical \( S \)-matrix is defined by

\[
S^o_{U,V} = \text{tr}(c_{U,V} \circ c_{V,U}) \in \mathbb{C},
\]

for any indecomposable modules \( U, V \). Its graphical representation is the Hopf link (1). By a modular tensor category \([T]\) we mean a fusion category which is ribbon, whose matrix \( S^o \) is invertible. The topological \( T \)-matrix in a modular tensor category is the diagonal matrix with entries \( T_{V,V} = \theta_V \) for any simple object \( V \). These categorical \( S \)- and \( T \)-matrices define a projective representation of \( SL(2, \mathbb{Z}) \). It satisfies \( S_{U,V}^o = S_{V,U}^o, S_{0,V}^o = \text{dim}(V) \), as well as Verlinde’s formula

\[
\sum_W N_{U,V}^{W_1W_2} S_{W_1X}^o S_{W_2X} = (\text{dim} X)^{-1} S_{U,X}^o S_{V,X}.
\]

When the VOA is strongly-rational, the categorical \( S \)- and \( T \)-matrices will agree up to scalar factors with the modular \( S \)- and \( T \)-matrices defined through the VOA characters.

Return for now to a finite tensor category \( C \). Assume it is ribbon. By a negligible morphism \( f \in \text{Hom}_C(U, V) \) we mean one for which the categorical trace \( \text{Tr}_C(g \circ f) = 0 \) for all \( g \in \text{Hom}_C(V, U) \). The negligible morphisms form a subspace \( \text{Id}_C(U, V) \) of \( \text{Hom}_C(U, V) \), closed under taking duals, arbitrary compositions, as well as arbitrary tensor products. Consider the category \( \overline{C} \) whose objects are the same as those of \( C \), but whose Hom-spaces are \( \text{Hom}_C(U, V) / \text{Id}_C(U, V) \). This category was originally defined in [BW]; see also p 236 of [EGNO]. If \( U, V \) are indecomposable in \( C \) and \( f \in \text{Hom}_C(U, V) \) is not an isomorphism, then \( f \) is negligible. Then \( \overline{C} \) is a ribbon category, whose simple objects are precisely the indecomposable objects of \( C \) with nonzero categorical dimension (the indecomposables with dimension 0 are in \( \overline{C} \) isomorphic to the 0-object). Moreover, two simple objects in \( \overline{C} \) are isomorphic iff they are isomorphic indecomposables in \( C \). So \( \overline{C} \) will generally have infinitely many inequivalent simples, but it is semi-simple in the sense that every object in \( \overline{C} \) is a direct sum of simples. By the semi-simplification \( C^{ss} \) of \( C \), we mean the full subcategory of \( \overline{C} \) generated by the simples of \( C \). Then \( C^{ss} \) will also be a semi-simple ribbon category. If in \( C \) the tensor product of simples is always a direct sum of simples and projectives (as it is for \( \mathcal{W}_p \) and the symplectic fermions),
then $\mathcal{C}^\text{ss}$ will be a fusion category, whose Grothendieck ring is the quotient of that of $\mathcal{C}$ by the modules in $\mathcal{C}$ which are both simple and projective. If $\mathcal{C}$ is ribbon, so is both $\mathcal{C}$ and $\mathcal{C}^\text{ss}$.

**Remark 2.1.** Note though that the semi-simplification of a nonsemi-simple finite tensor category $\mathcal{C}$ can never be unitary, as some quantum-dimensions must be negative in order that the projectives of $\mathcal{C}$ have 0 quantum-dimension. Many of the familiar strongly-rational VOAs (e.g. those coming from affine algebras at integral level) give rise to unitary modular tensor categories.

### 2.7. Lyubashenko’s modularity and the coend

Associated to a modular tensor category, are a tower of finite-dimensional projective actions of the mapping class groups of the surfaces. These correspond to the actions on the conformal blocks of an associated rational CFT [MS].

Lyubashenko explained how these (projective) actions also arise in the context of certain non semi-simple ribbon categories. The important object here is the coend of $\mathcal{C}$. It is denoted by

$$\text{coend}(\mathcal{C}) = \int_{X \in \mathcal{C}} X \otimes X^*$$

and defined by [Ly3]

$$\bigoplus_{f : A \to B \in \mathcal{C}} X \otimes X^* \xrightarrow{f \otimes B^* - A \otimes f'} X \otimes X^* \longrightarrow \text{coend}(\mathcal{C}) \longrightarrow 0$$

where $f' : B \to A^*$ is the transpose of $f : A \to B^*$. Lyubashenko defines 2-modular categories [Ly1] (see also definition 1.3.1 of [Ly3]). These are abelian ribbon categories with additional axioms. They imply the existence of a morphism $\mu : 1_{\mathcal{C}} \to \text{coend}(\mathcal{C})$ (called an integral) solving an equation relating open Hopf links dressed with twist to the inverse of the twist. This is called the quantum Fourier transform and is explicitly stated in section 1 of [Ly3]. The existence of the integral $\mu$ then allows one to show that a morphism $S : \text{coend}(\mathcal{C}) \to \text{coend}(\mathcal{C})$ defined via the integral and the monodromy together with another morphism $T$ that is defined via the twist define a projective representation of $SL(2; \mathbb{Z})$. More generally there is even a projective representation of the mapping class group of genus $g$ with $n$-punctures on $\text{Hom}(X_1 \otimes \cdots \otimes X_n, \text{coend}(\mathcal{C})^\otimes g)$. Clearly one expects and would like to understand a relation of these to conformal blocks.

The role of the coend in CFT has been investigated very recently in [FS3]. Very nice and probably also useful properties are that the coend has the structure of a Hopf algebra (more precisely a Hopf monad) and there is a braided equivalence between the category of coend $(\mathcal{C})$-modules in $\mathcal{C}$ and the center $Z(\mathcal{C})$ of $\mathcal{C}$ [DS, BrV].

### 2.8. Example 1: The triplet algebra $\mathcal{W}_p$

Our main reference on the triplet algebra is [TW1]. The triplet models $\mathcal{W}_p$ ($p \geq 2$ an integer) are a family of VOAs with central charge $c_p = 1 - 6(p - 1)/p$. They are logarithmic and strongly-finite [CF, AdM1]. As mentioned in section 2.4, this implies they have finitely many simple (ordinary) modules, whose (pseudo-)characters form a vector-valued modular function for $SL(2; \mathbb{Z})$, and which yields a braided tensor category $\text{Mod}^\otimes(\mathcal{W}_p)$. $\mathcal{W}_p$ has been studied by Adamovic and Milas in a series of papers; in [AdM4] Zhu’s algebra was described and in
[AdM3] it was shown that the space of 1-point functions is \(3p - 1\)-dimensional. The tensor product and rigidity for \(W_p\) were determined in [TW1] (the tensor product had been previously conjectured in [FHST, GabR]).

The simple modules are \(X_s^+\) and \(X_s^-\) for \(1 \leq s \leq p\). The tensor unit (vacuum) is \(X_1^+\). Their projective covers are \(P_i^+\) and \(P_i^-\), where \(P_p^\pm = X_p^\pm\) and
\[
0 \to Y_s^+ \to P_s^+ \to X_s^+ \to 0, \quad 0 \to Y_s^- \to P_{p-s}^- \to X_{p-s}^- \to 0
\]
for \(1 \leq s \leq p - 1\), where \(Y_j^\pm\) denotes the reducible but indecomposable modules
\[
0 \to X_s^+ \to Y_s^+ \to 2X_{p-s}^- \to 0, \quad 0 \to X_{p-s}^- \to Y_s^- \to 2X_s^+ \to 0.
\]
These 4\(p\) - 2 irreducible and/or projective \(W_p\)-modules are the most important ones. They are closed under tensor product and form the subring \(\text{Fus}^\text{simp}(W_p)\).

[NT] proved the equivalence as abelian categories of \(\text{Mod}^\text{fd}(W_p)\) with the category \(\text{Mod}^\text{fd}(\mathbf{U}_q(\mathfrak{sl}_2))\) of finite-dimensional modules of the restricted quantum group \(\mathbf{U}_q(\mathfrak{sl}_2)\) at \(q = e^{\pi i/p}\), and both [FGST2, KS] determined all indecomposables in \(\text{Mod}^\text{fd}(\mathbf{U}_q(\mathfrak{sl}_2))\) (although the key lemma, describing pairs of matrices up to simultaneous conjugation, is really due to Frobenius (1890)). Hence:

**Theorem 2.2 ([TW1]).** The complete list of indecomposable grading-restricted weak \(W_p\)-modules, up to equivalence, is:

1. for each \(1 \leq j \leq p\) and each sign, the simple modules \(X_j^\pm\) and their projective covers \(P_j^\pm\) (\(P_p^\pm = X_p^\pm\));
2. for each \(1 \leq j < p\), each sign, and each \(d \geq 1\), \(G_j^d\), \(H_j^d\), and \(I_j^d(\lambda)\), where \(\lambda \in \mathbb{CP}^1\).

Consider the full subcategory relevant to our discussion, whose objects are finite direct sums of \(X_j^\pm\), \(P_j^\pm\) for \(1 \leq s \leq p\). The fusion rules are completely determined from the following:

\[
X_1^+ \otimes X_1^- = X_1^+, \quad X_1^+ \otimes P_1^+ = P_1^-,
\]
\[
X_2^+ \otimes X_1^+ = \begin{cases} X_2^+ \otimes X_s^+ & s = 1 \\
X_{s-1}^+ \otimes X_{s+1}^+ & 2 \leq s \leq p, \\
P_{p-1}^+ & s = p \end{cases}
\]
\[
X_2^+ \otimes P_1^+ = \begin{cases} P_1^+ \otimes 2 \cdot X_p^- & s = 1 \\
P_{p-2}^+ \otimes P_{p+1}^+ & 2 \leq s \leq p - 1, \\
P_{p-2}^+ \otimes 2 \cdot X_p^- & s = p - 1 \end{cases}
\]

together with associativity and commutativity. For example one can show
\[
P_j^+ \otimes P_i^- = 2X_j^+ \otimes P_i^- \oplus 2X_{p-s}^- \otimes P_i^-.
\]

In \(W_2\), \(X_2^+ \otimes P_2^+\) must be replaced with \(X_2^+ \otimes P_2^+ = 2X_2^+ \oplus 2X_2^-\).

As always, the \(\mathbb{Z}\)-span of the projective modules \(\{P_p^\pm\} \cup \{X_p^\pm\}\) forms an ideal in \(\text{Fus}^\text{full}(W_p)\). The quotient of the tensor subring \(\mathbb{Z}\)-span of \(X_s^+\) by that ideal is easily seen to be two copies of the fusion ring of \(L_{p-2}(\mathfrak{sl}_2)\), more precisely it is isomorphic to the fusion ring of the rational VOA \(L_{1}(\mathfrak{sl}_2) \otimes L_{p-2}(\mathfrak{sl}_2)\), where \(X_1^-\) corresponds to the integrable highest weight module \(L(\Lambda_1)\) of \(L_{1}(\mathfrak{sl}_2)\) and \(X_s^+\) corresponds likewise to the module \(L((p - 1 - s)\Lambda_0 + (s - 1)\Lambda_1) of L_{p-2}(\mathfrak{sl}_2)\) for \(1 \leq s < p\). More importantly, this also persists categorically: the semi-simplification \((\text{Mod}^\text{fd}(W_p))^{\text{ns}}\) is a (nonunitary) modular tensor
category, namely the twist of that of $L_{p-2} (\mathfrak{sl}_2) \otimes L_1 (\mathfrak{sl}_2)$ by a simple-current of order 2 as discussed in [CG].

The graded-dimensions of $X^\pm_p$ are

$$\text{ch}[X^+_p](\tau) = \frac{1}{\eta(\tau)} \left( \frac{s}{p} \theta_{p-1,p}(\tau) + 2 \theta'_{p-1,p}(\tau) \right), \quad \text{ch}[X^-_p](\tau) = \frac{1}{\eta(\tau)} \left( \frac{s}{p} \theta_{p,p}(\tau) - 2 \theta'_{p,p}(\tau) \right)$$

where

$$\theta_{p,p}(\tau,z) = \sum_{j \in \mathbb{Z}^+ \sqrt{p} \mathbb{Z}} e^{2\pi i \tau j^2 / 2} e^{2\pi i jz}$$

is the theta series associated to the coset $\frac{1}{\sqrt{2p}} + L \in \mathbb{L}^*/\mathbb{L}$ of the lattice $L = \sqrt{2p} \mathbb{Z}$ and

$$\theta_{p,p}(\tau) = \theta_{p,p}(\tau,0), \quad \theta'_{p,p}(\tau) = \frac{1}{2\pi i} \frac{\partial}{\partial \tau} \theta_{p,p}(\tau,z) \bigg|_{z=0}.$$

The graded-dimensions of $P_{p}^+$ and $P_{p-1}^-$ are both

$$\text{ch}[P_{p}^+](\tau) = \text{ch}[P_{p-1}^-](\tau) = 2 \text{ch}[X^+_p](\tau) + 2 \text{ch}[X^-_{p-1}](\tau).$$

We define the pseudo-character to be the weight one part of the graded-dimensions (of course the choice of normalization is a convention)

$$\text{pch}[X^+_p](\tau) := -4i \pi \frac{\theta'_{p-1,p}(\tau)}{\eta(\tau)}, \quad \text{pch}[X^-_p](\tau) := 4i \pi \frac{\theta'_{p,p}(\tau)}{\eta(\tau)}.$$

Not all of these are linearly independent and a basis is given by

$$B := \left\{ \text{ch}[P_{p}^+], \text{ch}[X^+_p], \text{pch}[X^+_p], \text{ch}[X^-_p] \mid 1 \leq \ell \leq p - 1 \right\}.$$

Set $\xi := e^{\frac{2\pi i}{p}}$. The modular $S$-transformations are

$$\text{pch}[X^+_p] \left( -\frac{1}{\tau} \right) = \frac{4}{\sqrt{2p}} \sum_{\ell=1}^{p-1} (-1)^{\ell+p} (\xi^\ell - \xi^{-\ell}) \left( \text{ch}[X^+_p] - \frac{\ell}{2p} \text{ch}[P^+_\ell] \right)$$

and

$$\text{ch}[X^+_p] \left( -\frac{1}{\tau} \right) = \frac{1}{\sqrt{2p}} \sum_{\ell=1}^{p-1} (-1)^{\ell+p+1} (\xi^\ell - \xi^{-\ell}) \text{pch}[X^+_\ell]$$

and

$$\text{ch}[P^+_\ell] \left( -\frac{1}{\tau} \right) = \frac{(\ell)!}{\sqrt{2p}} \left( (-1)^{\ell} \text{ch}[X^+_\ell] + (-1)^{\ell} \text{ch}[X^-_\ell] + \sum_{\ell=1}^{p-1} (-1)^{\ell+s} (\xi^{\ell+s} + \xi^{-\ell+s}) \text{ch}[P^+_\ell] \right).$$

### 2.9. Example 2: Symplectic fermions

Another class of logarithmic and strongly-finite VOAs is $SF^+_d$ for $d \geq 1$, the even part of the symplectic fermions. The VOAs $SF^+_1$ and $\mathcal{V}_2$ coincide. The structure of $SF^+_2$ is inherited from $\mathcal{V}_2$ as $SF^+_d$ is an extension by an abelian intertwining algebra of the $d$-fold tensor product of $\mathcal{V}_2$.

$SF_1 = SF$ is a super VOA. It itself is its only simple module, and its projective cover $P$ has the form $0 \to T \to P \to T \to 0$, where $T$ satisfies $0 \to SF \to T \to SF \to 0$ (both non-split). $SF_d$ is its $d$-fold tensor product, it thus has also only one simple module and the projective cover is the tensor product of the projective covers of the components.
$SF_d^\pm$ has exactly 4 simple modules [Ab]: $SF_d^\pm$ and $SF^\pm(\theta)_d$, where $SF_d^\pm$ is the odd part of the symplectic fermion vertex operator superalgebra, and $SF^\pm(\theta)_d$ are the even/odd parts of a twisted module of the symplectic fermions. $SF_1^+$ is just $\mathcal{W}_2$ and its fusion is thus (for $\epsilon, \nu \in \{\pm\}$)

\[
SF_1^+ \otimes SF_d^\nu = SF_1^\nu,
\]

\[
SF_1^+ \otimes SF_1^+(\theta) = P_1^\nu,
\]

\[
SF_1^+(\theta) \otimes P_1^\nu = 2 SF_1^+(\theta) \oplus 2 SF_1^- (\theta),
\]

\[
P_1^+ \otimes P_1^- = 2 P_1^+ \oplus 2 P_1^-.
\]

Here $P_1^\pm$ are the even/odd part of $P$. The modules $SF_d^\pm, SF^{\pm}(\theta)_d$ and $P_1^\pm$ are the even/odd part of the $d$-fold tensor product of the corresponding $SF$-modules, that is they are the following induced module of a projective $\mathsf{SL}_2$-module:

\[
X_d^+ = \bigoplus_{\epsilon \in \{\pm\}} \bigotimes_{i=1}^d X_1^\epsilon,
\]

\[
X_d^- = \bigoplus_{\epsilon \in \{\pm\}} \bigotimes_{i=1}^d X_1^\epsilon,
\]

\[
X \in \{SF, SF(\theta), P\}.
\]

Their tensor ring can be deduced from the one of $(SF_1^+)^{\otimes d}$ and is isomorphic to the case of $d = 1$. For details on this procedure see section 5 of [AA]. However, note that the tensor ring is not the Klein four group. The symplectic fermions $SF_d^\pm$ are rigid (proposition 3.23 of [DR]).

The graded-dimensions of the simple modules are

\[
\text{ch}[SF_d^\pm](\tau) = \frac{1}{2} \left( \frac{(\eta(2\tau))^{2d}}{\eta(\tau)^{2d}} \pm 2 \eta(\tau)^{2d} \right),
\]

\[
\text{ch}[SF^{\pm}(\theta)_d](\tau) = \frac{1}{2} \left( \frac{(\eta(2\tau))^{2d}}{\eta(\tau/2)^{2d}} \pm \eta(\tau/2)^{2d} \right).
\]

The conformal weights are 0, 1, $-d/8, (4-d)/8$ respectively. We see that $SF^{\pm}(\theta)_d$ are projective in addition to being simple. Abbreviate these graded-dimensions as $\chi^\pm, \chi_d^\pm$. Modularity of the graded-dimensions of the simple modules is given by:

\[
\chi^\pm(\tau + 1) = e^{\pi \tau/6} \chi^\pm(\tau),
\]

\[
\chi^\pm(-1/\tau) = \frac{1}{2} \left( \chi^+(\tau) - \chi^-(\tau) \right) + \frac{1}{2d+1} \left( \chi_d^+(\tau) - \chi_d^-(\tau) \right),
\]

\[
\chi_d^\pm(\tau + 1) = \frac{1}{2} e^{-\pi \tau/12} \chi^\pm(\tau),
\]

\[
\chi_d^\pm(-1/\tau) = \pm 2^{d-1} \left( \chi^+(\tau) + \chi^-(\tau) \right) + \frac{1}{2} \left( \chi_d^+(\tau) + \chi_d^-(\tau) \right).
\]

Two indecomposable but nonsimple $SF_d^\pm$-modules $\overline{SF}_d^\pm$ are constructed in [Ab]. They both have characters $2^{d-1} \eta(2\tau)^{2d}/\eta(\tau)^{2d}$ and socles $SF_d^\pm$. They have Jordan–Hölder length 2d with composition factors $SF_d^\pm, SF_d^\pm \otimes \mathbb{C}^{2d}, \ldots, SF_d^\pm \otimes \mathbb{C}^{(2d)}, \ldots, SF_d^\pm$. Thus their images in the Grothendieck ring are both $2^{2d-1}[SF_d^\pm] + 2^{2d-1}[SF_d^\pm]$. L$_0$ for both these indecomposable modules acts by Jordan blocks of size $d + 1$ [AN]. [AN] show that the dimension of the $\mathsf{SL}(2, \mathbb{Z})$-representation is at least $2^{2d-1} + 3$, and conjecture this is the exact dimension.

$\overline{SF}_d^\pm$ are projective $SF_d$-modules and hence the projective covers of $SF_d^\pm$. This follows for example by viewing $SF_d^+$ as an algebra for $(SF^+)^{\otimes d}$ according to [HKL], then $\overline{SF}_d^+$ is the induced module of a projective $(SF^+)^{\otimes d}$-module and as such projective itself as an $SF_d^+$-module, see [EGNO, CKM]. We will explain this in detail in [CG].
Symplectic fermions, having automorphism group the symplectic Lie group, allow for many other orbifolds [CL3] than just by \( \mathbb{Z}_2 \). Since any orbifold of a \( C_2 \)-cofinite VOA by any solvable finite group (conjecturally, by any finite group) is itself \( C_2 \)-cofinite [Miy5], this would give rise to many more examples of strongly-finite VOAs.

3. The newer theory

3.1. Conjectures, observations and questions

Let \( V \) be a strongly-finite VOA. Based on the preceding remarks, one could hope that the following are true, at least in broad strokes.

The whole area needs more independent examples, to probe conjectures such as those collected below, and also so suggest new conjectures. We address this in section 3.4 below.

3.1.1. Characters and pseudo-characters. A problem is that, although the generalised trace functions of Miyamoto are very natural from the point of view of associative algebras, they are not natural from the point of view of VOAs. In quantum field theory, one should be able to recover all \( n \)-point functions from ordinary traces with insertions. In particular, we should be able to recover all modular forms in Miyamoto’s space \( F(V) \), through the 1-point functions \( Z^\tau(v, \tau) \) of (5), where \( V \) is some intertwiner. Let us make this idea more explicit.

Write \( P^i \) for the unique projective \( V \)-module whose socle is \( V \), i.e. \( P^i \) is indecomposable and contains \( V \) as a submodule. By conjecture 3.1 below, \( P^i \) should be the projective cover of \( V \). There is only one submodule of \( P^i \) isomorphic to \( V \), but typically \( P^i \) will have several other composition factors (i.e. subquotients) isomorphic to \( V \); by a \( V \)-lift we mean a subspace \( V \) of some submodule \( M \) in \( P^i \) such that \( M/N \) is isomorphic to \( V \) for some submodule \( N \) of \( M \) and \( V \cap N = 0 \).

Conjecture 3.1. Let \( \tilde{F}(V) \) be the span over \( \mathbb{C}[E_4(\tau), E_6(\tau)] \) of all functions \( \tau \mapsto Z^\tau(v, \tau) \), as \( \tau \) runs over any intertwiner of type \( (\rho^p, \rho) \) for \( P \) any projective cover of a simple \( V \)-module, and any \( v \) in any \( V \)-lift. Then \( \tilde{F}(V) \) equals the space of all \( F \in F(V) \) regarded as a function of \( \tau \) only.

The space \( F(V) \) is finite-dimensional over \( \mathbb{C}[E_4(\tau), E_6(\tau)] \) when the functions \( F \) are regarded as functions of 2 variables \( v, \tau \), but \( \tilde{F}(V) \) is infinite-dimensional, so information is lost. We would like to strengthen the conjecture: e.g. show that there are lifts to \( P^i \) of each subquotient of \( P^i \) isomorphic to \( V \), such that the 1-point functions restricted to those lifts are formal 1-point functions in Miyamoto’s space \( F(V) \).

Ordinary intertwining operators \( Y(v, z) \) (i.e. without any \( \log z \) terms) will commute with the nilpotent operator \( L_0^v = L_0 - L_0^v \), and so their 1-point functions \( Z^\tau_0(v, \tau) \) will be pure \( q \)-series by an argument given in section 2.5. But logarithmic intertwiners can have \( \log q \) contributions, as shown explicitly by Kausch [Kau] some time ago (see also section 7.1 of [Rui]).

This conjecture reduces to Zhu’s theorem when \( V \) is strongly-rational. A modular group action on the 1-point functions \( Z^\tau(v, \tau) \) in the strongly-finite context is studied in [Fi]. If one is interested only in weight-0 modular forms (generalised graded-dimensions) then one should restrict to \( v \) of minimal generalised conformal weight. We can expect, from [Miy1] (which was proved in the strongly-rational context), that choosing \( v \) with \( L(0)v = kv \) will get modular forms of weight \( k \).

Intertwiners are intimately related to tensor products. Suppose \( P \) is the projective cover of some simple module \( M_k \). Then its tensor product with \( P^i \) will be
\[ P^1 \otimes P \cong \sum_j \langle P^1, M_j \rangle M_j \otimes M_k \]

where we sum over all simple modules \( M_i \) and where \( \langle P^1, M_j \rangle \) denotes the multiplicity of \( M_j \) as a composition factor of \( P^1 \). In other words, we get an independent intertwiner \( \mathcal{V} \) of type \( \left( \frac{P}{P'}, \frac{P}{P'} \right) \), every time \( \mathcal{V} \) is a composition factor of \( P^1 \). The intertwiner associated to a subquotient \( M/N \cong \mathcal{V} \) in \( P^1 \) will vanish in \( N \).

Miyamoto’s space \( \mathcal{F}(\mathcal{V}) \) for the symplectic fermions is surprisingly large: of dimension at least (and probably exactly) \( 2^{2d-1} + 3 \). But each of the 4 possible indecomposable \( P \) will have \( 2^{2d-1} \) intertwiners (since \( P^1 \), the projective cover of \( \mathcal{V} \), has precisely \( 2^{2d-1} \) composition factors of type \( \mathcal{V} \)).

Let us make this more concrete by considering the \( \mathcal{W}_2 \) model. This is of the type discussed in section 2.4 (recall (10)). Take \( P = P^1 = P^1_+ \). Then \( M = X^+_1 \) and \( M' \) has socle \( X^+_1 \) and top \( X^-_1 \oplus X^-_1 \). There are 2 linearly independent intertwiners of type \( \left( \frac{P}{P'}, \frac{P}{P'} \right) \), corresponding to the 2 \( X^+_1 \)'s in \( P^1 \); the submodule \( \text{soc}(P^1) \) and the quotient \( \text{top}(P^1) \). Choosing \( v \) in the socle for the first intertwiner corresponds to \( \alpha_2(v) \) a nonzero scalar multiple of \( \rho_{M^1} \), so to an endomorphism \( \alpha_2(v) \sigma^{d_\mathfrak{e}} \) given in (10); the corresponding trace \( Z_2(v, \tau) \) will be a constant multiple of the ordinary 1-point function \( 2\chi[X^+_1](v, \tau) + 2\chi[X^-_1](v, \tau) \). The intertwiner corresponding to the top gives \( \alpha_2(v) \sigma^{d_\mathfrak{e}} \) of the matrix form

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & q^n & 0 \\
c & 0 & 0
\end{pmatrix}
\begin{pmatrix}
q^n & 0 & 2\pi i q^n \tau \\
0 & q^n & 0 \\
0 & 0 & q^n
\end{pmatrix}
\]

and so will give a \( Z_2(v, \tau) \) which is a multiple of \( \tau \).

It is highly desirable to understand better the relation with Lyubashenko—in particular, why should the \( \text{SL}(2, \mathbb{Z}) \) representations be the same?

An observation which always holds in the examples is that the graded-dimension \( \text{ch}[P](\tau) \) of each projective module is a modular form of weight 0 and trivial multiplier for some congruence group. Moreover, it seems in the examples that the \( \mathbb{C} \)-span of the characters \( \text{ch}[P](v, \tau) \) of the projective modules are closed under \( \text{SL}(2, \mathbb{Z}) \).

In the strongly-rational case, it is known that the \( \text{SL}(2, \mathbb{Z}) \)-representation contains a principal congruence group in the kernel. (The principal congruence groups are \( \Gamma(N) = \{ A \in \text{SL}(2, \mathbb{Z}) \mid A \equiv I \pmod{N} \} \); modular forms for groups containing them have by far the richest theory.) In fact, this is exactly what would be expected from the conjecture of Atkin–Swinnerton-Dyer, which says that if a modular form \( f(\tau) \) (with trivial multiplier) for some finite-index subgroup of \( \text{SL}(2, \mathbb{Z}) \) has a \( q \)-expansion \( f(\tau) = \sum a_n q^n \) where all coefficients \( a_n \) are algebraic integers, then \( f(\tau) \) is a modular form (with trivial multiplier) for a congruence group. Surely the principal congruence groups will continue to play a large role in the strongly-finite theory.

One natural extension of Atkin–Swinnerton-Dyer is as follows. Suppose \( \mathcal{S} \) is a finite-dimensional (over \( \mathbb{C} \)) space of functions \( f(\tau) = q^k \sum s_n \tau^n q^n \) which is closed under the action of \( \text{SL}(2, \mathbb{Z}) \) (at weight-0 say). Suppose in addition that \( \mathcal{S} \) has a basis consisting of functions whose coefficients \( s_n \) are all algebraic integers. Then there are finitely many functions \( f^{(j)}(\tau) \) in \( \mathcal{S} \), which are modular forms for some \( \Gamma(N) \) of some nonpositive weights \( k^{(j)} \in \mathbb{Z}_{\leq 0} \), and the modular closure of these \( f^{(j)}(\tau) \) equals \( \mathcal{S} \). (These \( f^{(j)}(\tau) \) would have pure \( q \)-expansions, but polynomials in \( \tau \) would arise through the \( (c\tau + d)^{-k^{(j)}} \) factors.) This is easily seen to be satisfied by all known examples of strongly-finite VOAs.

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The hope is that modularity arguments can help prove rationality, say, if \( V \) is already known to be \( C_2 \)-cofinite. This would be very useful, because \( C_2 \)-cofiniteness is easier to show than rationality. It seems reasonable to expect that if the \( C \)-span of the graded-dimensions \( \chi_{M_j}(\tau) \) of all simple modules of a strongly-finite VOA \( V \) is closed under \( \text{SL}(2,\mathbb{Z}) \), then \( V \) is in fact strongly-rational. In fact, it could be true that if \( V \) is strongly-finite and its vacuum graded-dimension \( \chi_V(\tau) \) is fixed by some principal congruence group, then \( V \) is strongly-rational. The reason this may hold is because from previous remarks it makes one suspect that \( V \) is then projective (as well as being the tensor unit), but as we will explain below the tensor product of a projective with any module should be projective, so for any module \( M, M \cong M \otimes V \) will be projective, and this quickly implies that \( V \) is strongly-rational.

Perhaps related is the interesting conjecture of Gaberdiel–Goddard [GabG], that a strongly-finite VOA is strongly-rational iff Zhu’s algebra \( A_0(V) \) is semi-simple.

3.1.2. A log-modular tensor category. Most fundamental is the question as to what replaces the role of modular tensor category for strongly-finite VOAs.

Definition 3.1. A log-modular tensor category \( C \) is a finite tensor category which is ribbon, whose double is isomorphic to the Deligne product \( C \otimes C^{opp} \).

These terms are defined in section 2.6. Modular tensor categories \( C \) are precisely the ribbon fusion categories whose double is isomorphic to \( C \otimes C^{opp} \), and a finite tensor category is the natural nonsemi-simple generalization of a fusion category, so this is a natural definition. There should be several superficially different but equivalent definitions.

Conjecture 3.2. The category \( \text{Mod}^{r \cdot f}(V) \) of \( V \)-modules is a log-modular category.

The centre (or quantum double) construction applied to a finite tensor category should yield a log-modular tensor category (as it does for modular tensor categories). Some authors have conjectured that any modular tensor category is realized by a strongly-rational VOA. Analogously, one should ask if any log-modular tensor category is realized by a strongly-finite VOA. As we explain in [CG], the work of [EG2] can be used to construct families of (nonsemi-simple) finite tensor categories, and hence log-modular tensor categories through the centre construction; this is interesting because it seems to be a way to generate fundamentally new examples.

Conjecture 3.2 is not merely abstract nonsense—it has practical consequences even for the most down-to-earth researchers. For instance, we now list some of its consequences, which we prove in section 5 of [CG].

Proposition 3.2. Suppose \( V \) is strongly-finite and logarithmic. Assume conjecture 3.2. Then:

(a) if \( M \) is simple then \( \text{End}(M) = \mathbb{C} \text{id} \); if \( M \) is indecomposable then all elements of \( \text{rad}(\text{End}(M)) \) are nilpotent and \( \text{End}(M) = \mathbb{C} \text{id} + \text{rad}(\text{End}(M)) \);
(b) \( V \) is not projective as a \( V \)-module; moreover, \( V \) is unimodular, in the sense that its projective cover \( P_V \) contains \( V \) as a submodule (and not merely as a quotient);
(c) \( V \) has infinitely many families of indecomposable \( V \)-modules, each parametrized by complex numbers;
(d) the projective \( V \)-modules form an ideal in \( \text{Mod}^{r \cdot f}(V) \), closed under tensor products and taking contragredient; in particular, for any \( V \)-module \( Y, P_i \otimes Y \cong Y \otimes P_i \cong \bigoplus_{j,k} \mathcal{N}_{ij}^k (Y : M_j) P_k \);
(e) the categorical dimension of any projective \( V \)-module \( P \) is 0;
(f) for any projective module \( P \) and any simple but non-projective module \( A \), the open Hopf link \( \Phi_{P,A} \) vanishes; if instead \( A \) is projective and indecomposable but not simple, then \( \Phi_{P,P'} \) is nilpotent.
In section 2.6 we define the ‘semi-simplification’ of a category, by quotienting by the negligible morphisms and restricting to the full subcategory generated by the simple modules.

**Conjecture 3.3.** The semi-simplification of category $\text{Mod}^F(\mathcal{V})$ is a modular tensor category, whose Grothendieck ring is the quotient of the Grothendieck ring of $\text{Mod}^F(\mathcal{V})$ by the image of the $\mathcal{V}$-modules which are both simple and projective.

This conjecture requires that, when the tensor product of simples in $\text{Mod}^F(\mathcal{V})$ is decomposed into a direct sum of indecomposables, any nonsimple indecomposable arising will have quantum-dimension 0. In the case of the triplet algebras and the even part of the symplectic fermions, the nonsimple indecomposables are in fact projective. Perhaps that stronger statement continues to hold in general.

To help make this conjecture more concrete, consider a simple $W$ in any braided finite tensor category (not necessarily satisfying the conjecture). Then $U \mapsto \phi_{U, W}$ will be a one-dimensional representation of the tensor ring,

$$
\sum_X N_{U, V}^X \phi_{X, W} = \Phi_{U, W} \circ \Phi_{V, W},
$$

(14)

where the $N_{U, V}^X$ are defined as usual by $U \otimes V = \oplus_X N_{U, V}^X X$, and the sum is over the (equivalence classes of) indecomposables $X$ in $\mathcal{C}$. Recall the Hopf link invariant (1):

$$
S^U_{V, W} = \text{tr}_W (\Phi_{V, W}) = S^W_{V, W} \in \mathcal{C}.
$$

Here we take the ordinary (quantum) trace. Since $\Phi_{V, W}$ is a number, $S_{V, W} = S_{V, W}^{1, W} \Phi_{V, W}$. Recall also the category $\mathcal{C}$ defined at the end of section 2.6, and write $\mathcal{C}$ for the image of $U \in \mathcal{C}$ in $\mathcal{C}$. Now, $S_{V, W}^W = 0$ if either the quantum-dimension $S_{V, W}^V$ or $S_{V, W}^W$ vanishes, in which case $W = 0$. Then for $W \in \mathcal{C}$, (14) becomes

$$
\sum_X N_{U, V}^X \frac{S_{X, W}^V}{S_{V, W}^V} = \frac{S_{U, W}^V}{S_{V, W}^V} \frac{S_{U, W}^W}{S_{V, W}^W}
$$

where the sum is now over all (equivalence classes of) indecomposable objects $X$ in $\mathcal{C}$, or equivalently all equivalence classes of indecomposable objects $X$ in $\mathcal{C}$ with nonvanishing quantum-dimension. This equation is true for any $\mathcal{C}$. If the matrix $S^v$ is invertible in $\mathcal{C}$ then one obtains the standard Verlinde formula

$$
N_{U, V}^X = \sum_{W \in \mathcal{C}} \frac{S_{U, W}^V}{S_{V, W}^V} \frac{S_{U, W}^W}{S_{V, W}^W} \frac{(S^v)^{-1}}{S_{V, W}^W}.
$$

(15)

But for this to happen, $S^v$ would have to be square. So we’d have to restrict $U, V$ in (14) to simples in $\mathcal{C}$, in which case $X$ appearing on the left-side with nonzero multiplicity would have to be either simple or have quantum-dimension 0. This is the situation of the previous conjecture, in which case $S^v$ would coincide with the modular matrix $S$ of the semi-simplification.

Let $\mathcal{V}$ be strongly-finite and write $P = P_0 \oplus \cdots \oplus P_n$ for the direct sum of projective covers of simple $\mathcal{V}$-modules. Then $A_V := \text{End}_\mathcal{V}(P)$ is a finite-dimensional associative algebra, and $\text{Mod}^F(\mathcal{V})$ is equivalent as an abelian category to $\text{Mod}^\text{fin}(A_V)$, where the $\mathcal{V}$-module $M$ corresponds to the right $A_V$-module $\text{Hom}_\mathcal{V}(P, M)$. Equivalence as abelian categories means we ignore tensor products and duals, but the equivalence preserves simples, projectives, indecomposables, composition series, dimensions of Hom-spaces, etc.

If we have in addition rigidity (e.g. if conjecture 3.2 holds), then $\text{Mod}^F(\mathcal{V})$ is in fact a finite tensor category, and we can say more. In particular, proposition 2.7 of [EO] says any finite tensor category can be regarded as the tensor category of modules of a weak quasi-Hopf algebra. These are not as nice though as quasi-Hopf algebras: ‘weak’ means the counit is not a homomorphism of algebras and comultiplication is not unit-preserving.
Recall the discussion of finite-type, tame and wild in section 2.1. The triplet algebras \( W_p \) are tame. It can be shown [CG] that the symplectic fermions \( SF_d \) (for \( d > 1 \)) are wild. Almost all finite-dimensional associative algebras have wild representation type. For these reasons, we’d expect almost all strongly-finite VOAs are wild type. Among other things, this means one should not try to classify their modules.

**Proposition 3.3.** Suppose \( \mathcal{V} \) is strongly-finite and its category is rigid. Then \( \mathcal{V} \) has finitely many indecomposable modules, if and only if it is strongly-rational. If \( \mathcal{V} \) is not strongly-rational, then \( \mathcal{V} \) has uncountably many indecomposable modules of arbitrarily high Jordan–Hölder length.

### 3.1.3. Open Hopf links and Verlinde

Most exciting is the question as to what replaces Verlinde’s formula (8). One complication is that there are (at least) two natural rings to consider: the Grothendieck ring \( \text{Fus}^{gr}(\mathcal{V}) \) and the simple-projective ring \( \text{Fus}^{simp}(\mathcal{V}) \), discussed in section 2.2. Verlinde is interested in its regular representation. This can be realised by the fusion matrices \( N^{gr}_{ij} = (N^{gr}_{ik}N^{ij}_{kj}) \), whose entries are the structure constants \( \langle [M_i] | [M_j] \rangle = \sum_k N^{gr}_{ik}N^{ij}_{kj} \) in \( \text{Fus}^{gr}(\mathcal{V}) \). If \( \text{Fus}^{simp}(\mathcal{V}) \) is finite-dimensional over \( \mathbb{Z} \) (and we expect it always is), then the analogous matrices \( N^{simp}_{M}M \) can be defined for all indecomposables \( M \) in the ring.

As explained in section 2.4, statement (V1)' is trivial: \( S^{gr} \otimes \) exists. The first big question is (V2)': interpreting the indecomposable \( \text{Fus}^{gr}(\mathcal{V}) \)-representations \( B^{gr} \). In the strongly-rational case, they are all 1-dimensional, and are given by ratios of Hopf link invariants as explained in section 2.3. But in the strongly-finite case, they are higher dimensional and a new idea is needed. That idea, we propose, is the open Hopf link invariant.

Let \( \mathcal{C} \) be a ribbon category (defined in section 2.6). Of course we are interested in \( \mathcal{C} \) being the category \( \text{Mod}^{g\cdot r}(\mathcal{V}) \). For each object \( W \), define the map

\[
\Phi_W : \mathcal{C} \rightarrow \text{End}(W), \quad U \mapsto \Phi_{U,W},
\]

where \( \Phi_{U,W} \) is the open Hopf link operator (2). Diagram (2) translates to the formula

\[
\Phi_{V,W} := \text{ptr}_L (c_{V,W} \circ c_{W,V}) \in \text{End}(W),
\]

where the left partial trace \( \text{ptr}_L : \text{End}(V \otimes W) \rightarrow \text{End}(W) \) is

\[
\text{ptr}_L (f) := (dv \otimes \text{Id}_W) \circ (\text{Id}_V \otimes f) \circ (b'_{V} \otimes \text{Id}_W),
\]

using maps defined in section 2.6. In fact [CG], each \( \Phi_{U,W} \) lies in the centre of \( \text{End}(W) \).

A fundamental property of \( \Phi_{U,W} \) is that \( U \mapsto \Phi_{U,W} \) defines a representation of the tensor ring in the finite-dimensional algebra \( \text{End}(W) \): \( \Phi_{U \otimes V,W} = \Phi_{U,W} \circ \Phi_{V,W} \). The graphical calculus proof of this statement is:

\[
\Phi_{U \otimes V,W} = U \otimes V \quad \Longrightarrow \quad \Phi_{V,W} \quad \Longrightarrow \quad U \otimes V = \Phi_{U,W} \circ \Phi_{V,W}.
\]

Here, the third equality holds because of naturality (12) of braiding, and the second one is (11) together with \( \text{Id}_{U \otimes W} = \text{Id}_U \otimes \text{Id}_W \). These \( \Phi_{U,W} \) obey several other properties. For example, if \( \mathcal{C} \) has direct sums (and ours always will), then \( \Phi_{V,W_1 \oplus W_2} = \Phi_{U,W_1} \oplus \Phi_{V,W_2} \). In other words, without loss of generality we can restrict to indecomposable \( W \).
In some cases at least, this representation \( U \mapsto \Phi_{U,W} \) of \( \text{Fus}^\text{full}(\mathcal{V}) \) descends to a representation of \( \text{Fus}^\text{gr}(\mathcal{V}) \) (we expect this always happens). We propose that \((V2)\)' is:

**Conjecture 3.4.** Each indecomposable subrepresentation \( M \mapsto B_M^W \) of the Grothendieck ring representation \( \mathcal{N}_M^W \) is equivalent to a subrepresentation of the open Hopf link representation \( M \mapsto \Phi_{M,P} \) for some projective cover \( P \) of a simple module \( M \). Each indecomposable subrepresentation of the regular representation of the Grothendieck ring \( \text{Fus}^\text{full}(\mathcal{V}) \) or the full tensor ring \( \text{Fus}^\text{full}(\mathcal{V}) \) occurs with multiplicity 1.

This is the complete story for \((V2)\)' for the Grothendieck ring, assuming each \( \text{Fus}^\text{full}(\mathcal{V}) \)-representation \( \Phi_{*,W} \) descends to \( \text{Fus}^\text{gr}(\mathcal{V}) \). We do not know which indecomposable subrepresentations should appear in the regular representation of the full tensor ring \( \text{Fus}^\text{full}(\mathcal{V}) \) or even the subring \( \text{Fus}^\text{simp}(\mathcal{V}) \) generated by the simple \( \mathcal{V} \)-modules \( M \) and their projective covers \( P \) — even for \( \mathcal{V} = \mathcal{W}_p \) this question is mysterious. This interesting question should be explored. We know for \( \mathcal{W}_p \) that open Hopf link operators do not suffice for \( \text{Fus}^\text{full}(\mathcal{V}) \) or \( \text{Fus}^\text{simp}(\mathcal{V}) \).

Now turn to \((V3)\)', which is the heart of Verlinde. For the \( \mathcal{W}_p \) models, [PRR] relate \( S^\text{gr} \otimes S^\text{gr} \) to the \(-1/\tau\) transformation of (pseudo-)characters — we describe this interesting observation later this section, but it is not clear at present how it generalizes to other strongly-finite \( \mathcal{V} \).

In any case, at present we have the following to add to the relation of \( \mathcal{W}_p \) models, [PRR] relate \( S^\text{gr} \otimes S^\text{gr} \) to the modular \( S \)-matrix and open Hopf links. Consider the following sum over all inequivalent projective objects:

\[
W = \bigoplus_{P \in \mathcal{P}} P,
\]

then the map

\[
\Phi_W : \mathcal{C} \to \text{End}(W), \quad U \mapsto \Phi_{U,W},
\]

is as said before a ring homomorphism. If \( \mathcal{C} \) was a modular tensor category then this map would have been one-to-one and it explained that Hopf links diagonalize the fusion rules, see e.g. theorem 3.1.12 of [BK]. In the log-modular setting, this map is neither expected to be injective nor surjective as can be seen from the \( \mathcal{W}_p \) example. Let \( \tilde{\mathcal{C}} \) be the quotient of \( \mathcal{C} \) by the kernel of \( \Phi_W \) and let \( X \) be the image of \( \Phi_W \). Then \( \Phi_W \) restricts to a ring isomorphism

\[
\tilde{\Phi}_W : \tilde{\mathcal{C}} \to X.
\]

On the other hand, by corollary 3.2 endomorphisms on each indecomposable projective object split into one-dimensional semi-simple part, the idempotents, and the nilpotent endomorphisms. In a modular tensor category, the idempotents form the diagonal basis for the fusion ring and we see that a natural generalization is that the idempotents and nilpotents block diagonalize the induced fusion ring on \( \tilde{\mathcal{C}} \). Nilpotent endomorphisms clearly form an ideal and quotienting by it leaves us with a semi-simple ring, fitting well with conjecture 3.3.

Unfortunately it turns out that in the case of \( \mathcal{W}_p \) one has \( 3 \times 3 \)-Jordan blocks while the map \( \tilde{\Phi}_W \) only notices a quotient of the tensor ring that has blocks up to size \( 2 \times 2 \). The reason is that indecomposable projective modules of \( \mathcal{W}_p \) have two-dimensional endomorphism algebras.

At least as subtle is to relate the open Hopf link operators to the modular \( S \)-matrix. Because the latter is numerical (once a basis of 1-point functions is chosen), it is worth investigating how to extract numerical invariants from the endomorphisms \( \Phi_{U,W} \).

Let us sketch one way which underlies one of our observations. Let \( \mathcal{P} \) be the (full) subcategory of \( \tilde{\mathcal{C}} \) consisting of the projective objects in \( \mathcal{C} \). Then among other things, \( \mathcal{P} \) is semi-simple (its simples are the projective covers of the simple objects of \( \mathcal{C} \)), and \( \mathcal{P} \) is a tensor ideal of \( \tilde{\mathcal{C}} \) (since the tensor product of a projective with any object of \( \mathcal{C} \) is projective). Suppose \( \mathcal{C} \)
contains a projective simple self-dual object \( \tilde{P} \). Then, provided \( \mathcal{C} \) is unimodular in the sense of proposition 3.2(b), there will be a family of linear functions
\[
\{ t_V : \text{End}(V) \to \mathbb{C} \mid V \in \mathcal{P} \}
\]
satisfying:

1. For \( U \) in \( \mathcal{P} \) and \( W \) in \( \mathcal{C} \) and any \( f \) in \( \text{End}(U \otimes W) \)
\[
t_{U \otimes W} (f) = t_W (\text{ptr}_U (f)) ;
\]
2. For \( U, V \) in \( \mathcal{P} \) and morphisms \( f : V \to U \) and \( g : U \to V \)
\[
t_V (g \circ f) = t_U (f \circ g) .
\]

This family \( t_V \) is called a modified trace on \( \mathcal{P} \), and has been developed by Geer, Patureau-Mirand and collaborators (see [GKP1, CGP] and references therein).

Given a modified trace on \( \mathcal{P} \), we can define the logarithmic Hopf link invariant
\[
S_{U,W}^{\mathcal{P},x} := t_W (\Phi_{U,W} \circ x) = t_W (x \circ \Phi_{V,W}) \in \mathbb{C},
\]
for any \( V \in \mathcal{C} \), \( W \in \mathcal{P} \), and \( x \) in \( \text{End}(W) \). For readability, we will drop the endomorphism \( x \) from the superscript if \( x \) is the identity, and we will drop the \( \mathcal{P} \) if \( x \) is present. The logarithmic Hopf link invariant is symmetric just like the non-logarithmic one: \( S_{V,W}^{\mathcal{P}} = S_{W,V}^{\mathcal{P}} \). Moreover, for any \( x \in \text{End}(W) \),
\[
S_{U,W}^{\mathcal{P},x} = t_W (\Phi_{U,W} \circ x) = t_W (\Phi_{1_{\mathcal{C}},W} \circ (\Phi_{U,W} \circ x)) = S_{1_{\mathcal{C}},W}^{\mathcal{P},x}.
\]

More generally, if \( \mathcal{P} \) is any tensor ideal of \( \mathcal{C} \) and \( \tilde{P} \) is a right ambidextrous object of \( \mathcal{P} \), then we will get a modified trace on \( \mathcal{P} \) and hence can define logarithmic Hopf links.

Incidentally, there is a relation between the pseudo-traces used by Miyamoto, and these modified traces, and we expect this to play a role in the general story. Let \( \tilde{A}_V \) be the finite-dimensional algebra of section 3.1.2 (see also theorem 2.3 of [CG]). Recall that given any symmetric linear functional \( \phi \) on \( \tilde{A}_V \), we get a pseudo-trace \( \phi_p : \text{End}_{\tilde{A}_V} (P) \to \mathbb{C} \) in the usual way for each projective finite-dimensional \( \tilde{A}_V \)-module \( P \). Given any modified trace \( \{ t_p \} \), there is one and only one symmetric linear functional \( \phi \) on \( \tilde{A}_V \), such that \( t_p = \phi_p \) for all projective \( \tilde{V} \)-modules \( P \). Conversely, the pseudo-traces \( \phi_p \) of any symmetric linear functional \( \phi \) on \( \tilde{A}_V \) necessarily satisfy property 2 of the definition of modified trace (but not in general property 1). See section 6.1 of [CG] for more details.

There is a striking relation between the modular matrix \( S^\chi \) and the logarithmic Hopf link invariants, for the \( \mathcal{W}_p \) models, and we expect this to generalize to any strongly-finite \( \mathcal{V} \). Recall the basis \( \mathcal{B} = \{ \text{ch}(P^\chi_+), \text{ch}[X^+_p], \text{pch}[X^+_p], \text{ch}[X^+_{\pm}] \} \) for the space of 1-point functions given in section 2.8. Let \( S^\chi \) denote the matrix corresponding to \( \tau \to -1/\tau \) in this basis. Then the entries of \( S^\chi \) appropriately normalized agree with logarithmic Hopf link invariants, appropriately normalized (see equations (19) and (20)).

Moreover, in section 3.3.2 we describe an expression due to [PRR] of \( S^\phi \) in terms of \( S^\chi \). This expression, together with the aforementioned interpretation of \( S^\chi \) in terms of \( S^\phi \), allows us to write \( S^\phi \) in terms of the (logarithmic) Hopf link invariants. And our (V2) conjecture tells us we can interpret the indecomposable subrepresentations \( B^\phi \) of the regular representation of \( \text{Fus}^\phi (V) \) in terms of \( \Phi_{U,W} \), i.e. in terms of the (logarithmic) Hopf link invariants. Putting these observations together, we get a Verlinde formula akin to (8), for the Grothendieck ring.

There are other ways to arrive at Verlinde-like formulas. One was discussed in section 3.1.2. For another example, assume that a projective module \( W \) has 2-dimensional endomorphism
(examples are $P^\pm_\nu$ of $\mathcal{W}_\nu$). Then it almost has a preferred basis: $\text{Id}_W$ and some $x = x_W$ with $x^2 = 0$. Then every open Hopf link operator is of the form

$$\Phi_{U,W} = a_{U,W} \text{Id}_W + b_{U,W} x_W$$

for complex numbers $a_{U,W}$ and $b_{U,W}$. These numbers relate to the tensor product structure constants via

$$\sum_x N_{U,V}^x a_{x,W} = a_{U,W} a_{V,W}, \quad \sum_x N_{U,V}^x b_{x,W} = a_{U,W} b_{V,W} + b_{U,W} a_{V,W}$$

(17)

where the sums are over all indecomposable objects. On the other hand these numbers also relate to the logarithmic Hopf link invariants, since

$$S_{U,V}^{\omega,\tau} = a_{U,W} S_{1,c,W}^{\omega,\tau} + b_{U,W} S_{1,c,W}^{\omega,\tau}, \quad S_{U,W}^{\omega,\tau} = a_{U,W} S_{1,c,W}^{\omega,\tau},$$

so that they can be expressed in terms of normalized logarithmic Hopf link invariants:

$$a_{U,W} = \frac{S_{U,W}^{\omega,\tau}}{S_{1,c,W}^{\omega,\tau}}, \quad b_{U,W} = \frac{S_{1,c,W}^{\omega,\tau}}{S_{1,c,W}^{\omega,\tau}} \left( \frac{S_{U,W}^{\omega,\tau}}{S_{1,c,W}^{\omega,\tau}} - \frac{S_{1,c,W}^{\omega,\tau}}{S_{1,c,W}^{\omega,\tau}} \right),$$

assuming the numbers $S_{1,c,W}^{\omega,\tau}$ do not vanish. One could then hope to invert (17) for some tensor subcategory of $\mathcal{C}$, expressing $N_{U,V}^x$ in terms of the logarithmic Hopf link invariants. For instance, in the triplet example, one recovers the standard Verlinde formula (15), for the subcategory generated by the simples. In that example we also get a Verlinde-type formula for the fusion coefficients of type simple with projective, see (21) below.

### 3.2. The picture in the locally finite setting

An abelian category is called **locally finite** if $\text{Hom}(X,Y)$ is finite-dimensional for any two objects in the category and if all objects have finite Jordan-Hölder length. Logarithmic CFTs with locally finite module categories are quite common and the recent development here has helped lead to our perspective. Older developments are already reviewed in the introduction to logarithmic CFT [CR4] and the reviews [RW2, C1].

The starting point had been logarithmic CFTs with a continuum of simple objects. Then characters are not modular in the sense that they are not components of a finite-dimensional vector-valued modular form. Nonetheless it is still possible to express the $S$-transformation $\tau \mapsto -1/\tau$ as an integral of characters:

$$\text{ch}[X]\left(-1/\tau\right) = \int_I S_{X,Y} \text{ch}[Y](\tau) d\tau.$$

Here $I$ is the index set parameterizing simple objects. The $S$-matrix is thus replaced by an $S$-kernel $S_{X,Y}$. What should then replace the Verlinde formula? It is a natural guess that this would be

$$N_{X,Y}^Z = \int_I S_{X,W} S_{Y,W} S_{W,Z}^{-1} dW.$$

$\mathcal{V}$ is the VOA itself. Here of course one needs to make sense of the inverse of the $S$-kernel. This is to be interpreted in a distributional sense and so are the fusion coefficients $N_{X,Y}^Z$. Further these formulas are subtle in the sense that one has to choose the integration measure wisely. However it turns out that in many cases, that is the free boson [CR4], the affine Lie
The superalgebra of \( gl(1|1) \) [CQS1, CR3], its extensions [AC] and especially in the fractional level WZW theories of \( sl_2 \) [CR1, CR2], these ideas work extremely well. Modular objects involved here include meromorphic Jacobi forms, distributions and mock modular forms.

The next natural question is now whether there is a good explanation for these experimental observations. In [CM1] the singlet algebra was studied, see also [RW1]. This is a \( U(1) \)-orbifold of the triplet and conversely the triplet is an infinite order simple current extension of the singlet. Here characters are false theta functions and there modularity had not even been known. A regularization, called \( \epsilon \), was needed to obtain such modular properties. It then turned out that regularized asymptotic dimensions of characters

\[
q\dim[X, \epsilon] = \lim_{\tau \to 0} \frac{\text{ch}[X, \epsilon](\tau)}{\text{ch}[V, \epsilon](\tau)}
\]

capture the tensor ring. Moreover, depending on the regime of the regularization parameter it actually either captures the semi-simplification or the full Grothendieck ring. This has then been further and more deeply investigated in [CMW] and generalized to higher rank analogues [CM2].

The for the moment final natural question is then to give a categorical meaning to these findings. The answer is: asymptotic dimensions in the continuous regularization parameter regime correspond to logarithmic Hopf links while the regime that captures the semi-simplification corresponds to logarithmic Hopf links with nilpotent endomorphism insertion [CMR]. This is to be taken as a sign that the interplay between modular-like tensor categories, modular forms and logarithmic CFT goes even beyond the strongly-finite setting and some kind of generalization to the locally-finite setting will be possible.

### 3.3. Example: the triplet algebra \( \mathcal{W}_p \)

In this section we verify explicitly many of our conjectures for the \( \mathcal{W}_p \) algebras. The other class of examples which can be studied this explicitly are the even part \( SF^+_d \) of the symplectic fermions. These two families nicely complement each other: \( \mathcal{W}_p \) has an unbounded number of simples but \( L_0 \) has nilpotent order 2, while \( SF^+_d \) always has 4 simples but its \( L_0 \) has arbitrarily high nilpotent order.

In most ways the symplectic fermions are simpler. But one place they differ is that the modular closure of the \( 2p \)-dimensional space spanned by the \( \mathcal{W}_p \)-characters recovers the full \( 3p - 1 \)-dimensional \( SL(2, \mathbb{Z}) \)-module \( \mathcal{F}(\mathcal{W}_p) \) expected from both the Miyamoto and Lyubashenko pictures. By contrast, the modular closure of the \( 4 \)-dimensional space spanned by the \( Sp^+_d \)-characters is \( d + 4 \)-dimensional, while the dimension of \( \mathcal{F}(SF^+_d) \) is expected to be \( 2^{2d-1} + 3 \). Another difference, as we saw in section 3.1.2, is that \( \mathcal{W}_p \) is tame while \( SF^+_d \), like most strongly-finite VOAs, is wild.

#### 3.3.1. Calculating the open Hopf link invariants

Crucial to our story are the open Hopf link invariants. These are difficult to compute in \( \text{Mod}^F(V) \) because we do not yet have a good grasp on the braiding. However, [FGST2] originally conjectured that the representation categories of \( \mathcal{W}_p \) and the restricted quantum group \( U_q(sl_2) \) at a \( 2p \)-th root of unity are equivalent as braided tensor categories. [NT] proved equivalence as abelian categories, and a direct comparison of [TW1, KS] shows that the corresponding simple-projective tensor product rings \( \text{Fus}^\text{simp} \) are isomorphic. Braided equivalence is however false [KS], but it is believed that the difference is minor, only lying in different associativity isomorphisms. This is proven for \( p = 2 \) by [GaiR1]. Although these braided tensor categories are not equivalent, it is expected that the open Hopf link invariants of greatest concern to us will be the same.
The quantum group $U_q(sl_2)$ is described for example in Murakami [Mur]. Let $q = e^{\pi i/p}$ be a primitive $2p$th root of unity, and write $\{n\}_q := q^n - q^{-n} = 2i \sin(\pi/p)$. The restricted quantum group $\widehat{U}_q(sl_2)$ does not have a universal $R$-matrix, i.e. it is not a quasi-triangular Hopf algebra. However it can be embedded in a quasi-triangular Hopf algebra $\widehat{D}$ with universal $R$-matrix [KS, FGST1]. All modules of $\widehat{U}_q(sl_2)$ that we are interested in, lift to $\widehat{D}$, and the universal $R$-matrix action of $\widehat{D}$ on them coincides with that of $\widehat{U}_q(sl_2)$.

The simple $U_q(sl_2)$ modules are $U^{\pm}_{s}$ for either sign and $1 \leq s \leq p$. Their projective covers are $R^{\pm}_{s}$, where $R^{\pm}_{p} = U^{\pm}_{p}$. The correspondence between the $U_q(sl_2)$- and $\mathcal{W}_p$-modules of interest to us are $U^{\pm}_{s} \leftrightarrow X^{\pm}_{s}$ and $R^{\pm}_{s} \leftrightarrow P^{\pm}_{s}$. The endomorphism ring of each $U^{\pm}_{s}$, $1 \leq s \leq p$, can be canonically identified with $\mathbb{C}$, while those of each projective module $R^{\pm}_{s}$, $1 \leq s < p$, is spanned by the identity (which we will call $e$) as well as a nilpotent one we will call $x$.

The open Hopf link operators $\Phi_{U^{\pm}_{s}_s, U^{\pm}_{s}_s}$ for $1 \leq s < p$ are easy to compute, and we find

$$\Phi_{U^{\pm}_{s}_s, U^{\pm}_{s}_s} = (-1)^{i'+1} \epsilon' \frac{x^{\sin(\pi ss'/p)}}{\sin(\pi s/p)}, \quad \Phi_{R^{\pm}_{s}_s, U^{\pm}_{s}_s} = 0,$$

for all $1 \leq s < p$, $1 \leq s' \leq p$, and $\epsilon, \epsilon' \in \{\pm 1\}$, for some sign $\epsilon''$ depending on $\epsilon, \epsilon'$.

The open Hopf links for the projectives is much more subtle. In [CG] we compute them using the idea of deformable families of modules, and the usefulness of this observation should be further explored.

In case $\epsilon = 0$, the open Hopf link invariant

$$\Phi_{U^{\pm}_{s}_s, R^{\pm}_{s}_s} = (-1)^{i'+1} \epsilon' \frac{x^{\cos(\pi ss'/p)}}{\sin(\pi s/p)} \frac{\sin(2 \pi i/j)}{\sin(2 \pi i/j)} (i', i) q^e + ((i+1)(i+1)/q) \pi \sin(\pi j/\pi) x,$$

and from these all logarithmic Hopf link invariants, such as

$$S_{U^{\pm}_{s}_s, R^{\pm}_{s}_s} = (-1)^{i'+1} \epsilon' \frac{x^{2 \cos(\pi ss'/p)}}{\sin(\pi s/p)}.$$

3.3.2. Block-diagonalizations for $\mathcal{W}_p$. In this section we review the block-diagonalization of the (regular representation of the) Grothendieck ring of $\mathcal{W}_p$ performed by [PRR], and the block-diagonalization of the tensor ring of $\mathcal{W}_p$ performed by [R].

Consider first the Grothendieck ring of $\mathcal{W}_p$. It has basis $X^+_i$, which we put in the order $X^+_1, X^+_2, X^+_3, \ldots, X^+_p, X^-_1$. As the tensor ring is generated by the simple-current $X^+_1$ and $X^+_2$, it suffices to block-diagonalize $F^R := N^R_{X^+_1}$ and $Y^R := N^R_{X^+_2}$.

Consider first $p = 2$. Then

$$Q^R \Phi^{R} = \text{diag}(1, -1, -1, 1), \quad Q^{R-1} Y^R = \text{diag}(2, (0, 1, 0, -2)).$$
for

\[
Q^{BF} = \begin{pmatrix}
1 & 4 & 0 & -1 \\
1 & -4 & 0 & -1 \\
2 & 0 & 4 & 2 \\
2 & 0 & -1 & 2
\end{pmatrix}.
\]

More generally [PRR], there is an invertible matrix \(Q^{BF}\) (discussed below) which simultaneously diagonalizes \(J^{BF}\) and puts \(Y^{BF}\) into Jordan canonical form. Those matrices \(Q^{BF-1}J^{BF}Q^{BF}\) and \(Q^{BF-1}Y^{BF}Q^{BF}\) fall naturally into two \(1 \times 1\) blocks and \(p-1 \times 2\) blocks:

\[
Q^{-1}JQ = \text{diag}(1; -I_2; \ldots; (-1)^{p-1}I_2; (-1)^p),
\]

\[
Q^{-1}YQ = \text{diag}(\lambda_0; B_{\lambda_1,2}; \ldots; B_{\lambda_p,2}; \ldots; B_{\lambda_{p-1},2}; \ldots; \lambda_p),
\]

where \(1 \leq j < p\), the eigenvalues are \(\lambda_j = 2 \cos(\pi j/p)\), and we write \(I_k\) for the \(k \times k\) identity matrix and \(B_{\lambda,k}\) for the canonical \(k \times k\) Jordan block with eigenvalue \(\lambda\).

Because any other matrix \(N^{BF}_{S,\pm}\) will be a polynomial in \(J^{BF}\) and \(Y^{BF}\), \(Q^{BF}\) also block-diagonalizes all \(N^{BF}_{S,\pm}\). The \(0\)th and \(p\)th blocks of \(Q^{-1}N^{BF}_{S,\pm}Q\) will be the numbers \(s\) and \((\pm 1)^p(-1)^{p-1}s\) respectively; its \(j\)th block, for \(1 \leq j < p\), will be \(2 \times 2\) upper-triangular, with diagonal entries (eigenvalues) \(\pm \sin(\pi j/s)/\sin(\pi s/p)\). Those \(2 \times 2\) blocks will not in general be canonical Jordan blocks, however.

Now turn to the tensoring \(Fus^{\text{simp}}(W_p)\) of \(W_p\). We are interested in the full subcategory spanned by the irreducible \(W_p\)-modules \(X^{\pm}\) and their projective covers \(P^{BF}_s\). The corresponding \(4p - 2\) tensor matrices \(N^{BF}_s\), each of size \((4p-2) \times (4p-2)\), were put into simultaneous block form in [R]. We will put these \(4p - 2\) \(W_p\)-modules in order \(X_1^-, X_1^+, \ldots, X_j^-, X_j^+, \ldots, X_{p-1}^-, X_{p-1}^+, \ldots, X_p^-, X_p^+\), \(p \geq 3\). Again, it suffices to block-diagonalize \(J := N^{BF}_{S,\pm}\) and \(Y := N^{BF}_{S,\pm}\).

Then [R] there is an invertible matrix \(Q\) (discussed below) which simultaneously diagonalizes \(J\) and puts \(Y\) into Jordan canonical form. Those matrices \(Q^{-1}JQ\) and \(Q^{-1}YQ\) fall naturally into two \(1 \times 1\) blocks and \(p - 1\) pairs of \(1 \times 1\) and \(3 \times 3\) blocks:

\[
Q^{-1}JQ = \text{diag}(1; 1, -I_2; \ldots; (-1)^{p-1}, (-1)^p-1, \ldots; (-1)^p),
\]

\[
Q^{-1}YQ = \text{diag}(\lambda_0; \lambda_1, B_{\lambda_1,3}; \ldots; \lambda_p, B_{\lambda_p,3}; \lambda_{p-1}; B_{\lambda_{p-1},3}; \ldots; \lambda_p).
\]

Because any other matrix \(N^{BF}_{S,\pm}\) is a polynomial in \(J\) and \(Y\), \(Q\) also block-diagonalizes them. The \(0\)th and \(p\)th blocks of \(Q^{-1}N^{BF}_{S,\pm}Q\) will be the numbers \(s\) and \((\pm 1)^p(-1)^{p-1}s\) respectively; the \(1 \times 1\) part of its \(j\)th block, for \(1 \leq j < p\), will be the number \((\pm 1)^{j-1} \sin(\pi j/s)/\sin(\pi s/p)\); the \(3 \times 3\) part of its \(j\)th block will be upper-triangular, with diagonal entries (eigenvalues) \((\pm 1)^{j-1} \sin(\pi j/s)/\sin(\pi s/p)\). Again, those \(3 \times 3\) blocks will not in general be canonical Jordan blocks, however. The block structure for \(Q^{-1}N^{BF}_{S,\pm}Q\) is similar.

[PRR] found an interesting interpretation of \(Q^{BF}\) in terms of the modular \(S\)-matrix. Choose the basis \(ch[X^+_s], pch[X^+_s], ch[X^-_s]\), for \(1 \leq s \leq p\) and \(1 \leq \ell < p\) and either sign. This is slightly different than the choice we make. Let \(\tilde{S}^X\) denote the corresponding matrix realizing the modular transformation \(\tau \mapsto -1/\tau\). Then

\[
Q^{BF} = \left(\tilde{S}^X_{\ell,X^+}, \tilde{S}^X_{\ell,X^-}, \tilde{S}^X_{\ell,X^+}, \tilde{S}^X_{\ell,X^-}\right)
\]

where we use the subscript \(0\) for the pseudo-character.
3.3.3. Verlinde (V2)’ for $\mathcal{W}_p$. In this section we identify the indecomposable representations contained in the regular representation of the Grothendieck and tensor rings $\text{Fus}^\text{rep}(\mathcal{W}_p)$, $\text{Fus}^\text{simp}(\mathcal{W}_p)$ of $\mathcal{W}_p$, with open Hopf link operators. We gave their decomposition last section, expressing them as direct sums of 1-, 2- and 3-dimensional subrepresentations.

The Grothendieck ring and tensor ring have the same two 1-dimensional subrepresentations, corresponding to $j = 0$ and $j = p$. The $j = 0$ one is uniquely determined by it sending $X_1^j$ to 1 and $X_2^j$ to 2; while the $j = p$ one sends $X_1^j$ to $(-1)^p$ and $X_2^j$ to $-2$. Using the open Hopf link calculations sketched in section 3.3.1, we identify these with $\Phi_{*,U_p^+}$ and $\Phi_{*,U_p^-}$ respectively. The Grothendieck ring also has a 2-dimensional subrepresentation for each $1 \leq j < p$, which sends $X_1^j$ to $(-1)^j I_2$ and $X_2^j$ to $B_{\lambda_2,2}$. We find this representation is isomorphic to $\Phi_{*,K_p^+]}$ or equivalently $\Phi_{*,K_p^-}$.

The tensor ring $\text{Fus}^\text{simp}(\mathcal{W}_p)$ has a 1-dimensional representation for each $1 \leq j < p$, sending $X_1^j$ to $(-1)^{j-1}$ and $X_2^j$ to $\lambda_j$. This is $\Phi_{*,U_p^j}$, where $\epsilon$ is the sign of $(-1)^{j-1}$.

What remains is to understand the 3-dimensional representations in the tensor ring. However:

Theorem 3.4 (CG). For any $\mathcal{W}_p$-modules $U, M$, the map $\Phi_{U,M} \in \text{End}(M)$ decomposes only into $1 \times 1$ and $2 \times 2$ Jordan blocks. In particular, no $\Phi_{*,M}$ can contain a subrepresentation equivalent to any of the 3-dimensional indecomposable tensor ring representations given in the previous section.

Incidentally, the 4-dimensional Grothendieck ring of $\text{SF}^+_p$ likewise decomposes into 1- and 2-dimensional indecomposables, which are in turn indecomposable summands in open Hopf link invariants of simples and projectives. Again, the indecomposables of the regular representation of $\text{SF}^+_p$ are not all subrepresentations of any open Hopf link invariant $\Phi_{*,W}$.

3.3.4. Verlinde (V3)’ for $\mathcal{W}_p$. Comparing the open Hopf link calculations in section 3.3.1 with the modularity calculations in section 2.8, we find that the (logarithmic) Hopf link invariants of the restricted quantum group and modular $S$-matrix coefficients for the (pseudo-)characters satisfy the remarkable formula

$$
\frac{S_{Y,Y'}^X}{S_{Y',Y}^X} = \frac{S_{\text{oo};Y';Y}^{\text{oo};Y}}{S_{Y';Y}^{\text{oo};Y'}^{\text{oo};Y}}
$$

(19)

for any $Y, Y' \in \{P^+_s, X^+_s, X^0_s, X^-_s\}$ (where $X^0_s$ refers to $\text{pchi}(X^+_s)$) and $Y_i$, the corresponding $\mathbb{U}_q(\mathfrak{sl}_2)$-module (i.e. $R^+_s$, $U^+_s$, $U^+_s$, $U^+_s$ respectively). Here $Y'' = X^+_s$ unless $(Y, Y') = (X^0_s, P^+_s)$ or $(X^+_s, X^+_s)$, in which case $Y'' = X^0_s$. Moreover, $\gamma = \gamma(Y, Y')$ is given by

<table>
<thead>
<tr>
<th>$Y/Y'$</th>
<th>$P^+_s$</th>
<th>$X^+_s$</th>
<th>$X^0_s$</th>
<th>$X^+_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^+_s$</td>
<td>$\mathbb{p}$</td>
<td>$\mathbb{p}$</td>
<td>$\emptyset$</td>
<td>$\mathbb{p}$</td>
</tr>
<tr>
<td>$X^+_s$</td>
<td>$\mathbb{p}$</td>
<td>$\mathbb{p}$</td>
<td>$\emptyset$</td>
<td>$\mathbb{p}$</td>
</tr>
<tr>
<td>$X^0_s$</td>
<td>$\mathbb{p}$</td>
<td>$\emptyset$</td>
<td>$\mathbb{p}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$X^+_s$</td>
<td>$\mathbb{p}$</td>
<td>$\mathbb{p}$</td>
<td>$\emptyset$</td>
<td>$\mathbb{p}$</td>
</tr>
</tbody>
</table>

where $1 \leq s, \ell < p$ (the choice of $\gamma$ is not always unique). For example

$$
\frac{S_{P^+_s, P^+_s}^{X_s}}{S_{X^+_s, P^+_s}^{X_s}} = \frac{S_{P^+_s, P^+_s}^{\text{oo};P^+_s}}{S_{X^+_s, P^+_s}^{\text{oo};P^+_s}}, \frac{S_{X^0_s, P^+_s}^{X_s}}{S_{X^+_s, P^+_s}^{X^0_s}}, \frac{S_{X^-_s, X^0_s}^{X^+_s}}{S_{X^+_s, X^0_s}^{X^-}}, \frac{S_{U^+_s, U^+_s}^{U^-}}{S_{U^-_s, U^+_s}^{U^-}} = 0.
$$
We thus see that the logarithmic Hopf link invariants correspond to the $S$-matrix of an $\text{SL}(2, \mathbb{Z})$-action, analogous to the Hopf link invariants in modular tensor categories.

In fact we can say more. Rescaling $x$ appropriately, it can be shown that there exist nonzero scalars $c^\chi(Y)$ and $c^\infty(\gamma)$ such that

$$c^\chi(Y')S^\chi_Y = c^\infty(\gamma)S^{\infty, \gamma}_{Y', Y}$$

for all $Y, Y'$. This carries a little more information than (19). At this point it is not clear to us whether the stronger (20) will continue to hold in other examples than $\mathcal{W}_p$, or whether (19) is the fundamental relation.

In section 3.1, we gave different ways to obtain Verlinde-like formulas for strongly-finite VOAs. Let us work out explicitly the final possibility mentioned there, for $\mathcal{W}_p$. It gives a Verlinde-type formula for some of the tensor ring (as opposed to Grothendieck ring) structure constants. Namely, the $S$-matrix restricted to the $R_i$ is the $S$-matrix of theta functions as we will see below. It is in particular invertible and hence

$$\mathcal{N}_{U^+_i, R_i} = \sum_{t=0}^{p-1} U^+_{i, t} S_{R_i, R_t} \left( S_{R_t, R_i}^{-1} \right) R_t, R_i.$$  

We conclude with the following observation from [CG]:

**Theorem 3.5.** The tensor subring of $\overline{U}_q(\mathfrak{sl}_2)$, spanned by the simple modules $U^\pm_1$ and their projective covers $R^\pm_1$, is completely determined from the following three data: the socle series of projective modules; the fact that the ambidextrous element $U^+_p$ is its own tensor dual; and the complete list of logarithmic Hopf link invariants.

### 3.4. Towards more examples

There is obviously a need for new examples, and currently much effort is being directed toward developing techniques to construct them [CKL, CKLR, CKM]. Given a VOA one can construct a new one as a VOA extension, as a kernel of a screening charge, as an orbifold or a coset. We will state three constructions of these types that are expected to yield new examples of $C_2$-cofinite logarithmic VOAs.

#### 3.4.1. Higher rank analogues of $\mathcal{W}_p$

The triplet algebra $\mathcal{W}_p$ is constructed as the kernel of a screening charge (the zero-mode of a specific intertwining operator) acting on the lattice VOA of the lattice $\sqrt{2}\mathbb{Z} = \sqrt{p}\mathbb{A}_1$. This has a fairly obvious generalization to root lattices of simply laced Lie algebras [AdM2, FT]. We follow [CM2]. Let $Q$ be a root lattice of type $A, D$ or $E$ and let $L = \sqrt{p}Q$ for positive integer $p$ at least equal to two. Further denote by $\alpha_1, \ldots, \alpha_n$ a set of simple roots of the Lie algebra $\mathfrak{g}$ corresponding to $Q$. Then one defines $\mathcal{W}(p)_Q$ as the intersection of intertwiners associated to the simple roots:

$$\mathcal{W}(p)_Q := \bigcap_{i=1}^n \text{Ker} \left( e_0^{-\alpha_i} : V_L \to V_L - \alpha_i \right).$$

Graded-dimensions of certain simple modules are known and they form a vector-valued modular form where the modular weights of homogeneous components ranges between zero and $|\Delta_+|$, the number of positive roots [BM]. $C_2$-cofiniteness is not proven but is believed to be true and the graded-dimensions are of course an indication for this.
Similar to the rank one case a close relation of the representation category of these VOAs and those of restricted quantum groups of type $\mathfrak{g}$ at 2pth root of unity is expected. As in the rank one case this will not be a braided equivalence. Since it seems much easier to study the quantum groups it is an instructive task to understand for example what happens in the case of $\mathfrak{g} = \mathfrak{sl}_3$ and compare it to $W(p)_{A_2}$.

3.4.2. Logarithmic parafermion or Z-algebras. Let $C_k$ be the coset VOA $C_k = \text{Com}(\mathcal{H}, L_k(\mathfrak{g}))$, where $L_k(\mathfrak{g})$ is the simple affine VOA of the simple Lie algebra $\mathfrak{g}$ at level $k$ and $\mathcal{H}$ is the Heisenberg sub algebra of rank equal to that of $\mathfrak{g}$. By a parafermion VOA we mean any VOA extension of $C_k$.

Parafermion algebras [FZ] have first been introduced by Lepowsky and Wilson under the name Z-algebras [LW]. They have been studied in the rational case, that is as Heisenberg cosets of rational affine VOAs. Under the assumption that the representation category of the parafermion VOA of non-integer admissible level is a vertex tensor category in the sense of [HLZ] a great deal of the representation theory of the parafermion algebra follows from the one of its affine parent theory [CKLR].

Let $\mathfrak{g}$ be a simple Lie algebra, then the WZW theory of $\mathfrak{g}$ at admissible rational negative level is surely not $C_2$-cofinite. However the space of highest weight vectors for the Heisenberg sub VOA corresponding to the Cartan subalgebra is an abelian intertwining algebra whose fusion corresponds to addition in the root lattice of $\mathfrak{g}$. Restricting to a sublattice with the property that all involved Heisenberg highest-weight vectors have integral conformal dimension for the Virasoro field of the Heisenberg VOA gives then a VOA (under the vertex tensor category assumption). The outcome of [CKLR] is that simple modules of this VOA are rare and we strongly believe that these VOAs furnish new strongly-finite logarithmic examples. In the case of $\mathfrak{g} = \mathfrak{sl}_2$ this indeed works on the level of graded-dimensions [ACR]. There, we have no access to open Hopf links so that its use as a further testing ground for our ideas is limited but will nonetheless be pursued.

Acknowledgments

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